Asymptotic properties and classification of bistable fronts with Lipschitz level sets

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Abstract. In this paper we study solutions to reaction-diffusion equations in the bistable case, defined on the whole space in dimension N. The existence of solutions with cylindric symmetry is already known. Here we prove the uniqueness of these cylindric solutions whose level sets are curved Lipschitz graphs. Using a centre manifold-like argument, we also give the precise asymptotics of these level sets at infinity. In dimenion 2, we classify all solutions under weak conditions at infinity. Finally, we also provide an alternative proof of the existence of these solutions in dimension 2, based on a continuation argument.

1 Introduction and main results

The purpose of this paper is the study of classical bounded solutions of the following elliptic equation:

(1.1) $\Delta u - c\partial_y u + f(u) = 0$ in $\mathbb{R}^N = \{z = (x, y),\ x = (x_1, \cdots, x_{N-1}) \in \mathbb{R}^{N-1},\ y \in \mathbb{R}\}$.

The function f is of class C^2 and it is assumed to be of the 'bistable' type. Namely, there exists $\theta \in (0,1)$ such that

$$\begin{cases} f(0) = f(\theta) = f(1) = 0,\ f < 0 \text{ on } (0,\theta) \cup (1, +\infty),\ f > 0 \text{ on } (-\infty, 0) \cup (\theta, 1), \\ f'(0) < 0,\ f'(1) < 1,\ f'(\theta) > 0. \end{cases}$$

Assume $\displaystyle\int_0^1 f > 0$. It is well-known that there is a unique $c_0 > 0$ such that the ordinary differential equation

$$U'' - c_0 U' + f(U) = 0 \text{ in } \mathbb{R}, \quad U(-\infty) = 0,\ U(+\infty) = 1 \tag{1.2}$$

has a solution ([4], [9]). Moreover the profile U is unique (namely, U is unique up to translation).

We deal with solutions of (1.1) converging to 1 and 0 respectively as $y \to +\infty$ and $-\infty$, uniformly away from a Lipschitz graph in the direction y. This problem is the natural extension of (1.2) in higher dimensions and it can also be interpreted in terms of geometrical movements [1], [7].

More precisely, we are interested here in solutions u with cylindric symmetry, i.e. such that $u(x,y) = \tilde{u}(|x|,y)$, and satisfying the following condition at infinity: there exists a globally Lipschitz-continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}$ such that

$$(1.3) \qquad \left\{ \begin{array}{rcl} \displaystyle\limsup_{A\to+\infty,\ y\geq A+\phi(|x|)} |u(x,y)-1| & = & 0, \\ \displaystyle\limsup_{A\to-\infty,\ y\leq A+\phi(|x|)} |u(x,y)| & = & 0. \end{array} \right.$$

The notation $|x|$ stands for the euclidean norm of x. We also note $\hat{x} = x/|x|$ for $x \neq 0$.

In [12], for given $N \geq 2$ and $\alpha \in (0, \pi/2]$, we proved the existence of solutions $(c,u) = (c_0/\sin\alpha, u)$ of (1.1) satisfying the following properties:

(P1) $0 < u < 1$ *in* $\mathbb{R}^N$,

(P2) $u(x,y) = \tilde{u}(|x|,y)$, $\partial_{|x|}\tilde{u} \geq 0$, $\partial_y u > 0$,

(P3) *the function* u *satisfies* (1.3) *with* $\phi(|x|) = \phi_\lambda(x)$, *for all* $\lambda \in (0,1)$, *where* $\{u(x,y) = \lambda\} = \{y = \phi_\lambda(x),\ x \in \mathbb{R}^{N-1}\}$,

(P4) $\hat{x}\cdot\nabla\phi_\lambda(x) \to -\cot\alpha$ *as* $|x| \to +\infty$,

(P5) *the function* u *is decreasing in any unit direction* $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1}\times\mathbb{R}$ *such that* $\tau_y < -\cos\alpha$,

(P6) *for any unit direction* $e \in \mathbb{R}^{N-1}$, *for any sequence* $r_n \to +\infty$ *and for any* $\lambda \in (0,1)$, $u(x + r_n e, y + \phi_\lambda(r_n e)) \to U((x\cdot e)\cos\alpha + y\sin\alpha + U^{-1}(\lambda))$ *in* $C^2_{loc}(\mathbb{R}^N)$,

(P7) *in dimension* $N = 2$, *we can choose* $\phi(|x|) = -|x|\cot\alpha$ *in* (1.3), *namely* u *satisfies*

$$(1.4) \qquad \left\{ \begin{array}{rcl} \displaystyle\limsup_{A\to+\infty,\ y\geq A-|x|\cot\alpha} |u(x,y)-1| & = & 0, \\ \displaystyle\limsup_{A\to-\infty,\ y\leq A-|x|\cot\alpha} |u(x,y)| & = & 0, \end{array} \right.$$

and we can shift u *so that* $u(x + x_n, y - |x_n|\cot\alpha) \to U(\pm x\cos\alpha + y\sin\alpha)$ *in* $C^2_{loc}(\mathbb{R}^2)$, *for any sequence* $x_n \to \pm\infty$.

In this paper, we first make explicit the asymptotic behaviour of the level sets of any solution u satisfying properties (P1)-(P6) above, and we distinguish the case $N = 2$ from the case $N \geq 3$. With these new results, we then prove some uniqueness and classification results in dimensions 2 and higher.

Theorem 1.1 (Asymptotics in dimension $N = 2$) *Let* $N = 2$, $\alpha \in (0, \pi/2)$ *and* $u(x,y) = \tilde{u}(|x|,y)$ *be a solution of* (1.1) *with* $c = c_0/\sin\alpha$ *and satisfying properties* (P1)-(P6) *above.*

Then there is exponential convergence of $u(x,y)$ *to the planar fronts* $U(\pm x\cos\alpha + y\sin\alpha)$ *in the directions* $(\pm\sin\alpha, -\cos\alpha)$; *moreover the slopes of the level lines of* u *converge exponentially, in the same directions, to* $\mp\cot\alpha$. *More precisely,* u *satisfies* (P7) *and if we set*

$$(1.5) \qquad X = x\sin\alpha - y\cos\alpha, \quad Y = x\cos\alpha + y\sin\alpha$$

and still denote $u(x,y)$ by $u(X,Y)$ with an obvious abuse of notations, then the level line $\{u(X,Y)=a\}$ is described in the half-plane $\{x\geq 0\}$, by the equation $\{Y=\psi_a(X)\}$, and there is $\omega=\omega(\alpha,f)>0$ such that, for all $a\in(0,1)$ and $X>0$,

$$|\psi_a'(X)|\leq C_a e^{-\omega|X|} \tag{1.6}$$

for some constant $C_a=C_a(a,\alpha,f,u)$. Also, for all Y such that the point $(X,Y+\psi_a(X))$ is in the half-plane $\{x>0\}$ then

$$|u(X,Y+\psi_a(X))-U(Y+U^{-1}(a))|\leq C_a e^{-\omega|X|}.$$

The constant C_a degrades as a converges to 0 or 1.

Theorem 1.2 (Asymptotics in dimension $N\geq 3$) *Let $N\geq 3$, $\alpha\in(0,\pi/2)$ and $u(x,y)=\tilde{u}(|x|,y)$ be a solution of* (1.1) *with $c=c_0/\sin\alpha$ and satisfying the properties* (P1)-(P6) *above.*

Then the the slopes of the level lines of u converge to $-\cot\alpha$ like $|x|^{-1}$ as $|x|\to+\infty$. More precisely, if we set

$$X=|x|\sin\alpha-y\cos\alpha,\quad Y=|x|\cos\alpha+y\sin\alpha$$

and still denote $\tilde{u}(|x|,y)$ by $u(X,Y)$ with an obvious abuse of notations, then the level surface $\{u=a\}$ is described by the equation $\{Y=\psi_a(X)\}$, and there is $k=k(N,\alpha)>0$ such that, for all $a\in(0,1)$ and $X>0$,

$$\left|\psi_a'(X)-\frac{k}{X}\right|\leq\frac{C_a}{X^2} \tag{1.7}$$

and

$$|u(X,Y+\psi_a(X))-U(Y+U^{-1}(a))|\leq\frac{C_a}{X}.$$

The constant $C_a=C_a(a,\alpha,f,u)$ degrades as a converges to 0 or 1.

These two asymptotics are obtained by a centre manifold-like argument.

As a consequence of these precise asymptotics we are able to prove the uniqueness (up to translations in y) of the cylindrical solutions satisfying properties (P1)-(P6) above. Actually, we can get uniqueness results under more general and weaker assumptions, like:

Hypothesis 1.3 There exists a globally Lipschitz function ϕ defined in $\mathbb{R}^+$ such that

$$\begin{cases}\displaystyle\liminf_{A\to+\infty,\ y\geq A+\phi(|x|)} u(x,y)>\theta,\\ \displaystyle\limsup_{A\to-\infty,\ y\leq A+\phi(|x|)} u(x,y)<\theta,\end{cases} \tag{1.8}$$

or

Hypothesis 1.4 The speed c is nonnegative, $\inf_{\mathbb{R}^N} u<\theta$, $\partial_y u\geq 0$ in $\mathbb{R}^N$, $u(x,y)=\tilde{u}(|x|,y)$ and $\partial_{|x|}\tilde{u}(|x|,y)\geq 0$ for all $(x,y)\in\mathbb{R}^{N-1}\times\mathbb{R}$.

In dimension $N\geq 2$ with cylindrical symmetry we prove the following uniqueness result:

Theorem 1.5 (Uniqueness in dimension $N \geq 3$) *Let $N \geq 3$ and let $u(x,y) = \tilde{u}(|x|, y)$ be a bounded nonconstant solution of* (1.1)*, with some speed c, and assume that Hypotheses 1.3 or 1.4 are satisfied. Then $c \geq c_0$. Furthermore, up to shift in y variable, u is the solution mentionned above and satisfying properties* (P1)-(P6) *for $\alpha = \arcsin(c_0/c) \in (0, \pi/2]$.*

In dimension $N = 2$, we can get a stronger classification result, without assuming symmetry, and only under the following assumption:

Hypothesis 1.6 There exists a globally Lipschitz function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\left\{ \begin{array}{l} \liminf\limits_{A \to +\infty,\ y \geq A + \phi(x)} u(x,y) > \theta, \\ \limsup\limits_{A \to -\infty,\ y \leq A + \phi(x)} u(x,y) < \theta. \end{array} \right. \tag{1.9}$$

Then we have

Theorem 1.7 (Classification and uniqueness in dimension $N = 2$) *Let $N = 2$ and let u be a bounded nonconstant solution of* (1.1)*, with some speed c, and assume that Hypothesis 1.6 is satisfied. Then $c \geq c_0$. Furthermore, up to shift in (x,y) variables, either u is a planar front $U(\pm x \cos\alpha + y \sin\alpha)$ with $\alpha = \arcsin(c_0/c) \in (0, \pi/2]$, or u is the unique solution of* (1.1) *satisfying properties* (P1)-(P7) *above.*

Notice that an immediate consequence of Theorems 1.5 and 1.7 is the non-existence of solutions of (1.1) satisfying (P1)-(P6) with an angle $\alpha \in (\pi/2, \pi)$ (see also Remark 1.7 in [12]). Also, notice that to look for solutions of equations of the type (1.1) with the additional constraint of having globally Lipschitz level sets is a rather natural one: other solutions to (1.1) that do not satisfy this assumption may exist; this will be studied in the forthcoming paper [6]. Also note that this condition already appears in Barlow, Bass, Gui [2]: in this paper, in addition to a proof of the de Giorgi Conjecture in spatial dimension 2, there is a Liouville theorem for the solutions of the Allen-Cahn equation in all space dimensions; it is commonly conjectured - although not at all proved yet - that nontrivial, truly multidimensionnal solutions to this equation may exist when the spatial dimension becomes large.

The plan of the paper is the following. In Section 2 we give in the explicit convergence rates along the level sets of the solutions: in the 2D case, the convergence is exponential, one of the consequences being the exponential convergence of the level sets to straight lines. In space dimensions $N \geq 3$ with cylindrical symmetry, the level lines differ from straight lines by a logarithmic function of x: thus the behaviour is radically different from the 2D case. The estimates are sufficient to allow the initialization of the sliding method, thus leading to the uniqueness Theorem 1.5, and to the classification Theorem 1.7 which are proved in Section 3. In this particular case, we are able to classify all the solutions of (1.1) whose level lines are Lipschitz graphs. In Section 4 we give an alternative proof of the existence of solutions of (1.1) satisfying properties (P1)-(P7) above in dimension $N = 2$. The proof is based on a continuation method with respect to the angle α. In doing so, we use some results on the linearized equation around a wave solution, that had also been discovered in [11]. In the appendix (Section 5) we mention without proofs, some comparison principles useful in dimension $N \geq 2$, which are easy adaptations of some results in [10] and [12].

2 Behaviour of the level sets at infinity

We prove Theorem 1.1 in dimension $N = 2$ in the first subsection, and Theorem 1.2 in dimension $N \geq 3$ in the second subsection.

2.1 Level sets in dimension 2

Here we deal with the case $N = 2$. Theorem 1.1 will not only help us to conclude to the uniqueness of nontrivial solutions of (1.1) up to shifts; it will also be of use in the forthcoming continuation argument of Section 4.

Proof of Theorem 1.1. Define the rotated variables (X, Y) just as in Theorem 1.1. It is enough to prove the desired result in the direction $e_X = (\sin\alpha, -\cos\alpha)$. We will note $e_Y = (\cos\alpha, \sin\alpha)$. The proof is really a centre manifold computation, the time variable being here the variable X; it includes some preparation steps that will transform the equation in an evolution problem in the variable X; then we will apply a centre manifold-like argument.

Step 1. Choose $a \in (0, 1)$ once and for all, and translate the planar front U in order to have $U^{-1}(a) = 0$. We may start with the following statements, which are consequences of properties (P1)-(P6) stated in Section 1:

$$\lim_{X\to+\infty} \psi_a'(X) = 0, \tag{2.1}$$

and

$$\lim_{X\to+\infty} u(X, Y + \psi_a(X)) = U(Y) \ \ \text{uniformly in } \{x \geq 0, Y \in \mathbb{R}\}. \tag{2.2}$$

Furthermore, the uniform limits also hold for the derivatives in Y up to the second order. With an abuse of notations, we use the same name u for the function u in both variables (x, y) and (X, Y).

Step 2: reduction to an evolution problem. Let $\gamma(x)$ be a smooth function satisfying

$$\gamma \in C^\infty(\mathbb{R}), \quad \gamma'(x) \geq 0, \quad \gamma(x) = 0 \quad \text{in} \quad (-\infty, 1], \quad \gamma(x) = 1 \quad \text{in} \quad [2, +\infty).$$

For any $X_0 > 0$ large enough (to be chosen later), let us define the function

$$\check{u}(x, y) = (1 - \gamma(X - X_0))U(Y - \psi_a(X_0)) + \gamma(X - X_0)\gamma(x)u(x, y) \tag{2.3}$$

in $\{X \geq 0\}$. With an abuse of notations, we will denote $\check{u}(x, y)$ by $\check{u}(X, Y)$. Then we define

$$\hat{u}(X, Y) = \check{u}(X + X_0, Y + \psi_a(X_0)).$$

This function $\hat{u}$ satisfies

$$\hat{u}(X, Y) = U(Y - \psi_a(X_0)) \quad \text{for} \quad 0 \leq X \leq 1, \quad \text{and} \quad \hat{u}_Y \geq 0 \quad \text{in} \quad \{X \geq 0\}. \tag{2.4}$$

The monotonicity property follows from the definition of $\hat{u}$ and the fact that the function u is nondecreasing in both directions x and y in the half-plane $\{x \geq 0\}$.

We obtain the following partial differential equation for the function $\hat{u}$:

$$\Delta \hat{u} + c\cos\alpha\ \hat{u}_X - c\sin\alpha\ \hat{u}_Y + f(\hat{u}) = g(X,Y) \quad (X>0,\ Y\in\mathbb{R}) \tag{2.5}$$

where

$$\begin{aligned} g(X-X_0, Y-\psi_a(X_0)) \;=\; & f(\Gamma_1 U + \Gamma_2 u) - \Gamma_1 f(U) - \Gamma_2 f(u) \\ & +(\Delta\Gamma_1)U + 2\nabla\Gamma_1\cdot\nabla U + U\ \ (-c_0 e_Y + c_0\cot\alpha\ e_X)\cdot\nabla\Gamma_1 \\ & +(\Delta\Gamma_2)u + 2\nabla\Gamma_2\cdot\nabla u + u\ \ (-c_0 e_Y + c_0\cot\alpha\ e_X)\cdot\nabla\Gamma_2 \end{aligned}$$

and $U = U(Y-\psi_a(X_0))$, $u = u(x,y)$, $\Gamma_1 = 1-\gamma(X-X_0)$, $\Gamma_2 = \gamma(X-X_0)\gamma(x)$.

In the set $\{X\geq 2\}$, the support of g is contained in the strip $S = \{1\leq x\leq 2\}\cap\{X\geq 2\}$. On the other hand, the function $\overline{u}(x,y) = U((y-\phi(x))\sin\alpha)$ with

$$\phi(x) = -\frac{1}{c_0\sin\alpha}\ln\left(2\cosh\left(xc_0\cos\alpha\right)\right)$$

is a supersolution in $\{\overline{u}<\theta\}$. Comparing $\overline{u}$ and u, and using the fact that u is non-increasing in any unit direction $\tau = (\tau_x,\tau_y)$ such that $\tau_y < -\cos\alpha$, we get that for $\{u=\theta\} = \{y = \phi_\theta(x)\}$:

$$u(x,y)\leq U\left(\left(y-\phi_\theta(x) + \frac{\ln 2}{c_0\sin\alpha}\right)\sin\alpha\right) \quad \text{in} \quad \{u<\theta\}.$$

But we know that there exists a constant $C>0$ such that

$$U(y)\leq Ce^{\mu y} \quad \text{in} \quad \{U<\theta\} \quad \text{for} \quad \mu = \frac{c_0+\sqrt{c_0^2-4f'(0)}}{2} > c_0$$

and then for some constant $C'>0$ we have

$$u(x,y)\leq C' e^{(\mu\sin\alpha)(y-\phi_\theta(x))} \quad \text{in} \quad \{u<\theta\}. \tag{2.6}$$

Then from (2.2) and (2.6), we deduce that for $\omega_0 = \mu\tan\alpha$ and for every $\varepsilon>0$, there exists X_0 large enough such that

$$|g(X,Y)| + |\nabla g(X,Y)| \leq \varepsilon e^{-\omega_0 X} \quad \text{in} \quad \{X\geq 0\}, \quad \text{and} \quad g(0,\cdot)=0.$$

Step 3: choosing a level curve. This is directly inspired from Fife-McLeod [9]. Define, for every couple $(X,\psi)\in\mathbb{R}_+\times\mathbb{R}$, the function

$$J(X,\psi) = \int_{\mathbb{R}} e^{-c_0 Y}(\hat{u}(X,Y+\psi) - U(Y))U'(Y)dY.$$

This integral is well defined because of the asymptotic behaviour of $U'(Y)$ in $-\infty$ ($U'(Y)\sim \kappa e^{\mu Y}$ as $Y\to-\infty$, with $\mu>c_0>0$ and $\kappa>0$). For every $X\geq 0$ we wish to find a zero of $J(X,.)$ in a neighbourhood of $\psi_a(X+X_0)-\psi_a(X_0)$. We have, for every $X\geq 0$:

$$\partial_\psi J(X,\psi) = \int_{\mathbb{R}} e^{-c_0 Y}\hat{u}_Y(X,Y+\psi)U'(Y)\ dY,$$

a strictly positive quantity because of (2.4). Moreover we have $0<\pm J(X,\pm\infty)<\infty$; hence $J(X,.)$ has a unique zero: call it $\psi(X)$. On the other hand,

$$\lim_{X\to+\infty} J(X,\psi_a(X+X_0)-\psi_a(X_0)) = 0 \text{ and } \liminf_{X\to+\infty}\partial_\psi J(X,\psi_a(X+X_0)-\psi_a(X_0)) > 0$$

because of (2.2). Therefore,

$$\lim_{X\to+\infty} |\psi(X) - (\psi_a(X+X_0) - \psi_a(X_0))| = 0. \tag{2.7}$$

Setting

$$v(X,Y) = \hat{u}(X, Y+\psi(X)) - U(Y), \tag{2.8}$$

we have:

$$\int_{\mathbb{R}} e^{-c_0 Y} v(X,Y)U'(Y)\, dY := < e^*, v(X,.) >= 0, \tag{2.9}$$

which is an orthogonality relation. By the Implicit Functions Theorem, ψ is C^2.

Step 4: the centre manifold setting. Introduce the linear operator

$$L = \Delta - c_0\partial_Y + c\cos\alpha\partial_X + f'(U). \tag{2.10}$$

In what follows, only the derivative of ψ (i.e. the slope of the "level curve of u") appears; therefore set

$$\varphi(X) = \psi'(X). \tag{2.11}$$

Equation (2.5) then becomes (for $v(X,Y), \varphi(X), U(Y)$)

$$\left\{\begin{array}{rcl} Lv - (\varphi' + c\cos\alpha\ \varphi)(U'+v_Y) & = & -(U''+v_{YY})\varphi^2 + 2\varphi v_{XY} \\ & & +Q(Y,v) + g(X,Y) \ \text{ in } \mathbb{R}_+\times\mathbb{R} \\ < e^*, v(X,.) > & = & 0 \ \text{ in } \{X>0\} \\ \varphi(0) & = & 0,\ v(0,.) \ = \ 0 \\ \lim\limits_{X\to+\infty}\varphi(X) & = & \lim\limits_{X\to+\infty}\|v(X,.)\|_{L^\infty(\mathbb{R}_Y)} = 0, \end{array}\right. \tag{2.12}$$

where

$$-Q(Y,v) = f(v+U(Y)) - f(U(Y)) - vf'(U(Y))$$

satisfies $Q\in C^1(\mathbb{R}^2)$ with $Q(Y,0)=0$, $D_vQ(Y,0)=0$. As usual, equation (2.12) splits into two equations projecting on e^* and on its orthogonal.

First, for every $X>0$, we project (2.12) onto U'; hence we get

$$\begin{array}{rcl} \varphi'(X) + (c\cos\alpha + \mathcal{K}(v,\varphi))\varphi(X) & = & \dfrac{< e^*, Q(\cdot, v(X,\cdot)) + g(X,.) >}{< e^*, U'(\cdot) + v_Y(X,\cdot) >}, \\ \varphi(0) \ = \ \varphi(+\infty) & = & 0, \end{array} \tag{2.13}$$

where

$$\mathcal{K}(v,\varphi) = \frac{< e^*, -(U''(\cdot)+v_{YY}(X,\cdot))\varphi(X) + 2v_{XY}(X,\cdot) >}{< e^*, U'(\cdot) + v_Y(X,\cdot) >}$$

is well-defined if X_0 is chosen large enough in Step 2.

We define the projection π by

$$\pi w = w - \frac{< e^*, w >}{< e^*, U' >}U'. \tag{2.14}$$

Applying π to (2.12), we get

$$(2.15)\qquad \begin{array}{rcl} Lv &=& -\varphi^2\pi(U''+v_{YY})+2\varphi\pi v_{XY}+\pi Q(Y,v)+\pi g+\mathcal{M}(v,\varphi)\pi v_Y,\\ v(0,.) &=& v(+\infty,.)\ =\ 0. \end{array}$$

with

$$-\mathcal{M}(v,\varphi)=\frac{< e^*,-(U''+v_{YY})\varphi^2+2v_{XY}\varphi+Q(Y,v)+g(X,.)>}{< e^*,U'+v_Y>}.$$

Our goal is to prove that (2.13), (2.15), with unknown (φ,v), has a unique solution. To this end we will apply the Implicit Functions Theorem, and the cornerstone of the argument is the following step.

Step 5. Properties of the operator L. For any β chosen in $(0,1)$, let E denote the space

$$E=\{v\in C^\beta(\{X\geq 0\}),\ v(0,.)=v(+\infty,.)=0,\ \forall X\geq 0,\ < e^*,v(X,.)>=0\}.$$

where the space $C^\beta(\{X\geq 0\})=C^\beta(\{(X,Y)\in\mathbb{R}^2,\ X\geq 0\})$ has to be understood as the closure of smooth functions with compact support in $\{X\geq 0\}$, for the Hölder norm.

We wish to prove that L is an isomorphism from $C^{2+\beta}(\{(X,Y)\in\mathbb{R}^2,\ X\geq 0\})\cap E$. For $f\in E$, let us first prove the existence of a solution u to

$$(2.16)\qquad Lu=f,\quad f\in E.$$

Set

$$(2.17)\qquad L_{1D}=\partial_{YY}-c_0\partial_Y+f'(U);\quad E_0=\{u=u(Y)\in C^0(\mathbb{R}):< e^*,u>=0\}.$$

The operator L_{1D} is sectorial in $C^0(\mathbb{R})$ and its spectrum lies in a cone of the left complex half-plane, bounded away from 0; see Sattinger [15]. Hence we may define - see, for instance, [14], Chap. 1 - the operator

$$B=\sqrt{-L_{1D}+\frac{c_0^2\cot^2\alpha}{4}I};$$

it is an isomorphism from $D(B)\cap E_0=D(I-L_{1D}^{1/2})$ to E_0, and its spectrum lies in a cone of the right half-plane which is once again bounded away from 0. A solution to (2.16) is readily given by (for $\lambda=c_0\cot\alpha/2$ and the function $f(X')$ denoting, for notational simplicity, the function $Y\mapsto f(X',Y)$)

(2.18)

$$\begin{array}{l} u(X,\cdot)=-\dfrac{1}{2}e^{-\lambda X}B^{-1}\Big(\displaystyle\int_0^X e^{-(X-X')B}e^{\lambda X'}f(X')\,dX'+\int_X^{+\infty}e^{(X-X')B}e^{\lambda X'}f(X')\,dX'\\ \qquad\qquad -\displaystyle\int_0^{+\infty}e^{-(X+X')B}e^{\lambda X'}f(X')\,dX'\Big); \end{array}$$

and the mapping $f\mapsto u$ is continuous, due to elliptic estimates. Therefore we only have to prove the uniqueness. To this end, set $f=0$ in (2.16); let u be a solution. First, notice that we have the existence of $\beta>c_0/2$ such that

$$|u(X,Y)|\leq Ce^{-\beta|Y|};$$

this simply comes from the maximum principle applied to u in a set of the type $\{X > 0,\ |Y| \geq A\}$. If A is large enough we then have indeed $f'(U) < 0$. Now, the change of unknown $u(X,Y) = e^{\frac{c_0}{2}Y} v(X,Y)$ symmetrizes (2.17) into

$$v_{XX} + c\cos\alpha\ v_X + \tilde{L}_{1D} v = 0; \quad \tilde{L}_{1D} = \partial_{YY} + \frac{c_0^2}{4} + f'(U).$$

We have now $v(X,.) \in \tilde{E}$, where

$$\tilde{E} = \{v \in C^\beta(\{X \geq 0\}),\ v(0,.) = v(+\infty,.) = 0,\ \forall X \geq 0,\ < \tilde{e}^*, v(X,.) >= 0\}$$

and

$$\tilde{e}^*(Y) = e^{c_0 Y/2} e^*(Y).$$

The space $\tilde{E}_0$ has a similar definition as E_0 with e^* replaced by $\tilde{e}^*$. Finally, take $\mu > 0$ small and set $w(X,Y) = e^{\mu X} v(X,Y)$; the equation for w is

$$w_{XX} + (c\cos\alpha - 2\mu) w_X + (\tilde{L}_{1D} + \mu^2) w = 0. \tag{2.19}$$

The function w is in every L^p, as well as all its derivatives: we may therefore multiply (2.19) by w and integrate by parts; taking into account the existence of $k > 0$ - [15] once again - such that

$$\forall w \in \tilde{E}_0 \cap H^1(\mathbb{R}), \quad \int -\tilde{L}_{1D} w\ w\ dY \geq k \|w\|^2_{L^2(\mathbb{R})},$$

we obtain

$$\int_{X>0} \left(w_X^2 + (k - \mu^2) w^2 \right) dX dY = 0,$$

which proves $w \equiv 0$.

Step 6. Conclusion. If $h(X) \in C^\beta(\mathbb{R}_+)$ with $h(0) = h(+\infty) = 0$, then problem

$$\varphi' + c\cos\alpha\ \varphi = h, \qquad \varphi(0) = \varphi(+\infty) = 0$$

has a unique solution in $C^{1+\beta}(\mathbb{R}_+)$, and the mapping $h \longmapsto \varphi$ is continuous. As a consequence, the implicit functions theorem is applicable to (2.13), (2.15) - at this stage it is standard to check that its assumptions hold - to yield a unique small solution (φ_1, v_1) in a ball B of 0 in $C^{1+\beta}(\mathbb{R}_+) \times C^{2+\beta}(\mathbb{R}^+ \times \mathbb{R})$. Now, notice that the size of B does not depend on the translation of the origin that we have chosen, i.e. the value of X_0; on the other hand, if u is the solution of (1.1) with "level curve" $\{Y = \psi(X)\}$, Step 1 implies that $(\psi', \pi(\hat{u}(X, . + \psi(X)) - U) \in B$ as long as X_0 is large enough. Hence

$$\psi' = \varphi_1, \quad \pi(\hat{u}(X, . + \psi(X)) - U) = v_1.$$

Now, we may also construct a small solution to (2.13), (2.15) that decays exponentially in X: to this end, simply look for a solution (φ, v) under the form (with ω small enough)

$$e^{-\omega X}(\overline{\varphi}, \overline{v}).$$

The effect of this change of variables is simply to modify the coefficients of the linear part by an amount of ω, to replace Q by $e^{-\omega X} Q$, and to remove an exponential decay of order

ω to the function g. Steps 1 to 5 apply integrally, and yield a small solution $(\overline{\varphi}, \overline{v})$; then once again by uniqueness: $(\varphi_1, v_1) = e^{-\omega X}(\overline{\varphi}, \overline{v})$. Using the fact that u_y is bounded from below on the level set $\{u = a\}$, we can estimate the difference between ψ and ψ_a, which is exponential. Finally using elliptic estimates on u, we control the derivatives of u, and we get the exponential estimate on ψ_a'. The estimate for u as stated in Theorem 1.1 is then a consequence.

This ends the proof of Theorem 1.1. □

2.2 Level sets in dimension $N \geq 3$

Let us now turn to the case $N \geq 3$, with cylindrical symmetry. Theorem 1.2 shows that the behaviour is radically different, and explains why the level sets of cylindrical solutions of (1.1) are not asymptotic to a cone at infinity, but have a logarithmic behaviour. Also, it will be crucial in the uniqueness proof.

Equation (1.1) is best expressed in terms of the variables $(r = |x|, y)$; it becomes

$$\begin{aligned} \Delta u + \frac{N-2}{r} u_r - c u_y &= -f(u), \quad r > 0,\ y \in \mathbb{R} \\ \partial_r u(0, y) &= 0. \end{aligned} \tag{2.20}$$

Define the rotated variables (X, Y) as

$$\tau_r = -\cos\alpha, \quad \tau_y = -\sin\alpha. \tag{2.21}$$

Proof of Theorem 1.2. It goes along the lines of Theorem 1.1; therefore we will follow the main steps of the latter and only indicate what changes.
Step 1. In the rotated coordinates, equation (2.20) becomes

$$\Delta u + \frac{N-2}{\sin\alpha X + \cos\alpha Y}(\sin\alpha u_X + \cos\alpha u_Y) + c(-\cos\alpha u_X + \sin\alpha u_Y) + f(u) = 0. \tag{2.22}$$

The definition of $\psi_a(X)$, as well as equations (2.1) and (2.2), remain unchanged.

Step 2. The exponential decay of u in the vertical direction is still valid, as well as the exponential decays of u_r in both directions r and y. Consequently we introduce the same function $\hat{u}(r, y)$ as in equation (2.3); the equation for $\hat{u}$ has the form

$$\begin{aligned} & \Delta\hat{u} + c\cos\alpha\ \hat{u}_X - c\sin\alpha\ \hat{u}_Y \\ + \quad & \frac{N-2}{\sin\alpha(X + X_0) + \cos\alpha(Y + \psi_a(X_0))}(\sin\alpha u_X + \cos\alpha u_Y) + f(\hat{u}) = g(X, Y) \end{aligned} \tag{2.23}$$

valid for $(X > 0,\ Y \in \mathbb{R})$. The function g is modified by the addition of the curvature terms; they do not, however, modify the exponentially small character of g.

In what follows, X_0 will be assumed to be large enough so that the function $u(X, Y)$ is small.

Step 3. Identical to Step 2. We retrieve a function $\psi(X)$ satisfying (2.7) and (2.9).

Step 4. Same change of coordinates leading to expressions (2.8) and (2.10). Also we use the notation (2.11): $\varphi := \psi'$; note that

$$\psi(X) = o(X) \text{ as } X \to +\infty. \tag{2.24}$$

As will be clear in the sequel, the function ψ will be treated as a datum, whereas its derivative φ will be the real unknown. We may afford to do that because $\psi(X)$ only comes up in expressions of the form $aX+\psi(X)$, $a>0$; hence it is only a perturbation. The new system derived from (2.22) is the following:

$$
(2.25)\qquad \left\{\begin{array}{rcl}
Lv-(\varphi'+c\cos\alpha\varphi)U' & = & \dfrac{(N-2)U'}{(X+X_0)\sin\alpha+(Y\psi_a(X_0)+I\varphi(X))\cos\alpha} \\
 & & -(U''+v_{YY})\varphi^2+2\varphi v_{XY} \\
 & & +Q(X,\varphi,v)+\varepsilon g(X,Y)\ \text{ in } \mathbb{R}_+\times\mathbb{R} \\
<e^*,v(X,.)> & = & 0\ \text{ in } \{X>0\} \\
\varphi(0) & = & 0,\ v(0,.)\ =\ 0 \\
\lim\limits_{X\to+\infty}\varphi(X) & = & \lim\limits_{X\to+\infty}\|v(X,.)\|_{L^\infty(\mathbb{R}_Y)}=0,
\end{array}\right.
$$

where, in order to emphasize the fact that the main unknown is φ:

$$
I\varphi(X)=\psi(0)+\int_0^X\varphi(X')\ dX'.
$$

$$
(2.26)\qquad \begin{array}{rcl}
\|Q(X,\varphi,v)\|_{C^1(\mathbb{R}_Y)} & = & O\Bigg(\dfrac{|\varphi|+\|v\|_{C^1(\mathbb{R}_Y)}}{1+X}+|\varphi|^2+\|v\|^2_{C^1(\mathbb{R}_Y)}\Bigg) \\
\|D_{\varphi,v}Q(X,\varphi,v)\|_{C^1(\mathbb{R}_Y)} & = & O\Bigg(\dfrac{1}{1+X}+|\varphi|+\|v\|_{C^1(\mathbb{R}_Y)}\Bigg).
\end{array}
$$

Once again, equation (2.26) splits into two equations: first, for every $X>0$, the projection onto U' yields

$$
(2.27)\qquad \begin{array}{rcl}
\varphi'+(c\cos\alpha+\mathcal{K}(v,\varphi)\varphi)\varphi & = & p_\alpha(X)+\dfrac{<e^*,Q(v)+\varepsilon g(X,.)>}{<e^*,U'>} \\
\varphi(0)\ =\ \varphi(+\infty) & = & 0
\end{array}
$$

where the function $p_\alpha(X)$ has the expression

$$
(2.28)\qquad p_\alpha(X)=-\frac{(N-2)}{<e^*,U'>}\int_{\mathbb{R}}\frac{e^{-c_0Y}(U'(Y))^2}{(X+X_0)\sin\alpha+(Y+I\varphi(X))\cos\alpha}\ dY;
$$

one may check the existence of a universal constant $k_\alpha>0$ such that

$$
(2.29)\qquad p_\alpha(X)=-\frac{k_\alpha}{X}+o(\frac{1}{X})\ \text{as}\ X\to+\infty.
$$

Furthermore, $\mathcal{K}$ linear in v, C^1 in φ, and for some small $\omega>0$:

$$
(2.30)\qquad |\mathcal{K}(v,\varphi)|\leq C(\varphi^2+\int_{\mathbb{R}}e^{-\omega|Y|}|v_{YY}|\ dY).
$$

Then, if π has the expression (2.14) we have

$$
(2.31)\qquad \begin{array}{rcl}
Lv & = & \pi\left(\dfrac{(N-2)U'}{X\sin\alpha+(Y+\psi(X))\cos\alpha}\right) \\
 & & -\pi(U''+v_{YY})\varphi^2+2\varphi\pi v_{XY}+\pi Q(v)+\varepsilon\pi g,
\end{array}
$$

still with the condition $v(0,.) = v(+\infty,.) = 0$.

Step 5. Identical to the step 5 of the proof of Theorem 1.1.

Step 6. First, one proves the existence of a unique couple (φ, v) in the space of all all bounded, Hölder-continuous functions (φ, v) such that $X(\varphi(X), v(X,.))$ is also Hölder-continuous. This is done just as in Step 6 of the above proposition, and what really is of interest to us is (i) the exact decay of $\varphi(X)$, (ii) an estimate $\|v(X,.)\|_\infty$ as $O(\frac{1}{X})$. To do this, the only additional step is to set

$$\varphi(X) = -\frac{k_\alpha}{X} + \varphi_1(X),$$

the function $-\frac{k_\alpha}{X}$ being an asymptotic solution of (2.27) when $\mathcal{K}$ has been set to 0.

It only then suffices to write down system (2.27-2.31) in terms of φ_1 and v, and to argue as in the preceding Step 6, but this time in the space of functions (φ, v) behaving respectively like $o(\frac{1}{X})$ and $O(\frac{1}{X})$ as $X \to +\infty$. The details from then on are tedious, but standard and left to the reader. Simply notice that the expression (2.18) is especially suited to this purpose. $\square$

3 Uniqueness and classification

3.1 Uniqueness in dimension $N \geq 3$ with cylindrical symmetry

The goal of this subsection is to prove Theorem 1.5.

Proof of Theorem 1.5. At that point *we not only have all the benefits of Theorem 1.2,* but from Theorem 1.6 in [12] on general qualitative properties of the solutions, we know that every bounded, nonconstant and cylindrically symmetric solution $v(x,y)$ of (1.1) for some $c \in \mathbb{R}$, and satisfying moreover either Hypothesis 1.3 or 1.4 is a solution satisfying the properties (P1)-(P6) stated in Section 1 with $c \geq c_0$ and $\alpha = \arcsin(c_0/c) \in (0, \pi/2]$.

We will compare this solution v to the cylindrical solution u of (1.1) for the same angle α, satisfying properties (P1)-(P6) of Section 1: the existence of such a solution was recalled in Section 1 and proved in [12].

Let us immediately adopt the $(r = |x|, y)$ coordinates; both functions u and v are solutions of (1.1) with $c = c_0/\sin\alpha$. We wish to prove the existence of $y_0 \in \mathbb{R}$ such that $u(r,y) = v(r, y+y_0)$. To this end, we use the sliding method (see [5]). Consider a small $\theta_1 \in (0, \theta)$ such that

$$\exists \delta > 0 : \quad \forall s \in [0, \theta_1] \ \cup \ [1-\theta_1, 1] : \quad f'(s) \leq -\delta.$$

Denote by $\{(r, \phi(r)),\ r \geq 0\}$ (resp. $\{(r, \psi(r)),\ r \geq 0\}$) the level sets $\{v = \theta_1\}$ (resp. $\{v = 1-\theta_1\}$). We first claim the existence of a large $y_0 > 0$ such that

$$\forall (r,y) \in \mathbb{R}_+ \times \mathbb{R}, \quad u(r, y-y_0) \leq v(r,y). \tag{3.1}$$

Indeed, Theorem 1.2 being valid for u and v, and the constant k being universal, we have for some $y_0 \in \mathbb{R}$,

$$\forall r \in \mathbb{R}_+, \quad \phi(r) \leq y \leq \psi(r) \implies u(r, y - y_0) \leq v(r, y). \tag{3.2}$$

Lemmata 5.1 and 5.2 in the appendix are then valid and yield (3.1).

We may now consider the smallest $\underline{y}_0$ such that (3.1) holds for every $y_0 \geq \underline{y}_0$. Still from Theorem 1.2, there is $t_u \leq t_v$ such that we have the following uniform convergence results:

$$\begin{aligned} |u(r, y - \underline{y}_0) - U(r\cos\alpha + y\sin\alpha - k\log r \ + \ t_u)| &= \ O(\frac{1}{r}) \\ |v(r, y) - U(r\cos\alpha + y\sin\alpha - k\log r \ + \ t_v)| &= \ O(\frac{1}{r}). \end{aligned} \tag{3.3}$$

Two cases are to be considered.

Case 1. We have $t_u < t_v$. Then there is a small constant $d > 0$ such that we have $u(r, y - \underline{y}_0) \leq v(x, y) - d$ in $\{\phi(r) \leq y \leq \psi(r)\}$. Consequently there is a small $t_0 > 0$ such that (3.2) holds for all $y_0 \geq \underline{y}_0 - t_0$. Lemmata 5.1 and 5.2 imply that in fact (3.1) holds for all $y_0 \geq \underline{y}_0 - t_0$, contradicting the minimality of $\underline{y}_0$.

Case 2. We have $t_u = t_v$. Arguing as above, we claim the existence of a minimal $\bar{y}_0$ such that

$$\forall y_0 \geq \bar{y}_0, \ \forall (r, y) \in \mathbb{R}_+ \times \mathbb{R}, \quad u(r, y + y_0) \geq v(r, y).$$

At that point, a statement similar to (3.3) holds, but this time $t_u \leq t_v$ being replaced by two different constants $s_u \geq s_v$. If $s_u > s_v$, then argue as in Case 1 above. If $s_u = s_v$, we have by construction: $t_u = t_v = s_u = s_v$ and $\bar{y}_0 = -\underline{y}_0$. This implies in turn $u(r, y + \bar{y}_0) = v(r, y)$. □

3.2 Classification in space dimension $N = 2$

This subsection is devoted to the proof of Theorem 1.7. First, let us recall that we have the following result proved in Section 2.2 of [12]:

Proposition 3.1 ([12]) *Let v be a bounded and nonconstant solution of* (1.1) *on* $\mathbb{R}^2$ *satisfying Hypothesis 1.6. Then* $0 < v < 1$ *in* $\mathbb{R}^2$, $c \geq c_0$ *and each level set* $\{v = \lambda\}$ *(for* $\lambda \in (0,1)$*) is a graph of a globally Lipschitz function* ϕ_λ *whose Lipschitz norm is equal to* $\cot\alpha$ *with* $\alpha = \arcsin(c_0/c) \in (0, \pi/2]$. *Moreover v is decreasing in any unit direction* $(\tau_x, \tau_y) \in \mathbb{R}^2$ *such that* $\tau_y < -\cos\alpha$, *and*

$$\left\{ \begin{array}{lll} \limsup\limits_{A \to +\infty, \ y \geq A + \phi_\lambda(x)} |u(x, y) - 1| & = & 0, \\ \limsup\limits_{A \to -\infty, \ y \leq A + \phi_\lambda(x)} |u(x, y)| & = & 0. \end{array} \right.$$

for any $\phi_\lambda(x)$ *with* $\lambda \in (0,1)$ *(the same property also holds with* ϕ*).*

Proof of Theorem 1.7. Let $N = 2$, let v be a bounded nonconstant solution of (1.1) and assume that Hypothesis 1.6 is satisfied.

Proposition 3.1 applies and in particular we get that then $v(0,y) \to 0$ as $y \to -\infty$. Since v is nonincreasing in the cone of unit directions τ with $\tau_y \le -\cos\alpha$, we get that

$$\limsup_{A\to-\infty,\ y\le A-|x|\cot\alpha} v(x,y) = 0.$$

Call now $u(x,y) = u(-x,y)$ the solution of (1.1) for $N = 2$ with the same angle α, satisfying properties (P1)-(P7) stated in Section 1. The existence of such a solution is proved in [12] (see also an alternative proof in Section 4). The comparison principle Theorem 1.4 in [10] can then be applied to $\underline{u} = v$ and $\overline{u} = u$. Therefore, there exists $t_0 \in \mathbb{R}$ such that $v(x,y) \le u(x,y+t_0)$ for all $(x,y) \in \mathbb{R}^2$ and

$$\inf_{y=B-|x|\cot\alpha} (u(x,y+t_0) - v(x,y)) = 0 \tag{3.4}$$

for all $B \in \mathbb{R}$. But

$$\begin{cases} u(x, B-|x|\cot\alpha + t_0) & \to & U((B+t_0)\sin\alpha) > 0 \\ v(x, B-|x|\cot\alpha) & \to & v_\pm(B) \end{cases} \quad \text{as } x \to \pm\infty. \tag{3.5}$$

where the limits $v_\pm(B)$ exist because v is nonincreasing in both directions $(\pm\sin\alpha, -\cos\alpha)$ and satisfy $v_\pm(B) \in [0,1)$ because $v(0,B) < 1$. From now on, let us fix $B \in \mathbb{R}$, and define $v_\pm = v_\pm(B)$. According to the values of $v_\pm$, four cases may occur:

Case 1: $v_- = v_+ = 0$. It follows from (3.4) and (3.5) that $v(x_0, y_0+t_0) = u(x_0,y_0)$ for some $(x_0,y_0) \in \mathbb{R}^2$. Since both functions $v(\cdot,\cdot+t_0)$ and u are ordered and satisfy the same equation (1.1), the strong maximum principle then yields $v(x,y+t_0) = u(x,y)$ for all $(x,y) \in \mathbb{R}^2$. This is impossible because of (3.5).

Case 2: $0 < v_- < 1$ and $v_+ = 0$. Choose any real number ρ_0 and call $w(x,y) = u(x+\rho_0, y+\rho_0\cot\alpha)$. With the same arguments as previously, there exists then a real number $t = t(\rho_0)$ such that $v(x,y) \le w(x,y+t)$ for all $(x,y) \in \mathbb{R}^2$ and such that (3.4) holds, with w instead of u. Since $v(x,y) \not\equiv w(x,y+t)$ and because of the different asymptotic limits in the direction $(\sin\alpha, -\cos\alpha)$, it then follows that

$$v(x, B-|x|\cot\alpha) - w(x, B-|x|\cot\alpha + t) \to 0 \quad \text{as } x \to -\infty,$$

whence $U((B+t)\sin\alpha) = v_- = U((B+t_0)\sin\alpha)$, i.e. $t = t_0$.

As a consequence, $t = t_0$ does not depend on ρ_0 and

$$v(x,y) \le u(x+\rho_0, y+\rho_0\cot\alpha + t_0)$$

for all $(x,y) \in \mathbb{R}^2$ and $\rho_0 \in \mathbb{R}$. Passing to the limit as $\rho_0 \to -\infty$ implies, that $v(x,y) \le U(-x\cos\alpha + y\sin\alpha)$ for all $(x,y) \in \mathbb{R}^2$.

On the other hand, since u is nonincreasing in the direction $(-\sin\alpha, -\cos\alpha)$, one gets that

$$v(x,y) \ge \lim_{r_n\to-\infty} v(x+r_n, y-|r_n|\cot\alpha) = V(-x\cos\alpha + y\sin\alpha).$$

Because V satisfies $V'' - c_0V' + f(V) = 0$, $V(+\infty) = 1, V(-\infty) = 0$, we deduce that $V(-x\cos\alpha + y\sin\alpha) = U(-x\cos\alpha + y\sin\alpha + t_-)$ for some $t_- \in \mathbb{R}$. We then redefine the coordinates $\tilde{Y} = -x\cos\alpha + y\sin\alpha$ and $\tilde{X} = -x\sin\alpha - y\cos\alpha$, and denoting $v(x,y)$ by $v(\tilde{X},\tilde{Y})$ (with a slight abuse of notations), we remark that v satisfies

$$\begin{cases} \limsup_{A\to+\infty,\ \tilde{Y}\ge A} |v(\tilde{X},\tilde{Y}) - 1| & = & 0, \\ \limsup_{A\to-\infty,\ \tilde{Y}\le A} |v(\tilde{X},\tilde{Y})| & = & 0 \end{cases}$$

and v satisfies in coordinates $(\tilde{X}, \tilde{Y})$ an equation similar to (1.1) with different first order terms. Then from Theorem 2 in [3], we get that $v(\tilde{X}, \tilde{Y})$ only depends on $\tilde{Y}$, and then one concludes that

$$\forall (x, y) \in \mathbb{R}^2, \quad v(x, y) = U(-x \cos \alpha + y \sin \alpha + t_-).$$

Case 3: $v_- = 0$, $0 < v_+ < 1$. The same arguments as in Case 2 yield the existence of $t_+ \in \mathbb{R}$ such that $v(x, y) = U(x \cos \alpha + y \sin \alpha + t_+)$ for all $(x, y) \in \mathbb{R}^2$ and some $t_+ \in \mathbb{R}$.

Case 4: $0 < v_\pm < 1$. It then follows that

$$\sup_{x \in \mathbb{R}} |\phi_\lambda(x) + |x| \cot \alpha| < \infty$$

for all $\lambda \in (0, 1)$. Hence from Proposition 3.1, the function v satisfies (1.3) with $\phi(x) = -|x| \cot \alpha$, namely v satisfies (1.4). Then from Theorem 1.1 of [12], one concludes that v is unique, and then equal (up to a shift) to the solution u.

That completes the proof of Theorem 1.7. □

4 An existence result in dimension $N = 2$ via a continuation method

In this section, we give an alternative proof of the existence of solutions of (1.1) satisfying properties (P1)-(P7) stated in Section 1, in dimension $N = 2$.

Proposition 4.1 *There exists $\varepsilon > 0$ such that, for all $\alpha \in [\pi/2 - \varepsilon, \pi/2]$, problem* (1.1), (1.4) *has a solution (c, u), where $c = c_0/\sin \alpha$ and u satisfies all properties* (P1)-(P7) *stated in Section 1.*

Proof. This is one of the simplest cases of a paper by Haragus and Scheel [13]; see also Fife [8]. Let us give a brief account of what happens: Theorem 1 of [13] yields, for $\alpha < \pi/2$ close enough to $\pi/2$, the existence of a solution $u(x, y) = \tilde{u}(|x|, y)$ of (1.1) of the form

$$u(x, y) = U(y + \xi(x)) + v(x, y), \quad \|\xi'\|_\infty + \|v\|_{L^\infty \cap H^1} = O(\frac{\pi}{2} - \alpha), \quad \lim_{x \to \pm\infty} \xi'(x) = \mp \cot \alpha.$$

From the results in Section 2 in [12] and from Theorem 1.1, one has $c = c_0/\sin \alpha$ and u satisfies all properties (P1)-(P7) stated in Section 1. Proposition 2.1 in [12] implies that u is therefore the unique (up to shift) solution of (1.1), (1.4) with the reference angle α.□

Proposition 4.2 *Let $\alpha_* \in (0, \pi/2)$ be given and assume that there exists a solution $(c_0/\sin \alpha_*, u)$ of* (1.1), (1.4), *satisfying properties* (P1)-(P7) *stated in Section 1 with angle α^*. Then there exists $\delta_0 > 0$ such that, for all $\alpha \in [\alpha_* - \delta_0, \alpha_* + \delta_0]$, problem* (1.1), (1.4) *has a solution $(c, u) = (c_0/\sin \alpha, u)$ satisfying properties* (P1)-(P7).

Proof. For the sake of simplicity, set

$$NL_\alpha(u) = \Delta u - \frac{c_0}{\sin \alpha} \partial_y u + f(u).$$

Let us give ourselves some $\alpha_* \in (0, \pi/2)$ for which there exists a nontrivial solution $u_*(x, y)$ of $NL_{\alpha_*}(u) = 0$ with the conical boundary conditions (1.4) at $\alpha = \alpha_*$. Let us consider a function $h(x)$, smooth and nondecreasing, such that $h(0) = 0$, $h \equiv -1$ in $(-\infty, -1]$ and $h \equiv 1$ in $[1, +\infty)$. Consider some small $|\delta|$ with $\delta \in \mathbb{R}$.

We define the transformation $T = (T_1, T_2)$ of the plane $\mathbb{R}^2$ as follows:

$$\begin{array}{rcrcl} T_1(x,y) & = & x\cos\delta & + & yh(x)\sin\delta \\ T_2(x,y) & = & -xh(x)\sin\delta & + & y\cos\delta \end{array}$$

and we look for a solution under the form

$$u(x,y) = u_*(T(x,y)) + v(x,y) =: \tilde{u}(x,y) + v(x,y) \quad \text{with } |v(x,y)| = O(e^{-\omega|(x,y)|})$$

for some small $\omega > 0$. Notice that the function $\tilde{u}$ has the right conical conditions (1.4) with angle $\alpha_* + \delta$ at infinity: the transformation T brings the level lines of asymptotic slopes $\pm\cot(\alpha_* + \delta)$ to level lines of asymptotic slope $\pm\cot\alpha_*$.

Let us evaluate $NL_{\alpha_*+\delta}(\tilde{u})$; for that we consider the cases $0 \le x \le 1$ and $x > 1$; the remaining cases following from symmetry of u_* with respect to the y axis.

1. $0 \le x \le 1$. By elementary algebra, we have

$$\begin{aligned} NL_{\alpha_*+\delta}\tilde{u} = \ & \delta\ [k_{1,1}(x,y)\partial^2_{xx}u_* + k_{1,2}(x,y)\partial^2_{xy}u_* + k_{2,2}(x,y)\partial^2_{yy}u_* \\ & +k_1(x,y)\partial_x u_* + k_2(x,y)\partial_y u_*], \end{aligned}$$

where $u_* = u_*(T(x,y))$, k_{ij} and k_i belong to $C^1([0,1] \times \mathbb{R}, \mathbb{R})$ and satisfy $|k_{ij}(x,y)|$, $|k_i(x,y)| \le C_1 + C_2|y|$ in $[0,1] \times \mathbb{R}$ for some constants C_1 and C_2. From standard elliptic estimates and Lemmata 2.16 and 2.17 in [12], the following exponential bounds hold:

$$|NL_{\alpha_*+\delta}\tilde{u}(x,y)| \le C|\delta|e^{-\omega|y|} \le C'|\delta|e^{-2\omega|(x,y)|} \quad \text{in } [0,1] \times \mathbb{R} \tag{4.1}$$

for some small $\omega > 0$ and for some constants C and C'.

2. $x > 1$. Then $h(x) = 1$ and it is straightforward to check that

$$\begin{aligned} NL_{\alpha_*+\delta}\tilde{u} &= -\frac{c_0 \sin\delta}{\sin\alpha_* \sin(\alpha_* + \delta)}(\sin\alpha_*, -\cos\alpha_*) \cdot \nabla u_*(T(x,y)) \\ &= -\frac{c_0 \sin\delta}{\sin\alpha_* \sin(\alpha_* + \delta)}\partial_X u_*(T(x,y)), \end{aligned}$$

where the rotated variables (X, Y) are defined as in (1.5) with α_*. It follows from Theorem 1.1 that, for a possibly smaller $\omega > 0$:

$$|NL_{\alpha_*+\delta}\tilde{u}| \le |\delta|Ce^{-2\omega|X|} \le C'|\delta|e^{-2\omega|(x,y)|}$$

as long as X remains outside a cone of axis the Y axis with small aperture. However, for such a cone, estimate (4.1) is valid because of Lemmata 2.16 and 2.17 in [12].

Let L be the linearized operator around u_*:

$$L = \Delta - c\partial_y + f'(u_*) = \Delta - \frac{c_0}{\sin\alpha_*}\partial_y + f'(u_*).$$

Equation $NL_{\alpha_*+\delta}(u) = 0$ with the conical asymptotic conditions (1.4) at $\alpha = \alpha_* + \delta$ reduces to

$$Lv = q(x,y,v)v^2 + \delta f_*(x,y), \qquad v(x,y) = O(e^{-\omega|(x,y)|}) \text{ as } |(x,y)| \to +\infty. \tag{4.2}$$

Here the function q is C^1, bounded as well as its derivatives on $\mathbb{R}^2 \times [-1,1]$ and the function f_* belongs to the space

$$E = \{w \in BUC(\mathbb{R}^2), \quad e^{\omega|(x,y)|}w \in BUC(\mathbb{R}^2)\},$$

where BUC stands for the set of bounded uniformly continuous functions. From the arguments used in [11], L is an isomorphism from its domain $D(L)\cap E$ to E. This allows the application of the implicit functions theorem to equation (4.2). Hence, there exists $\delta_0 \in (0,\pi/2-\alpha_*)$ such that for all $|\delta| \le \delta_0$, there is a solution u of $NL_{\alpha_*+\delta}(u) = 0$ with the conical conditions (1.4) with angle $\alpha = \alpha_* + \delta$ (this solution then satisfies all properties (P1)-(P7) stated in Section 1, from Proposition 2.1 in [12]). □

Proposition 4.3 *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\pi/2)$ such that $\alpha_n \to \alpha \in (0,\pi/2)$ as $n \to +\infty$. Assume that, for each $n \in \mathbb{N}$, problem* (1.1), (1.4) *has a solution $(c_0/\sin\alpha_n, u_n)$ satisfying all properties* (P1)-(P7) *stated in Section 1 with angle α_n. Then problem* (1.1), (1.4) *has a solution $(c_0/\sin\alpha, u)$ satisfying properties* (P1)-(P7) *with angle α.*

Proof. Remember that each function u_n is even in x, and is such that $\partial_x u_n(x,y) > 0$ for all $x > 0$ and $y \in \mathbb{R}$. Furthermore, each u_n is decreasing in any unit direction (τ_x,τ_y) such that $\tau_y < -\cos\alpha_n$. One can also assume up to shift in y that $u_n(0,0) = \theta/2$.

Up to extraction of some subsequence, the functions u_n converge in $C^2_{loc}(\mathbb{R}^2)$ to a solution $0 \le u \le 1$ of (1.1) with the speed $c = c_0/\sin\alpha$. Furthermore, $u(0,0) = \theta/2$ (the strong maximum principle then implies that $0 < u < 1$ in $\mathbb{R}^2$), u is even in x, $\partial_x u(x,y) \ge 0$ for all $y \in \mathbb{R}$ and $x \ge 0$, and u is nonincreasing in any unit direction (τ_x,τ_y) such that $\tau_y \le -\cos\alpha$. It then follows that Hypothesis 1.4 is satisfied and Theorems 1.6 in [12] and 1.1 above imply that u solves problem (1.1) and all properties (P1)-(P7) stated in Section 1. □

Let us now turn to the

Proof of the existence of solutions of (1.1) satisfying (P1)-(P7). Call

$$\alpha_* = \inf\{\alpha \in (0,\pi/2], \quad \text{problem (1.1), (1.4) has a solution } (c,u) \text{ for all } \alpha' \in [\alpha,\pi/2]\}.$$

From Proposition 2.1 in [12], any solution (c,u) of (1.1), (1.4) with an angle $\alpha \in (0,\pi/2]$ satisfies all properties (P1)-(P7) stated in Section 1, up to shift in variables (x,y). It then follows from Propositions 4.1, 4.2 and 4.3 that $\alpha_* = 0$. More precisely, for each $\alpha \in (0,\pi/2]$, problem (1.1), (1.4) has a solution (c,u) with $0 < u < 1$ and Proposition 2.1 in [12] implies that $c = c_0/\sin\alpha$, and that u is unique up to a shift in the (x,y) variables and satisfies all properties (P1)-(P7). □

5 Appendix: some comparison results

Trivial adaptations from dimension $N = 2$ to dimension $N \ge 2$ of Lemmata 2.6 and 2.7 in [12] (see also Lemmata 5.1 and 5.2 in [10]) allow us to get the following results:

Lemma 5.1 *Let $\underline{u}$ and $\overline{u}$ be two bounded $C^{2,\beta}$ functions (with $\beta > 0$) in the set $\overline{\Omega}$, where $\Omega = \{y > \phi(x)\}$ for some globally Lipschitz-continuous function ϕ, satisfying*

$$\Delta\underline{u} - c\partial_y\underline{u} + g(\underline{u}) \ge \Delta\overline{u} - c\partial_y\overline{u} + g(\overline{u}) \quad \text{in } \Omega,$$

$\underline{u} \le \overline{u}$ on $\partial\Omega$ and $\liminf_{A\to+\infty,\ y\ge A+\phi(x)}(\overline{u}(x,y)-\underline{u}(x,y)) \ge 0$. Furthermore, assume that there exists $\rho \in \mathbb{R}$ such that g is Lipschitz continuous on $\mathbb{R}$, nonincreasing on $[\rho,+\infty)$, and $\overline{u} \ge \rho$ in Ω. Then $\underline{u} \le \overline{u}$ in $\overline{\Omega}$.

Lemma 5.2 *Let $\underline{u}$ and $\overline{u}$ be two bounded $C^{2,\beta}$ functions (with $\beta > 0$) in the set $\overline{\Omega}$, where $\Omega = \{y < \phi(x)\}$ for some globally Lipschitz-continuous function ϕ, satisfying*

$$\Delta\underline{u} - c\partial_y\underline{u} + g(\underline{u}) \ge \Delta\overline{u} - c\partial_y\overline{u} + g(\overline{u}) \quad \text{in } \Omega,$$

$\underline{u} \le \overline{u}$ on $\partial\Omega$ and $\limsup_{A\to-\infty,\ y\le A+\phi(x)}(\underline{u}(x,y)-\overline{u}(x,y)) \le 0$. Furthermore, one assumes that there exists $\rho \in \mathbb{R}$ such that g is Lipschitz continuous on $\mathbb{R}$, nonincreasing on $(-\infty,\rho]$, and $\underline{u} \le \rho$ in Ω. Then $\underline{u} \le \overline{u}$ in $\overline{\Omega}$.

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