Traveling Waves with Paraboloid Like Interfaces for Balanced Bistable Dynamics *†

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Abstract. Cylindrically symmetric traveling waves with paraboloid like interfaces are constructed for reaction-diffusion equations with balanced bistable nonlinearities. It is shown that the interface (a level set) is asymptotically a paraboloid $z = \frac{c}{2(n-1)}|x|^2$, where $(x, z) \in \mathbb{R}^n \times \mathbb{R}$ $(n \geq 2)$ is the space variable and c is the speed that the wave travels upwards in the vertical z-direction. In the two dimensional case (i.e., $n = 1$), the interface is asymptotically a hyperbolic cosine curve $z = A\cosh(\mu x)$ for some positive constants A and μ.

1 Introduction

Consider the Allen-Cahn equation [2], for $u = u(x, z, t)$,

$$u_t = u_{zz} + \Delta u - f(u)\ , \qquad x \in \mathbb{R}^n,\ z \in \mathbb{R},\ t > 0 \tag{1.1}$$

where t is the time variable, subscripts denote partial derivatives, $(x, z) = (x_1, \cdots, x_n, z)$ is the spatial coordinates with dimension $n + 1 \geqslant 2$, and $\partial_{zz} + \Delta$ with $\Delta = \sum_{i=1}^{n} \partial_{x_i x_i}$ is the Laplacian. In the original Allen-Cahn dynamics [2], the forcing term f is the derivative of a double-equal-well, also called **balanced bistable**, potential; more precisely,

(A) $\quad f = F' \in C^2(\mathbb{R}),\ F(\pm 1) = 0 < F(s)\ \forall s \neq \pm 1,\ F''(\pm 1) > 0.$

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A typical example is the cubic function $f(u) = 4[u^3 - u]$ with potential $F = [1 - u^2]^2$.

Here the constants $\pm\mathbf{1}$ represent two **stable phase states**. **Phase regions** at each time t are represented by the sets $\{(x,z) \mid u(x,z,t) \sim \pm 1\}$. Typically the unit length in (1.1) is relatively tiny in comparing to a sample size, so that the **interfacial region**, defined as the complement of the phase regions, is very thin and can be roughly regarded as a hypersurface called the **interface**. It is a common practice to use a level set $\gamma(t) = \{(x,z) \mid u(x,z,t) = \alpha\}$ to denote the interface at time t, where $\alpha \in (-1,1)$ is a number chosen at one's convenience.

We are interested in solutions having interfaces that travel upwards in the vertical z direction with a constant speed c. Mathematically, this renders to a solution of the form $u(x,z,t) = U(x, z - ct)$, where (c, U), called a **traveling wave** with **speed** c and **profile** U, satisfies the differential equation and the "boundary values"

$$(1.2) \qquad \begin{cases} cU_z + U_{zz} + \Delta U = f(U) & \forall x \in \mathbb{R}^n, z \in \mathbb{R}, \\ \lim\limits_{z \to \pm\infty} U(x,z) = \pm 1 & \forall x \in \mathbb{R}^n. \end{cases}$$

In this paper, we shall work in the class of cylindrically symmetric solution; that is, U depends only on z and $r = |x|$. Without cylindrical symmetry, problem (1.2) for U is extremely hard, and we leave it as a challenging open problem.

Since we shall look for cylindrically symmetric solutions which are monotone decreasing along the radial (i.e., $r = |x|$) direction, U must have the boundary value $\lim_{|x| \to \infty} U(x,z) = -1$ on the "lateral boundary" $|x| = \infty$.

In the sequel, subscripts denote partial/ordinary derivatives; in particular, U_r denotes the directional derivative in the radial direction of x.

Theorem 1 *Assume* **(A)**. *For any $c > 0$, (1.2) admits a cylindrically symmetric solution U with the* **monotonicity property***:*

$$(1.3) \qquad U_z > 0 \quad \text{on} \quad \mathbb{R}^{n+1} \quad \text{and} \quad U_r < 0 \quad \text{on} \quad (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}.$$

Notice that the function $V(x,z) = U(x,-z)$ is a solution of (1.2) with speed c replaced by $-c < 0$; it satisfies $V(x, \pm\infty) = \mp 1$, $V_z < 0$ on $\mathbb{R}^{n+1}$ and $V_r < 0$ on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$.

When $F(1) \neq F(-1)$, the existence of a traveling wave with asymptotic planar interface was proved by Fife [20] in dimension $n+1 = 2$ (see also [32]). Solutions having asymptotic conical level sets with any positive aperture angle were constructed by Ninomiya and Taniguchi [36, 37] in dimension $n+1 = 2$ and by Hamel, Monneau, and Roquejoffre [29] in any dimension $n+1 \geq 2$, where the nonlinearities f is assumed to have exactly one zero in $(-1,1)$. See also the works of Bonnet and Hamel [8] and Hamel, Monneau, and Roquejoffre [28] for the combustion case (i.e., $f = 0$ in $[-1, \theta]$ and $f > 0$ in $(\theta, 1)$ for some $\theta \in (-1,1)$) in dimension $n+1 = 2$, and Hamel and Nadirashvili [31] for the mono-stable case (i.e., $f > 0$ in $(-1,1)$) and for solutions with general level sets in any dimension $n+1 \geq 2$. Other related works can be found in [26, 27, 32, 34, 35].

Another motivation of our study of (1.2) is the De Giorgi conjecture [16] which asserts that

when $c = 0$ and $f(U) = U^3 - U$, all z-monotonic solutions of (1.2) are planar at least in dimension $n \leq 8$. Here planar means that all level sets are planes, i.e., there exist a unit vector $\mathbf{a} \in \mathbb{R}^{n+1}$ and a function $\Phi : \mathbb{R} \to [-1,1]$ such that $U(x,z) = \Phi(\mathbf{a} \cdot (x,z))$ for all (x,z); in this conjecture, the radial symmetry in x is not assumed. This conjecture was proven recently by Savin [39] (see also [1, 3, 5, 24]). More general nonlinearities of type (**A**) can also be considered in the spirit of [24, 39].

In view of the De Giorgi conjecture, a natural extension is to ask *whether planar solutions are the only solutions to the corresponding parabolic equation*

$$u_t = u_{zz} + \Delta u + u - u^3, \qquad (x,z) \in \mathbb{R}^n \times \mathbb{R},\ t \in \mathbb{R} \tag{1.4}$$

subject to the monotonicity conditions

$$\lim_{z\to\pm\infty} u(x,z,t) = \pm 1, \qquad u_z(x,z,t) > 0 \qquad \forall (x,z,t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}. \tag{1.5}$$

In the literature, a solution to a parabolic equation that is defined for all $t \in \mathbb{R}$ is called an **entire solution**. Since traveling waves are special entire solutions, our Theorem 1 clearly provides an example, when $n \geqslant 1$, of an entire solution that satisfies the monotonicity conditions (1.5) and that is not planar. Thus, for the elliptic equation (1.2) with $c \neq 0$ or for the parabolic equation (1.4) additional conditions are needed for an entire monotone solution to be planar. Our Lemma 2.2 below in §2 provides one such condition in that direction.

The monotonicity property (1.3) and the boundary values of U imply that the interface can be represented as a graph $z = H(|x|)$ or $|x| = R(z)$ where R is the inverse of H. We can describe the asymptotic shape of the interface as follows.

Theorem 2 *Assume* (**A**). *Let (c,U) be as in Theorem 1 and Γ be the 0-level set of U.*

(i) If $n > 1$, Γ is asymptotically a paraboloid, i.e.

$$\lim_{z\to\infty,\, U(x,z)=0} \frac{|x|^2}{2z} = \frac{n-1}{c}.$$

(ii) If $n = 1$, Γ is asymptotically a hyperbolic cosine curve, i.e., for some $A = A(f) > 0$,

$$\lim_{z\to\infty,\, U(x,z)=0} \frac{\cosh(2\mu x)}{\mu z} = \frac{A}{c}, \qquad \mu := \sqrt{f'(1)}.$$

We remark that if we choose an α–level set $\{(x,z) \mid U(x,z) = \alpha\}$ as the interface where $\alpha \in (-1,1)$, the limit value in the case $n > 1$ is unchanged, whereas in the case $n = 1$, A is a function of α, given by the formula (6.2).

It is well-known that the interface (level set) of solutions of (1.1) evolves, in an appropriate space and time scale, according to the motion by mean curvature flow; see [2, 11, 17, 19, 33] and references therein. For a traveling wave solution of (1.2), after shrinking the space by a factor of $R(\hat{z})$, the interface near $\mathbb{R}^n \times \{\hat{z}\}$ is asymptotically, as $\hat{z} \to \infty$, a circular cylinder $\mathbb{S}(1) \times \mathbb{R}$ where $\mathbb{S}(r)$ represents the sphere in $\mathbb{R}^n$ with radius r and center origin. As a hypersurface in $\mathbb{R}^{n+1}$, $\mathbb{S}(1) \times \mathbb{R}$ has a sum of all principal curvatures equal to $n-1$. Thus, when $n > 1$, the interface moves, in a certain scaled space-time, with a normal velocity equal to $n-1$. Translating into the original space-time, this motion should represent a constant vertical velocity c motion. In the moving coordinates, this renders to the approximation equation $cR' \sim (n-1)/R$, from which the asymptotic behavior $z \approx c|x|^2/(2[n-1])$ for the interface follows.

In comparing with the traveling wave solutions for the mean curvature flow $z = z(R)$:

$$\frac{c}{\sqrt{1+(z')^2}} = \frac{z''}{[1+(z')^2]^{3/2}} + \frac{(n-1)}{R}\frac{z'}{\sqrt{1+(z')^2}}, \tag{1.6}$$

our asymptotics $cR' \sim (n-1)/R$ is exactly $c \sim z'(n-1)/R$ for z (and R) large for each given c. This is because the radial curvature term $z'(n-1)/[R\sqrt{1+(z')^2}]$ becomes dominant in the curvature term for z (and R) large (see §5 for the rigorous derivation). Since we are mainly concerned with the asymptotics of the interface for z (and R) large, we do not analyze the solution of (1.6) for "not large" z (and R).

In the two ($n = 1$) space dimension case, the scaled interface is asymptotically two lines $\{\pm 1\} \times \mathbb{R}$, for which the curvature effect is negligible. To discover the dynamics, we compare (1.1) with its one space dimensional version $u_t = \varepsilon^2 u_{\xi\xi} - f(u)$ ($\varepsilon = 1/R(\hat{z}), \xi = x/R(\hat{z})$). It has been discovered more than a decade ago by Carr and Pego [10], Fusco [22], and Fusco and Hale [23] that for well-developed initial profile in a bounded domain with Neumann or periodic boundary conditions, the speed that two interfaces of distance d approach each other is of order $e^{-2\mu d/\varepsilon}$. Such a result was recently extended with simplified proofs by Chen [13] to arbitrary initial data and on the whole real line (see also Ei [18]). In particular, if initially there are two interfaces of distance d, the velocity that the two interfaces approach each other is $Ae^{-2\mu d/\varepsilon + o(1)}$, after an initiation which processes an arbitrary initial data into a special wave profile. The time needed for such an initiation is significantly short in comparing to the exponentially slow motion of the interface. If this size of normal velocity should produce a vertical velocity c motion, the shape of interface for solutions of (1.2) should be asymptotically governed by the equation $cR' = Ae^{-2\mu R}$, resulting a hyperbolic cosine curve, as describes in Theorem 2.

From another point of view, formally, for large z we have $c\,R'' = -2\mu A e^{-2\mu R} R' = o(1)R'$, so the U_{zz} term in (1.2) can be expected to be dropped without causing any significant change (for large z). Then (1.2) becomes $cU_z + U_{xx} = f(U)$. A change of variables $s = z/c$ gives $U_s + U_{xx} = f(U), (s,x) \in \mathbb{R}^2$. A recent result of Chen, Guo, and Ninomiya [14] shows that *there is a unique (up to a translation) entire solution having two interfaces located asymptotically on the hyperbolic cosine curve described in Theorem 2.*

Thus, Theorem 2 verifies the following speculation: *when $n > 1$, it is the pure curvature effect that contributes to the vertical velocity c motion of the interface; when $n = 1$, the curvature effect is insignificant and it is the interaction of the two branches of the interface that drives the effective vertical velocity c motion of the combined interface.*

Finally, we remark that the uniqueness of the solution given in Theorem 1 is not known. We leave it here as another challenging open problem.

2 Preparation

2.1 Basic Notation

Throughout this paper, the condition **(A)** and only the condition **(A)** is assumed. It implies the existence of constants $\alpha \in (0,1)$ and $\hat\alpha \in (-1,0)$ satisfying

$$f' = F'' > 0 \quad \text{on} \quad [-1,\hat\alpha] \cup [\alpha,1], \qquad F(\alpha) = F(\hat\alpha) < F(s) \ \forall s \in (\hat\alpha,\alpha). \tag{2.1}$$

In the sequel, α and $\hat\alpha$ are thus fixed. Also fixed is the wave speed $c > 0$.

Note that all wells (roots to $f(\cdot) = 0$) other than ± 1 lie either in $(\hat\alpha,\alpha)$ or in $(-\infty,-1) \cup (1,\infty)$ where the latter is not our concern at all. The depth (the value of F) of any well in $(-1,1)$ is higher than $F(\alpha) > 0 = F(\pm 1)$.

For definiteness, we use notation

$$x \in \mathbb{R}^n, \quad z \in \mathbb{R}, \quad y = (x,z) \in \mathbb{R}^{n+1}, \quad r = |x|, \quad \rho = |y| = \sqrt{z^2 + |x|^2}.$$

For a function ψ of $x \in \mathbb{R}^n$, radial symmetry means that ψ depends only on $r = |x|$ so $\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r$. A function $W(y)$ is called cylindrically symmetric if $W(x,z) = \tilde W(|x|,z)$ for some $\tilde W$. For simplicity, we shall not distinguish W and $\tilde W$; i.e., we abuse the notation $W(x,z) = W(|x|,z)$. A function $W(y)$ is radially symmetric if $W(y) = \tilde W(|y|)$ for some $\tilde W$. For radially symmetric functions of $y = (x,z)$, $\partial_{zz} + \Delta = \frac{n}{\rho}\partial_\rho + \partial_{\rho\rho}$.

2.2 Stationary Waves

The one-dimensional stationary wave Φ used in this paper is the unique solution to

$$\Phi'' = f(\Phi) \quad \text{on} \quad \mathbb{R}, \qquad \Phi(\pm\infty) = \pm 1, \quad \Phi(0) = \alpha \tag{2.2}$$

where α is as in (2.1). Since $[\Phi'^2 - 2F(\Phi)]' = 0$, one derives that

$$\Phi' = \sqrt{2F(\Phi)}, \qquad \int_\alpha^{\Phi(\xi)} \frac{ds}{\sqrt{2F(s)}} = \xi \quad \forall \xi \in \mathbb{R}.$$

2.3 Traveling Waves

For any $\varepsilon > 0$, Φ is also the profile of a 1-D speed ε traveling wave to

$$\varepsilon\Phi' + \Phi'' = f_\varepsilon(\Phi) \quad \text{on} \quad \mathbb{R}; \qquad f_\varepsilon := f + \varepsilon\sqrt{2F}. \tag{2.3}$$

Note that f^ε is unbalanced; in particular $\int_{-1}^s f_\varepsilon(u)du > 0$ for all $s \in (-1,1]$. Furthermore, up to shift, Φ is the only solution to (2.3) such that $\Phi(\pm\infty) = \pm 1$ (cf. [4]). The family $\{f_\varepsilon\}_{\varepsilon>0}$ will be used to construct solutions approximating that of (1.2).

2.4 Radially Symmetric Stationary Waves

For definiteness, in the sequel $\zeta \in C^3(\mathbb{R})$ is a fixed function satisfying

$$\zeta = 0 \quad \text{on} \quad \{-1\} \cup [\hat\alpha,1], \qquad \zeta > 0 \text{ in } (-1,\hat\alpha), \qquad \int_{-1}^1 \Big\{\zeta(s) - \sqrt{2F(s)}\Big\} ds > 0.$$

For each $\varepsilon > 0$, we define

$$g_\varepsilon(s) = f_\varepsilon(s) - \varepsilon\zeta(s) = f(s) + \varepsilon\sqrt{2F(s)} - \varepsilon\zeta(s) \quad \forall\, s \in [-1,1].$$

For each sufficiently small positive ε, notice the following:

(i) both wells ± 1 of g_ε are stable, i.e., $g'_\varepsilon(\pm 1) > 0 = g_\varepsilon(\pm 1)$;

(ii) all wells of g_ε in $(-1,1)$ lies in $(\hat{\alpha}, \alpha)$;

(iii) 1 is the only deepest well of g_ε on $[-1,1]$, i.e. $\int_1^s g_\varepsilon(u)du > 0$ for all $s \in [-1,1)$.

Using a standard shooting argument [7, 15, 38] one can show the following:

Lemma 2.1 *For each sufficiently small positive ε, there exists a unique solution w^ε to*

$$\frac{n}{\rho} w^\varepsilon_\rho + w^\varepsilon_{\rho\rho} - g_\varepsilon(w^\varepsilon) = 0 > w^\varepsilon_\rho \text{ in } (0,\infty), \quad w^\varepsilon_\rho(0) = 0, \quad w^\varepsilon(\infty) = -1. \tag{2.4}$$

The solution satisfies $w^\varepsilon(0) < 1 = \lim_{\varepsilon \searrow 0} w^\varepsilon(0)$.

These solutions will be used as subsolutions to establish the boundary values of U obtained from a limit process.

2.5 Planar Waves

In studying the asymptotic behavior of the interface, a limiting procedure leads to the following, for $\Psi = \Psi(\xi, z)$, $\xi \in \mathbb{R}$, $z \in \mathbb{R}$:

$$c\,\Psi_z + \Psi_{zz} + \Psi_{\xi\xi} = f(\Psi), \quad |\Psi| \leqslant 1, \quad \Psi_z \geqslant 0 \geqslant \Psi_\xi \quad \text{on } \mathbb{R}^2, \quad \Psi(0,0) = \alpha. \tag{2.5}$$

Lemma 2.2 *Assume* (**A**) *and $c > 0$. Then $\Psi(\xi, z) = \Phi(-\xi)$, $(\xi, z) \in \mathbb{R}^2$, is the only solution to (2.5).*

The proof will be given at the end of §4. In particular, this result implies that $\lim_{z\to\infty} \|U_z(\cdot, z)\|_{L^\infty(\mathbb{R}^n)} = 0$ and that the interface is asymptotically vertical.

2.6 Energy Functionals

The Allen-Cahn equation (1.1) is a gradient flow of an energy functional with the density function $u_z^2 + |\nabla u|^2 + 2F(u)$. For the traveling wave problem (1.2), there are certain variational structures. For this, we introduce the following notation. For functions ψ, ψ_1, ψ_2 of $r = |x|$ and a cylindrically symmetric function W on $\mathbb{R}^n \times (-\infty, 0]$, we define

$$\|\psi\| := \sqrt{\langle \psi\,,\, \psi \rangle}, \quad \langle \psi_1\,,\, \psi_2 \rangle = \int_0^\infty r^{n-1} \psi_1(r)\psi_2(r)\, dr,$$

$$X(l) := \Big\{ \psi \in C((0,\infty)) \;\Big|\; \psi \geqslant \alpha \quad \text{on} \quad (0,l], \quad \psi(\infty) = -1 \Big\} \quad \forall l > 0,$$

$$\mathbf{E}(\psi) := \int_0^\infty r^{n-1} \Big\{ \tfrac{1}{2}\psi_r^2 + F(\psi) \Big\} dr,$$

$$\mathbf{J}(W) := \int_{-\infty}^0 \Big\{ \tfrac{1}{2}\|W_z\|^2 + \mathbf{E}(W) \Big\} c\, e^{cz}\, dz.$$

Here the function spaces used are those that makes the norms or functionals finite.

The following is a consequence of the Euler-Lagrange equation for energy minimizers:

Lemma 2.3 *Suppose for each $z \in \mathbb{R}$, $U(\cdot, z+\cdot)$ on $\mathbb{R}^n \times (-\infty, 0]$ is a minimizer of $\mathbf{J}$ subject to the boundary condition $W(\cdot, 0) = U(\cdot, z)$ on $\mathbb{R}^n \times \{0\}$. Then $cU_z + U_{zz} + \Delta U = f(U)$ in $\mathbb{R}^{n+1}$.*

Since the interface is asymptotically vertical, from a dynamical point of view, for each large enough z, there is enough time for $u(\cdot, z+ct, t)$ to merge to an almost optimal shape that consumes energy as small as possible. That is, for $z \gg 1$, the wave profile $U(\cdot, z)$ should be close to a minimizer $\phi(\cdot, l)$ of the energy $\mathbf{E}$ in the set $X(l)$ where $l = R(z)$. For this purpose, it is natural to consider the minimization problem

$$\phi \in X(l), \qquad \mathbf{E}(\phi) = E(l) := \min_{\psi \in X(l)} \mathbf{E}(\psi). \tag{2.6}$$

Here we establish a basic property of the minimizers. More details will be given in §6.2.

Lemma 2.4 *For each $l > 0$, (2.6) admits at least one solution. Any solution satisfies $\phi_r < 0$ in $(0, l) \cup (l, \infty)$ and $\phi|_{r=l} = \alpha$. In addition, for each ψ satisfying $\psi(\infty) = -1$,*

$$\mathbf{E}(\min\{\psi, \phi\}) \leqslant \mathbf{E}(\psi). \tag{2.7}$$

Furthermore, $\lim_{l\to\infty} \phi(\cdot, l) = 1$ uniformly in any compact subset of $[0, \infty)$.

Proof. Fix $l > 0$. It is easy to show that there is at least a minimizer of the energy $\mathbf{E}$ in $X(l)$. Let $\phi(\cdot)$ be an arbitrary minimizer. Here we only prove (2.7). The rest will be proven in §6.2. Given ψ satisfying $\psi(\infty) = -1$, set $w_1 = \min\{\phi, \psi\}$ and $w_2 = \max\{\phi, \psi\}$. Then, $\mathbf{E}(w_1) + \mathbf{E}(w_2) = \mathbf{E}(\phi) + \mathbf{E}(\psi)$. Since $w_2 \in X(l)$ we see that $\mathbf{E}(w_2) \geqslant \mathbf{E}(\phi)$, so that $\mathbf{E}(w_1) \leqslant \mathbf{E}(\psi)$. ∎

3 The Existence of Cylindrically Symmetric Traveling Waves

We shall prove the existence by two totally different methods.

The first method uses the fact (cf. [28, 29, 36]) that the existence of a positive speed 1-D traveling wave implies the existence of cylindrically symmetric traveling waves of any speed. We shall construct a sequence of such waves to approximate a solution to (1.2).

In the second approach, the solution is approximated by the $l \to \infty$ limit of appropriately vertically lifted energy $\mathbf{J}$ minimizers subject to the boundary values being energy minimizers of $\mathbf{E}$ in $X(l)$. The proof is presented in a self-contained manner and the solution established satisfies an extra energy minimum principle that cannot be derived from the first method.

3.1 Approximation by Traveling Waves of Unbalanced Potentials

For any $\varepsilon > 0$, $f_\varepsilon := f + \varepsilon\sqrt{2F}$ gives an unbalanced potential

$$F_\varepsilon(u) := \int_{-1}^{u} f_\varepsilon(s)ds$$

which attains its deepest well only at $u=-1$. Chosen in such a manner, the solution Φ to (2.2) is also a traveling wave of speed ε to $\varepsilon\,\Phi'+\Phi''=f_\varepsilon(\Phi)$ that connects the equilibrium states $-\mathbf{1}$ and $\mathbf{1}$. One assumes that $\varepsilon>0$ is small enough so that $f_\varepsilon'(\pm 1)>0$ and the profile of Φ is then unique. Hence, according to [36] when $n+1=2$, and [29] when $n+1\geq 3$, for any given speed $c>0$, there exists a cylindrically symmetric traveling wave $U^\varepsilon=U^\varepsilon(x,z)$ satisfying

$$(3.1)\qquad \begin{cases} c\,U_z^\varepsilon+U_{zz}^\varepsilon+\Delta U^\varepsilon=f_\varepsilon(U^\varepsilon) \quad \text{on} \quad \mathbb{R}^{n+1}, \\ U^\varepsilon(0,0)=\alpha, \quad U^\varepsilon(\cdot,\pm\infty)\equiv\pm 1, \quad U_z^\varepsilon>0\geqslant U_r^\varepsilon \quad \text{on} \quad \mathbb{R}^{n+1}, \end{cases}$$

where $r=|x|$. Since $|U^\varepsilon|\leqslant 1$, by a standard elliptic estimate [25], $\{U^\varepsilon\}_{0<\varepsilon\ll 1}$ is a bounded family in $C^3(\mathbb{R}^{n+1})$. Thus it is a compact family in $C^2_{\rm loc}(\mathbb{R}^{n+1})$. Along a sequence $\varepsilon\searrow 0$, it converges to a cylindrically symmetric solution U to

$$(3.2)\quad c\,U_z+U_{zz}+\Delta U=f(U), \quad |U|\leqslant 1, \quad U_z\geqslant 0\geqslant U_r \quad \text{on} \quad \mathbb{R}^{n+1}, \quad U(0,0)=\alpha.$$

In the next subsection, we shall identify the boundary values of U at $|x|+|z|=\infty$.

Remark 1 1. The cylindrically symmetric solution U^ε has many special properties. Beside uniqueness ([30]), one of the characteristic property is that asymptotically, as $z\to\infty$, the interface (e.g. the α-level set of U^ε) is a cone of the form $|x|=z\tan\theta_\varepsilon$ with $\theta_\varepsilon=\arcsin(\varepsilon/c)$; see [29, 30, 36] for more details. Though we shall not use such a special property, we have in mind that in the limit, as $\varepsilon\searrow 0$, the open angle of the cone becomes zero, so the interface of U should look like a paraboloid.

2. One can use any approximation sequence $\{f_\varepsilon\}$, as long as $\lim_{\varepsilon\searrow 0} f_\varepsilon=f$ and there are positive speed 1-D traveling waves connecting steady states ± 1. We point out that the unbalanced condition $\int_{-1}^1 f_\varepsilon(s)ds>0$ is necessary but not sufficient for such an existence since other stable wells of f_ε in $[\hat\alpha,\alpha]$ may prevent a connection from -1 to 1; see, for example, [21]. By choosing f_ε as in the proof, we avoided unnecessary complications.

3. In [29], the result used in our proof (the existence of U^ε) is proven under the additional assumption that $f_\varepsilon=0$ has only one root in $(-1,1)$, and that the root is non-degenerate. This technical condition can indeed be removed. We shall, however, not present a modification of the proof of [29] to make the result in [29] be valid purely under the assumption (**A**). Instead, we provide an alternative proof (cf. §3.3) which does not rely on any analysis in [29, 36].

3.2 The "Boundary Values"

To show that solutions to (3.2) has the right boundary value, we shall distinguish the case $n=1$ from $n>1$. In the former case no extra condition is needed whereas in the latter case we have to assume that either U is the limit of U^ε or $f=0$ has only one root in $(-1,1)$.

Lemma 3.1 **(1)** *Suppose $n=1$. Then any symmetric (about x) solution U to* (3.2) *satisfies*

$$(3.3)\qquad \lim_{z\to\pm\infty} U(x,z)=\pm 1\ \forall\, x\in\mathbb{R}^n, \qquad \lim_{|x|\to\infty} U(x,z)=-1\ \forall\, z\in\mathbb{R}.$$

(2) *Suppose $n>1$. Let U be a limit, along a sequence $\varepsilon\searrow 0$, of the cylindrically symmetric family $\{U^\varepsilon\}$ of solutions to* (3.1)*. Then U has the boundary value* (3.3).

Proof. **(i) The Limit Equations.** As $U_z \geqslant 0 \geqslant U_r$ and $|U| \leqslant 1$, there exist

$$\varphi^{\pm}(x) := \lim_{z\to\pm\infty} U(x,z) \quad \forall\, x \in \mathbb{R}^n, \qquad \varphi(z) := \lim_{|x|\to\infty} U(x,z) \quad \forall\, z \in \mathbb{R}.$$

Consequently, $\lim_{z\to\pm\infty}(|U_z| + |U_{zz}|) = 0$ and $\lim_{|x|\to\infty} \Delta U = 0$, by the boundedness of the $C^3(\mathbb{R}^{n+1})$ norm of U and the interpolation

$$\|\cdot\|_{C^1(D)} \leq 5\|\cdot\|^{1/2}_{C^2(D)}\|\cdot\|^{1/2}_{C^0(D)} \tag{3.4}$$

for any cubic domain D with side length $\geqslant 1$. Thus,

$$\begin{aligned}
&\Delta\varphi^{\pm} - f(\varphi^{\pm}) = 0 \geqslant \varphi^{\pm}_r \quad \text{on } \mathbb{R}^n, && \varphi^{+}(0) \geqslant \alpha \geqslant \varphi^{-}(0),\\
&c\,\varphi_z + \varphi_{zz} - f(\varphi) = 0 \leqslant \varphi_z \quad \text{on } \mathbb{R}, && \varphi(0) \leqslant \alpha.
\end{aligned}$$

To complete the proof, we need show that $\varphi^{\pm} \equiv \pm 1$ and $\varphi \equiv -1$. In the sequel, the proofs for the case $n = 1$ and the case $n > 1$ differ only at the proofs of $\varphi^{+} \equiv 1$.

(iia) The $z \to \infty$ limit when $n = 1$. Integrating $\varphi^{+}_x\{\varphi^{+}_{xx} - f(\varphi^{+})\} = 0$ over $[0,\infty)$ and using $\varphi^{+}_x(0) = 0$ gives $F(\varphi^{+}(\infty)) = F(\varphi^{+}(0))$. Since $\varphi^{+}(0) \geqslant \alpha$, the definition of α in (2.1) implies that $\varphi^{+}(\infty) \in [-1,\hat{\alpha}] \cup [\alpha, 1]$. As $f(\varphi^{+}(\infty)) = 0$, we can only have either $\varphi^{+}(\infty) = -1$ or $\varphi^{+}(\infty) = 1$. The former case cannot happen, since F is a balanced potential with its deepest well at ± 1 so that $\psi \equiv -1$ is the only solution to

$$\psi_{xx} - f(\psi) = 0 \geqslant \psi_x \quad \text{on } [0,\infty), \;\; \psi_x(0) = 0, \;\; \psi(\infty) = -1.$$

Thus $\varphi^{+}(\infty) = 1$. Consequently, since $\varphi^{+}_x \leqslant 0$ on $[0,\infty)$, $\varphi^{+} \equiv 1$.

(iib) The $z \to \infty$ limit when $n > 1$. We write $\varphi^{+}(x)$ as $\varphi^{+}(r)$ where $r = |x|$.

Suppose $\varphi^{+} \not\equiv 1$. Since $\varphi^{+}_r(0) = 0$, we must have $\alpha \leqslant \varphi^{+}(0) < 1$. Set $\beta := \varphi^{+}(\infty)$. Then $f(\beta) = 0$. Integrating

$$\varphi^{+}_r\left[\varphi^{+}_{rr} + \frac{n-1}{r}\varphi^{+}_r - f(\varphi^{+})\right] = 0$$

over $r \in [0,\infty)$, and using $\varphi^{+}_r(0) = 0$, we obtain

$$F(\beta) - F(\varphi^{+}(0)) = \int_0^\infty \frac{n-1}{r}(\varphi^{+}_r)^2 > 0.$$

This implies that $\beta \in (\hat{\alpha}, \alpha)$.

Next, consider the solution w^ε of (2.4). Since $\lim_{\varepsilon\searrow 0} w^\varepsilon(0) = 1$, there exists $\varepsilon_0 > 0$ such that $w^{\varepsilon_0}(0) > \varphi^{+}(0)$. Also, since $w^{\varepsilon_0}(\infty) = -1$, there exists $R_0 > 0$ such that $w^{\varepsilon_0}(R_0) = \hat{\alpha}$. Set $\delta := \frac{1}{3}\min\{w^{\varepsilon_0}(0) - \varphi^{+}(0),\ \beta - \hat{\alpha}\} > 0$.

Now, since $\lim_{z\to\infty} U(\cdot, z) = \varphi^{+}(|\cdot|)$ locally uniformly, there exists $z_0 \in \mathbb{R}$ such that $|U(x,z) - \varphi^{+}(|x|)| < \delta$ for all $|x| \leqslant R_0$ and $z \geqslant z_0$. Also, by the assumption, along a sequence $\varepsilon \searrow 0$, $U^\varepsilon \to U$ uniformly on any compact subset of $\mathbb{R}^{n+1}$. There exists $\varepsilon \in (0, \varepsilon_0)$ such that $|\, U^\varepsilon(x,z) - U(x,z)\,| \leqslant \delta$ for all $|z - z_0| + |x| \leqslant 2R_0$. Hence

$$|\, U^\varepsilon(x,z) - \varphi^{+}(|x|)\,| \leqslant 2\delta \quad \text{if } z_0 \leqslant z \leqslant z_0 + R_0, |x| \leqslant R_0.$$

We shall compare $U^\varepsilon((0,z)+\cdot)$ with $w^{\varepsilon_0}(|\cdot|)$ on $B(R_0):=\{y\in\mathbb{R}^{n+1}\mid |y|<R_0\}$. Since $\lim_{z\to\pm\infty}U^\varepsilon(\cdot,z)=\pm1$ locally uniformly, we can define z^* by

$$z^*:=\min\{z\in\mathbb{R}\ \mid\ U^\varepsilon((0,z)+y)\geqslant w^{\varepsilon_0}(|y|)\quad\forall\, y\in\bar{B}(R_0)\ \}.$$

Upon noting that for every $z\leqslant z_0$, $U^\varepsilon(0,z)\leqslant U^\varepsilon(0,z_0)\leqslant\varphi^+(0)+2\delta<w^{\varepsilon_0}(0)$, we see that $z^*>z_0$.

Let $y_0\in\bar{B}(R_0)$ be a point such that

$$0=U^\varepsilon((0,z^*)+y_0)-w^{\varepsilon_0}(|y_0|)=\min_{y\in\bar{B}(R_0)}\Big\{U^\varepsilon((0,z^*)+y)-w^{\varepsilon_0}(|y|)\Big\}.$$

Since $U^\varepsilon(x,z)$ is monotonic in z and in $|x|$,

$$w^{\varepsilon_0}(|y_0|)=U^\varepsilon((0,z^*)+y_0)\geqslant U^\varepsilon((0,z_0)+y_0)\geqslant\varphi^+(|y_0|)-2\delta\geqslant\beta-2\delta>\hat{\alpha}=w^{\varepsilon_0}(R_0).$$

It follows that y_0 is an interior point of $\bar{B}(R_0)$. Consequently, $(\partial_{zz}+\Delta)U^\varepsilon((0,z^*)+y_0)\geqslant(\partial_{zz}+\Delta)w^{\varepsilon_0}(|y_0|)$. Also, as $w^{\varepsilon_0}(|y_0|)>\hat{\alpha}$, we have $\zeta(w^{\varepsilon_0}(|y_0|))=0$. Hence

$$\begin{aligned}0&=&cU^\varepsilon_z+(\partial_{zz}+\Delta)U^\varepsilon-f_\varepsilon(U^\varepsilon)\Big|_{(0,z^*)+y_0}>0+(\partial_{zz}+\Delta)w^{\varepsilon_0}-f_{\varepsilon_0}(w^{\varepsilon_0})\Big|_{y_0}\\&=&(\partial_{zz}+\Delta)w^{\varepsilon_0}-g_{\varepsilon_0}(w^{\varepsilon_0})\Big|_{y_0}=0,\end{aligned}$$

which is impossible. This impossibility shows that $\varphi^+\equiv1$.

(iii) The $z\to-\infty$ behavior. We show that $\varphi^-\equiv-1$. Write $\varphi^-(x)$ as $\varphi^-(|x|)$.

Again, set $\beta:=\varphi^-(\infty)$. First we show that $\beta=-1$. Suppose not. Since $\varphi^-(0)\leqslant U(0,0)=\alpha$, we must have $f(\beta)=0$ and $\beta\in(\hat{\alpha},\alpha)$. Furthermore, since $U_z\geqslant0\geqslant\varphi^-_r$,

$$U(x,z)\geqslant U(x,-\infty)=\varphi^-(|x|)\geqslant\varphi^-(\infty)=\beta\quad\forall\,(z,x)\in\mathbb{R}^{n+1}.$$

Now let $\varepsilon_0>0$ be such that $w^{\varepsilon_0}(0)>\alpha$. Let R_0 be such that $w^{\varepsilon_0}(R_0)=\hat{\alpha}$. Since $\lim_{z\to\infty}U(\cdot,z)=\varphi^+(|\cdot|)\equiv1$ locally uniformly and since $w^{\varepsilon_0}(0)>\alpha=U(0,0)$, the following constant z^* is well defined:

$$z^*:=\min\{z\in\mathbb{R}\mid w^{\varepsilon_0}(|y|)\leqslant U((0,z)+y)\quad\forall\, y\in\bar{B}(R_0)\ \}.$$

Same as before, there is a point $y_0\in\bar{B}(R_0)$ such that $U((0,z^*)+\cdot)-w^{\varepsilon_0}(|\cdot|)$ obtains a zero minimum on $\bar{B}(R_0)$ at y_0. Since $U\geqslant\beta$ on $\mathbb{R}^{n+1}$ and $w^{\varepsilon_0}(|\cdot|)=\hat{\alpha}<\beta$ on $\partial B(R_0)$, y_0 is an interior point of $B(R_0)$. Upon noting that $g_{\varepsilon_0}(w^{\varepsilon_0}(y_0))=f_{\varepsilon_0}(w^{\varepsilon_0}(y_0))>f(w^{\varepsilon_0}(y_0))=f(U((0,z^*)+y_0))$, a similar argument as before leads a contradiction. Thus, $\varphi^-(\infty)=-1$.

Finally, integrating

$$\varphi^-_r\left[\varphi^-_{rr}+\frac{n-1}{r}\varphi^-_r-f(\varphi^-)\right]=0$$

over $|x|=r\in[0,\infty)$, we obtain

$$\int_0^\infty\frac{n-1}{r}\varphi_r^2(r)dr=F(\varphi^-(\infty))-F(\varphi^-(0))=F(-1)-F(\varphi^-(0)).$$

Since $F(-1)-F(\varphi^-(0))\leqslant0$, we have $\varphi^-_r\equiv0$ and so $\varphi^-\equiv-1$.

(iv) The limit as $|x| \to \infty$. Finally, we show that $\varphi \equiv -1$.

As $U_r \leqslant 0$, $\varphi(z) \leqslant U(x,z)$ so that $\varphi(-\infty) \leqslant U(x,-\infty) = -1$; i.e., $\varphi(-\infty) = -1$.

Since $\varphi_z \geqslant 0$, it remains to show that $\varphi(\infty) = -1$. Suppose not. Then $\beta := \varphi(\infty) > -1$. Since F is a balanced potential and φ solves $c\varphi_z + \varphi_{zz} = f(\varphi)$ on $\mathbb{R}$ where $c > 0$, we must have $\beta < 1$. Thus $\beta \in (\hat{\alpha}, \alpha)$; i.e., $\varphi(\infty) \in (\hat{\alpha}, \alpha)$.

Take $z_0 > 0$ such that $\varphi(z_0) > \hat{\alpha}$. Then $U > \hat{\alpha}$ on $\mathbb{R}^n \times [z_0, \infty)$.

Take positive ε_0 and R_0 such that $w^{\varepsilon_0}(0) > \alpha$ and $w^{\varepsilon_0}(R_0) = \hat{\alpha}$. Since $\lim_{z\to\infty} U = 1$ locally uniformly, there exists $z^* > z_0 + R_0$ such that $U((0,z^*)+y) > w^{\varepsilon_0}(|y|)$ for all $|y| \leqslant R_0$. Now consider

$$R^* = \sup\{\, |x| \mid U((x,z^*)+y) \geq w^{\varepsilon_0}(|y|)\ \forall\, y \in \bar{B}(R_0)\}.$$

Since $\beta > \varphi(z^*) = \lim_{|x|\to\infty} U(x,z^*)$ and $w^{\varepsilon_0}(0) = \alpha > \beta$, R^* must be finite. Consequently, there exists $x_0 \in \mathbb{R}^n$ and $y_0 \in \bar{B}(R_0)$ such that $U((x_0,z^*)+\cdot) - w^{\varepsilon_0}(|\cdot|)$ attains its zero minimum on $\bar{B}(R_0)$ at y_0. Since $U > \hat{\alpha}$ on $\mathbb{R}^n \times [z^* - R_0, \infty)$ and $w^{\varepsilon_0} = \hat{\alpha}$ on $\partial B(R_0)$, y_0 must be an interior point of $B(R_0)$. A similar argument as before gives a contradiction.

This contradiction shows that $\varphi(\infty) = -1$ so that $\varphi \equiv -1$. ◻

Finally, the monotonicity property $U_z > 0$ on $\mathbb{R}^{n+1}$ and $U_r < 0$ for all $r = |x| > 0$ follows from a strong maximum principle. This completes proof of Theorem 1. ◻

Remark 2 If in addition to (**A**) one assumes that $f = 0$ has only one root in $(-1,1)$, then a similar proof leads to the following conclusion (see [29]):

any cylindrically symmetric solution to (3.2) satisfies the boundary value (3.3).

3.3 Approximation by Energy Minimizers

In this subsection, we shall use another approach to establish the existence of a traveling wave claimed in Theorem 1. The solution obtained will satisfy a certain energy minimum principle. We use the notation introduced in §2.6.

Theorem 3.1 *Assume* (**A**). *There exists a cylindrically symmetric solution U to* (1.2) *and* (3.3) *that satisfies the monotonicity property* (1.3) *and the* **minimum energy principle**

$$\mathbf{J}(U(\cdot, z+\cdot\,)\,) = \min_{W(\cdot,0)=U(\cdot,z)} \mathbf{J}(W) \quad \forall\, z \in \mathbb{R}. \tag{3.5}$$

This kind of variational principle has already been used in [1] to validate the conjecture of De Giorgi for general nonlinearities f in dimension $n+1 = 3$.

Proof. **1.** For each $l > 0$, let $\phi(\cdot, l)$ be a minimizer of the energy $\mathbf{E}$ in $X(l)$, claimed in Lemma 2.4. Consider the minimization problem

$$\min_{W(\cdot,0)=\phi(\cdot,l)} \mathbf{J}(W).$$

It is easy to show that there is a minimizer, which we denote by U^l. Since for each W,

$$\mathbf{J}(W) = e^{ch}\mathbf{J}(W(\cdot, h+\cdot\,)\,) + \int_h^0 \Big\{\tfrac{1}{2}\|W_z\|^2 + \mathbf{E}(W)\Big\} ce^{cz}dz \quad \forall\, h < 0,$$

$$\mathbf{J}(U^l(\cdot,h+\cdot)) = \min_{W(\cdot,0)=U^l(\cdot,h)} \mathbf{J}(W) \quad \forall h<0. \tag{3.6}$$

Also, the Euler-Lagrange equation shows that U^l is a (classical) solution to the boundary value problem

$$cU^l_z + U^l_{zz} + \Delta U^l = f(U^l) \quad \text{on } \mathbb{R}^n \times (-\infty,0), \quad U^l(\cdot,0) = \phi(\cdot,l).$$

Integrating over $\mathbb{R}^n \times \{z\}$ the equation for U^l multiplied by U^l_z we obtain

$$\begin{aligned} \frac{d}{dz}\mathbf{E}(U^l) &= \tfrac{1}{2}\frac{d}{dz}\|U^l_z\|^2 + c\|U^l_z\|^2, \\ \frac{d}{dz}\Big(e^{cz}\mathbf{E}(U^l)\Big) &= \tfrac{1}{2}\frac{d}{dz}\Big(e^{cz}\|U^l_z\|^2\Big) + ce^{cz}\Big(\tfrac{1}{2}\|U^l_z\|^2 + \mathbf{E}(U^l)\Big). \end{aligned}$$

Integrating the last equation over $z \in (-\infty,0]$ and using the boundary value we obtain

$$\mathbf{J}(U^l) = E(l) - \tfrac{1}{2}\|U^l_z(\cdot,0)\|^2. \tag{3.7}$$

2. By a rearrangement in the $r=|x|$ direction, one can show that $U^l_r \leqslant 0$. To show $U^l_z \geqslant 0$, the rearrangement technique does not work, due to the e^{cz} term in the energy functional. Instead, we use the following argument.

First of all, in view of (2.7), we see that $\mathbf{J}(\min\{\phi,U^l\}) \leqslant \mathbf{J}(U^l)$, where equal sign holds if and only if $\min\{\phi,U^l\} = U^l$. As an energy minimizer, we have

$$U^l = \min\{U^l,\phi\}, \quad \text{i.e.,} \quad U^l \leqslant \phi \quad \text{for } z \leqslant 0.$$

For $\varepsilon>0$, set $U^{l,\varepsilon} = U^l(\cdot,\cdot-\varepsilon)$, $w_1 = \min\{U^l,U^{l,\varepsilon}\}$ and $w_2 = \max\{U^l,U^{l,\varepsilon}\}$. Then $\mathbf{J}(w_1)+\mathbf{J}(w_2) = \mathbf{J}(U^l)+\mathbf{J}(U^{l,\varepsilon})$. Since $w_1(\cdot,0) = U^{l,\varepsilon}(\cdot,0)$ and $w_2(\cdot,0) = U^l(\cdot,0)$, we have $\mathbf{J}(w_1) \geqslant \mathbf{J}(U^{l,\varepsilon})$ and $\mathbf{J}(w_2) \geqslant \mathbf{J}(U^l)$. This implies that $w_1 = U^{l,\varepsilon}$ and $w_2 = U^l$; that is, $U^l(\cdot,z) \geqslant U^l(\cdot,z-\varepsilon)$ for every $\varepsilon>0$, so that $U^l_z \geqslant 0$.

The same argument also shows that the minimizer U^l is unique, since otherwise if there is a different minimizer $\tilde{U}^l$, then both $w_1 = \min\{U^l,\tilde{U}^l\}$ and $w_2 = \max\{U^l,\tilde{U}^l\}$ are minimizers. This contradicts (3.7), since, by the Hopf Lemma, $w_{2z}(\cdot,0) > w_{1z}(\cdot,0) \geqslant 0$.

A similar argument shows that

$$l_2 > l_1 > 0 \quad \Rightarrow \quad \phi(\cdot,l_2) > \phi(\cdot,l_1) \quad \Rightarrow \quad U^{l_2} > U^{l_1}. \tag{3.8}$$

Note that $U^l_z \geqslant 0$ implies the existence of $\varphi^- := \lim_{z\to-\infty} U^l(\cdot,z)$. Since $\varphi^- \leqslant \phi(\cdot,l)$ we see that $\varphi^-(\infty) = -1$. After integrating

$$\varphi^-_r\Big[\varphi^-_{rr} + \frac{n-1}{r}\varphi^-_r - f(\varphi^-)\Big] = 0$$

over $r \in [0,\infty)$ we see that $\varphi^- \equiv -1$. The strong maximum principle then gives $U^l_z > 0$.

3. Thus, there exists a unique constant $H(l)>0$ such that $U^l(0,-H(l)) = \alpha$. In view of (3.8), we see that $H(l)$ is monotonic in $l>0$. We claim that $\lim_{l\to\infty} H(l) = \infty$. To do this, we define

$$E^c(h) = \min_{\psi(-h)\leqslant\alpha\leqslant\psi(0)} \int_{-\infty}^0 \Big\{\tfrac{1}{2}\psi_z^2 + F(\psi)\Big\} ce^{cz}dz \quad \forall\, h \geqslant 0.$$

Note that $E^c(\cdot)$ is continuous and positive on $[0,\infty)$. Also,

$$E(l) > \mathbf{J}(U^l) \geqslant \int_0^l r^{n-1} E^c(H(l))dr = \frac{l^n}{n}\, E^c(H(l)).$$

This implies that $\lim_{l\to\infty} H(l) = \infty$, since $E(l) = O(1)l^{n-1}$.

Finally, set $U^{l,H(l)}(x,z) = U^l(x, z - H(l))$ and consider the family $\{U^{l,H(l)}\}_{l>0}$. This family is bounded in C^3, so along a subsequence, it converges to a limit U which solves the differential equation in (1.2). It is cylindrically symmetric and $U_z \geqslant 0 \geqslant U_r$.

Following an argument similar to that in the previous section, one can show that such limit U has the right boundary value (3.3). Indeed, the only modification is to replace U^ε in (iib) part of the proof by $U^{l,H(l)}$ and use the fact that

$$\lim_{l\to\infty} U^{l,H(l)}(\cdot, H(l)) = \lim_{l\to\infty} \phi(|\cdot|, l) = 1$$

uniformly in any compact subset of $\mathbb{R}^n$.

In the next section we shall show the exponential decay of $U+1$ as $|r| \to \infty$. It then follows from (3.6) that U satisfies (3.5). ◻

4 Qualitative Behavior of Solutions

In this section, we provide a rough description of the cylindrically symmetric traveling wave solution stated in Theorem 1. Also, we make a crude estimate about the behavior of the interface; that is, we show that it is paraboloid like.

4.1 Basic Properties of the Interface and Wave Profile

Lemma 4.1 *For each $m \in (-1,1)$, there exists a unique root $Z(m)$ to $U(0,\cdot) = m$ and there is a unique function $R(m,\cdot)$ defined on $[Z(m),\infty)$ such that*

$$\Gamma(m) := \Big\{(x,z) \mid U(x,z) = m\Big\} = \Big\{(x,z) \mid z \geqslant Z(m), |x| = R(m,z)\Big\}.$$

In addition, $R_z(m,z) > 0$ for all $z > Z(m)$ and

$$\lim_{z\to\infty} R(m,z) = \infty, \qquad \lim_{z\to\infty} R_z(m,z) = 0, \tag{4.1}$$

$$\lim_{z\to\infty} \Big\| U(\cdot,z) - \Phi(R(\alpha,z) - |\cdot|)\Big\|_{L^\infty(\mathbb{R}^n)} = 0, \tag{4.2}$$

$$\lim_{z\to\infty} \Big\{\|U_z(\cdot,z)\|_{L^\infty(\mathbb{R}^n)} + \|U_{zz}(\cdot,z)\|_{L^\infty(\mathbb{R}^n)}\Big\} = 0. \tag{4.3}$$

Proof. For convenience, we write $U(x,z)$ as $U(r,z)$ where $r = |x|$.

The existence of unique $Z(m)$ and $R(m,\cdot)$ follows from the monotonicity property and the boundary values of U. As $\lim_{z\to\infty} U = 1$, we must have $\lim_{z\to\infty} R(m,z) = \infty$. Applying the Implicit Function Theorem to the equation $U(R,z) = m$ we see $R(m,\cdot)$ is smooth in $(Z(m),\infty)$ and $R_z = -U_z(R,z)/U_r(R,z) > 0$ for every $z > Z(m)$.

Next we use a blow-up technique to show the rest assertions of the Lemma.

Consider the family $\{U(R(\alpha,\hat z)+\cdot,\hat z+\cdot)\}_{\hat z>Z(\alpha)}$. This family is bounded in C^3. For any sequence $\{z_i\}$ satisfying $\lim_{i\to\infty} z_i=\infty$, there exists a subsequence converging locally uniformly to a function $\Psi=\Psi(\xi,z)$, defined for all $(\xi,z)\in\mathbb{R}^2$ and satisfying (2.5), which, by Lemma 2.2, is given by $\Psi(\xi,z)=\Phi(-\xi)$. The uniqueness of the limit Ψ implies that the whole sequence $\{U(R(\alpha,\hat z)+\cdot,\hat z+\cdot)\}_{\hat z>Z(\alpha)}$ approaches, in $C^2_{loc}(\mathbb{R}^2)$, to $\Phi(-\xi)$ as $\hat z\to\infty$. Consequently, locally uniformly as $\hat z\to\infty$,

$$|U_z(R(\alpha,\hat z)+\cdot,\hat z+\cdot)|+|U_{zz}(R(\alpha,\hat z)+\cdot,\hat z+\cdot)|\to 0,$$
$$U_r(R(\alpha,\hat z)+\cdot,\hat z)\to-\Phi'(-\cdot).$$

Denote by Φ^{-1} the inverse function of Φ. This in particular implies that for each $m\in(-1,1)$,

$$\lim_{z\to\infty}\{R(m,z)-R(\alpha,z)\}=\Phi^{-1}(m),\qquad \lim_{z\to\infty}R_z(m,z)=-\lim_{z\to\infty}U_z/U_r=0.$$

To prove (4.2) and (4.3), set, for every $\varepsilon>0$, $R_\varepsilon=\Phi^{-1}(1-\varepsilon)-\Phi^{-1}(-1+\varepsilon)$. Since $\lim_{\hat z\to\infty}U(R(\alpha,\hat z)+\xi,\hat z+z)=\Phi(-\xi)$ uniformly on $[-R_\varepsilon,R_\varepsilon]^2$, there exists $z_\varepsilon>0$ such that for every $\hat z>z_\varepsilon$, $|U(R(\alpha,\hat z)+\xi,\hat z+z)-\Phi(-\xi)|\le\varepsilon$ for all $|z|+|\xi|\leqslant 2R_\varepsilon$.

Fix any $|z|\leqslant R_\varepsilon$. When $\xi\in[0,R(\alpha,\hat z)-R_\varepsilon]$, using the monotonicity of U in $r=|x|$, we have

$$1\geqslant U(\xi,\hat z+z)\geqslant U(R(\alpha,\hat z)-R_\varepsilon,\hat z+z)\geqslant\Phi(R_\varepsilon)-\varepsilon>1-2\varepsilon.$$

Thus, $|U(\xi,\hat z+z)-\Phi(R(\alpha,\hat z)-\xi)|\leqslant 2\varepsilon$ for all $\xi\in[0,R(\alpha,z)-R_\varepsilon]$. Similarly, $|U(\xi,\hat z+z)-\Phi(R(\alpha,\hat z)-\xi)|\leqslant 2\varepsilon$ for all $\xi\geqslant R(\alpha,z)+R_\varepsilon$. Thus

$$\max_{\xi\geqslant 0,|z|\leqslant R_\varepsilon}\Big|U(\xi,\hat z+z)-\Phi(R(\alpha,\hat z)-\xi)\Big|\le 2\varepsilon\quad\forall\,\hat z\geqslant z_\varepsilon,\xi\geqslant 0.$$

Sending $\varepsilon\searrow 0$ we see that

$$\lim_{\hat z\to\infty}\max_{|z|\le M}\Big\|U(\cdot,\hat z+z)-\Phi(R(\alpha,\hat z)-|\cdot|)\Big\|_{L^\infty(\mathbb{R}^n)}=0\quad\forall\,M\geqslant 0.$$

In particular this implies (4.2). Finally, (4.3) follows from an interpolation of the form (3.4). This completes the proof of the Lemma. □

4.2 The Exponential Decay

For convenience, we denote the phase domains $\Omega^\pm$, interfacial region Γ, and interface γ by

$$\Omega^+:=\{(z,x)\mid U(x,z)>\alpha\},\qquad\Omega^-:=\{(z,x)\mid U(x,z)<\hat\alpha\}.$$
$$\Gamma:=\cup_{m\in[\hat\alpha,\alpha]}\Gamma(m),\quad\gamma:=\Gamma(\alpha).$$

Lemma 4.2 *There exist positive constants M and $\nu>0$ such that*

$$|U^2-1|+|U_r|+|U_z|+|U_{zz}|\leqslant Me^{-\nu|r-R(\alpha,z)|}\quad\forall z\geqslant 0,x\in\mathbb{R}^n,$$
$$|U+1|+|U_r|+|U_z|+|U_{zz}|\leqslant Me^{-\nu(|z|+|x|)}\quad\forall z<0,x\in\mathbb{R}^n.$$

Proof. For $y \in \mathbb{R}^{n+1}$, let $d(y)$ be the distance from y to Γ. Let

$$k = \min_{s\in[-1,\hat{\alpha}]\cup[\alpha,1]} f'(s).$$

Note that $k > 0$ and by the definition of $\Omega^{\pm}$, $f'(U) \geqslant k$ for all $y \in \Omega^{\pm}$. Consequently, the positive function $1 \mp U$ satisfies

$$\begin{aligned} &\{-c\,\partial_z - \partial_{zz} - \Delta + k\}(1 \mp U) \\ = \quad &(1 \pm U)\Big\{k - \frac{f(\pm 1) - f(U)}{\pm 1 - U}\Big\} \leqslant 0 \quad \text{in } \Omega^{\pm}. \end{aligned}$$

Let λ_1, λ_2 be positive constants uniquely determined by

$$\lambda_1 = \lambda_2 + c/2, \quad \lambda_1^2 + n\lambda_2^2 = c^2/4 + k.$$

Consider the function

$$B(x,z) = 4e^{-\lambda_2 L - cz/2}\cosh(\lambda_1 z)\prod_{i=1}^{n}\cosh(\lambda_2 x_i)$$

where $x = (x_1, \cdots, x_n)$. Set $D := \{(x_1, \cdots, x_n, z) \mid |z| < L, |x_i| < L\}$. Note that

$$\min_{\partial D} B \geq 1, \qquad -c\,B_z - B_{zz} - \Delta B + kB = 0 \quad \text{in } D.$$

For each $y \in \Omega^{\pm}$, set $L = d(y)/\sqrt{n+1}$ and compare $1 \mp U(y + \cdot)$ with $B(\cdot)$ on D. Since $y + D \subset \Omega^{\pm}$ and $0 < 1 \mp U(y + \cdot) < 1$ on ∂D, the comparison gives $1 \mp U(y + \cdot) \leqslant B(\cdot)$ on D. In particular, $1 \mp U(y) \leqslant B(0,0) \leqslant 4e^{-d(y)\lambda_2/\sqrt{n+1}}$. Using interpolation, one also obtains the estimates for U_z, U_{zz}, U_r and U_{rr}.

Finally, since $\lim_{z\to\infty} R_z(m,z) = 0$, we have $d(y) > \frac{1}{2}(|z| + |r|) - M_1$ for all $z < 0, x \in \mathbb{R}^n$ and some constant $M_1 > 0$. Also, since $\lim_{z\to\infty}\{R(\hat{\alpha}, z) - R(\alpha, z)\} = \Phi^{-1}(\alpha) - \Phi^{-1}(\hat{\alpha})$, we see that for $y = (x,z)$ where $z > 0$, $d(y) > \frac{1}{2}|r - R(\alpha,z)| - M_2$ for some positive constant M_2 that is independent of y. The assertion of the Lemma thus follows with $\nu = \lambda_2/[2\sqrt{n+1}]$. $\square$

4.3 Energy Identities

Once we know the exponential decay of $|U^2 - 1|$ and U_r in the phase domains, we see immediately that for each $z \in \mathbb{R}$, the energy $\mathbf{E}(U(z,\cdot))$ is bounded. In addition, both $U_z(z,\cdot)$ and $U_{zz}(z,\cdot)$ are $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ functions.

Lemma 4.3 *There holds the energy identities, for every $z \in \mathbb{R}$,*

$$\begin{aligned} \frac{d}{dz}\mathbf{E}(U(z,\cdot)) &= \tfrac{1}{2}\frac{d}{dz}\|U_z\|^2 + c\|U_z\|^2, \\ \mathbf{E}(U(z,\cdot)) &= \tfrac{1}{2}\|U_z(z,\cdot)\|^2 + c\int_{-\infty}^{z}\|U_z\|^2 dz \\ &= \tfrac{1}{2}\|U_z(z,\cdot)\|^2 + c\int_{-\infty}^{z} e^{c(\hat{z}-z)}\Big\{\tfrac{1}{2}\|U_z\|^2 + \mathbf{E}(U)\Big\}d\hat{z}. \end{aligned}$$

Also, denote by $|x| = R(z)$ the interface defined by $U(z,x) = \alpha$ for $z > 0$,

$$\lim_{z\to\infty} \frac{\mathbf{E}(U(z,\cdot))}{R(z)^{n-1}} = \sigma := \int_{-1}^{1} \sqrt{2F(s)}ds, \qquad \lim_{z\to\infty} \frac{\|U_z\|^2}{R(z)^{n-1}} = 0.$$

In particular, in the case when $n = 1$,

$$\lim_{z\to\infty} \int_0^\infty \Big\{\tfrac{1}{2}U_x^2 + F(U)\Big\}dx = \sigma, \qquad \int_{\mathbb{R}}\int_0^\infty U_z^2 dxdz = \frac{\sigma}{c}.$$

Proof. Multiplying the equation for U by U_z and integrating over $\{z\} \times \mathbb{R}^n$ we obtain the first energy identity. From the exponential decay, we know that $\lim_{z\to-\infty}\{\mathbf{E}(U)+\|U_z\|^2\} = 0$. Hence, integrating the first identity over $(-\infty, z]$, we obtain the second identity in the Lemma.

Note that the energy identity can also be written as

$$\frac{d}{dz}\Big(e^{cz}\mathbf{E}(U)\Big) = \tfrac{1}{2}\frac{d}{dz}\Big(e^{cz}\|U_z\|^2\Big) + ce^{cz}\Big(\tfrac{1}{2}\|U_z\|^2 + \mathbf{E}(U)\Big).$$

After integrating the identity over $(-\infty, z]$ we obtain the third identity in the Lemma.

From the limit behavior, as $z \to \infty$, of $U(z,r) \sim \Phi(R(\alpha,z) - |x|)$ and the uniform exponential decay of $|U^2 - 1| + |U_z|$ with respect to $|r - R(\alpha,z)|$ we can derive the remaining limits stated in the Lemma. □

4.4 Proof of Lemma 2.2

Lemma 2.2 is analogous to the De Giorgi conjecture. The proof, however, is much simpler than that in [24, 39] since the monotonicity is assumed in both ξ and z directions. We divide the proof into several steps.

1. First, we show that $\lim_{\xi\to-\infty}\Psi(\xi,z) = 1$ for all $z \in \mathbb{R}$. Set $\varphi_-(z) := \lim_{\xi\to-\infty}\Psi(\xi,z)$. By (2.5), $\varphi_-(0) \geqslant \alpha$ and $\varphi_-' \geqslant 0$ on $\mathbb{R}$. Thus $\varphi_-(\infty) = 1$ and $c\,\varphi_-' + \varphi_-'' - f(\varphi_-) = 0$ on $\mathbb{R}$ with $c > 0$, which implies $\varphi_- \equiv 1$.

Let ε_0 and R_0 be positive constants such that $w^{\varepsilon_0}(0) > \alpha$ and $w^{\varepsilon_0}(R_0) < \hat{\alpha}$. Also, define

$$\Gamma := \{(\xi,z) \mid \hat{\alpha} \leqslant \Psi(\xi,z) \leqslant \alpha\}, \qquad \kappa := \inf\{-\Psi_\xi(\xi,z) \mid (\xi,z) \in \Gamma\}.$$

2. We claim that $\kappa > 0$. Suppose otherwise. Then there exists a sequence $\{(\xi_i, z_i)\}$ in Γ such that $\lim_{i\to\infty}\Psi_\xi(\xi_i,z_i) = 0$. As the family $\{\Psi(\xi_i + \cdot, z_i + \cdot)\}$ is compact, along a subsequence it converges to a function W satisfying

$$\begin{aligned} &c\,W_z + W_{zz} + W_{\xi\xi} = f(W) \quad \text{on } \mathbb{R}^2, \quad W_\xi \leqslant 0 \leqslant W_z, \\ &W_\xi(0,0) = 0, \quad W(0,0) \in [\hat{\alpha}, \alpha]. \end{aligned}$$

The maximum principle implies that $W_\xi \equiv 0$. Thus $W(\xi,z) = \varphi(z)$ for some φ satisfying

$$c\,\varphi' + \varphi'' = f(\varphi).$$

Since $c > 0$ and $\varphi(0) = W(0,0) \in [\hat{\alpha}, \alpha]$, we must have $\varphi(\infty) \in (\hat{\alpha}, \alpha)$. Let z_0 be a number such that $\varphi(z_0 - 2R_0) > \hat{\alpha}$. Then $\varphi(z) > \hat{\alpha}$ for all $z \geqslant z_0 - 2R_0$. Consequently, for some large enough i,

$$\hat{\alpha} \leqslant \Psi(\xi_i + \xi, z_i + z_0 + z) \leqslant \varphi(\infty) \qquad \forall\, |\xi| \leqslant 2R_0, |z| \leqslant 2R_0.$$

Now consider the constant

$$\xi^* = \sup\{\hat{\xi} \mid w^{\varepsilon_0}(\cdot) \leqslant \Psi((\xi_i + \hat{\xi}, z_i + z_0) + \cdot) \quad \text{on} \quad \bar{B}(R_0)\}.$$

Since $\Psi_\xi \leqslant 0$ and $\lim_{\xi \to -\infty} \Psi = 1$, we see that ξ^* is well-defined. In addition, since $w^{\varepsilon_0}(0) > \alpha > \varphi(\infty)$, we have $\xi^* < 0$.

Let $y_0 \in \bar{B}(R_0)$ be the point such that $\Psi((\xi_i + \xi^*, z_i + z_0) + \cdot) - w^{\varepsilon_0}(\cdot)$ obtains its zero minimum on $\bar{B}(R_0)$ at y_0. Noting that

$$\Psi((\xi_i + \xi^*, z_i + z_0) + y) \geqslant \Psi((\xi_i, z_i + z_0) + y) \geqslant \hat{\alpha}$$

for $y \in \bar{B}(R_0)$, we see that y_0 is an interior point of $B(R_0)$ which will lead us a contradiction. Thus, we must have $\kappa > 0$. The strong maximum principle then yields $\Psi_\xi < 0$ in $\mathbb{R}^2$.

3. Set $K = \|\Psi_z\|_{L^\infty}$. Let $R(\cdot)$ and $\hat{R}(\cdot)$ be solutions to $\Psi(R, \cdot) = \alpha$ and $\Psi(\hat{R}, \cdot) = \hat{\alpha}$, respectively. Then

$$0 < \hat{R}(z) - R(z) \leqslant \frac{\alpha - \hat{\alpha}}{\kappa}, \qquad 0 \leqslant R_z = -\frac{\Psi_z}{\Psi_\xi} \leqslant \frac{K}{\kappa} \quad \forall z \in \mathbb{R}.$$

Notice that $R(z)$ and $\hat{R}(z)$ are then automatically well-defined for all $z \in \mathbb{R}$. If we denote by $d(y)$ the distance from $y = (\xi, z)$ to Γ, then

$$d(y) \geqslant \frac{\kappa}{\sqrt{K^2 + \kappa^2}} \left(|\xi - R(z)| - \frac{\alpha - \hat{\alpha}}{\kappa} \right).$$

Following the same proof as before, we see that there exist two positive constants M and ν such that

$$|\Psi^2 - 1| + |\Psi_z| + |\Psi_\xi| + |\Psi_{zz}| + |\Psi_{\xi\xi}| \leq M e^{-\nu|\xi - R(z)|} \qquad \forall (\xi, z) \in \mathbb{R}^2.$$

This implies that

$$\sup_{z \in \mathbb{R}} \left(\mathbf{E}_2(\Psi(\cdot, z)) + \|\Psi_z(\cdot, z)\|^2 \right) < \infty,$$

where

$$\mathbf{E}_2(\phi(\cdot)) = \int_{\mathbb{R}} [\phi'(\xi)^2/2 + F(\phi(\xi))] d\xi \quad \text{and} \quad \|\phi\|^2 = \int_{\mathbb{R}} \phi^2(\xi) d\xi.$$

4. Now we have the energy identity

$$c\|\Psi_z(\cdot, z)\|^2 = \frac{d}{dz} \left(\mathbf{E}_2(\Psi(\cdot, z)) - \tfrac{1}{2}\|\Psi_z(\cdot, z)\|^2 \right).$$

Integrating this identity over $z \in (-h, h)$ and sending $h \to \infty$ we see that $\int_{\mathbb{R}} \|\Psi_z(\cdot, z)\|^2 dz < \infty$. This implies that $\lim_{z \to \pm\infty} \|\Psi_z(\cdot, z)\|^2 = 0$, which further implies that $\lim_{z \to \pm\infty} \|\Psi_{zz}(\cdot, z)\|^2 = 0$. Using the differential equation for W and the positivity of κ, we can show

$$\lim_{z \to \pm\infty} \Psi(R(z) + \cdot, z) = \Phi(\cdot).$$

The exponential decay estimate further implies that $\lim_{z \to \pm\infty} \mathbf{E}_2(\Psi(\cdot, z)) = \sigma$. After integrating the energy identity we derive that $\int_{\mathbb{R}} \|\Psi_z(\cdot, z)\|^2 dz = 0$, i.e., Ψ is independent of z. This completes the proof of Lemma 2.2. ∎

5 The case $n > 1$: The Curvature Dominant Effect

In this section we study the asymptotic expansion, in the leading order, of the interface function $R(z) := R(\alpha, z)$ when $n > 1$. In the higher space dimensional case, the interfacial dynamics of (1.1) is governed by the mean curvature flow. Since $\lim_{z\to\infty} R_z = 0$, we see that asymptotically, the shape of the interface is locally a circular cylinder of radius $R(z)$, which has a total curvature $(n-1)/R$. That a normal velocity $(n-1)/R$ produces a vertical velocity c gives us an approximation equation $cR' = (n-1)/R$. Here we shall make such a formal calculation rigorous.

Lemma 5.1 *Assume that $n > 1$. Then the interface is asymptotically a paraboloid. More precisely, let $r = R(z)$ be the function for the interface, i.e. $U(x, z) = \alpha$ at $|x| = R(z)$. Then*

$$\lim_{|z|\to\infty} \frac{R^2(z)}{z} = \frac{2(n-1)}{c}.$$

Proof. Here we use the inverse of a blow-up technique.

To study the behavior of the interface, it is convenient to consider the interface from the time evolution point of view. Namely, consider $u(x, z, t) = U(x, z - ct)$ as a solution of (1.1).

For each fixed $\hat{z} \geqslant 1$, set

$$\varepsilon = \frac{1}{R(\hat{z})}, \quad (\xi, \eta, \tau) = (\varepsilon x, \varepsilon(z - \hat{z}), \varepsilon^2 t), \quad w^\varepsilon(\xi, \eta, \tau) = U(x, z - ct).$$

Then denote by $\tilde{\Delta}$ the Laplacian with respect to the (ξ, η) variables, we have

$$w^\varepsilon_\tau = \tilde{\Delta} w^\varepsilon - \frac{1}{\varepsilon^2} f(w^\varepsilon) \quad \text{in } \mathbb{R}^{n+1} \times (0, \infty). \tag{5.1}$$

For convenience, we quote the following theorem from Chen [11]; see also, De Montonni-Schatzman [17], Evans-Soner-Souganidis [19], and Ilmanen [33].

Proposition 5.1 [11] *Assume* **(A)**. *Let $\{w^\varepsilon\}$ be solutions to (5.1) with initial values satisfying $|w^\varepsilon(\cdot, 0)| \leqslant 1$ on $\mathbb{R}^{n+1}$ and $\lim_{\varepsilon\to 0} w^\varepsilon(\cdot, 0) = 1$ (or -1) uniformly on Ω^0 where Ω^0 is a bounded domain with smooth boundary γ^0. Let $\{\gamma^\tau\}_{0\leqslant\tau\leqslant\tau^*}$ be the classical solution of the mean curvature flow and Ω^τ, $0 \leqslant \tau \leqslant \tau^*$, be the interior of γ^τ. Then*

$$\lim_{\varepsilon\to 0} w^\varepsilon(\xi, \eta, \tau) = 1 \ (or - 1) \quad \textit{uniformly in any compact subset of } \cup_{0\leqslant\tau\leqslant\tau^*} (\Omega^\tau \times \{\tau\}).$$

We now investigate the initial value of w^ε. For this note that by the mean value theorem and the relation between $(z, \hat{z}, x)$ and (ε, ξ, η),

$$\begin{aligned} |x| - R(z) &= R(\hat{z})|\xi| - R(z) = R(\hat{z})\{|\xi| - 1\} + \{R(\hat{z}) - R(z)\} \\ &= R(\hat{z})[|\xi| - 1] + R_z(\tilde{z})[\hat{z} - z] = R(\hat{z})\Big\{|\xi| - 1 - R_z(\tilde{z})\eta\Big\} \end{aligned}$$

where $\tilde{z} \in [\min\{\hat{z}, z\}, \max\{\hat{z}, z\}] \subset [\hat{z} - |\eta|R(\hat{z}), \hat{z} + |\eta|R(\hat{z})\}$.

Since $\lim_{z\to\infty} R_z(z) = 0$ and $\lim_{z\to\infty} R(z) = \infty$, it follows from the exponential estimate in Lemma 4.2 that

$$\lim_{\varepsilon\searrow 0} w^\varepsilon(\xi, \eta, 0) = \begin{cases} 1 & \text{if } |\xi| < 1, \\ -1 & \text{if } |\xi| > 1, \end{cases}$$

where the limit is uniform in any compact subset away from the initial interface positioned at $\mathbb{S}(1)\times\mathbb{R}$ where $\mathbb{S}(r)$ is the unit sphere in $\mathbb{R}^n$ of radius r.

First of all, the solution to the motion by mean curvature starting from initial surface $\mathbb{S}(1)\times\mathbb{R}$ is given by $\{\mathbb{S}(R^*(\tau))\times\mathbb{R}\}_{0\leqslant\tau<1/[2(n-1)]}$ where

$$\frac{d}{d\tau}R^*(\tau) = -\frac{n-1}{R^*(\tau)}, \quad R^*(0)=1; \quad \text{or} \quad R^*(\tau)=\sqrt{1-2(n-1)\tau}.$$

Although the initial interface $\mathbb{S}(1)\times\mathbb{R}$ is unbounded and Proposition 5.1 does not apply directly, we can approximate the interior of $\mathbb{S}(1)\times\mathbb{R}$ by the ellipsoids

$$\gamma^0(\delta) := \{(\xi,\eta)\mid |\xi|^2+\delta^2\eta^2 = 1-\delta^2\}, \qquad 0<\delta\ll 1.$$

Denote by $R_1(\delta,\tau)$ the radius of the circle being the intersection of the $\{\eta=0\}$ plane with the mean curvature curvature flow starting from $\gamma^0(\delta)$. Then $\lim_{\varepsilon\to 0} w^\varepsilon(\xi,0,\tau)=1$ uniformly in $|\xi|\leqslant R_1(\delta,\tau)-\delta$. One can also show that $\lim_{\delta\to 0}R_1(\delta,\tau)=\sqrt{1-2(n-1)\tau}$. Hence, sending $\delta\to 0$, we obtain $\lim_{\varepsilon\to 0} w^\varepsilon(\xi,0,\tau)=1$ uniformly in any compact subset of $\{(\xi,\tau)\mid |\xi|<\sqrt{1-2(n-1)\tau}, 0\leqslant\tau<1/[2(n-1)]\}$.

Similarly, from outside, the mean curvature flow starting from $\mathbb{S}(1)\times\mathbb{R}$ can be approximated by that starting from the torus

$$\{(\xi,\eta)\mid (|\xi|-1-K)^2+\eta^2=K^2\}, \quad K\gg 1.$$

First sending $\varepsilon\to 0$ and then $K\to\infty$ we conclude that $\lim_{\varepsilon\to 0} w^\varepsilon(\xi,0,\tau)=-1$ uniformly in any compact subset of $\{(\xi,\tau)\mid |\xi|>\sqrt{1-2(n-1)\tau}, 0\leqslant\tau<1/[2(n-1)]\}$.

Thus, for sufficiently small positive ε, the α-level set of $w^\varepsilon(\cdot,0,\tau)$ is near the set $|\xi| = \sqrt{1-2(n-1)\tau}$; that is,

$$w^\varepsilon(\xi,0,\tau)=\alpha \;\text{ when }\; |\xi|=\sqrt{1-2(n-1)\tau+o(1)}$$

where $\lim_{\varepsilon\to 0} o(1)=0$ uniformly in τ in any compact interval of $\tau\in[0,1/[2(n-1)])$. Since $w^\varepsilon(\xi,0,\tau)=U(x,\hat z-ct)$ and $t=\tau/\varepsilon^2=R(\hat z)^2\tau$, this implies, translating to the original variables, that

$$\text{(5.2)} \qquad \begin{gathered}\frac{R(\hat z - c\tau R(\hat z)^2)}{R(\hat z)} = \sqrt{1-2(n-1)\tau+o(1)},\\ \lim_{\hat z\to\infty} o(1)=0 \;\text{ uniform in }\; \tau\in[0,\tfrac{1}{2[n-1]}-\delta]\;\forall\,\delta>0.\end{gathered}$$

We shall now derive an asymptotic expansion of R from such an asymptotic relation.

Let $z_0\geqslant 1$ be a constant large enough so that the quantity $o(1)$ in (5.2) is $\leqslant 1/8$ for all $\hat z\geqslant z_0$ and $\tau\in[0,3/8[n-1]]$. Let $\{z_i\}_{i=1}^\infty$ be a sequence defined by

$$R(z_i)=2^{i/2}R(z_0), \qquad i=0,1,2,\cdots.$$

Note that $R(z_{i+1})/R(z_i)=\sqrt{2}$. Then for $\hat z=z_{i+1}$, let τ_i be such that $1-2(n-1)\tau_i+o(1)=1/2$ in (5.2) . Then $\tau_i=1/[4(n-1)]+o(1)$ and

$$z_i = z_{i+1}-c\tau_i R(z_{i+1})^2 = z_{i+1}-\frac{c+o(1)}{4(n-1)}2^{i+1}R^2(z_0).$$

This implies that

$$z_{i+1}-z_0=R^2(z_0)\sum_{j=1}^{i}\frac{c+o(1)}{4(n-1)}2^{j+1}=\frac{(2^{i+1}-1)[c+o(1)]R^2(z_0)}{2(n-1)}=\frac{(1-2^{-i-1})[c+o(1)]R^2(z_{i+1})}{2(n-1)}.$$

A simple interpolation then provides the basic estimate $z=O(1)R^2(z)$ and $R^2(z)=O(1)z$ where $O(1)$ is uniformly bounded as $z\to\infty$.

Now pick any small positive δ and set $\tau=1/(2[n-1]+\delta])$. We see from the square of (5.2) that

$$R^2\Big(z-\tfrac{c}{2[n-1]+\delta}R(z)^2\Big)=\Big\{\frac{\delta}{2[n-1]+\delta}+o(1)\Big\}R^2(z)$$

Since z and $R^2(z)$ are comparable, this implies that

$$\begin{aligned} z-\tfrac{c}{2(n-1)-\delta}R(z)^2 &= O(1)R^2\Big(z-\tfrac{c}{2(n-1)-\delta}R(z)^2\Big)\\ &= O(1)\Big\{\frac{\delta}{2[n-1]+\delta}+o(1)\Big\}R^2(z)=O(1)[\delta+o(1)]z. \end{aligned}$$

This implies that

$$\frac{R^2(z)}{z}=\frac{2(n-1)}{c+O(\delta)+o(1)}.$$

First sending $z\to\infty$ and then sending $\delta\searrow 0$ we obtain the required limit. $\square$

6 The Center Manifold Approach

Apparently the method in the previous section does not yield information accurate enough to describe the interface position when $n=1$. Since the position of the interface is believed (and proven) to be governed by the interaction of the two interfaces at $x=\pm R(z)$, we take a classical geometric approach.

The center manifold approach was first used by Carr and Pego [10], Fusco [22], and Fusco and Hale [23] for the 1-D Allen-Cahn equation $u_t=\varepsilon^2u_{\xi\xi}-f(u)$ for $\xi\in[0,1]$ with Neumann or periodic boundary conditions. Recently, the method was revisited by Chen and Ei in [13, 18] for the same problem for $\xi\in\mathbb{R}$. Here we follow the setting and idea in [13], but since (1.2) is of elliptic nature, there is a key difference in the method developed here.

6.1 An Overview

The manifold will be defined in the space of radially symmetric functions in $\mathbb{R}^n$. With notation as in §2.6, the **quasi-invariant** or **center manifold M** is defined in this paper by

$$\mathbf{M}:=\Big\{\phi(\cdot,l)\in X(l)\;\Big|\; l\geq l_0,\ \mathbf{E}(\phi(\cdot,l))=\min_{\psi\in X(l)}\mathbf{E}(\psi)\Big\}.$$

Here $l_0\geqslant 0$ is a number chosen so that $\mathbf{M}$ is a 1-dimensional manifold.

For each $z>0$, we project $U(z):=U(\cdot,z)$ onto $\mathbf{M}$ and write

$$U(z)=\phi(\ell(z))+V,\qquad \mathrm{dist}(U(z),\phi(\ell(z)))=\mathrm{dist}(U(z),\mathbf{M}),\quad V\perp\mathbf{T}_\phi\mathbf{M}$$

where $\mathbf{T}_\phi\mathbf{M}$ is the one-dimensional tangent space of $\mathbf{M}$ at $\phi = \phi(\ell) = \phi(\cdot, \ell(z))$.

It is easy to imagine that $\ell(z) \approx R(z)$ is the location of the interface, that V is negligible for large z, and, regarding z as time, that the dynamics of $U(z)$ is accurately described by the dynamics of $\ell(z)$. We shall rigorously derive that $\ell(\cdot)$ satisfies the 1-d ODE equation

$$[c + o(1)]\,\frac{d\ell}{dz} + \frac{d^2\ell}{dz^2} = \frac{1+o(1)}{\|\phi_l(\ell)\|^2}\,\frac{dE(\ell)}{d\ell}, \qquad \lim_{z\to\infty} o(1) = 0.$$

Note that $\|\phi_l\| dl$ is the length element of $\mathbf{M}$, $E(l) = \mathbf{E}(\phi(\cdot, l))$ is the energy function defined on $\mathbf{M}$, and $\frac{dE}{\|\phi_l\|\,dl}$ is the gradient of E. Since $\ell'' \ll \ell'$ for large z, we can ignore the ℓ'' term. Also regarding $z \to \phi = \phi(\ell(z))$ as the motion of a particle on $\mathbf{M}$, we can express the limit of the projected dynamics in terms of the motion of the particle on $\mathbf{M}$. In the language of differential geometry, it can be written as

$$c\,\dot{\phi} = \text{grad } E, \qquad \text{or} \quad c\,\frac{dE}{dt} = \Big|\text{grad } E\Big|^2.$$

Here $\dot{\phi}$ is a tangent vector with magnitude

$$|\dot{\phi}| := \lim_{h\to 0} \frac{\|\phi(h+z) - \phi(z)\|}{h}$$

measuring the distance that the particle moves per unit time, grad E is a co-vector with magnitude

$$|\text{grad } E| = \limsup_{\tilde{\phi}\in\mathbf{M}, \tilde{\phi}\to\phi} \frac{|\mathbf{E}(\tilde{\phi}) - \mathbf{E}(\phi)|}{\|\phi - \tilde{\phi}\|}$$

measuring the maximum change of E per unit length, and c in the first equation is a tensor mapping a vector to a co-vector. Here the first equation is for velocity whereas the second one is for speed. When $\mathbf{M}$ is one dimensional, velocity and speed are the same, so both equations are equivalent but the second equation has clear advantages over the first.

6.2 The Quasi-invariant (Center) Manifold

The Euler-Lagrange equation for a minimizer ϕ of $\mathbf{E}$ in $X(l)$ leads to the problem

$$\text{(6.1)} \qquad \begin{cases} \phi_{rr} + \frac{n-1}{r}\phi_r - f(\phi) = 0 > \phi_r & \text{in } (0,l)\cup(l,\infty), \\ \phi_r(0) = 0, \quad \phi(l) = \alpha, \quad \phi(\infty) = -1, \quad \phi(0) < 1. \end{cases}$$

Since the solution is not smooth across $r = l$, we introduce notation for jumps and averages:

$$[\![\psi]\!] := l^{n-1}\{\psi(l+) - \psi(l-)\}, \qquad \bar{\psi} := \tfrac{1}{2}[\psi(l+) + \psi(l-)].$$

Note that for functions $\psi_1(\cdot, l), \psi_2(\cdot, l) \in C^1([0,l]) \cup C^1([l,\infty))$,

$$\frac{d}{dl}\langle \psi_1\,,\, \psi_2\rangle = \langle \psi_{1l}\,,\, \psi_2\rangle + \langle \psi_1\,,\, \psi_{2l}\rangle - [\![\psi_1\psi_2]\!], \qquad [\![\psi_1\psi_2]\!] = \bar{\psi}_1[\![\psi_2]\!] + [\![\psi_1]\!]\bar{\psi}_2.$$

Lemma 6.1 *For every $l > 0$, let $\phi = \phi(\cdot, l)$ be a minimizer of the energy $\mathbf{E}$ in $X(l)$, claimed in Lemma 2.4. Then ϕ satisfies (6.1). When $n = 1$ or $\{n > 1, l \gg 1\}$, $\phi(\cdot, l)$ is unique and*

$$\phi_l = \frac{\partial\phi}{\partial l} > 0, \quad E_l := \frac{dE(l)}{dl} = \tfrac{1}{2}[\![\phi_r^2]\!] > 0.$$

Furthermore, there exist $M > 0$ and $l_0 \geqslant 0$ such that for all $l > l_0$,

$$\begin{gathered}
\|\phi_l\|_{L^1(\mathbb{R}^n)} + \|\phi_{ll}\|_{L^1(\mathbb{R}^n)} + \|\phi_{ll}\|^2 + \|\phi_{lll}\|^2 \le M\|\phi_l\|^2, \\
|[\![\phi_l]\!]| + |[\![\phi_{ll}]\!]| + |[\![\phi_{rl}]\!]| + |E_{ll}| + |E_{lll}| \le ME_l, \\
\lim_{l\to\infty} \frac{\|\phi_l\|^2}{l^{n-1}} = \lim_{l\to\infty} \frac{\|\phi_r\|^2}{l^{n-1}} = \sigma = \int_{-1}^{1} \sqrt{2F(s)}ds, \\
\lim_{l\to\infty} \sup_{r>0} |\phi(r,l) - \Phi(l-r)| = 0.
\end{gathered}$$

Proof. **1.** First we show that $\phi(l) = \alpha$. For this, let $l^* = \max\{r > 0 \mid \phi \geqslant \alpha \text{ on } [0,r]\}$. Then $l^* \geqslant l$, since $\phi \in X(l)$. Define $\tilde{\phi}(r) = \phi(r + l^* - l)$ for $r \geqslant 0$. Then $\tilde{\phi} \in X(l)$ and $\tilde{\phi}(l) = \alpha$. It is easy to verify that $\mathbf{E}(\phi) \ge \mathbf{E}(\tilde{\phi})$ where the equal sign holds only if $l^* = l$. Since ϕ is an energy minimizer, we must have $l^* = l$. Hence, $\phi(l) = \alpha$.

Similarly, for any $r_1 \geqslant r_2 \geqslant l$ satisfying $\phi(r_1) = \phi(r_2)$, we can compare the energy of ϕ with that of $\hat{\phi}$ defined by $\hat{\phi} = \phi$ on $[0, r_1]$ and $\hat{\phi}(r) = \phi(r + r_2 - r_1)$ to conclude that $r_1 = r_2$. Thus, ϕ is strictly decreasing on $[l, \infty)$.

In (l, ∞) there is no restriction of ψ in $X(l)$ so ϕ must be smooth and satisfy the differential equation $\phi_{rr} + (n-1)\phi_r/r = f(\phi)$ with the boundary conditions $\phi(l) = \alpha$ and $\phi(\infty) = -1$. As $\phi_r \leqslant 0$ on $[l, \infty)$, the strong maximum principle implies that $\phi_r < 0$ on $[l, \infty)$. Solving the ode problem, one can show that ϕ on $[l, \infty)$ is also unique if $n = 1$ or $\{n > 1 \text{ and } l \gg 1\}$.

To find the behavior of ϕ on $[0, l]$, let $\tilde{F}$ be a C^1 function such that $\tilde{F} = F$ on $[\alpha, \infty)$ and $\tilde{F}'(s) = f(\alpha)$ for all $s < \alpha$. Then restricted on the interval $[0, l]$, ϕ is a minimizer, without constraint, of the energy with F replaced by $\tilde{F}$. It then follows that on $[0, l]$, ϕ is the unique solution to the differential equation $\phi_{rr} + (n-1)\phi_r/r = f(\phi)$ on $(0, l)$ with the boundary condition $\phi_r(0) = 0$ and $\phi(l) = \alpha$. It satisfies $\alpha < \phi < 1$ on $[0, l)$ and $\phi_r < 0$ on $(0, l]$.

In conclusion, ϕ is a solution to (6.1); there is $l_0 \geqslant 0$ such that ϕ is unique when $l \geqslant l_0$.

2. Next, we calculate the variation of $E(l)$ with respect to l. For this we denote a (generic) minimizer by $\phi(\cdot, l)$. Suppose $l_2 > l_1 > 0$. Set $w_1 = \min\{\phi(\cdot, l_1), \phi(\cdot, l_2)\}$ and $w_2 = \max\{\phi(\cdot, l_1), \phi(\cdot, l_2)\}$. Then $\mathbf{E}(w_1) + \mathbf{E}(w_2) = \mathbf{E}(\phi(\cdot, l_1)) + \mathbf{E}(\phi(\cdot, l_2))$. Since $w_i \in X(l_i)$, we see that w_i is a minimizer of $\mathbf{E}$ in $X(l_i)$ and hence is smooth. This implies that $w_1 = \phi_1 < \phi_2 = w_2$. Consequently, except a possible set of measure zero, ϕ is smooth in l.

For each $\varepsilon > 0$, $X(l + \varepsilon) \subset X(l)$ so that $E(l + \varepsilon) > E(l)$ for all $l > 0$. Also, using

$$\tilde{\phi}(r, l) = \begin{cases} \phi(0, l), & \text{if } 0 < r < \varepsilon, \\ \phi(r - \varepsilon, l), & \text{if } r \geqslant \varepsilon, \end{cases}$$

as a test function in estimating $E(l + \varepsilon)$ one can derive that $E(l + \varepsilon) < E(l) + O(\varepsilon)$; we omit the details. Hence, $E(\cdot)$ is a strictly decreasing and Lipschitz continuous function.

Now consider $l \geqslant l_0$, so $\phi(\cdot, l)$ is unique. Differentiating $\phi(l, l) = \alpha$ we see that $\phi_l + \phi_r|_{r=l\pm} = 0$. Now identifying $\nabla\phi$ with ϕ_r and $\Delta\phi$ with $\phi_{rr} + (n-1)\phi_r/r$, we calculate

$$\begin{aligned}
E_l &= \langle \nabla\phi \,,\, \nabla\phi_l \rangle + \langle f(\phi) \,,\, \phi_l \rangle - \tfrac{1}{2}[\![\phi_r^2]\!] \\
&= \langle -\Delta\phi + f(\phi) \,,\, \phi_l \rangle - [\![\phi_r \phi_l]\!] - \tfrac{1}{2}[\![\phi_r^2]\!] = \tfrac{1}{2}[\![\phi_r^2]\!] = \tfrac{1}{2}[\![\phi_l^2]\!].
\end{aligned}$$

The proof of the last assertion is omitted. We only point out the following:

$$\begin{aligned}
\phi(r,l) &= \Phi(l-r)+o(1), & \phi_r(l\pm,l) &= -\sqrt{2F(\alpha)}+o(1),\\
[\![\phi_l]\!] &= -[\![\phi_r]\!] = O(1)E_l, & [\![\phi_{rr}]\!] &= \tfrac{1-n}{l}[\![\phi_r]\!],\\
\|\phi_l\|^2 = E(l)+o(1) &= [\sigma+o(1)]l^{n-1}, & E_l &= [(n-1)\sigma+o(1)]l^{n-2}
\end{aligned}$$

where $o(1)\to 0$ as $l\to\infty$.

6.3 The Center Manifold When $n=1$

As an illustration and also for later applications, we provide detailed calculation for $n=1$. The equation for ϕ is

$$\phi_{xx}=f(\phi)\quad\text{on}\quad (0,l)\cup(l,\infty),\qquad \phi_x(0,l)=0,\quad \phi(l,l)=\alpha,\quad \phi(\infty,l)=-1.$$

It follows, denoting $b=\phi(0,l)$, that

$$\begin{aligned}
\phi_x^2 &= \begin{cases} 2F(\phi)-2F(b) & \text{when}\ \ x\in[0,l),\\ 2F(\phi) & \text{when}\ \ x\in(l,\infty),\end{cases}\\
l &= \int_\alpha^b \frac{ds}{\sqrt{2F(s)-2F(b)}} = \frac{1}{\sqrt{f'(1)}}\ln\frac{2(1-\alpha)}{1-b}+A_1+o(1)
\end{aligned}$$

where $\lim_{l\to\infty}o(1)=0$ and

$$\begin{aligned}
A_1 = &\quad \lim_{b\nearrow 1}\int_\alpha^b \Big\{\frac{1}{\sqrt{2F(s)-2F(b)}}-\frac{\sqrt{1/f'(1)}}{\sqrt{(1-s)^2-(1-b)^2}}\Big\}ds\\
&= \int_\alpha^1\Big\{\frac{1}{\sqrt{2F(s)}}-\frac{1}{\sqrt{f'(1)}(1-s)}\Big\}ds.
\end{aligned}$$

Consequently, we have the identities and the expansion

$$E_l(l)=\tfrac12[\![\phi_r^2]\!]=F(b)=\sigma A e^{-2\mu l+o(1)}$$

where $\lim_{l\to\infty}o(1)=0$, $\mu=\sqrt{f'(1)}$ and

$$A=\frac{2\,(1-\alpha)^2\,f'(1)}{\int_{-1}^1\sqrt{2F(s)}ds}\exp\Big(\int_\alpha^1\Big\{\frac{\sqrt{2f'(1)}}{\sqrt{F(s)}}-\frac{2}{1-s}\Big\}ds\Big). \tag{6.2}$$

6.4 Projection onto the Center Manifold

Lemma 6.2 (i) *For every large enough z, there is a unique $\ell(z)\geqslant l_0$ such that*

$$\|\phi(\cdot,\ell(z))-U(z,\cdot)\| = \min_{l\geqslant l_0}\|\phi(\cdot,l)-U(\cdot,z)\| = \operatorname{dist}(U(\cdot,z),\mathbf{M}).$$

(ii) *For some $z_0\geqslant 0$, the function $z\to\ell(z)$ is smooth on $[z_0,\infty)$ and*

$$\ell_z>0\quad on\quad [z_0,\infty),\qquad \lim_{z\to\infty}\ell_z(z)=0,\qquad \lim_{z\to\infty}[\,R(z)-\ell(z)\,]=0.$$

(iii) *Write $V(\cdot,z)=U(\cdot,z)-\phi(\cdot,\ell(z))$. Then $V(\cdot,z)\perp\phi_l(\cdot,\ell(z))$, i.e. $\langle\phi_l\,,V\rangle=0$, and*

$$\lim_{z\to\infty}\|V,V_r,V_z\|_{L^\infty(\mathbb{R}^n)}=0,\qquad \lim_{z\to\infty}\frac{\|V(\cdot,z)\|}{\|\phi_l(\cdot,\ell(z))\|}=0.$$

Proof. Set $G(l,z) := \frac{1}{2}\|U(\cdot,z)-\phi(\cdot,l)\|^2$. Then

$$G_l = \langle \phi - U\,,\,\phi_l\rangle, \qquad G_{lz} = -\langle U_z\,,\,\phi_l\rangle < 0.$$

Also, using $\|\phi_{ll}\|_{L^1(\mathbb{R}^n)} + |[\![\phi_l]\!]| = O(1)\|\phi_l\|^2$ we have

$$\begin{aligned} G_{ll} &= \|\phi_l\|^2 + \langle \phi - U\,,\,\phi_{ll}\rangle - [\![(\phi-U)\phi_l]\!] \\ &= \|\phi_l\|^2\Big\{1 + O(1)\|U-\phi\|_{L^\infty(\mathbb{R}^n)}\Big\} > 0 \end{aligned}$$

as long as $\|U-\phi\|_{L^\infty(\mathbb{R}^n)} < \delta_0$ for some fixed $\delta_0 > 0$. From the limit behavior of U in Lemma 4.1, one sees that the minimum of $G(\cdot,z)$ is achieved at $\ell(z) \sim R(z)$. The implicit function theorem gives the uniqueness of $\ell(z)$ and the monotonicity $\ell_z = -G_{lz}/G_{ll} > 0$.

The last assertion (iii) follows from Lemma 4.1 and the previous assertions. □

Since $\Delta\phi = f(\phi)$ in $\mathbb{R}^n \setminus \mathbb{S}(\ell)$, the decomposition $U = \phi + V$ gives, for $z \geqslant z_0$ and $x \in \mathbb{R}^n \setminus \mathbb{S}(\ell(z))$,

$$(c\,\ell_z + \ell_{zz})\phi_l + \ell_z^2\phi_{ll} + cV_z + V_{zz} = f(\phi+V) - f(\phi) - \Delta V. \tag{6.3}$$

6.5 Projection of the Dynamics onto the Manifold

In the sequel, $\phi = \phi(\cdot,\ell(z))$. Taking the inner product of ϕ_l with (6.3) we obtain

$$(c\,\ell_z + \ell_{zz})\,\|\phi_l\|^2 = J_0 + J_1 - J_2 - \ell_z^2\langle \phi_{ll}\,,\,\phi_l\rangle \tag{6.4}$$

where

$$\begin{aligned} J_0 &= \langle f'(\phi)V - \Delta V\,,\,\phi_l\rangle, \\ J_1 &= \langle f(\phi+V) - f(\phi) - f'(\phi)V\,,\,\phi_l\rangle, \\ J_2 &= c\langle V_z\,,\,\phi_l\rangle + \langle V_{zz}\,,\,\phi_l\rangle. \end{aligned}$$

1. The main term. The major contribution on the right-hand side of (6.4) is J_0. Using integration by parts and the equation $\Delta\phi_l - f'(\phi)\phi_l = 0$ in $\mathbb{R}^n \setminus \mathbb{S}(\ell)$ we obtain

$$J_0 := \langle f'(\phi)V - \Delta V\,,\,\phi_l\rangle = [\![V_r\phi_l - V\phi_{lr}]\!].$$

Since V and ϕ are continuous, $[\![V_r]\!] = -[\![\phi_r]\!] = [\![\phi_l]\!]$. Hence,

$$[\![V_r\phi_l]\!] = [\![V_r]\!]\bar{\phi}_l + \bar{V}_r[\![\phi_l]\!] = [\![\phi_l]\!]\bar{\phi}_l + \bar{V}_r[\![\phi_l]\!] = \tfrac{1}{2}[\![\phi_l^2]\!] + \bar{V}_r[\![\phi_l]\!].$$

Using $E_l = \frac{1}{2}[\![\phi_l^2]\!]$, we have

$$J_0 = E_l + \bar{V}_r[\![\phi_l]\!] - \bar{V}[\![\phi_{lr}]\!] = E_l\Big\{1 + O(\bar{V}_r) + O(\bar{V})\Big\},$$

since the sizes of $[\![\phi_l]\!]$ and $[\![\phi_{lr}]\!]$ are majorized by $E_l = \frac{1}{2}[\![\phi_r^2]\!] = \bar{\phi}_r[\![\phi_r]\!]$. Here we note that the estimates in Lemmas 4.1 and 4.2 and the definition of ℓ imply that

$$\lim_{z\to\infty}\Big\{|\bar{V}| + |\bar{V}_r|\Big\} = 0.$$

2. The remainder term. We use the mean value theorem to conclude that

$$J_2 := \langle f(\phi+V) - f(\phi) - f'(\phi)V\,,\,\phi_h\rangle \;=\; \langle O(1)V^2\,,\,\phi_l\rangle.$$

3. The z-derivative Terms. Differentiating $\langle \phi_l\,,\,V\rangle = 0$ with respect to z gives

$$\begin{aligned}
\langle V_z\,,\,\phi_l\rangle &= \ell_z\Big\{[\![V\phi_l]\!] - \langle V\,,\,\phi_{ll}\rangle\Big\},\\
\langle V_{zz}\,,\,\phi_l\rangle &= \ell_{zz}\Big\{[\![V\phi_l]\!] - \langle V\,,\,\phi_{ll}\rangle\Big\}\\
&\qquad +\ell_z\Big\{[\![V_z\phi_l]\!] + [\![V\phi_l]\!]_z - 2\langle V_z\,,\,\phi_{ll}\rangle - \ell_z\langle V\,,\,\phi_{lll}\rangle\Big\}.
\end{aligned}$$

4. Conclusion. Combining all these calculations, the equation can be written as

$$\begin{aligned}
&\Big(c\,\ell_z + \ell_{zz}\Big)\Big(\|\phi_l\|^2 + [\![V\phi_l]\!] - \langle V\,,\,\phi_{ll}\rangle\Big)\\
&\qquad = E_l\{1+o(1)\} + \langle O(V^2)\,,\,\phi_l\rangle + o(1)\ell_z\|\phi_l\|^2. \qquad (6.5)
\end{aligned}$$

It can also be written as

$$[c+o(1)]\,\ell_z + \ell_{zz} = \frac{[1+o(1)]E_l + \langle O(1)V^2\,,\,\phi_l\rangle}{\|\phi_l\|^2}. \qquad (6.6)$$

This equation is "self-contained" if we can show that $\langle O(1)V^2\,,\,\phi_l\rangle = o(1)E_l$. For this we need to study the dynamics of $\|V\|^2$.

6.6 The Dynamics in the Direction Normal to the Manifold

Taking the inner product of (6.3) with V and using $\langle V\,,\,\phi_l\rangle = 0$ we obtain

$$\langle V\,,\,cV_z + V_{zz}\rangle \;=\; \langle V\,,\,f'(\phi)V - \Delta V\rangle + \langle V\,,\,f'(\phi+V) - f(\phi) - f'(\phi)V\rangle - \ell_z^2\langle \phi_{ll}\,,\,V\rangle.$$

Note that

$$\begin{aligned}
\langle V\,,\,V_z\rangle &= \tfrac{1}{2}\frac{d}{dz}\|V\|^2,\\
\langle V\,,\,V_{zz}\rangle &= \frac{d}{dz}\langle V\,,\,V_z\rangle - \|V_z\|^2 + \ell_z[\![VV_z]\!]\\
&= \tfrac{1}{2}\frac{d^2}{dz^2}\|V\|^2 - \|V_z\|^2 + \ell_z^2\bar V[\![\phi_r]\!].
\end{aligned}$$

Here we have used the fact that $0 = [\![V]\!]_z = [\![V_r\ell_z + V_z]\!]$ so $[\![V_z]\!] = -\ell_z[\![V_r]\!] = \ell_z[\![\phi_r]\!]$. Also

$$\begin{aligned}
\langle f'(\phi)V - \Delta V\,,\,V\rangle &= \|\nabla V\|^2 + \langle f'(\phi)V\,,\,V\rangle + [\![VV_r]\!]\\
&= \|V_r\|^2 + \langle f'(\phi)V\,,\,V\rangle - \bar V[\![\phi_r]\!].
\end{aligned}$$

Hence, we have the equation

$$\begin{aligned}
\tfrac{1}{2}\Big(c\,\frac{d}{dz} + \frac{d^2}{dz^2}\Big)\|V\|^2 &= \|V_z\|^2 + \|V_r\|^2 + \langle f'(\phi)V\,,\,V\rangle\\
&\quad -\bar V[\![\phi_r]\!] - \ell_z^2\{\langle V\,,\,\phi_{ll}\rangle + \bar V\,[\![\phi_r]\!]\}.
\end{aligned}$$

Lemma 6.3 *There exist positive constants l_0 and ν such that for any $l \geqslant l_0$ and $\phi = \phi(\cdot, l)$,*

$$\|\psi_r\|^2 + \langle f'(\phi)\psi\,,\,\psi\rangle \geqslant 2\nu\Big\{\|\psi_r\|^2 + \|\psi\|^2 + |\bar\psi|^2\Big\} \qquad \forall \psi \perp \phi_l.$$

See, for example, Chen [12].

It then follows that

$$\Big(c\,\frac{d}{dz} + \frac{d^2}{dz^2}\Big)\|V\|^2 \;\geqslant\; \nu\Big\{\|V_z\|^2 + \|V_r\|^2 + \|V\|^2 + |\bar V|^2\Big\} + O(E_l^2) + O(\ell_z^4)\|\phi_l\|^2.$$

6.7 Rigorous Derivation of the Projected Dynamics When $n = 1$

For simplicity, we focus only on the one dimensional case. The key is to bound $\|V\|$ by E_l.

1. For some large constant $M > 1$, all $z \geqslant z_0 \gg 1$:

$$\Big(c\,\frac{d}{dz} + \frac{d^2}{dz^2}\Big)\|V\|^2 \quad \geqslant \quad \nu\|V\|^2 - M(\ell_z^4 + E_l^2).$$

Since $|E_{ll}| + |E_{lll}| = O(1)E_l$, taking larger z_0 if necessary, we have for all $z \geqslant z_0$,

$$\begin{aligned}\Big(c\frac{d}{dz} + \frac{d^2}{dz^2}\Big)E_l^2(\ell(z)) \quad &= \quad 2(c\ell_z + \ell_{zz})E_l E_{ll} + 2(E_l E_{lll} + E_{ll}^2)\ell_z^2 \\ &\leqslant \quad M\Big\{E_l + \ell_z^2 + \|V\|^2\Big\}E_l^2 \leq \frac{\nu}{4}\,E_l^2\end{aligned}$$

by using (6.5). Thus, for $B(z) := \|V\|^2 - \frac{4M}{\nu}E_l^2 - e^{c(z_0-z)/4}$,

$$\Big(c\frac{d}{dz} + \frac{d^2}{dz^2}\Big)B(z) \geqslant \nu\|V\|^2 - M(2E_l^2 + \ell_z^4) + \tfrac{3c^2}{16}e^{c(z_0-z)/4}. \tag{6.7}$$

2. Set $B^* = \sup_{z\geqslant z_0} B(z)$. We want to show that $B^* \leqslant 0$ when $z_0 \gg 1$.

Suppose $B^* > 0$. Since $B(z_0) < 0$ and $\lim_{z\to\infty} B(z) = 0$, there exists $z_1 > z_0$ such that

$$B^* = B(z_1) = \|V(z_1)\|^2 - \tfrac{4M}{\nu}E_l^2(\ell(z_1)) - e^{c(z_1-z_0)/4}.$$

Note that $B_z(z_1) = 0$ and $B_{zz}(z_1) \leqslant 0$. It follows from (6.7) that,

$$\|V(z_1)\|^2 \leq \tfrac{M}{\nu}\Big(2E_l^2(\ell(z_1)) + \ell_z^4(z_1)\Big) - \tfrac{3c^2}{16\nu}e^{c(z_0-z_1)/4}. \tag{6.8}$$

Substituting this estimate into the definition of B^* gives

$$0 < B^* \leqslant \frac{M}{\nu}\ell_z^4(z_1) - \frac{2M}{\nu}E_l^2(\ell(z_1)) - \Big(1 + \tfrac{3c^2}{16\nu}\Big)e^{c(z_0-z_1)}.$$

or

$$\begin{aligned}\frac{2M}{\nu}E_l^2(\ell(z_1)) \quad &\leqslant \quad \frac{M}{\nu}\ell_z^4(z_1) - \Big(1 + \tfrac{3c^2}{16\nu}\Big)e^{c(z_0-z_1)}, \\ E_l(\ell(z_1)) \quad &< \quad \sqrt{2}\ell_z^2(z_1).\end{aligned}$$

Consequently, by (6.8),

$$\|V(z_1)\|^2 \leqslant \frac{2M}{\nu}\ell_z^4(z_1) - \Big(1 + \tfrac{6c^2}{16\nu}\Big)e^{c(z_0-z_1)/4}.$$

3. Also, taking larger z_0 if necessary, we have for any $z \geqslant z_0$,

$$\begin{aligned}\frac{d}{dz}\Big(e^{cz/4}E_l\Big) &= e^{cz/4}E_l\Big\{\tfrac{c}{4} + \frac{E_{ll}\ell_z}{E_l}\Big\} > 0, \\ \frac{d}{dz}\Big(e^{cz/4}E_l^2\Big) &= e^{cz/4}E_l^2\Big\{\tfrac{c}{4} + \frac{2E_{ll}\ell_z}{E_l}\Big\} > 0.\end{aligned}$$

Namely, both $e^{cz/4}E_l$ and $e^{cz/4}E_l^2$ are increasing functions on $[z_0, \infty)$.

Since $\ell_z > 0$, we have from (6.6) that, for all $z \geq z_0$, $\frac{c}{2}\ell_z + \ell_{zz} \leq M(E_l + \|V\|^2)$. Using the definition of B^*, this implies that

$$\begin{aligned} \frac{d}{dz}\Big(e^{c(z-z_1)/2}\ell_z\Big) &\leqslant Me^{c(z-z_1)/2}(E_l + \|V\|^2) \\ &\leqslant Me^{c(z-z_1)/2}\Big\{E_l + \tfrac{4M}{\nu}E_l^2 + e^{c(z_0-z)/4} + B^*\Big\}. \end{aligned}$$

Integrating this inequality over $[z_0, z_1]$ and using the monotonicity of $e^{c(z-z_1)/4}E_l$ and $E^{c(z-z_1)/4}E_l^2$ we obtain

$$\begin{aligned} \ell_z(z_1) - \ell_z(z_0)e^{c(z_0-z_1)/2} &\leqslant \tfrac{4M}{c}\Big\{E_l(\ell(z_1)) + \tfrac{4M}{\nu}E_l(\ell(z_1))^2 + e^{c(z_0-z_1)/4} + B^*\} \\ &= \tfrac{4M}{c}\Big(E_l(\ell(z_1)) + \|V(z_1)\|^2\Big) \\ &\leqslant \tfrac{4M}{c}\Big(\sqrt{2}\ell_z^2(z_1) + \tfrac{2M}{\nu}\ell_z^4(z_1) - \Big(1 + \tfrac{6c^2}{16\nu}\Big)e^{c(z_0-z_1)/4}\Big). \end{aligned}$$

This implies that

$$\Big(1 - \tfrac{4\sqrt{2}M}{c}\ell_z(z_1) - \tfrac{8M^2}{\nu c}\ell_z^3(z_1)\Big)\ell_z(z_1) \leqslant e^{c(z_0-z_1)/4}\Big(\ell_z(z_0) - \tfrac{4M}{\nu}\Big(1 + \tfrac{6c^2}{16\nu}\Big)\Big).$$

This is impossible for large z_0 since $\ell_z > 0$ and $\lim_{z\to\infty}\ell_z = 0$. Hence $B^* = 0$ and we have

Lemma 6.4 *Assume $n = 1$. There exists $z_0 \gg 1$ such that $\|V\|^2 \leqslant \frac{4M}{\nu}E_l^2 + e^{-c(z-z_0)/4}$ $\forall\, z \geq z_0$.*

Now the equation for ℓ_z can be written as

$$[c + o(1)]\ell_z + \ell_{zz} = \frac{1 + o(1)}{\|\phi_l\|^2}E_l + O(1)e^{-cz/4}.$$

Since $E_l = \sigma A e^{-2\mu\ell + o(1)}$ and $\ell(z) \sim R(z) = o(1)z$, we see that $e^{-cz/4} = o(1)E_l$ for large z. Hence, using $\|\phi_l\|^2 = \sigma + o(1)$, we can summarize our result as follows.

Lemma 6.5 *Assume $n = 1$. Then the function ℓ defined Lemma 6.2 satisfies*

$$[c + o(1)]\ell_z + \ell_{zz} = [A + o(1)]e^{-2\mu\ell} \qquad \forall z \gg 1 \tag{6.9}$$

where A is as in (6.2) and $\lim_{z\to\infty} o(1) = 0$.

Define $Q(z) = e^{2\mu\ell}$. The equation can be written as

$$2\mu A + o(1) = [c + o(1)]Q' + Q'' - 2\mu\ell_z Q' = [c + o(1)]Q' + Q''.$$

With an integrating factor this gives

$$Q'(z) \quad = \quad Q'(z_0)e^{-\int_{z_0}^{z}[c+o(1)]d\hat{z}} + \int_{z_0}^{z} e^{-\int_{\hat{z}}^{z}[c+o(1)]d\tilde{z}}\Big[2\mu A + o(1)\Big]d\hat{z} = 2\mu A/c + o(1).$$

Thus, $Q(z) = 2\mu Az/c + o(z)$. From the definition $Q = e^{2\mu\ell}$ we then conclude that

$$2\mu\ell = \ln\Big\{\frac{[2\mu A + o(1)]z}{c}\Big\} \quad \text{or} \quad \frac{\cosh(2\mu\ell)}{\mu z} = \frac{A}{c} + o(1).$$

Here we used the fact that $\cosh s = [\frac{1}{2} + o(1)]e^{|s|}$ for $|s| \gg 1$.

We remark that we can obtain iteratively more and more accurately expansions.

6.8 The Case $n > 1$

Assume $n > 1$. Following a similar analysis as above, we can derive the projected dynamics

$$[c + o(1)]\,\ell_z + \ell_{zz} = \frac{[1 + o(1)]E_\ell}{\|\phi_l\|^2} = \frac{n - 1 + o(1)}{\ell}.$$

This gives

$$z = \frac{c + o(1)}{2(n-1)}\,\ell^2, \quad \lim_{z\to\infty} o(1) = 0.$$

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