Existence of pulsating waves in a model of flames in sprays

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Abstract

A one-dimensional system describing the propagation of low Mach number flames in sprays is studied. We show that pulsating waves may exist when the droplet distribution in the unburnt region is spatially periodic. The range of possible propagation speeds may be either bounded or unbounded, depending on the threshold temperatures of the burning and vaporization rates.

1 Introduction and main results

In view of the breadth of their potential applications and complexity, models for flame propagation in sprays present significant analytical and computational challenges. It is therefore desirable to have a good understanding of the basic phenomena when flame propagation involves both burning of the gas and droplet evaporation that converts them into the flammable gas and provides an additional source of fuel for the flame. We consider a simplified model for the propagation of one-dimensional flames in sprays (see [16] for the background on this model) that involves three unknowns (T, Y, S):

- $T(t,x)$ is the temperature of the mixture,
- $Y(t,x)$ is the mass fraction of the gaseous reactant,
- $S(t,x)$ represents the surface of the droplets that are, at time t, located at x. We are thus making the – simplistic but not irelevant – assumption that, at every point in time and space, the droplets have only one radius. Such a spray is called *monodisperse*.

The evolution of T, Y and S is described by the following system:

$$\left\{ \begin{aligned} T_t - T_{xx} &= \; Yf(T) \\ Y_t - Y_{xx} &= \; -Yf(T) - \partial_t(S^{3/2}) \qquad (t,x) \in \mathbb{R}^2 \\ S_t &= \; -\varphi(T)S \end{aligned} \right. \tag{1.1}$$

The first two equations in (1.1) are the usual thermo-diffusive system but for the last term in the equation for Y that represents the gas production due to evaporation of droplets. The last equation describes the decrease in the number of droplets available because of the same process. The reaction rate f and evaporation rate φ are smooth, and satisfy the following assumptions:

- we have $f(0) = \varphi(0) = 0$; moreover f and φ are nonnegative nondecreasing functions, uniformly bounded from above:

$$0 \le \varphi(T), f(T) \le C, \quad \text{for } 0 \le T < +\infty; \tag{1.2}$$

- there exists $0 \le \theta_v \le \theta_i < +\infty$ such that f (resp. φ) is positive on $(\theta_i, +\infty)$ (resp. $(\theta_v, +\infty)$) and zero outside. If $\theta_v = 0$ we assume $\varphi'(0) > 0$.

The thresholds θ_v and θ_i are, respectively, called the boiling and ignition temperatures. The ignition temperature can (and sometimes will) be taken equal to 0.

The system (1.1) has been considered in [7] where travelling fronts have been shown to exist when the density of droplets in the unburnt region is uniform. In this paper we consider a different physical setting: a periodic distribution of liquid droplets is located "ahead of the front "at $-\infty$ (the convention is that the flame propagates from the right to the left). This leads to the following boundary conditions as $x \to -\infty$:

$$T(t, -\infty) = 0, \; Y(t, -\infty) = Y_u, \quad \lim_{x \to -\infty} (S(t,x) - S_u(x)) = 0, \tag{1.3}$$

where

- Y_u is a nonnegative quantity, that may be – and sometimes will be – 0;
- $S_u(x)$ is a smooth, positive, 1-periodic function.

The boundary conditions in the burned region, as $x \to +\infty$ are as follows. If we ask – which is quite natural – that everything is burnt at $+\infty$, then both S and Y shall be 0 at $+\infty$:

$$S(+\infty) = Y(+\infty) = 0. \tag{1.4}$$

We will see that the value T_b of T as $x \to +\infty$ will be determined automatically by the values of Y_u and the average of $S_u(x)$ over the period:

$$T_b = T(+\infty) = Y_u + \langle S_u^{3/2} \rangle. \tag{1.5}$$

Here $\langle g \rangle$ denotes the average of a function g over its period.

As we have mentioned, should the droplet distribution S_u at $-\infty$ be constant, the system under study would admit travelling waves [7]. The periodicity of the radius distribution at $-\infty$ leads us to replace this notion by the wider notion of pulsating waves [6, 17], namely solutions of (1.1)-(1.3) that are time-periodic in some galilean reference frame. In more mathematical terms these are solutions of (1.1) of the form $T(t,x) = U(x + ct, x)$ with a function U that is periodic in the second variable. Alternatively, there exists a speed $c > 0$ such that

$$(T, Y, S)(t, x - ct) \text{ is } \frac{1}{c} \text{ periodic in } t.$$

Our first result deals with the case of a positive ignition temperature.

Theorem 1.1 *(Nonzero ignition temperature) Assume $\theta_i > 0$. There exists a constant $q_0 \in (0,1)$ such that, if we have*

$$\frac{\theta_v}{\theta_i} \le q_0 \quad \textit{and} \quad \max(Y_u, \min_{x \in [0,1]} S_u(x)) \ge \frac{1}{q_0} \tag{1.6}$$

then the problem (1.1)-(1.3) has a pulsating wave solution.

The first condition in (1.6) makes sure that the evaporation produces fuel at sufficiently low temperatures on the left. The second assumption means that there is enough fuel, either liquid or gaseous, in the unburnt region. We also show that there exist two constants c_0 and c_1 that depend only on the data of the problem so that the pulsating front speed c satisfies

$$0 < \underline{c}_0 \le c \le \overline{c}_1 < +\infty.$$

This means that the possible range of speeds is a priori bounded in the ignition case.

The case of a zero ignition temperature is treated separately. Following the terminology of the thermo-diffusive systems we refer to it as the 'KPP case', although the underlying physics, and, potentially, the behavior of solutions, may vary according to the respective proportion of droplets and gaseous fuel at $-\infty$.

Theorem 1.2 *Assume $\theta_i = \theta_v = 0$. There exists $c_0 > 0$ such that the problem* (1.1)-(1.3) *has a pulsating wave solution with the speed c_0.*

It is well known that, in the case of scalar reaction-diffusion equations

$$u_t - u_{xx} = uf(u)$$

with, for instance, $f > 0$ on $[0, 1)$ and $f(1) = 0$, there exists $c_0 > 0$ such that, for all $c \geq c_0$, the above problem has travelling wave solutions that move with the speed c and connecting $u = 0$ to $u = 1$. We would therefore like to see if this is the case here; we have a partial result in this direction.

Theorem 1.3 *Assume, for simplicity, that $f(T) = \varphi(T) = T$ for $0 \leq T \leq 1$ and otherwise satisfy the aforementioned assumptions.*

(i) There exists $c_0 > 0$ such that any pulsating wave solution to (1.1)-(1.3) *has a velocity larger than c_0.*

(ii) Assume in addition that $Y_u > 0$. There exists $c_1 \geq c_0$ such that, for all $c \geq c_1$, the problem (1.1)-(1.3) *has a pulsating wave solution that moves with the speed c.*

We do not know whether $c_1 = c_0$, that is, if the range of speeds is a semi-infinite interval. In particular, we do not know if there exists a pulsating front that moves with the minimal speed. It is also not known whether the velocity spectrum is unbounded when $Y_u = 0$.

We mention that while existence of pulsating fronts for a single reaction-diffusion equation has been extensively studied [6, 17], the only result for reaction-diffusion systems that we are aware of, is that of [15], where small (but non-trivial) perturbations of a planar travelling front have been considered.

The paper is organized as follows. Sections 2 and 3 are devoted to the proof of Theorem 1.1: in Section 2, we perform *a priori* estimates on potential pulsating wave solutions to (1.1)-(1.3); in Section 3 we prove the actual existence by a degree argument allowed by the estimates of the preceding section. In Section 4 we prove Theorem 1.2 by approximation by a sequence of problems with ignition temperature. Section 5 is devoted to Theorem 1.3: it is done *via* a direct estimate on the velocity, an additional weighted estimate for the temperature and a homotopy argument. Possible extensions are discussed in Section 6.

2 The case of nonzero ignition temperature: a priori estimates

The main difficulty in deriving uniform bounds will be to bound Y. Indeed, as opposed to what happens in a purely gaseous flame, there is a positive source term in the equation for Y due to droplet evaporation. This may in turn cause a (mild) unboundedness for the function $Y(t, x)$. We do not see any convincing reason why the Cauchy problem for (1.1) would produce a solution whose Y-component would be unbounded, but we are not able to prove it at the moment. It turns out, however, that finding a lower bound for the velocity will help us, because this puts us in the framework of periodic functions with a bounded period $L = 1/c$. This is a large restriction to the set of global solutions that we have to investigate, and this will help us in finding the upper bounds that will in turn allow us to set up a degree argument.

It also turns out that the bounds that we shall find strongly depend on the value of Y_u. If Y_u is large enough – larger than the ignition temperature θ_i – we will be faced with a flame in which the presence of the spray neither helps, nor prevents, the propagation; the physics of the phenomenon is that of a purely gaseous flame. As opposed to that, when Y_u is small or 0, the propagation cannot take place without the help of the evaporation process; in the limit $Y_u \to 0$ it is really this process that governs the propagation. Thus we are confronted with two very different physics; a situation which is described in detail for travelling waves in [7].

In what follows, we consider a pulsating wave solution (T, Y, S) to (1.1), with a speed c. System (1.1) is then written in the reference frame of the wave; thus the system under study is now:

$$\left\{\begin{array}{rl} T_t - T_{xx} + cT_x = & Yf(T) \\ Y_t - Y_{xx} + cY_x = & -Yf(T) - (\partial_t + c\partial_x)(S^{3/2}) \qquad (t,x) \in \mathbb{R}^2. \\ S_t + cS_x = & -\varphi(T)S \end{array}\right. \tag{2.1}$$

As we have mentioned, the conditions at $+\infty$ for temperature T cannot be chosen arbitrarily. Adding up the equations for T and Y, then integrating the whole lot on $(0, \frac{1}{c}) \times \mathbb{R}$ yields

$$(T + Y)(+\infty) = Y_u + \langle S_u^{3/2} \rangle.$$

This last value will be denoted by $T_b(Y_u, S_u)$ – the burnt gas temperature; these considerations are therefore summed up in the conditions at $\pm\infty$:

$$\left\{\begin{array}{rlrl} T(t,-\infty) = & 0, & T(t,+\infty) & = T_b(Y_u, S_u) \\ Y(t,-\infty) = & Y_u, & Y(t,+\infty) & = 0 \\ \lim\limits_{x \to -\infty}(S(t,x) - S_u(x)) = & 0, & S(t,+\infty) & = 0 \end{array}\right. \tag{2.2}$$

and the periodicity condition

$$(T, Y, S)(t + \frac{1}{c}, x) = (T, Y, S)(t, x). \tag{2.3}$$

2.1 A lower bound for the flame speed

The main result of this section is the following proposition. Hereafter we set

$$\underline{S}_u = \inf_{x \in [0,1]} S_u(x). \tag{2.4}$$

Proposition 2.1 *If* $\max(Y_u, \underline{S}_u)$ *is large enough, there exists* $\underline{c_0} > 0$*, depending only on the data, such that any solution* (c, T, Y, S) *of* (2.1)-(2.3) *satisfies:* $c \geq \underline{c_0}$.

PROOF. Two cases should be distinguished.
1. The case of large Y_u. Namely, we assume that we have

$$Y_u > \theta_i. \tag{2.5}$$

As is classical, introduce the enthalpy function

$$W(t,x) = T(t,x) + Y(t,x); \tag{2.6}$$

we have

$$W_t - W_{xx} + cW_x = -\partial_t(S^{3/2}) \geq 0; \quad W(t, \pm\infty) \geq Y_u. \tag{2.7}$$

Letting $t \to +\infty$ yields $W(t \to +\infty) \geq Y_u$ – moreover, this is true for all times as W is periodic in time. Therefore we have

$$T_t - T_{xx} + cT_x \geq (Y_u - T)f(T) := g(T). \tag{2.8}$$

Take any $\delta > 0$ such that $Y_u - \delta > \dfrac{Y_u + \theta_i}{2}$ and consider the travelling wave $\underline{T}_\delta(x + c_\delta t)$, solution of

$$u_t - u_{xx} = g_\delta(u), \quad u(t, -\infty) = -\delta, \ u(t, +\infty) = Y_u - \delta.$$

Here $g_\delta(u) = (Y_u - \delta - u)f(u)$ with the convention that $f(u) = 0$ for $u < 0$. The front speed c_δ is controlled from below by some $\underline{c_0} > 0$ depending only on Y_u, the smoothness of g and the size of δ. We claim that $c \geq c_\delta$; indeed assume the contrary: this makes $\underline{T}_\delta(x)$ a sub-solution to (2.8); moreover, due to its limits at $\pm\infty$ we may, up to the correct translation, assume that $\underline{T}_\delta(x) \leq T(0, x)$ with a contact point x_0. The maximum principle implies $T(\dfrac{1}{c}, x) \geq \underline{T}_\delta(x)$; moreover the periodicity in t implies that x_0 is still a contact point between $T(\dfrac{1}{c}, .)$ and $\underline{T}_\delta$. This contradicts the strong maximum principle.

2. The case of a large $\underline{S}_u$. We give the proof in the case of zero boiling temperature; the case $\theta_v > 0$ differs only by computational details. As $\phi'(0) > 0$, there exist $0 < \underline{\varphi} < \overline{\varphi}$ such that

$$\underline{\varphi} T \leq \varphi(T) \leq \overline{\varphi} T \text{ on } [0, \theta_i]. \tag{2.9}$$

The non-increase of the lap number for a parabolic equation [1], combined with the fact that $T(t,x) - \theta_i$ is a time-global solution for the (appropriately re-written to subtract θ_i) first equation of (2.1), there exists a smooth, c^{-1}-periodic function $x_i(t)$ such that

$$\forall t \in \mathbb{R}, \quad x < x_i(t) \implies T(t,x) < \theta_i, \qquad x > x_i(t) \implies T(t,x) > \theta_i. \tag{2.10}$$

Let x_i^- be the minimum value of x_i; we may assume it to be 0. The function $x \mapsto \theta_i e^{cx}$ satisfies the equation for T for $x < 0$ and is above T for $(t,x) \in \mathbb{R} \times \{0\}$; letting $t \to +\infty$ implies – after shifting the time origin: $T(t,x) \leq \theta_i e^{cx}$ for $(t,x) \in \mathbb{R} \times \mathbb{R}_-$. Then we have

$$S_t + cS_x \geq -\overline{\varphi} TS \geq -\overline{\varphi}\theta_i e^{cx} S, \quad x < 0.$$

This implies – once again let $t \to +\infty$ and use the fact that S is $\dfrac{1}{c}$-periodic in t:

$$S(t,x) \geq \underline{S}_u \exp\left(-\frac{\overline{\varphi}\theta_i}{c^2} e^{cx}\right), \quad x < 0. \tag{2.11}$$

Our goal is to find a sub-solution to the equation for Y in $\mathbb{R} \times \mathbb{R}_-$. Note that in this domain Y satisfies

$$Y_t + cY_x - Y_{xx} = \frac{3}{2}\phi(T)S^{3/2} \geq \frac{3}{2}\underline{\varphi} TS^{3/2}.$$

To find an explicit sub-solution for Y we would be glad to replace $T(t,x)$ by $\theta_i e^{cx}$; unfortunately it is an upper bound for T; not a lower bound. To make up for that we first assume $c < 1$ – otherwise the proof would end here, then re-scale t and x in a parabolic fashion:

$$\tau = c^2 t, \quad \xi = cx. \tag{2.12}$$

Then T satisfies

$$T_\tau - T_{\xi\xi} + T_\xi = 0 \text{ on } \mathbb{R}\times\mathbb{R}_-; \ T(\tau+c,\xi) = T(\tau,\xi).$$

In particular, there exists $\tau_0 \in [0,1]$ such that $T(\tau_0+nc,0) = \theta_i$ for all $n\in\mathbb{Z}$ – these are the times when the curve $x_i(t)$ hits zero. It follows that $T(\tau_0+nc,\xi) > \theta_i$ for all $\xi > 0$. Without loss of generality we may assume that $\tau_0 = 0$. Moreover, T is a super-solution to the advection equation everywhere. This implies

$$\forall (t,x) \in [0,1]\times\mathbb{R}_+, \quad T(\tau,\xi) \geq \underline{T}(\tau,\xi)$$

where

$$\begin{cases} \underline{T}_\tau - \underline{T}_{\xi\xi} + \underline{T}_\xi = & 0 \qquad (0<\tau<1, \ \xi>0) \\ \underline{T}(\tau,0) = & 0 \qquad (0<\tau<1) \\ \underline{T}(0,\xi) = & \theta_i \qquad (\xi>0) \end{cases}$$

The parabolic Harnack inequality implies the existence of $C>0$ such that

$$\forall \tau \in [0,1], \quad \underline{T}(\tau,1) \geq C\theta_i. \tag{2.13}$$

We may now construct yet another sub-solution $\underline{\Phi}$ for T as

$$\begin{cases} \underline{\Phi}_\tau - \underline{\Phi}_{\xi\xi} + \underline{\Phi}_\xi = & 0 \qquad (0<\tau<1, \ \xi>0) \\ \underline{\Phi}(\tau,1) = & C\theta_i \qquad (0<\tau<1) \\ \underline{\Phi}(0,\xi) = & \theta_i \qquad (0\leq\xi<1) \\ \underline{\Phi}(0,\xi) = & 0 \qquad (\xi<0). \end{cases}$$

The maximum principle implies that $\underline{\Phi} \leq T$, hence applying the parabolic Harnack inequality once again we obtain, for a possibly different $C>0$:

$$\forall \tau \in \mathbb{R}_+, \quad T(\tau,0) \geq C\theta_i.$$

Still working in the (τ,ξ) variables we end up with – let $\tau\to+\infty$; use the time-periodicity of T:

$$\forall (\tau,\xi)\in\mathbb{R}\times\mathbb{R}_-, \quad T(\tau,\xi) \geq C\theta_i e^\xi. \tag{2.14}$$

Consequently we obtain, for $(\tau,\xi)\in\mathbb{R}\times\mathbb{R}_-$:

$$Y_\tau - Y_{\xi\xi} + Y_\xi \geq \frac{C\varphi\theta_i e^\xi}{c^2}\underline{S}_u^{3/2}\exp\Big(-\frac{3\overline{\varphi}\theta_i}{2c^2}e^\xi\Big). \tag{2.15}$$

An eventual sub-solution to (2.15) is obtained by setting the time-derivative equal to 0 and solving the differential equation with the zero boundary conditions at $\pm\infty$; recall that the solution of

$$-u'' + u' = f \text{ on } \mathbb{R}_-, \qquad u(0) = u(-\infty) = 0$$

is

$$u(\xi) = \int_\xi^0 (e^{\xi-\zeta} - e^\xi) f(\zeta)\, d\zeta + (1-e^\xi)\int_{-\infty}^\xi f(\zeta)\, d\zeta. \tag{2.16}$$

Applying formula (2.16) with $f(\xi) = \dfrac{C\overline{\varphi}\theta_i e^\xi}{c^2}\exp\Big(-\dfrac{3\overline{\varphi}\theta_i}{2c^2}e^\xi\Big)$ we obtain

$$Y(\tau,-1) \geq C\underline{S}_u^{3/2}\int_{-\infty}^{-1}\frac{\overline{\varphi}\theta_i e^\zeta}{c^2}\exp\Big(-\frac{3\overline{\varphi}\theta_i}{2c^2}e^\zeta\Big)d\zeta \geq C\underline{S}_u^{3/2}\int_0^{C/c^2} e^{-\eta}d\eta \geq C\underline{S}_u^{3/2} \tag{2.17}$$

for a constant $C>0$ under control, as we have assumed that $c<1$.

Come back to the (t,x) variables. As a consequence of (2.17) we have $Y(t,-1/c)\geq C\underline{S}_u^{3/2}$, and the function $W(t,x)$ – recall that it is given by (2.6) – satisfies

$$W_t-W_{xx}+cW_x\geq 0;\ \ W(t,+\infty)\geq C\langle S_u^{3/2}\rangle,\ \ W(t,-1/c)\geq C\underline{S}_u^{3/2}. \tag{2.18}$$

Therefore – once again, passing to $t\to+\infty$ and using the periodicity of W in time we conclude that if $\underline{S}_u$ is large enough, there is $\delta>0$ such that $W(t,x)\geq\theta_i+\delta$ for $x\geq -1/c$. We conclude as in Part 1: let T^δ be a travelling wave solution of

$$c_\delta T_x^\delta=T_{xx}^\delta+(\Pi-T^\delta)f(T^\delta),\quad T^\delta(-\infty)=-\delta, T^\delta(+\infty)=\Pi$$

with $\theta_i<\Pi<C\underline{S}_u^{3/2}$. Then c_δ is bounded below uniformly in δ for δ small. Moreover, if $c<c_\delta$, then as $T_x^\delta>0$, we have

$$cT_x^\delta<T_{xx}^\delta+(W-T^\delta)f(T^\delta),\ \ x>-1/c, \tag{2.19}$$

and of course,

$$T_t+cT_x=T_{xx}+(W-T)f(T) \tag{2.20}$$

holds. Now, shift T^δ to the right so much that $T^\delta(x)<T(0,x)$ for all x and start moving it back to the left. There will be x_0 so that the shifted $T^\delta(x)$ and $T(0,x)$ touch for the first time. Then we would still have $T^\delta\leq T(0,x)$ everywhere, which would mean that (2.19) holds even for $x\leq -1/c$ simply because $f(T^\delta)=0$ there. Therefore, for this particular shift we have both that T^δ and T touch at exactly one point and that T^δ satisfies (2.19). Then we get the contradiction as in Part 1 of this proof.

If $\theta_v\neq 0$ we only have to replace $\varphi(T)$ by $\underline{\varphi}(T-\theta_v)$, for a θ_v slightly above the actual vaporization temperature and use the same argument. □

2.2 The upper bounds

This paragraph is devoted to L^∞ bounds for all the unknown functions, as well as an upper bound for the front speed. The lower bound for c obtained in the previous section will in particular be of use to us. The result of this section is best stated in terms of the enthalpy function W defined by (2.7).

Proposition 2.2 *Assume that $\theta_v\leq q_0\theta_i$ with a constant $q_0<1$. There exists a constant $\overline{W}>0$, depending only on the data, the lower bound $\underline{c_0}$ of Proposition 2.1, and q_0, such that $\|W\|_{C^1(\mathbb{R}^2)}\leq\overline{W}$.*

An immediate corollary is

Corollary 2.3 *There exists $c_1>0$, depending only on the data and $\underline{c_0}$, so that $c\leq c_1$.*

PROOF. Assuming Proposition 2.2 to be true, the temperature $T(t,x)$ satisfies

$$T_t-T_{xx}+cT_x\leq(\overline{W}-T)f(T);\quad T(t,-\infty)=0,\ T(t,+\infty)\leq\overline{W}.$$

Arguing as in Proposition 2.1, part 1 we may prove that, if $\delta>0$ is below θ_i and $(c_1,\overline{T}(x))$ solves

$$-\overline{T}''+c_1\overline{T}'=(\overline{W}+\delta-\overline{T})f(\overline{T}),\ \ \overline{T}(t,-\infty)=\delta,\ T(t,+\infty)=\overline{W}+\delta,$$

then $c\leq c_1$. □

PROOF OF PROPOSITION 2.2. We do not know yet any upper bound for c; to make up for that let us come back to the parabolic scaling $\tau = c^2 t$, $\xi = cx$. The equation for W is then

$$W_\tau - W_{\xi\xi} + W_\xi = \frac{3\varphi(T)S^{3/2}}{2c^2}. \tag{2.21}$$

We wish to find an eventual super-solution to W. To do so, let us define $\bar\xi$ such that

$$\forall \tau \in \mathbb{R}, \forall \xi \geq \bar\xi, \qquad \begin{array}{rl} T(\tau,\xi) \geq & \dfrac{T_b(Y_u + \langle S_u^{3/2}\rangle) + \theta_i}{2}, \\ W(\tau,\xi) \leq & 2T_b(Y_u, S_u). \end{array} \tag{2.22}$$

If $x_i(t)$ is defined by (2.10), let $\xi_i(\tau)$ be its counterpart in the (τ,ξ) system. Without loss of generality we may assume that the function ξ_i takes its minimum at $\xi = 0$. Two cases are to be discussed.
The region $\xi \leq 0$. We simply have $T(\tau,\xi) \leq \theta_i e^\xi$, hence

$$\varphi(T)S^{3/2} \leq C\|S\|_\infty^{3/2}\theta_i e^\xi. \tag{2.23}$$

The region $\xi \geq 0$. Let us find an upper bound for $S^{3/2}\varphi(T)$. We have – see Part 2 of Proposition 2.2 – a constant $q_1 \in (0,1]$ such that

$$\forall \tau \in \mathbb{R}, \ \forall \xi > 0, \quad T(\tau,\xi) \geq q_1\theta_i.$$

We take $q_0 < q_1$. Then there is $\bar\varphi > 0$ such that

$$\forall \tau \in \mathbb{R}, \ \forall \xi > 0, \quad \varphi(T(\tau,\xi)) \geq \bar\varphi.$$

An eventual super-solution for S in $\{\xi \geq 0\}$ is

$$\overline{S}(\tau,\xi) = \|S\|_\infty e^{-\overline{\varphi}\xi/c^2}. \tag{2.24}$$

This will bound $S(\tau,\xi)$ for $\xi \geq 0$. Gathering (2.23) and (2.24) we realize that there is $C > 0$ depending only on data such that

$$\forall \tau > 0, \quad \frac{1}{c^2}\|S^{3/2}(\tau)\varphi(T(\tau))\|_{L^1_\xi(\mathbb{R})} \leq C. \tag{2.25}$$

An eventual super-solution for W is the function $\overline{W}(\xi)$, which satisfies, on $\{\xi \leq \bar\xi\}$:

$$-\overline{W}'' + \overline{W}' = \frac{C}{c^2}\|S\|_\infty^{3/2}\theta_i e^\xi \mathbf{1}_{\mathbb{R}_-} + \frac{\|\varphi\|_\infty \|S\|_\infty^{3/2} e^{-\overline{\varphi}\xi/c^2}}{c^2}, \tag{2.26}$$

with $\overline{W}(-\infty) = Y_u$ and $\overline{W}(\bar\xi) = 2T_b(Y_u, S_u)$. A simple ODE integration shows that $\overline{W}(\xi)$ is bounded independently of c. The C^1 bounds follow from parabolic regularity. □

2.3 Uniform exponential decay

Assume that the solution (c, T, Y, S) of (2.1)-(2.3) additionally satisfies the normalization condition:

$$T(0,0) = \theta_i. \tag{2.27}$$

The main result of this subsection is the following.

Proposition 2.4 *There exist $\rho_0 > 0$ and $C > 0$, depending only on the data, such that we have, for all $t \in \mathbb{R}$:*

$$\begin{array}{ll} \forall x \in \mathbb{R}_-, & |(T(t,x), Y(t,x) - Y_u, S(t,x) - S_u(x-ct))| \leq Ce^{\rho_0 x} \\ \forall x \in \mathbb{R}_+, & |(T(t,x) - Y_u - \langle S_u^{3/2} \rangle, Y(t,x), S(t,x))| \leq Ce^{-\rho_0 x}. \end{array} \tag{2.28}$$

By parabolic regularity, it is sufficient to prove the following

Lemma 2.5 *There exist $\rho_0 > 0$ and $C > 0$, depending only on the data, such that we have, for all $t \in \mathbb{R}$:*

$$\begin{array}{l} \|e^{-\rho_0 x}(T(t,.), Y(t,.) - Y_u, S(t,.) - S_u(. - ct))\|_{L^2(\mathbb{R}_-)} \leq C \\ \|e^{\rho_0 x}(T(t,.) - Y_u - \langle S_u^{3/2} \rangle, Y(t,.), S(t,.))\|_{L^2(\mathbb{R}_+)} \leq C. \end{array} \tag{2.29}$$

An important intermediate step is

Lemma 2.6 *Let $x_i(t)$ be the function defined by* (2.10). *There is $\bar{x}_0 > 0$ such that, for all $t \in \mathbb{R}$, we have $|x_i(t)| \leq \bar{x}_0$.*

Proof of Lemma 2.6. Let $-x_0 \leq 0$ be the minimal value of x_i; the proof of Proposition 2.1 – see Part 2 – yields the existence of $C > 0$ such that $T(t, -x_0) \geq C\theta_i$. Also, remember the existence – as in the proof of Proposition 2.1 – of $\underline{W} > \theta_i$ - this real number only depends on data – such that $W(t,x) \geq \underline{W}$. Consequently we have

$$T_t - T_{xx} + cT_x \geq (\underline{W} - T)f(T); \quad T(t, +\infty) \geq \underline{W}. \tag{2.30}$$

For any $W > \theta_i$, let $c(W)$ be the unique speed of the travelling wave connecting 0 to W by the equation

$$-u'' + cu' = (W - u)f(u).$$

Two cases are to be investigated.

- We have $c \leq c(\underline{W})$. Then there exists - see [4] - a unique solution $\underline{T}(x)$ to

$$\begin{array}{rl} -\underline{T}'' + c\underline{T}' = & (\underline{W} - \underline{T})f(\underline{T}) \quad (x > x_0) \\ \underline{T}(x_0) = C\theta_i, \ \underline{T}(+\infty) = & \underline{W} \end{array} \tag{2.31}$$

  which is an eventual sub-solution to (2.29). Moreover, because of the boundedness of c from above, there is an absolute constant $\rho_0 > 0$ such that

$$\forall x > x_0, \quad |\underline{T}(x) - \underline{W}| \leq Ce^{-\rho_0(x - x_0)}. \tag{2.32}$$

- We have $c \geq c(\underline{W})$. Then - Aronson-Weinberger [2], [3] - we have

$$T(t,0) \geq \underline{W}.$$

Then Y decays, in both cases, exponentially to 0 as $x \to +\infty$, at a uniform rate. This implies an inequality for T of the form (2.32).
All this is enough to prove the lemma – the minimal value $-x_0$ can not be too negative since $T(0,0) \geq \theta_i$, and the maximal value of $x_i(t)$ is bounded directly by (2.32). □
Proof of Lemma 2.5. Once x_0 is known to be bounded, the first part of (2.29) – the bounds on the left – is easy: indeed, we have $0 \leq T(t,x) \leq \theta_i e^{c(x - x_0)}$; this is enough due to the uniform boundedness of c from below. Then, using the boundedness of S by $\|S_u\|_\infty$ we have the existence of a constant $C > 0$ such that

$$|Y_t - Y_{xx} + cY_x| \leq Ce^{c(x - x_0)};$$

an eventual, exponentially decreasing super-solution is easily found and left to the reader. The equation for S is treated in the same fashion.

Consider any $\delta > 0$. From Lemma 2.6, there is $x_1 > 0$, uniformly bounded, such that $T \geq \theta_i + \delta$ as soon as $x \geq x_1$. This forces an exponential decay for S due to the boundedness of c from below; this in turn forces an exponential decay for Y from the maximum principle. It remains to prove the L^2 bound for W; to do so we argue as follows. First, by commodity, re-scale the time: $\tau = ct$; the new function W is hence 1-periodic in t. Decompose $W(\tau, x)$ in a Fourier series:

$$W(\tau, x) = \sum_{n \in \mathbb{Z}} w_n(x) e^{2i\pi n\tau}.$$

An equation for $w_0(x)$ is

$$-w_0'' + cw_0' = \frac{3}{2} \int_0^1 \varphi(T) S(\tau, x)^{3/2} \, d\tau,$$

which implies

$$w_0(x) = T_b(Y_u, S_u) - \frac{3}{2} \int_x^{+\infty} e^{c(x-y)} \int_{-\infty}^y \int_0^1 \varphi(T) S(\tau, z)^{3/2} d\tau dz dy. \tag{2.33}$$

The limits for w_0 are Y_u and $T_b(Y_u, S_u)$, because $3\varphi(T)S(\tau, z)^{3/2}/2$ is exactly $-(\partial_t + \partial_x)S^{3/2}$; the desired L^2 bound for w_0, is obtained, at the possible expense of decreasing ρ_0 a bit, by recalling the exponential decay of S at $+\infty$ and the fact that $\varphi(T)$ vanishes for large negative x.

For $n \neq 0$ an equation for w_n is

$$-w_n'' + cw_n' + 2i\pi cn w_n = -(2i\pi cn + c\partial_x) \int_0^1 e^{-2i\pi n\tau} S^{3/2}(\tau, x) \, d\tau \ \ (x \in \mathbb{R}); \quad w_n(\pm\infty) = 0.$$

This equation has two characteristic roots $r^{\pm}$, whose real part is above (resp. below) $C(1 + \sqrt{n})$ (resp. $-C(1 + \sqrt{n})$) where C is a constant under uniform control. This implies, by an elementary computation, the existence of a small, uniform $\rho > 0$, such that

$$\|e^{\rho_0 |x|} w_n\|_{L^2(\mathbb{R})} \leq \frac{C}{n}. \tag{2.34}$$

This in turn implies an $L^2([0, 1], L^2(\mathbb{R}))$ bound for $e^{\rho_0 x} W(t, x)$. $\Box$

3 Nonzero ignition temperature: construction of the wave

The uniform L^2 bounds for W and T will allow us to perform directly a topological degree argument for the system (2.1)-(2.3) on the whole real line, without the approximation step on a finite interval taken in [5]. The system will first be reduced to a fixed point problem; then we shall introduce a homotopy bringing it onto the problem of finding a travelling wave for the 1D thermo-diffusive scalar equation – for which everything is known.

3.1 Strategy

To explain how we wish to proceed, let us start by recalling some basic facts. Assume that we are given a Banach space X and a sectorial operator A such that $\|e^{-A}\| < 1$. For $\alpha \in (0, 1)$, consider a function $f(t) \in C^{\alpha}(\mathbb{R}, X)$ which is 1-periodic. The Cauchy problem for

$$\dot{u} + Au = f(t) \tag{3.1}$$

is well-posed, in the sense that, for every initial datum $u_0 \in X$, it admits a unique strong solution $u(t)$ such that $u(0) = u_0$. We are interested in finding some 1-periodic solutions for (3.1); to do so it is sufficient to look for the initial datum; it is uniquely given by

$$u_0 = (I - e^{-A})^{-1} \int_0^1 e^{-(1-s)A} f(s)\, ds,$$

and the (unique) 1-periodic solution of (3.1) is given by

$$u(t) = e^{-tA}(I - e^{-A})^{-1} \int_0^1 e^{-(1-s)A} f(s)\, ds + \int_0^t e^{-(t-s)A} f(s)\, ds. \tag{3.2}$$

Let us denote by $\mathcal{F}f$ the right side of (3.2). If the right side of (3.1) is replaced by a nonlinear function $f(t,u)$ which is, say, Hölder in its first variable and Lipschitz in its second variable; and which is moreover 1-periodic in time, the problem of finding periodic solutions to

$$\dot{u} + Au = f(t,u)$$

reduces to

$$u = \mathcal{F}f(.,u). \tag{3.3}$$

It is this very simple fact that we wish to use in order to reduce (2.1)-(2.3) to a fixed point problem, the major point that we will have to care about being that $u \mapsto \mathcal{F}f(.,u)$ should be compact if we wish to have a chance to apply a degree argument.

3.2 Fixed point setting

Let us try to apply the above strategy, namely to define a subspace X of $C^{\alpha,\alpha/2}_{per}(\mathbb{R}^2) \times C^{\alpha,\alpha/2}_{per}(\mathbb{R}^2) \times C_{per}(\mathbb{R}^2) \times \mathbb{R}$ and a compact nonlinear mapping $\mathcal{F}$ of X such that (2.1)-(2.3) reduces to finding a fixed point of $\mathcal{F}$. Here the subscript $_{per}$ means that we are dealing with functions of the variables (t,x) that are 1-periodic in t.

In what follows, we take the boiling and ignition temperatures θ_i and θ_v **to be positive,** in agreement with the assumption $\theta_i > 0$ of Theorem 1.1. The other data are also assumed to be in agreement with the assumptions of this theorem. The limit $\theta_v \to 0$ will be considered at the end of this section.

The first step is to renormalize the time so that the period in (2.3) becomes 1. We take Y, W, S, c as our principal unknowns instead of T, Y, S, c; the reason for this choice will become clear as the discussion goes on. The set of equations that we have to satisfy then becomes

$$\left\{ \begin{array}{rl} cY_t - Y_{xx} + cY_x = & -Yf(W-Y) - c(\partial_t + \partial_x)(S^{3/2}) \\ cW_t - W_{xx} + cW_x = & -c(\partial_t + \partial_x)(S^{3/2}) \\ S_t + S_x = & -\dfrac{1}{c}\varphi(W-Y)S \end{array} \right. \qquad (t,x) \in \mathbb{R}^2 \tag{3.4}$$

together with the conditions at $\pm\infty$,

$$\left\{ \begin{array}{rlll} Y(t,-\infty) = & Y_u, & Y(t,+\infty) & = 0 \\ W(t,-\infty) = & Y_u, & W(t,+\infty) & = T_b(Y_u, S_u) \\ \lim\limits_{x \to -\infty} (S(t,x) - S_u(x)) = & 0, & S(t,+\infty) & = 0, \end{array} \right. \tag{3.5}$$

the periodicity condition

$$(Y, W, S)(t+1, x) = (Y, W, S)(t, x) \tag{3.6}$$

and the normalization condition

$$(W-Y)(0,0)=\theta_i. \tag{3.7}$$

Let us then define the space X_r as

$$X_r=\{u\in C^{\alpha,\alpha/2}_{per}:\quad e^{r|x|}u\in C^{\alpha,\alpha/2}_{per}\}. \tag{3.8}$$

We first choose once and for all $\alpha\in(0,1)$ which will measure the Hölder character of Y and W. Next, we recall the real number ρ_0 defined in Lemma 2.5, the lower bound $\underline{c_0}$ for the velocity, its upper bound: $\overline{c}_0$, and we fix $r>0$ such that

$$r<\frac{1}{5}\min\left(\rho_0,\underline{c_0},\frac{-\overline{c}_0+\sqrt{\overline{c}_0^2+4f(T_b(Y_u,S_u))}}{2}\right). \tag{3.9}$$

In fact, the function S will not be a principal unknown: we will compute it directly from W and Y. Assume therefore that Y and W are known; then the equation for S in (3.4) has a unique 1-periodic solution in t that is given by

$$S(t,x)=S_u(x-t)\exp\left(-\frac{1}{c}\int_{-\infty}^{0}\varphi_4(t+s,x+s)\ ds\right) \tag{3.10}$$

where we have set $\varphi_4(s,y)=\varphi(W-Y)(s,y))$. We denote the right side of (3.10) by $\mathcal{F}_4(c,Y,W)(t,x)$.

Let $\gamma(x)$ be a smooth nonnegative function that is equal to 0 on $(-\infty,-1]$ and to 1 on $\mathbb{R}_+$. In order to obtain unknowns that decay exponentially at $\pm\infty$, let $u_0(x)$ and $w_0(x)$ be defined as

$$u_0(x)=Y_u(1-\gamma(x));\quad w_0(x)=Y_u+\langle S_u^{3/2}\rangle\gamma(x). \tag{3.11}$$

Then look for Y and W in the form

$$Y=u_0+u,\quad W=w_0+w. \tag{3.12}$$

Examination of (3.10) and of the definition of ρ_0 in Lemma 3.1 yields the following estimate for $\mathcal{F}_4$, that we cast in the

Lemma 3.1 *Consider two functions Y and W of the form (3.12) with u, w in a bounded subset of X_r, and $c\in[\underline{c_0}/2,2\overline{c}_0]$. Then $\mathcal{F}_4(c,Y,W)(t,x)$ has the form*

$$\mathcal{F}_4(c,Y,W)(t,x)=S_u(x-t)(1-\gamma(x))+\tilde{\mathcal{F}}_4(c,Y,W)(t,x)$$

with $\tilde{\mathcal{F}}_4(c,Y,W)\in X_r$; moreover there is a constant $C(c,Y,W)>0$ such that

$$|\tilde{\mathcal{F}}_4(c,Y,W)(t,x)|\leq Ce^{-\rho_0|x|}.$$

Next we turn to the equation for Y. The first equation in (3.4) is rewritten as – for short we redefine T as $W-Y$, and we keep the notations W and Y when we do not want to underline a specific decay property:

$$\begin{array}{rl} & cu_t-u_{xx}+cu_x+\gamma(x)f(T_b(Y_u,S_u))u\\ = & -(f(T)-\gamma(x)f(T_b(Y_u,S_u))Y-\gamma(x)f(T_b(Y_u,S_u))u_0\\ & -c(\partial_t+\partial_x)\mathcal{F}_4(c,Y,W)^{3/2}+u_0''-cu_0'.\end{array}$$

Let A_c denote the differential operator

$$A_c=-c^{-1}\frac{d^2}{dx^2}+\frac{d}{dx}+c^{-1}\gamma(x)f(T_b(Y_u,S_u)).$$

If $UC(\mathbb{R})$ is the space of all bounded, continuous functions of $\mathbb{R}$, we define, for all $\rho>0$:

$$Y_\rho=\{u\in UC(\mathbb{R}):\quad e^{-\rho|x|}u\in UC(\mathbb{R})\}$$

We extract from (3.9) and [9], Chap. 5, the following

Lemma 3.2 *There is $\delta_0 > 0$, depending on $\underline{c_0}$ and ρ_0 such that, for every $c \in [\underline{c_0}/2, 2\overline{c}_0]$, and for all $\delta \in [0, \delta_0]$, the operator A_c is sectorial in $Y_{r+\delta_0}$; its spectrum is moreover within the right side of the complex plane, bounded away from the imaginary axis. This statement is uniform with respect to $\delta \in [0, \delta_0]$.*

The proof is standard and omitted. Thus e^{-A_c} has norm < 1 in $X_{r+\delta}$, for every $\delta \in [0, \delta_0]$; hence we may define $\mathcal{F}_2(c, u, w)$ as follows:

- the underlying space is the space Y_r;
- the quantity $\mathcal{F}_2(c, u, w)$ is defined by (3.2), with f being the RHS of 3.13.

Rephrasing Lemma 2.5, we have the

Lemma 3.3 *For the quantity δ_0 of Lemma 3.2, the mapping $(c, u, w) \mapsto \mathcal{F}_2(c, u, w)$ is C^1 and compact from $[\underline{c_0}/2, 2\overline{c}_0] \times X_r \times X_r$ into $X_{r+\delta_0}$.*

This implies the following

Proposition 3.4 *The mapping $\mathcal{F}_2$ is C^1 and compact and compact from $[\underline{c_0}/2, 2\overline{c}_0] \times X_r \times X_r$ into X_r.*

PROOF. Straightforward – but lengthy – by Lemma 3.3 and parabolic regularity. We omit it. □

We would now like to do the same operation for the W-equation, but we do not have here a term that ensures some coercivity at $+\infty$. What is, however, true is the following: set, for all $\rho > 0$:

$$\tilde{Y}_\rho = \{u \in UC(\mathbb{R}) : \quad e^{-\rho x} u \in UC(\mathbb{R})\}.$$

Then, for all $c \in [\frac{\underline{c_0}}{2}, 2\overline{c}_0]$, the operator

$$B_c = -c^{-1} \frac{d^2}{dx^2} + \frac{d}{dx}$$

satisfies: $\|e^{-B_c}\|_{\mathcal{L}(\tilde{Y}_r)} < 1$, uniformly with respect to the parameter c. The verification is even simpler than Lemma 3.2, and therefore omitted. Consequently, a mapping can be constructed as before: first, the equation for w is

$$cw_t - w_{xx} + cw_x = -c(\partial_t + \partial_x)(\mathcal{F}_4(c, Y, W))^{3/2} - w_0'' + cw_0'. \tag{3.13}$$

Then, by writing formula (3.2) with f as the RHS of (3.13), and $A = B_c$; we obtain a mapping that we call $\mathcal{F}_3(c, Y, W)$. Now, Lemma 2.5 together with parabolic regularity imply the

Proposition 3.5 *The mapping $\mathcal{F}_3$ is C^1 and compact and compact from $[\frac{\underline{c_0}}{2}, 2\overline{c}_0] \times X_r \times X_r$ into X_r.*

Finally, we define the mapping

$$\mathcal{F}_1(c, Y, W) = c - (W(0,0) - Y(0,0) - \theta_i). \tag{3.14}$$

Clearly, $\mathcal{F}_1$ is compact.

We are now ready to state the fixed point problem. Let $C > 0$ be such that, if (c, T, Y, S) solves (3.4)-(3.7), then the corresponding functions u and w defined by (3.12) are – due to all the estimates of Section 2 – in the open ball of X_r with center 0 and radius C, which we denote by $B_r(0, C)$. Now, define the open subset of $\mathbb{R} \times X_r \times X_r$ and the mapping $\mathcal{F}$ from Ω to $\mathbb{R} \times X_r \times X_r$ by

$$\Omega = \left(\frac{\underline{c_0}}{2}, 2\overline{c}_0\right) \times B_r(0, C) \times B_r(0, C); \quad \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3). \tag{3.15}$$

Clearly, $\mathcal{F}$ is compact from Ω to $X_r \times X_r \times \mathbb{R}$. Moreover, a fixed point of $\mathcal{F}$ cannot be on $\partial\Omega$.

3.3 The homotopy

To do the homotopy, we simply perturb the values at $-\infty$. Without loss of generality, we assume that $\theta_i < 1/2$. We then replace Y_u as the left limit of Y by $\tau Y_u + 1 - \tau$, and S_u by τS_u – that is, we replace $T_b(Y_u, S_u)$ by $T_b(\tau Y_u + 1 - \tau, \tau S_u)$. We note that, for $\tau \in [0,1]$, the estimates of Section 2 apply to these new conditions at $-\infty$; we call $\mathcal{F}^\tau$ the corresponding mappings defined in the preceding section. Clearly, $(\tau, c, u, w) \mapsto \mathcal{F}^\tau(c, u, w)$ is C^1 and compact and compact. Also, any fixed point of $\mathcal{F}^\tau$ is inside Ω according to the a priori estimates. We may therefore define $\deg(I - \mathcal{F}^\tau, \Omega, 0)$; it is constant with respect to τ. This triggers the last step of the

Proof of Theorem 1.1. It suffices to prove that $\deg(I - \mathcal{F}^\tau, \Omega, 0) \neq 0$. However, for $\tau = 0$, we have the usual thermo-diffusive system with the Lewis number equal to one – there are no droplets. Hence, a fixed point of $I - \mathcal{F}^\tau$ is such that the corresponding function W is exactly equal to 1, and the corresponding function Y is a solution of

$$Y_t - Y_{xx} + cY_x = -Yf(1-Y) := g(Y). \tag{3.16}$$

The only time-periodic solution of (3.16) such that $1 - Y$ is equal to θ_i at $(0,0)$ and goes to 0 at $+\infty$ is the 1D wave that we call Y_0 with the speed called c_0. Let us quickly prove that $I - \partial_{c,u,w}\mathcal{F}^0$ at the wave is an isomorphism of $\mathbb{R} \times X_r \times X_r$; notice that, because of what preceeds, $\partial_{c,u,w}\mathcal{F}^\tau$ is a compact operator of $\mathbb{R} \times X_r \times X_r$. Hence it suffices to prove that $I - \partial_{c,u,w}\mathcal{F}^0$ has a zero null space. By the definition of $\mathcal{F}^\tau$, it suffices to solve the following equation, with unknowns $(\tilde{c}, \tilde{u}, \tilde{w})$:

$$\left\{ \begin{array}{rl} \tilde{u}(0,0) = & 0 \\ (\partial_t - \partial_{xx} + c_0\partial_x - g'(Y_0))\tilde{u} = & 0, \quad u(t+1,.) = u(t,.) \\ \tilde{w} = & 0 \end{array} \right. \tag{3.17}$$

From [9], Chap. 5 – the operator

$$L_0 = -\frac{d^2}{dx^2} + c_0\frac{d}{dx} - g'(Y_0)$$

with domain in X_r, is non-degenerate, in the sense that the geometric and algebraic multiplicity of the eigenvalue 0 is 1, with associated eigenfunction Y_0'. Consequently, the second line of (3.17) implies that $u(t,x)$ is proportional to Y_0', and the first equation in turn implies that $u \equiv 0$.

To summarize, $\mathcal{F}^0$ has a unique zero in Ω, and $\partial_{c,u,w}\mathcal{F}^0$ is an isomorphism of $\mathbb{R} \times X_r \times X_r$, the consequence of which being [11]: $\deg(I - \mathcal{F}^\tau, \Omega, 0)$ is nonzero. This ensures the existence of a wave solution to (2.1)-(2.3) as soon as the evaporation temperature θ_v is positive.

It remains to send θ_v to 0. However, all the bounds that are proved in Section 2 are uniform with respect to θ_v; as soon as θ_i is fixed – in fact, $\theta_v > 0$ was only required to obtain some compactness for W. The passage to the limit $\theta_v > 0$ is therefore standard; see for instance [6] for the details. $\Box$

4 The KPP limit

As in [6], the strategy that we shall use here for obtaining the wave of lower velocity is to send the ignition temperature to 0. Our main problem is that the bounds devised so far are not uniform with respect to the ignition temperature. On the other hand, what we are now aiming at is the existence of waves when both ignition and vaporization temperatures are zero. This leaves us some freedom for the approximating sequences, and we will use one that will generate a painless estimate for Y – something that we had to work for in Section 2. This in turn will allow us a – less easy – estimate for the enthalpy. Also, we will in a first approximation keep the mass fraction of the unburnt gases nonzero: this will give us a free lower estimate, for in this case, Part 1 of the proof of Proposition 2.1 applies. All this is summed up in the following

Proposition 4.1 *Let $(f_\theta)_{\theta>0}$ and $(\varphi_\theta)_{\theta>0}$ be two sequences of Lipschitz functions, defined for small $\theta > 0$ having θ as ignition – resp. vaporization – temperatures. Assume moreover the ratio φ_θ/f_θ to be uniformly bounded from above, and bounded away from 0. Consider $Y_u > 0$ and a positive, smooth, 1-periodic function $S_u(x)$. Then there exists a family of solutions $(c_\theta, T_\theta, Y_\theta, S_\theta)$ to the problem (2.1)-(2.3). Moreover, the following properties hold*

- *the sequence (c_θ) is bounded away from 0,*
- *if W_θ is the enthalpy, then the sequence $\|W_\theta\|_\infty$ is bounded,*
- *the exponential estimates of Section 2.3 hold uniformly with respect to θ.*

Proof. It is clear that, given the considerations of Sections 2 and 3, a solution $(c_\theta, T_\theta, Y_\theta, S_\theta)$ to the problem (2.1)-(2.3) exists as soon as the estimates stated in the proposition hold, and the proof reduces to proving these estimates. In what follows, we consider (c, T, Y, S) a solution to (2.1)-(2.3); we have – and continue to do so in the course of the proof – deleted the subscript $_\theta$ in order to alleviate the notations.

1. Upper bound for Y. Break $Y(t,x)$ into $Y_1(t,x) + Y_2(t,x)$ where

$$\left\{\begin{array}{rl} (\partial_t - \partial_{xx} + c\partial_x + f(T))Y_1 = & 0 \quad (t>0,\ x\in\mathbb{R}) \\ Y_1(0,x) = & Y(0,x) \end{array}\right.$$

and

$$\left\{\begin{array}{rl} (\partial_t - \partial_{xx} + c\partial_x + f(T))Y_2 = & \varphi(T)S \quad (t>0,\ x\in\mathbb{R}) \\ Y_2(0,x) = & 0 \end{array}\right.$$

We have – by an elementary computation for the advection-diffusion equation:

$$\limsup_{t\to+\infty} Y_1(t,x) \le Y_u;$$

moreover, if the function $(s,x) \mapsto Y_2(s,x)$ has a maximum point (t_0, x_0), we have at that point

$$Y_2(t_0,x_0) \le \frac{\varphi(T(t_0,x_0))}{f(T(t_0,x_0))} S(t_0,x_0) \le C\|S_u\|_\infty,$$

as the ratio ϕ/f is uniformly bounded by assumption. If there is no maximum, we always may consider a maximizing sequence (t_n, x_n), consider the suitably translated sequence $Y(t+t_n, x+x_n)$ and send n to $+\infty$, to get the same estimate. In any case, this bounds Y from above.

2. Lower bound for c. Similar to Part 1 of the proof of Proposition 2.1.

In the next two steps we revert – for commodity, and without change of notation – to the original reference frame; thus the functions (T, Y, S) satisfy (1.1)-(1.3). Of course, the benefit of the estimates of the previous two steps is kept.

3. L^∞ bound for T_t and T_{xx}. We start from the Duhamel formula for $W(t,x)$:

$$W(t,x) = e^{t\partial_{xx}}W(0,x) + \int_0^t \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}} (-\partial_s S^{3/2})\ ds.$$

The free term $e^{t\partial_{xx}}W(0,x)$ has the eventual bound $\dfrac{T_b(Y_u,S_u)}{2}$ and is of no concern. The remaining term, that we denote by $W_1(t,x)$, is broken into

$$W_1(t,x) = \int_0^{t-1} + \int_{t-1}^t := W_{11} + W_{12},$$

which we study separately.
[i]. Because $\varphi(T)$ and S are both bounded (see (1.2)), we immediately infer from the parabolic regularity that, for every $p \in (1, +\infty)$, there is $C_p > 0$ such that, for every bounded interval I of length $1/2$, and for every $(t, x) \in (2, +\infty) \times \mathbb{R}$ we have

$$\|\partial_t W_{12}\|_{L^p((t,x)+I^2)} + \|\partial_{xx} W_{12}\|_{L^p((t,x)+I^2)} \leq C_p. \tag{4.1}$$

Indeed W_{12} solves

$$(\partial_s - \partial_{xx}) W_{12} = -\partial_s S^{3/2} = \frac{3}{2}\varphi(T) S^{3/2} \text{ for } (s,x) \in (t-1, t) \times \mathbb{R}; \quad W_{12}(t-1, x) = 0. \tag{4.2}$$

[ii]. The term W_{11} has the expression

$$\begin{aligned} W_{11}(t,x) = & \int_{\mathbb{R}} \left(\frac{e^{-\frac{(x-y)^2}{4}}}{\sqrt{4\pi}} S^{3/2}(t-1, y) - \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} S^{3/2}(0, y) \right) dy \\ + & \int_0^{t-1} \int_{\mathbb{R}} \frac{1}{t-s} \left(1 - \frac{(x-y)^2}{2(t-s)} \right) \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}} S^{3/2}(s,y)\, dsdy \\ := & \; W_{111}(t,x) + W_{112}(t,x) \end{aligned}$$

The term $\partial_t W_{111}$ is uniformly bounded, just because S_t is uniformly bounded. As for W_{112} there is a polynomial $P(X)$ – easily explicitly computed, but whose expression is no use to us – such that

$$|\partial_t W_{112}(t,x)| \leq \int_0^{t-1} \frac{1}{(t-s)^2} P\left(\frac{(x-y)^2}{t-s} \right) e^{-\frac{(x-y)^2}{4(t-s)}} S^{3/2}(s,y)\, dsdy.$$

This bounds $\partial_t W_{112}$.

Now, remembering (4.1) and using the equation for W and the boundedness of its right side, we conclude that the outcome of the two sub-paragraphs [i] and [ii] is

$$\|\partial_t W\|_{L^p((t,x)+I^2)} + \|\partial_{xx} W\|_{L^p((t,x)+I^2)} \leq C_p, \tag{4.3}$$

for all $p \in (1, +\infty)$. This is not quite enough; we would in fact wish to bound $\partial_t W$ and $\partial_{xx} W$ in some Hölder space. However we are now in a relatively good shape, and we may argue as follows: first, the boundedness of the coefficients of the equation for Y, as well as the boundedness of Y, imply a bound for Y of the type (4.3). This in turn imply a similar bound for T, because $W = T - Y$. Consequently, because $\varphi(T)$ and $f(T)$ are bounded as well as their derivatives, there is $\alpha \in (0,1)$ such that

$$\|f(T)\|_{C^{\alpha,\alpha/2}} + \|\varphi(T)\|_{C^{\alpha,\alpha/2}} \leq C.$$

If we now set $u(t,x) = T_t(t,x)$ we have, from the previous considerations:

$$u_t - u_{xx} - f'(T) Y u = Y_t f(T);$$

the coefficients and RHS of the above equation are bounded in $C^{\alpha,\alpha/2}$. Moreover, because of the L^p_{loc} bound for u, there is $t_0 \in (0,1)$ such that $\|u(t_0, .)\|_{L^p(I)}$ is bounded uniformly on all intervals I of length 1. Parabolic regularity implies a Hölder bound for u_t, which is enough to infer a Hölder bound for W_t. Hence the outcome of this step is

$$\|T_t\|_\infty + \|T_{xx}\|_\infty \leq C. \tag{4.4}$$

4. L^∞ **bound for** W**.** If C is the bound of (4.4), and $\underline{c_0}$ a lower bound for c, consider $K > 0$ large enough so that

$$K \geq 1 + 2\frac{C + 3\sqrt{CK}}{\underline{c_0}}; \tag{4.5}$$

let x_0 the smallest point x for which there is $t \in \mathbb{R}_+$ such that $T(t,x) = K$; without loss of generality we may take it to be equal to $(t,x) = (0,0)$. Hence we have $T(k,-ck) = K$ – recall that at the moment we are working in the original variables. Now, by interpolation, using (4.5) and (4.4) we have $T(t,-ct) \geq 1$ for all $t \in \mathbb{R}$. Letting $t \to +\infty$ yields

$$T(t,x) \geq \min(1, T_b(Y_u, S_u),$$

for $x > -ct$, which in turn implies that $\varphi(T)$ is bounded away from 0 by a constant $\underline{\varphi}$ independent of θ. Consequently we have, for $t > 0$ and $x > -ct$:

$$S(t,x) \leq \|S_u\|_\infty \exp\left\{-\underline{\varphi}\left(t - \frac{x}{c}\right)\right\}, \tag{4.6}$$

and $u(t,x) := W(t, x - ct)$ satisfies, for $x > 0$:

$$|u_t - u_{xx} + cu_x| \leq C\|S_u\|_\infty e^{-\underline{\varphi}t}.$$

Due to the upper bound for Y, it is bounded for $x = 0$ and $x = +\infty$. Letting $t \to +\infty$ yields a uniform bound for u and W.

Once the L^∞ bound for W holds, the upper bound for c and the exponential bounds follow as in Section 2.2. □

Proposition 4.1 readily implies Theorem 1.2. Indeed, one only has to consider a sequence of approximating solutions $(c_\theta, T_\theta, Y_\theta, S_\theta)$. The uniform L^2 estimate plus the lower bound on c_θ ensure that the limiting triple (T,Y,S) converges to the right limit at $\pm\infty$: the details are as in [5].

5 Existence of waves with higher velocities

A first trivial observation to support this fact – which also has the merit of clearly pointing out where the $Y_u > 0$ assumption is needed – is the following: any solution $(T,Y,S)(t,x)$ of the Cauchy problem for (1.1)-(1.3) has $W(t,x) \geq Y_u/2$ as soon as we wait long enough; consequently, if $f(T) = T$ we have

$$W - T \geq \left(\frac{Y_u}{2} - T\right) T$$

for $t > 0$ large. The RHS of the above equation is – once again – a KPP term, which generates travelling waves connecting 0 to $Y_u/2$ travelling at any speed larger than $\sqrt{Y_u/2}$. We therefore may have arbitrarily high burning rates – in the sense of [8], hence arbitrary large propagation velocities might be expected.

Let us now try to give some substance to this observation. To do so, we will be guided by the following toy problem

$$u_t - u_{xx} = (Y_u - u)u := f_0(u), \quad u(t,-\infty) = 0,\ u(t,+\infty) = Y_u. \tag{5.1}$$

For every $K > 0$ and $c > 2\sqrt{K}$, let us define the two quantities $r_-(c,K) < r_+(c,K)$ by

$$r_\pm(c,K) = \frac{c \pm \sqrt{c^2 - 4K}}{2}. \tag{5.2}$$

For every $c > 2\sqrt{Y_u}$, a travelling wave solution of (5.1) $\phi_c(x)$, that moves with the speed c, exists and decays at $-\infty$ according to the *minimal* rate [13], i.e. for every $r < c$ there exists $k_r > 0$ such that

$$\phi_c(x) = k_r e^{r_-(c,Y_u)x} + O(e^{rx}). \tag{5.3}$$

Moreover, still for every $r \in (0, c)$, define the weighted space

$$B_r = \{u(x) \in BUC(\mathbb{R}) : \quad (1 + e^{-rx})u \in BUC(\mathbb{R})\};$$

then the operator L, with a suitable domain in B_r, defined as

$$L = -\frac{d^2}{dx^2} + c\frac{d}{dx} - f_0'(\phi_c)$$

is an isomorphism from its domain into B_r; see once again [13].

For the construction of waves with higher velocities, we are going to use a degree argument similar in spirit to the one of Section 3, up to the fact that the velocity is now prescribed. Note that, once the velocity is prescribed, L^∞ bounds for T, W and S can be obtained by arguing as in, for instance, the proof of Proposition 4.1. We will use freely these bounds, without writing them in the form of a theorem.

5.1 Direct lower bound on the velocity

Theorem 1.2 does not by any means imply Theorem 1.3. Indeed, it yields a pulsating wave solution whose velocity is bounded in terms of the data; however it does not say that *all* pulsating wave solutions to (1.1)-(1.3) satisfy this estimate. Of course, it also says nothing about the boundedness or unboundedness of the velocity spectrum.

The first task in proving Theorem 1.3 is to prove a direct lower bound on the wave speeds. We start with a qualitative property of the temperature analogous to – but weaker than – the lap number decay principle, that will be useful to us in the sequel.

Proposition 5.1 *If (c, T, Y, S) is a solution to (2.1)-(2.3) then, for all $t \in \mathbb{R}$, the function $x \mapsto T(t, x)$ is nondecreasing on the set where it is below $T_b(Y_u, S_u)$.*

Proof. Assume the contrary. Then there is a value $l \in (0, T_b(Y_u, S_u))$ that is, at some time t_0, taken twice by $x \mapsto T(t_0, x)$. By Sard's Theorem, we may assume this value to be noncritical for the function T. The level set $\{T(t, x) = l\}$ consists, therefore, of a finite set of ordered, non-intersecting smooth curves $\{t, y_i(t)\}$ in space-time. Take any i such that $T(t, x) < l$ if $x \in (y_i(t), y_{i+1}(t))$. This defines an open subset Ω in space-time in which, by periodicity of T, a minimum is attained. However T is a super-solution to an advection-diffusion equation, which contradicts the strong maximum principle. □

Proof of Theorem 1.3, Step 1: a lower bound on velocities. It shares many common points with the proof of Proposition 2.1, up to the fact that we may not get an upper bound for the temperature in the unburnt region – for the simple reason that there is no unburnt region, as there is no ignition temperature. We argue by contradiction, that is, assume the existence of a sequence $c_n \to 0$ – that we immediately relabel simply c – giving raise to pulsating wave solutions. We use yet for another time the parabolic scaling

$$\tau = c^2 t, \quad \xi = cx. \tag{5.4}$$

Consider $A > 0$ large; the size of A – independent of c – will be made precise later. From Proposition 5.1 there exists a – possibly discontinuous, but at least lower semi-continuous – function $\xi_A(\tau)$ such that

$$T(\tau, \xi) \geq Ac^2 \quad \text{if and only if} \quad \xi \geq \xi_A(\tau). \tag{5.5}$$

We may assume that the minimum of ξ_A is 0. For $\xi \geq 0$ we have

$$(\partial_\tau - \partial_{\xi\xi} + \partial_\xi)T \geq 0. \tag{5.6}$$

Arguing as in the proof of Proposition 2.1 we infer the existence of of $\delta > 0$, independent of A and c, such that

$$\forall (\tau, \xi) \in \mathbb{R} \times \mathbb{R}_+, \qquad T(\tau, \xi) \geq \delta Ac^2. \tag{5.7}$$

Now, remember that (5.6) also holds for $\xi \leq 0$; as a consequence we have

$$\forall (\tau, \xi) \in \mathbb{R} \times \mathbb{R}_-, \qquad T(\tau, \xi) \geq \delta Ac^2 e^\xi. \tag{5.8}$$

In particular, we have

$$T(\tau, -2) \geq \frac{\delta Ac^2}{e^2}. \tag{5.9}$$

Turn now to $S(\tau, \xi)$; recall the equation

$$(\partial_\tau + \partial_\xi)S = -\frac{\varphi(T)S}{c^2},$$

and the fact that $C_1 T \leq \varphi(T) \leq C_2 T$ – together they imply

$$\forall (t, \xi) \in \mathbb{R} \times [-2, +\infty), \qquad S(\tau, \xi) \leq \|S_u\|_\infty \exp\left(-\frac{C\delta A}{e^2}(\xi + 2)\right). \tag{5.10}$$

Finally, turn to the function $W(\tau, \xi)$. The time period of the wave in the rescaled coordinates is c; however, we may also consider the wave as being $\tau_0 = Nc$-periodic, with $N = [1/c]$. Note that when c is small – which is the case here, τ_0 is a number in the interval $[1/2, 1]$. This consideration will be useful to us when we wish to apply parabolic regularity. Let $w_0(\xi)$ be the zeroth Fourier mode of W – that is, its average over a time period; we have, for $-2 \leq \xi \leq 0$:

$$\begin{aligned} w_0(\xi) = \;& Y_u + \langle S_u^{3/2} \rangle - \int_\xi^{+\infty} e^{\xi - \zeta} \langle S^{3/2} \rangle(\zeta)\, d\zeta \\ \geq \;& \langle S_u^{3/2} \rangle - \frac{\|S_u\|_\infty^{3/2}}{(1 + C\delta A)} \quad \text{by (5.10)} \\ \geq \;& \frac{\langle S_u^{3/2} \rangle}{2} \quad \text{as soon as } A \text{ is large enough.} \end{aligned} \tag{5.11}$$

Consider now A to be chosen so that the last inequality of (5.11) holds; we have, for $-2 \leq \xi \leq 0$, since $\varphi(T)$ is uniformly bounded

$$(\partial_\tau - \partial_{\xi\xi} + \partial_\xi)W = \frac{3}{2c^2} S^{3/2} \varphi(T) = O(1).$$

By parabolic regularity – and also by letting $\tau \to +\infty$, so that we only keep the effect of the right side, for all $p > 1$, there is $C_p > 0$ independent of c such that

$$\|W_\tau\|_{L^p([0,1] \times [-2,0])} + \|W_{\xi\xi}\|_{L^p([0,1] \times [-2,0])} \leq C_p.$$

Take p large enough so that the above estimate implies a $C^{\alpha,\alpha/2}$ estimate for some $\alpha \in (0,1)$. This, combined with (5.11), and the fact that W is c-periodic in τ – and c is a small number – implies

$$\forall \tau \in \mathbb{R}, \qquad W(\tau,-1) \geq \frac{\langle S_u^{3/2}\rangle}{3}. \tag{5.12}$$

Next, we recall that $T(\tau,\xi) \leq Ac^2$ for all τ and $\xi \leq 0$. Consider now a some small number θ and a smooth function $g(T)$ having θ as an ignition temperature, and such that $g(T) \leq T$. We therefore have $Ac^2 \leq \theta$ if c is too small; hence, for our pulsating wave we have

$$(W-T)T \geq \left(\frac{\langle S_u^{3/2}\rangle}{3} - T\right) g(T) \quad \text{on } \mathbb{R}^2. \tag{5.13}$$

Indeed, (5.13) holds for $\xi > -1$ because of (5.12), while for $\xi \leq -1$ the right side of (5.13) vanishes because T is below the ignition temperature for $g(T)$. The velocity of the pulsating wave is therefore larger than the velocity c_θ^0 of the travelling wave of

$$u_t - u_{xx} = \left(\frac{\langle S_u^{3/2}\rangle}{3} - u\right) g(u), \quad u(t,-\infty) = 0,\ u(t,+\infty) = \frac{\langle S_u^{3/2}\rangle}{3}.$$

When $\theta > 0$ is small enough, we have $c > c_\theta^0 \sim 2\sqrt{\frac{\langle S_u^{3/2}\rangle}{3} f'(0)}$ - the KPP velocity. This contradicts the smallness of c. $\square$

5.2 Uniform decay bounds and homotopy

The general idea is the following: perform a homotopy from Problem (1.1)-(1.3) to Problem (5.1). The deformation is done through the droplet distribution at $-\infty$; namely we go from $S(t,x) = S_u(x)$ at $-\infty$ to $S(t,x) = 0$ at $-\infty$. This means that we simply forget, in the end, the effect of the droplets, and this is understandable: combustion will occur in this situation whether or not droplets are present in the picture.

What we will need to complete the degree argument is not only the classical L^2 bound on T at $+\infty$, but a uniform control of T in the X_r norm, for some $r \in (r_-(c,Y_u), c)$. This will allow us to reduce the issue to the problem of finding a fixed point of an operator which is a sum of a contracting and a compact one.

Proposition 5.2 *There exist $K > 0$ and $\delta_0 > 0$ satisfying $r_-(c,Y_u) + \delta_0 < r_+(c,Y_u)$, depending only on the data, such that, if (T,W,S) is a pulsating wave solution to (1.1)-(1.3) with velocity $c \geq K$, and such that*

$$T(t,x) - e^{r_-(c,Y_u)x} = O(e^{-(r_-(c,Y_u)+\delta_0)x}) \qquad \text{as } x \to -\infty, \tag{5.14}$$

then there exist $C > 0$ also depending only on the data, such that

$$|T(t,x) - e^{r_-(c,Y_u)x}| \leq Ce^{(r_-(c,Y_u)+\delta_0)x} \qquad \text{for } (t,x) \in \mathbb{R}^2. \tag{5.15}$$

Proof. The proof of this proposition is really a stability argument; instead of comparing directly $T(t,x)$ to $e^{r_-(c,Y_u)x}$ we will compare it to the only (KPP) wave $\phi_0(x)$ of (5.1), satisfying

$$\phi_0(x) \sim e^{r_-(c,Y_u)x} \text{ as } x \to -\infty. \tag{5.16}$$

1. We claim that, if $T(t,x) \sim \phi_0(x)$ for $x \to -\infty$, then we have $T(t,x) \geq \phi_0(x)$. Indeed, we have $W(t,x) \geq Y_u$, implying

$$(\partial_t - \partial_{xx} + c\partial_x)T \geq (Y_u - T)T; \quad T(t,+\infty) \geq Y_u. \tag{5.17}$$

Let $u(t,x)$ be the only solution of (5.1) with the initial datum $T(0,x)$. Because of (5.14) we have – see [14] –

$$\lim_{t\to+\infty} u(t,x) = \phi_0(x).$$

This, together with (5.17), proves the claim.
2. Consider $\delta > 0$ small, to be chosen later. Assume the function $T(t,x)$ has been translated in time and space such that

$$T(0,0) = \min_{(t,x)\in\mathbb{R}\times\mathbb{R}_+} T(t,x) = \delta. \tag{5.18}$$

This implies, through Step 1, that the corresponding $\phi_0(x) \leq \delta$ for $x \leq 0$. It also follows that $\varphi(T) = f(T) = T$ for $\xi \leq 0$. Now, for all $\xi \in \mathbb{R}_-$, denote by $w(t,\xi)$ the function $W(t,\xi)$ and decompose $W(t,x)$ – for $t \in \mathbb{R}$, $x \leq \xi$ – as $W[T] + W^0$, where both functions $W[T]$ and W^0 are $\dfrac{1}{c}$-periodic in t, and where

$$\begin{array}{rlrl} (\partial_t - \partial_{xx} + c\partial_x)W[T] = & \dfrac{3}{2}S^{3/2}T, & & W[T](t,\xi) = 0 \\ (\partial_t - \partial_{xx} + c\partial_x)W^0 = & 0, & & W^0(t,\xi) = w(t,\xi). \end{array} \tag{5.19}$$

We recall the existence of $C > 0$ such that

$$0 \leq W^0(t,x) - Y_u \leq Ce^{c(x-\xi)}, \tag{5.20}$$

simply because the right side of (5.20) is a super-solution to the equation (5.19) for W^0, and because of the L^∞ bounds for W. Now, set

$$r_\delta = r(c, Y_u) + \delta, \quad v(t,x,\xi) = e^{-r_\delta(x-\xi)}(T(t,x) - \phi_0(x)); \tag{5.21}$$

the real number δ is not chosen yet – simply remember that it will be small. The function $v(t,x,\xi)$ solves, for $(t,x) \in \mathbb{R} \times (-\infty, \xi)$:

$$\begin{array}{l} v_t - v_{xx} + (c - 2r_\delta)v_x + (cr_\delta - r_\delta^2 - Y_u)v \\ = -(\phi_0 + T)v + (W^0 - Y_u)e^{-r_\delta(x-\xi)}T + e^{-r_\delta(x-\xi)}W[T]T \\ \leq e^{-r_\delta(x-\xi)}\Big(C\delta e^{(c-r_\delta)(x-\xi)} + W[T]T\Big) \end{array} \tag{5.22}$$

In the last inequality of (5.22), we have used the positivity of v to drop the first term on the second line, while in the second term we used the decay of $T \sim \phi_0(x)$, and (5.20). Set

$$V_\delta(t,\xi) = \|v(t,.,\xi)\|_{L^\infty((-\infty,\xi))}; \tag{5.23}$$

recall that an eventual super-solution for $W[T]$ is the solution $\overline{W}(t,x)$ solving

$$-\overline{W}'' + c\overline{W}' = \frac{3}{2}\|S_u\|_\infty^{3/2}(\phi_0 + e^{r_\delta(x-\xi)}V_\delta(t,\xi)).$$

We invoke the following three facts:

• (i) we have $r < c$,
• (ii) formula (2.16) holds and gives an expression of $\overline{W}$,
• (iii) we have $\phi_0(x) \leq C\delta e^{r_-(c,Y_u)(x-\xi)}$ – see step 1 of this proof. The constant C is independent of $\delta > 0$.

Points (i) to (iii) above imply, after a computation, the following bound for $W[T]T$ on $\mathbb{R}\times(-\infty,\xi)$:

$$0 \leq e^{-r_\delta(x-\xi)}W[T]T \leq C(e^{(r_-(c,Y_u)-\delta)(x-\xi)} + e^{r_\delta(x-\xi)}V(t,\xi)^2)\delta^2 \tag{5.24}$$

Now, we start shrinking δ. First, ask the amount $cr_\delta - r_\delta^2 - Y_u$ to be positive; from (5.22)-(5.24), the normalization condition (5.18) and the maximum principle, we have

$$V_\delta(t,\xi) \leq \delta + e^{-(cr_\delta - r_\delta^2 - Y_u)t}V_0 + C\delta^2(1 + \int_0^t e^{(cr_\delta - r_\delta^2 - Y_u)(t-s)}V_\delta^2(s,\xi)\ ds).$$

Then, ask δ to be small enough so that the equation $C\delta X^2 - X + C\delta$ has two positive roots; one that is $O(\delta)$, the other one that is $O(\frac{1}{\delta})$. Fix such a δ, and call it δ_0.

Letting $t \to +\infty$ and using the $\frac{1}{c}$ periodicity of $V_\delta(t,\xi)$ we get, for all $\delta \leq \delta_0$, and for a constant C once again independent of δ:

$$\begin{aligned} \|V_\delta(.,\xi)\|_\infty \leq\ & C(\delta + \delta^2 + \frac{\delta^2}{cr_\delta - r_\delta^2 - Y_u}\|V_\delta(.,\xi)\|_\infty^2) \\ \leq\ & C\delta(1 + \|V_\delta(.,\xi)\|_\infty^2). \end{aligned}$$

This implies: either $V_\delta(t,\xi) = O(\delta)$, or $V_\delta(t,\xi) = O(\frac{1}{\delta})$. The first solution prevails for large negative ξ and small δ; so by continuity: $V_{\delta_0}(t,0) = O(\delta_0)$. This proves our proposition. □

The fact that $T \geq \phi_0$ readily implies the exponential convergence of S and W to their limits at $-\infty$.

Lemma 5.3 *There is $\rho_0 > 0$ and $C > 0$, depending only on data, such that*

$$\forall (t,x) \in \mathbb{R}\times\mathbb{R}_-, \quad 0 \leq S_u(x) - S(t,x) \leq Ce^{\rho_0 x}, \quad 0 \leq W(t,x) - Y_u \leq Ce^{\rho_0 x}.$$

The proof is at this stage routine and is omitted. The last ingredient that we need for the homotopy is a quantitative, uniform decay to the right for the functions $T - Y_u - < S_u^{3/2} >$, $W - Y_u - < S_u^{3/2} >$. This is provided by the

Proposition 5.4 *There is $\rho_0 > 0$ such that, if $T(t,x)$ satisfies the assumptions of Proposition 5.2, then we have, for some $C > 0$ depending only on data:*

$$\forall (t,x) \in \mathbb{R}\times\mathbb{R}_+, \quad |(T(t,x), W(t,x)) - T_b(Y_u, S_u)(1,1)) > | \leq Ce^{-\rho_0 x}. \tag{5.25}$$

PROOF. Come back to a pulsating wave that satisfies the normalization condition (5.18). Arguing as in Proposition 2.1 we have $T(t,x) \geq C\delta_0$ for $x \geq 0$; on the other hand we have

$$S_t + cS_x + S\varphi(T) = 0;$$

together these facts imply a uniform exponential decay for S. Then $Y(t,x)$ satisfies

$$Y_t + cY_x - Y_{xx} + f(T)Y = \frac{3}{2}S^{3/2}\varphi(T);$$

using the lower bound for T on $\mathbb{R}\times\mathbb{R}_+$, and the exponential bound for S on the same set, implies the exponential bound for Y. The function W is then treated as in Proposition 2.4. $\square$

PROOF OF THEOREM 1.3 (END). Let us pick c satisfying the assumptions of Proposition 5.2. Let γ be a smooth, nonnegative function, equal to 1 on $\mathbb{R}_-$ and 0 on $[1,+\infty)$. Let δ_0 satisfy the conclusions of Proposition 5.2, and set once and for all

$$r = r(c, Y_u) + \delta_0. \tag{5.26}$$

The fixed point setting that we are going to devise here is simpler than in Section 3, because c is not to be looked for. Let us define the space X_{r,δ_0} as

$$X_{r,\delta_0} = \{u \in C^{\alpha,\alpha/2}_{per}: \quad (e^{-rx}(u-\phi_0), e^{\delta_0 x}(u - Y_u - \langle S_u^{3/2}\rangle)) \in C^{\alpha,\alpha/2}_{per}\}. \tag{5.27}$$

Here the space $C^{\alpha,\alpha/2}_{per}$ means the classical Hölder functions which are additionally $\dfrac{1}{c}$ in t. Now, for $T\in X_r$, let respectively $\mathcal{F}_1[T]$ be defined as (3.10) - i.e. $S(t,x)$ is defined in terms of T, and let the only $\dfrac{1}{c}$ periodic solution of

$$W_t - W_{xx} + cW_x = \frac{3}{2}(\mathcal{F}_1[T])^{3/2}\varphi(T), \quad W(t,-\infty) = Y_u$$

be defined as

$$W(t,x) = \gamma(x)Y_u + (1-\gamma(x))T_b(Y_u,S_u) + \mathcal{F}_2[T]. \tag{5.28}$$

The equation that we are solving is then

$$\begin{aligned} (\partial_t - \partial_{xx} + c\partial_x)T = &\ \ (Y_u + (1-\gamma)\langle S_u^{3/2}\rangle + (\mathcal{F}_2[T]-T)f(T)\ (\mathbb{R}^2) \\ T(t,x) = &\ \ \phi_0(x) + O(e^{rx}) \ \ (x\to-\infty) \\ T(t,+\infty) = &\ \ T_b(Y_u,S_u) \end{aligned} \tag{5.29}$$

As soon as we have found a $\dfrac{1}{c}$ periodic solution to (5.29), we will be done.

Once the estimates are at hand, the rest of the proof of the theorem only resorts to putting (5.29) under the right functional form, so we are not going to dwell on it too much, all the more as it was detailed in Section 3. The unknown $T(t,x)$ is best looked for under the form

$$T(t,x) = (1-\gamma(x)T_b(Y_u,S_u) + \Big(\gamma(x)(\phi_0(x)+e^{rx}) + (1-\gamma(x))e^{-\rho x}\Big)u(t,x) \tag{5.30}$$

where the exponent r is defined in (5.26) and the exponent ρ is, say, less than $\dfrac{\rho_0}{2}$; where ρ_0 is defined in Lemma 5.3. This complicated-looking expression simply says that we wish T to be asymptotic to $\phi_0(x)$ as $x\to-\infty$, and to converge to $T_b(Y_u,S_u)$ at some small exponential rate. The equation for $u(t,x)$ therefore reads

$$u_t + Lu = f_0(x,u,u_x) + \phi_1(x) + \frac{(\mathcal{F}_2[T]-T)f(T)}{\gamma(x)(\phi_0(x)+e^{rx}) + (1-\gamma(x))e^{-\rho x}}. \tag{5.31}$$

where the quantities L, f_0, and ϕ_1 have the following features.

1. The operator L is defined as

$$\begin{aligned} L = &\ \ -\frac{d^2}{dx^2} + a(x)\frac{d}{dx} + b(x) \\ a(x) = &\ \ \gamma(x)(c-2r) - (1-\gamma(x))(c-2\rho) \\ b(x) = &\ \ \gamma(x)(Y_u + cr - r^2) + (1-\gamma(x))(T_b(Y_u,S_u) - c\rho - \rho^2) \end{aligned} \tag{5.32}$$

Notice that $b(x)$ is controlled from below by a positive constant. By the maximum principle we have therefore the inequality

$$\|e^{-L/c}\|_{\mathcal{L}(UC(\mathbb{R}))} < 1. \tag{5.33}$$

2. The function f_0 is smooth in all its variables; moreover we have

$$\forall x \notin [0,1],\ \forall (u,p) \in \mathbb{R}^2, \qquad f(x,u,p) = 0. \tag{5.34}$$

3. There is δ_0 depending only on data such that we have

$$\forall x \in \mathbb{R}, \quad |\phi_1(x)| \le Ce^{-\delta_0 |x|}. \tag{5.35}$$

4. Recall that T is - for short - defined by equation (5.30). From Proposition 5.2, Lemma 5.3 and Proposition 5.4, there exists $\delta_1 < \min(r, \dfrac{\rho_0}{2})$ such that, if $T - (Y_u - \langle S_u^{3/2} \rangle)(1-\gamma)$ belongs to X_{r,δ_0}, then we have, for some C depending on the X_r norm of $T - (Y_u - \langle S_u^{3/2} \rangle)(1-\gamma)$:

$$\forall x \in \mathbb{R}, \quad \left| \frac{(\mathcal{F}_2[T] - T)f(T)}{\gamma(x)(\phi_0(x) + e^{rx}) + (1-\gamma(x))e^{-\rho x}} \right| \le Ce^{-\delta_0 |x|}. \tag{5.36}$$

Consequence: call $\mathcal{K}[u]$ the right side of (5.31). Then the operator $\mathcal{F}$ acting on $C_{per}^{\alpha,\alpha/2}$, and defined by $\mathcal{F}[u] = e^{-L/c}\mathcal{K}[u]$ is compact in $C_{per}^{\alpha,\alpha/2}$.

Now, it suffices to perform the homotopy consisting in replacing once again S_u by τS_u, while keeping Y_u fixed. For $\tau = 0$, this reduces to the study of the operator $-\dfrac{d^2}{dx^2} + c\dfrac{d}{dx} - f'(\phi_0)$, which is an isomorphism between its domain and the set of all functions u decaying like e^{rx} on the left side; see [13]. This implies the existence result. □

6 Extensions

Clearly, we have not in this paper discussed the thermo-diffusive propagation of spray flames in its full generality, although we believe that we have captured some of its main features in the study that we have presented. Some extensions of the theory developed so far can be thought about; some of them are simple generalizations; others seem less obvious to us. Let us list three of them.

- *Holes in the distributions of droplets at $-\infty$.* We only have treated droplets distributions at $-\infty$ that never vanished. This is truly a convenience assumption, that is only needed in the lower bound for the velocity in the ignition temperature case. A more accurate proof would have revealed that what matters is $\langle S_u \rangle$. This generalization is omitted.

- *Polydisperse sprays.* A more general description of the spray would describe the droplet distribution by a probability density $f(t,x,s)$ accounting – roughly speaking – for the number of droplets that, between the times t and $t + dt$, and between the positions x and $x + dx$, have sizes comprised between s and $s + ds$. The governing equations are then

$$\begin{aligned} &T_t - T_{xx} = Yf(T) \\ &Y_t - Y_{xx} = -Yf(T) + \varphi(T) \int_0^{+\infty} s^{3/2} f(t,x,s)\ ds \\ &f_t - \partial_s(\varphi(T)sf) = 0. \end{aligned} \tag{6.1}$$

Such a general spray is said to be *polydisperse*, as opposed to the monodisperse case that we have treated so far: we simply have plugged into (6.1) the measure $\delta_{s=S(t,x)}$.

Although (6.1) looks much more formidable than (1.1), the investigation of pulsating waves is essentially the same as for (1.3). In fact, the relevant quantity to be considered is the maximal size of the droplets at $-\infty$. See [7].

- *Several space dimensions* What prevents us from going to several space dimensions is the lap number decay - which is,in Section 4, replaced by another typically 1D result. Most certainly, one could by-pass the use of this result.

A less obvious point to study is what happens when there is no gaseous fuel at $-\infty$, i.e. $Y_u = 0$. The main question to be answered is: is there an unbounded range of possible velocities? Preliminary computations of the *travelling wave* problem seem to indicate that this is impossible, and that the system would be qualitatively close to the one with ignition temperature. This, however, needs a more serious study than these considerations. Also, a complete study of the burning rate, in the spirit of [8] or [12], would be very welcome. This might not be an easy task, for we would have to drop the pulsating wave assumption - an assumption that we have heavily used in several crucial instances. In particular, we still do not know how to derive an upper bound for the enthalpy - although sharp upper bound results for the purely gaseous thermo-diffusive system, such as [10] - might help. We hope to give – even partial – answers to these last two questions in the future.

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