INTRODUCTION

The purpose of this paper is to provide a rigorous analysis of a bifurcation phenomenon occurring in the thermo-diffusive model for flame propagation, which is relevant in the low Mach numbers regimes. The situation under study is the propagation of a flame in an infinite channel, under the assumptions of simple chemistry $A \to B$, with Lewis number close to unity, and in the large activation energy framework. Namely, let Σ be the two-dimensional strip

$$\Sigma = \{(x,y) \in I\!R \times [0,1]\};$$

The temperature of the mixture will be denoted by $T(t,x,y)$, and the mass fraction of the reactant $Y(t,x,y)$. After some non-dimensionalizing process, one obtains - see [6], [13], [33] - the following evolution problem:

$$\begin{cases} T_t - \Delta T = f(T)Y \\ Y_t - \Lambda \Delta Y = -f(T)Y \end{cases} \quad (\Sigma) \tag{1}$$

$$T_y(t,x,0) = T_y(t,x,1) = Y_y(t,x,0) = Y_y(t,x,1) = 0 \tag{2}$$

The boundary conditions (2) express that the cylinder is adiabatic. The real number Λ - which will soon be taken to be close to unity - is the Lewis number, i.e the rate between the thermal and molecular diffusion; the term $Yf(T)$, which will also soon be specified, is the rate at which the reaction proceeds.

Travelling wave solutions of (1) are solutions of the form $(T(x+ct,y), Y(x+ct,y))$; therefore they satisfy the following elliptic system, where c is an additional unknown:

$$\begin{cases} -\Delta T + cT_x = f(T)Y \\ -\Lambda \Delta Y + cY_x = -f(T)Y \end{cases} \quad (\Sigma) \tag{3}$$

$$T_y(x,0) = T_y(x,1) = Y_y(x,0) = Y_y(x,1) = 0 \tag{4}$$

Conditions at $\pm\infty$ have to be imposed; we will choose

$$\begin{cases} T(-\infty,y) = Y(+\infty,y) = 0 \\ T(+\infty,y) = Y(-\infty,y) = 1 \end{cases} \tag{5}$$

In the whole sequel, ε will denote a small real number. The Lewis number will be chosen to read $\Lambda = 1 + \varepsilon l$, with $l > 0$. The nonlinearity will be of "high activation energy" type, with an ignition temperature to avoid the cold boundary difficulty. Namely, let us set

$$f(T) = \frac{1}{\varepsilon^2}\Phi_\varepsilon(\frac{T-1}{\varepsilon}), \tag{6}$$

where Φ_ε is a smooth function which is additionnally required to have

$$\Phi_\varepsilon(\sigma) = 0 \quad \text{if } \sigma \le \frac{\theta-1}{\varepsilon} \tag{7}$$
$$\Phi_\varepsilon(\sigma) = e^\sigma \quad \text{if } \sigma \ge \frac{\theta-1}{2\varepsilon}$$

A more general form of (3)-(5) with nonhomogeneous convection and ε-independent source term has been studied by Berestycki, Larrouturou, Lions and the second author [8], in the vincinity of $\Lambda = 1$, following a previous paper by the first three authors [7]. The same problem in one space dimension has been solved by Berestycki, Nicolaenko and Scheurer [10], who have proved the existence of planar wave solutions to (3)-(5) - i.e of the form $(T_\varepsilon(x+c_\varepsilon t), Y_\varepsilon(x+c_\varepsilon t))$; in fact they proved existence for a general $\Lambda > 0$, and general positive, ignition-temperature type f. These functions satisfy

$$\begin{cases} -T_\varepsilon'' + c_\varepsilon T_\varepsilon' = f(T_\varepsilon)Y_\varepsilon \\ -\Lambda Y_\varepsilon'' + c_\varepsilon Y_\varepsilon = -f(T_\varepsilon)Y_\varepsilon \end{cases} \quad (I\!R) \tag{8}$$

with conditions at $\pm\infty$

$$\begin{cases} T_\varepsilon(-\infty) = Y_\varepsilon(+\infty) = 0 \\ T_\varepsilon(+\infty) = Y_\varepsilon(-\infty) = 1. \end{cases} \tag{9}$$

For $\Lambda = 1$, then $Y_\varepsilon = 1 - T_\varepsilon$ and the system (8) is reduced to differential equation

$$-T_\varepsilon'' + c_\varepsilon T_\varepsilon' = g(T_\varepsilon)$$

where $g(T) = f(T)(1-T)$. In this case, there is uniqueness of this solution. There is also uniqueness of system (8)-(9) only for Lewis greater than 1 (see Marion [25]), but for Lewis lower than 1, the system can have multiple solutions for special choices of f (see Bonnet [11]). The singular limit of $(T_\varepsilon, Y_\varepsilon, c_\varepsilon)$ as $\varepsilon \to 0$ has been investigated in [10] for self-similar f of type (6) : it is shown there that if $(T_\varepsilon, Y_\varepsilon, c_\varepsilon)$ is a a solution of (8)-(9) such that $T_\varepsilon(0) = \theta$, then $(T_\varepsilon, Y_\varepsilon, c_\varepsilon) \to (T_\infty, Y_\infty, \sqrt{2})$ in $H^1 \times H^1 \times I\!R$ as $\varepsilon \to 0$, where

$$T_\infty(x) = \theta e^{\sqrt{2}x} \quad \text{for} \quad x < \bar{x}, \quad T_\infty(x) = 1 \quad \text{for} \quad x > \bar{x}$$
$$Y_\infty(x) = 1 - \theta e^{\sqrt{2}x} \quad \text{for} \quad x < \bar{x}, \quad Y_\infty(x) = 0 \quad \text{for} \quad x > \bar{x}$$

with $\bar{x} = -\dfrac{\mathrm{Log}\theta}{\sqrt{2}}$. Singular limits for general types of f have been investigated by the first author [20] and non-uniqueness phenomena for these limits by Bonnet and the first author [12].

In this paper, uniqueness of the planar fronts - i.e. solutions of (3)-(5) - for small ε will be proved; this has not, to our knowledge, been done before except in [11] for a Heaviside-like f.

What we really are interested in is the stability of the planar waves. As a matter of fact, it is well-known by physicists that they are stable for $\Lambda = 1$, but that they quickly lose their stability as Λ leaves the very close vincinity of 1, giving birth eiher to a Hopf bifurcation for $\Lambda < 1$, or a steady-state bifurcation if $\Lambda > 1$. The latter phenomenon is known as "cellular instability" - due to the particular form of the new profile - and this is the one which will be focused on throughout the paper.

Barenblatt, Zeldovich, Istratov [4] were the first to observe that the planar wave could be destabilized by long-wave transversal perturbations. However the general dispersion relation is due to Sivashinsky. In [33], he introduced a general procedure, based on matched asymptotic expansions. Plugging in (1) solutions of the form

$$(T(t,x,y), Y(t,x,y)) = (T_\varepsilon + e^{iky-\sigma t}u_0(x), \quad Y_\varepsilon + e^{iky-\sigma t}v_0(x)), \tag{10}$$

and performing two-scaled developmets at the front, he obtained a dispersion relation $D(\sigma, k, l) = 0$. For l increasing, he found a real eigenvalue becoming negative as l crossed the value $l_0^k = 2(1 + 2k^2)$, hence a destabilisation of the planar flame. This destabilization process led him to conclude to a steady-state bifurcation. Sivashinsky's method was subsequently used in many works - see or example [23], [24].

Numerical experiments have been made by Benkhaldoun and Larrouturou [5]. They solved the Cauchy Problem for (1) with a multi-dimensional perturbation of the planar wave. They observed, in the cases consistent with Sivashinsky's theory, that the initially plane flame front oscillated for a while, and then was stabilized into a curved one. Simulations were also carried out by Denet [17].

For $\Lambda = 1$, the plane wave has rigorously been shown to be stable [31]; it is in fact stable for much more general reaction terms than (7). The first rigorous result for $\Lambda \neq 1$ is proved in an important paper of Terman [36]. In the framework of (7), he provided a justification for the expansions of the solutions of equation (3) around solutions of the type (10), up to the first order in ε. Then he showed that the Evans function (see [2], [18]) of the linearized system, denoted by $D_\varepsilon(\sigma, k, l)$ satisfied, uniformly on every compact in σ, k, l:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} D_\varepsilon(\sigma, k, l) = D(\sigma, k, l).$$

This result allowed him to prove rigorously the instability of the planar flame for l larger than some l_ε^k, with

$$\lim_{\varepsilon \to 0} l_\varepsilon^k = l_0^k.$$

The problem of analyzing the bifurcation remained open, due precisely to the lack of estimates on σ and k. For the same reason, he could not prove that the planar flame solutions were stable for $l < l_\varepsilon^k$, and stable with respect to one-dimensional perturbation for every $l > 0$.

Mathematical investigations were also led by Volpert, Volpert and Merzhanov [38] for general systems; under suitable assumptions on the plane wave and the linearized operator they gave Hopf bifurcation results. However they did not apply their theory to System (1).

The purpose of our work is therefore three-fold. First, we will provide the needed estimates. Then we will set up a mathematical framework, in which we will give a rigorous proof of Sivashinsky's results. Finally we shall use this mathematical framework to the three-dimensional situation, which will lead us to find new behaviors of the bifurcation related to the section ω of the domain - this does not seem to have been noticed up to now.

To sum up, our main result is the following:

THEOREM. *For every $\varepsilon > 0$ small enough, there exists $l_\varepsilon^0 > 0$, with $\lim_{\varepsilon \to 0} l_\varepsilon^0 = 2(1+2\pi^2)$, such that:*

- *for $0 < l < l_\varepsilon^0$, the planar wave solutions of (3)-(5) are stable with respect to the evolution problem (1), with small multidimensional perturbations. Further, for $l > l_\varepsilon^0$, they are unstable with respect to multi-dimensional perturbations, but stable with respect to small one-dimensional perturbations.*
- *For every $l > l_\varepsilon^0$ close enough to l_ε^0, system (3)-(5) has two non-planar solutions, each being the symmetric of the other with respect to the line $\{y = \frac{1}{2}\}$. These multidimensional solutions are stable with respect to small multidimensional perturbations.*

This Theorem has - at least - one shortcoming: using Crandall-Rabinowitz's theory, we obtain a local branch of nontrivial equilibria, parametrized by some real number s, as well as a local expansion near $s = 0$. However, we do not get the precise dependence of the range of validity of the expansion with respect to ε; we expect it to be singular. Such an information would open the way to more qualitative properties of the bifurcating state.This question will be addressed in a future work.

Let us now describe the organization of this paper. Section 1 is devoted to setting up the problem in the framework of a bifurcation problem of the form $\mathcal{F}_\varepsilon(l, T, Y) = 0$, which will make it amenable to a Crandall-Rabinowitz-type treatment.

In Section 2, upper bounds on σ and k will be derived. Then, by matching at a suitable point various derivatives, we will obtain uniqueness of the planar fronts as well as the dispersion relation. Once the upper bound derived, we could rely on Terman's result [36]; however we will provide an alternative proof to his rigorous derivation of the dispersion relation by showing that indeed the method of matched asymptotics converges; a proof which is only based on the paper of Berestycki-Nicolaenko-Scheurer.

The existence of the bifurcation will be addressed in Section 3. With the aid of the approximate dispersion relation and basic facts of functional analysis, we will prove that the assumptions of the classical bifurcation theorem of [15] are satisfied.

In order to investigate the stability of the bifurcating branch, one needs to know the behaviour of $sl_\varepsilon'(s)$ in the vincinity of 0. Since $l_\varepsilon'(0) = 0$, Section 4 is entirely devoted to the computation of $l_\varepsilon''(0)$. The translational invariance of the problem causes this quantity to cancel at the orders 0 and 1 in ε, leading us to push the expansions up to the order 2 in ε. This makes that section the most technical of the paper, and the reader who is only interested in the result should skip it.

In Section 5, we solve the stability questions. Linear stability, due to the location of the eigenvalues in the 1D case, and Crandall-Rabinowitz's theorem on linearized stability, is not difficult to obtain. Nonlinear stability, however, requires closer investigation: unlike the purely scalar case, restricting the perturbations to weighted spaces does not push the essential spectrum away from 0. The convergence results that we obtain are not exponential, and this is in agreement with the results of [31] for the system with $\Lambda = 1$. Finally, through a centre manifold theorem, we give a mathematical explanation to the phenomenon of calculated by Benkhaldoun and Larrouturou [5]. We get the full description of the cellular instability.

In Section 6, we investigate the three-dimensional version of (1). In generic cases, the local branch is shown to have $l'_\varepsilon(0) \neq 0$; this implies local uniqueness of the stable branch of curved solutions, and also in certain cases the appearence of a turning point.

1. REDUCTION TO A BIFURCATION PROBLEM, DEFINITIONS AND NOTATIONS

The nonplanar solutions that we seek will be proved to bifurcate from the Berestycki-Nicolaenko-Scheurer waves $(c_\varepsilon, T_\varepsilon, W_\varepsilon)$. This section is aimed at setting up the functional framework that will be used, as well as introducing some technical devices which will be useful.

Function spaces

The choice of the spaces between which we are going to operate will be done in two steps. In the process, several intermediate spaces that will appear at various places of the paper will also be defined.

1. For $r > 0$, set $\mu_r(x) := 1 + e^{-rx}$, and $\mu_r^\#(x) = e^{-rx}$. Let also $UC(\overline{\Sigma})$ be the space of all bounded, uniformly continuous functions of $\overline{\Sigma}$. Define the space B_{μ_r} by the set of all functions u of $UC(\overline{\Sigma})$ such that $\mu_r u$ still belongs to $UC(\overline{\Sigma})$, and the space $B_{\mu_r^\#}$ by the set of all functions u such that $\mu_r^\# u$ still belongs to $UC(\overline{\Sigma})$. Let X_r and $X_r^\#$ be defined as

$$
\begin{aligned}
X_r &= \{u \in B_{\mu_r} \cap H^2(\Sigma),\ \Delta u \in B_{\mu_r},\quad u_y(x,0) = u_y(x,1) = 0\} \\
X_r^\# &= \{u \in B_{\mu_r^\#}\ ,\ \Delta u \in B_{\mu_r^\#},\quad u_y(x,0) = u_y(x,1) = 0\}
\end{aligned}
\tag{1.1}
$$

The spaces X_r and $X_r^\#$ will be endowed with the norms

$$
\begin{aligned}
\|u\|_{X_r} &= \|\Delta u\|_{B_{\mu_r}} + \|u\|_{H^2(\Sigma)} \\
\|u\|_{X_r^\#} &= \|\Delta u\|_{B_{\mu_r^\#}} + \|u\|_{B_{\mu_r^\#}}
\end{aligned}
\tag{1.2}
$$

In the sequel, the real number $r > 0$ will be chosen once and for all to be less than $\dfrac{\sqrt{2}}{2}$.

Some more notations will be used. For $u \in X_r^\#$, set:

$$
< u > (x) = \int_0^1 u(x,y)\ dy; \qquad \tilde{u} = u - < u > . \tag{1.3}
$$

let $\tilde{X}_r$ (resp. $\tilde{X}_r^\#$) be defined as

$$
\tilde{X}_r(\text{resp. } \tilde{X}_r^\#) = \{u \in X_r(\text{resp. } \tilde{X}_r^\#) : < u >= 0\} \tag{1.4}
$$

Let

$$
X = X_r \times X_r \quad \text{and} \quad X^\# = X_r^\# \times X_r^\#. \tag{1.5}
$$

Trivially, X and $X^\#$ are Banach spaces.

2. Let the x-Fourier transform be defined in the classical way. It has already been successfully used by Amick [3] to investigate Dirichlet problems in strips. It is also a key ingredient in [8]. For $u(x,y) \in L^2(\Sigma)$; its x-Fourier transform will be denoted by $\hat{u}(\xi, y)$; whenever the expression makes sense we take

$$\hat{u}(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\xi} u(x,y)\ dx.$$

For $f \in B_{\mu_r} \cap L^2(\Sigma)$, set

$$N(f) := \sqrt{\int_{-\infty}^{+\infty} \Big| \int_\omega \hat{f}(\xi, y) dy \Big|^2 \ \frac{d\xi}{\xi^2}},$$

and set

$$Y_2 = \{f \in B_{\mu_r} \cap L^2(\Sigma) : \ N(f) < +\infty\}.$$

The quantities defined by (1.3) are still well-defined in Y_2. Let us set

$$\tilde{Y}_2 = \{f \in Y_2 : <f> (x) = 0, \quad \forall x\} \tag{1.6}$$

We then introduce the spaces

$$Y = L^2(\Sigma) \cap B_{\mu_r} \times Y_2, \quad \text{and} \quad Y^\# := B_{\mu_r^\#} \times B_{\mu_r^\#}; \tag{1.7}$$

Y is endowed with the norm

$$\|(f_1, f_2)\|_Y = \|(f_1, f_2)\|_{L^2(\Sigma)} + \|(f_1, f_2)\|_{B_{\mu_r}} + N(f_2). \tag{1.8}$$

The spaces Y and $Y^\#$ are Banach spaces.

The mapping $\mathcal{F}_\varepsilon$

Let us first deal with the normalization. Let us impose $T_\varepsilon(0) = 1 - \varepsilon$. We then introduce, as in [36], the points x_1^ε, x_2^ε and x_3^ε such that

$$\begin{aligned} T_\varepsilon(x_1^\varepsilon) &= \theta \\ T_\varepsilon(x_3^\varepsilon) &= 1 + 100\varepsilon \mathrm{Log}\varepsilon \\ T_\varepsilon(x_2^\varepsilon) &= 1 - \frac{\varepsilon}{2} \end{aligned} \tag{1.9}$$

Thus for small $\varepsilon > 0$, $x_1^\varepsilon < x_3^\varepsilon < x_2^\varepsilon$; further the reaction term $f(T_\varepsilon)Y_\varepsilon$ will not exceed $O(\varepsilon^{90})$ on $[x_1^\varepsilon, x_3^\varepsilon]$ and will vanish for $x < x_1^\varepsilon$.

The real coefficient d_ε will be chosen to satisfy

$$c_\varepsilon = d_\varepsilon \int_{-\infty}^{x_3^\varepsilon} T_\varepsilon(x)\ dx;$$

from [10], we have $d_\varepsilon = c_\varepsilon^2(1 + o(1))$. In order to eliminate the x translation invariance and to get rid of the speed as an unknown, let us set

$$\mu_\varepsilon(T) = d_\varepsilon \int_0^1 \int_{-\infty}^{x_3^\varepsilon} T(x,y)\ dxdy. \tag{1.10}$$

It will be convenient to introduce the energy $W = T + Y - 1$. Recall also that we define the function g as

$$g(T) = f(T)(1-T) \tag{1.11}$$

Therefore, (1) is equivalent to solving, in the space X:

$$\mathcal{F}_\varepsilon(l, T_\varepsilon + u, W_\varepsilon + w) = 0, \tag{1.12}$$

with

$$\mathcal{F}_\varepsilon(l, T, W) = \begin{pmatrix} -\Delta T + \mu_\varepsilon(T)T_x - g(T) + f(T)W \\ -\Lambda\Delta W + \mu_\varepsilon(T)W_x - (1-\Lambda)\Delta T \end{pmatrix} \tag{1.14}$$

Many more normalisations of type (1.14) can be used; see for instance [10], [37] for a different formulation. Ours has the advantage of somewhat simplifying the computations which sometimes will be heavy.

Thus we have $\mathcal{F}_\varepsilon(l, T_\varepsilon, W_\varepsilon) = 0$ for all l, and we have reduced (1) to a bifurcation problem. In order to apply Crandall-Rabinowitz theory [15], [16], we have to study the differential of $\mathcal{F}_\varepsilon$ with respect to T and W, namely

$$\begin{aligned} &\partial_{T,W}\mathcal{F}_\varepsilon(l, T, W)(u, w) = \\ &\begin{pmatrix} -\Delta u + \mu_\varepsilon(T)u_x + \mu_\varepsilon(u)T_x - (g'(T) + f'(T)W)u - f(T)w \\ -\Lambda\Delta w + \mu_\varepsilon(T)w_x + \mu_\varepsilon(u)W_x - (1-\Lambda)\Delta u \end{pmatrix} \end{aligned} \tag{1.15}$$

and the eigenvalue problem

$$\partial_{T,W}\mathcal{F}_\varepsilon(l, T_\varepsilon, W_\varepsilon)\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \sigma \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}; \tag{1.16}$$

Fourier expansions of (1.16)

Let us notice that elliptic regularity allows us to seek solutions of (1.16) under the form of a Fourier expansion with respect to the transversal variable. Namely, let us set $k_n = n\pi$; the function $\cos k_n y$ is the n-th eigenvalue of the Neumann problem

$$-u''(y) = k^2 u(y), \qquad u'(0) = u'(1) = 0. \tag{1.17}$$

Let us also write the unknowns (Φ, Ψ) of (1.16) under the form

$$\begin{pmatrix} \Phi(x,y) \\ \Psi(x,y) \end{pmatrix} = \sum_{n=0}^{+\infty} \begin{pmatrix} \phi_n(x) \\ \psi_n(x) \end{pmatrix} \cos k_n y \tag{1.18}$$

This leads us a sequence of differential systems, valid for $n \geq 0$, with unknowns $(\phi_n, \psi_n) \in L^2(I\!R)^2$. For $n = 0$ one has

$$\left\{ \begin{aligned} -\phi'' + c_\varepsilon\phi' + \mu_\varepsilon(\phi){T_\varepsilon}' &= \big(g'(T_\varepsilon) + f'(T_\varepsilon)W_\varepsilon\big)\phi + f(T_\varepsilon)\psi + \sigma\phi \\ -\Lambda\psi'' + c_\varepsilon\psi' + \mu_\varepsilon(\phi)W_\varepsilon' &= (1-\Lambda)\phi'' + \sigma\psi \end{aligned} \right. \tag*{$(1.18)_0$}$$

For $n \geq 0$ the contributions of $\mu_\varepsilon(u)T'_\varepsilon$ and $\mu_\varepsilon(u)W'_\varepsilon$ disappear, and we end up with

$$(1.18)_n \qquad \begin{cases} -\phi''_n + c_\varepsilon\phi'_n + k_n^2\phi_n = \big(g'(T_\varepsilon) + f'(T_\varepsilon)W_\varepsilon\big)\phi_n + f(T_\varepsilon)\psi_n + \sigma\phi_n \\ -\Lambda\psi''_n + c_\varepsilon\psi'_n + \Lambda k_n^2\psi_n = -(1-\Lambda)(-\phi''_n + k_n^2\phi_n) + \sigma\psi_n \end{cases}$$

For commodity, we will write $(1.18)_n$ and $(1.18)_0$ under the form

$$(1.19)\ \ \mathcal{L}_0\begin{pmatrix}\phi\\ \psi\end{pmatrix} + \mu_\varepsilon(\phi)\begin{pmatrix}T'_\varepsilon\\ W'_\varepsilon\end{pmatrix} = \sigma\begin{pmatrix}\phi\\ \psi\end{pmatrix}\ \ (n=0), \quad \mathcal{L}_n\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \sigma\begin{pmatrix}\phi\\ \psi\end{pmatrix}\ \ (n \geq 1).$$

The operators $\mathcal{L}_n$ are unbounded, with obvious domains of definition. In the sequel, the problems of interest will be $(1.18)_n$, with $n \geq 1$.

Eigenvalues and related quantities

The differential operators $\mathcal{L}_n$ depend on Λ, and are sectorial [21]. Their essential spectra are located, for $\frac{1}{2} \leq \Lambda \leq \frac{3}{2}$, inside parabolas that can be computed as in [21], page 136. In a more precise way, let us set

$$(1.20) \qquad P_\Lambda = \{\sigma \in \mathbb{C}\ :\ \mathrm{Re}\big(\frac{c_\varepsilon - \sqrt{c_\varepsilon^2 - 4\Lambda\sigma}}{2\Lambda}\big) \geq -1\}.$$

The large eigenvalues σ in $(1.18)_0$ and $(1.18)_n$ will be searched outside this region of the complex plane. Their union $\bigcup_{1/2\leq\Lambda\leq 3/2} P_\Lambda$ is contained in a cone of the complex plane with vertex on the real line, and with aperture $< \pi$.

The problem that will occupy us in the next section will be to derive a common upper bound for them, independently of ε.

Let us introduce for $\sigma \notin \bigcup\limits_{1/2\leq\Lambda\leq 3/2} P_\Lambda$ and $k \in I\!R$ the following quantities:

$$(1.21) \qquad \begin{gathered} \Gamma(k,\sigma) = \sqrt{1 - 2(\sigma - k^2)} \\ \Delta_\varepsilon(k,\sigma) = \sqrt{c_\varepsilon^2 - 4\Lambda(\sigma - \Lambda k^2)} \qquad \delta_\varepsilon(k,\sigma) = \sqrt{c_\varepsilon^2 - 4(\sigma - k^2)} \end{gathered}$$

$$(1.22) \qquad \begin{aligned} r_1^\varepsilon(k,\sigma) &= \frac{c_\varepsilon + \Delta_\varepsilon(k,\sigma)}{2\Lambda}, \qquad & r_2^\varepsilon(k,\sigma) &= \frac{c_\varepsilon - \Delta_\varepsilon(k,\sigma)}{2\Lambda} \\ \rho_1^\varepsilon(k,\sigma) &= \frac{c_\varepsilon + \delta_\varepsilon(k,\sigma)}{2}, \qquad & \rho_2^\varepsilon(k,\sigma) &= \frac{c_\varepsilon - \delta_\varepsilon(k,\sigma)}{2} \end{aligned}$$

When $\sigma = 0$, we will simply denote them by $r_i^\varepsilon(k)$ and $\rho_i^\varepsilon(k)$. When $\varepsilon = 0$, we will drop the superscript ε. Notice that in this case, we have $r_i = \rho_i$.

To end the section, let us introduce the Sivashinsky dispersion function:

$$(1.23) \qquad D(\sigma,k,l) = 2\Gamma^2(\Gamma - 1) - l(\sigma + \Gamma - 1).$$

Notice that, in comparison to [36], l (resp. σ) has been changed to $(-l)$ (resp. (σ)).

Stretching of the variables

The natural variable for the study of the problem is : $\xi = \dfrac{x}{\varepsilon}$. We then introduce

$$p_\varepsilon(\xi) = \frac{1 - T_\varepsilon(x)}{\varepsilon}, \quad q_\varepsilon(\xi) = \frac{Y_\varepsilon(x)}{\varepsilon}, \quad r_\varepsilon(\xi) = \frac{W_\varepsilon(x)}{\varepsilon}. \tag{1.24}$$

These quantities blow up for $x < x_3^\varepsilon$ as $\varepsilon \to 0$, but they are useful on the interval $[x_3^\varepsilon, +\infty[$. For $i \in \{1,2,3\}$ let us define the points $\xi_i^\varepsilon := \dfrac{x_i^\varepsilon}{\varepsilon}$; recall that x_i^ε are defined by (1.9). Consider the following operators $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\varepsilon^*$

$$\begin{aligned} \mathcal{T}_\varepsilon &= -\frac{d^2}{d\xi^2} + \varepsilon c_\varepsilon \frac{d}{d\xi} - h'(p_\varepsilon) \\ \mathcal{T}_\varepsilon^* &= -\frac{d^2}{d\xi^2} - \varepsilon c_\varepsilon \frac{d}{d\xi} - h'(p_\varepsilon) \end{aligned} \tag{1.25}$$

Let $UC([\xi_3^\varepsilon, +\infty[)$ the set of all bounded, uniformly continuous functions on $[\xi_3^\varepsilon, +\infty[$. The operators $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\varepsilon^*$ act in $UC_0^2([\xi_3^\varepsilon, +\infty[)$, i.e the set of all continuous functions u such that their derivatives up to order 2 belong to $UC([\xi_3^\varepsilon, +\infty[)$, and such that $u(\xi_3^\varepsilon) = 0$. In (1.25) we have set:

$$h(p) = -\left(1 - \frac{\varepsilon l}{\Lambda}\right) p e^p. \tag{1.26}$$

We notice that, for $l = 0$ and $T > T_\varepsilon(x_3^\varepsilon)$, there holds

$$h\left(\frac{T-1}{\varepsilon}\right) = \varepsilon g(T) = -\frac{T-1}{\varepsilon} e^{\frac{T-1}{\varepsilon}}.$$

These operators will repeatedly be used in the sequel for the analysis of various eigenvalue problems.

Final notation

In the whole sequel, C will denote a generic constant, independent of ε.

2. UNIFORM SPECTRAL ESTIMATES, UNIQUENESS AND THE DISPERSION RELATION

This part is devoted to the examination of systems $(1.18)_n$ and $(1.18)_0$. The new contribution in this section is the bound on $\sigma + k_n^2$ for σ not belonging to the P_Λ. In this scope we use a bootstrap method: any information on the size of $|\sigma| + k_n^2$ results in an expansion of ϕ_n with respect to ε, which itself yields a more precise bound for $|\sigma| + k_n^2$, and so on.

Once $|\sigma| + k_n^2$ is bounded, matching the expansions of ϕ_n and of its first derivative at x_3^ε will yield the dispersion relation. It only remains to show that this relation is necessary and sufficient; this is done by a simple fixed point argument.

We therefore obtain an alternative proof to Terman's [36], which is slightly shorter than his. Let us point out that matching derivatives works here because of the very particular form of systems $(1.18)_n$; it is not an alternative method to the Evans function in more general cases. Finally, let us mention that the methods we use are not unlike the SLEP method, used in [27] by Nishiura, Mimura, Ikeda, Fujii for bistable systems. The scales which are involved in our problem are however different, which renders the upper bound for the eigenvalues delicate to obtain.

Let us therefore investigate the following general system, with $\sigma \in \mathbb{C}$ and $k \in \mathbb{R}$:

$$\left\{\begin{array}{r}-\phi'' + c_\varepsilon \phi' + k^2\phi = \big(g'(T_\varepsilon) + f'(T_\varepsilon)W_\varepsilon\big)\phi + f(T_\varepsilon)\psi + \sigma\phi \\ -\Lambda\psi'' + c_\varepsilon\psi' + \Lambda k^2\psi = -(1-\Lambda)(-\phi'' + k^2\phi) + \sigma\psi\end{array}\right. \tag{2.1}$$

The first result is the upper bound result, which settles the problem left open in [36]:

THEOREM 2.1. *Let* $\sigma \notin \bigcup_{1/2 \le \Lambda \le 3/2} P_\Lambda$ *and* $k \in \mathbb{R}$ *such that (2.1) has a nontrivial solution* $(\phi, \psi) \in \big(L^2(\mathbb{R})\big)^2$. *There exists a constant* $C > 0$*, independent of* ε*, such that* $k^2 + |\sigma| \le C$.

Let us notice that, if $\sigma \notin \bigcup_{1/2\le\Lambda\le3/2} P_\Lambda$, then $(-k^2+\sigma) \notin \bigcup_{1/2\le\Lambda\le3/2} P_\Lambda$. The complex number σ will from then on always be assumed to satisfy this property; this will not be recalled again.

Let us now state our uniqueness result.

THEOREM 2.2. *Given* $M > 0$*, there exists a real number* $\varepsilon_0(M) > 0$ *such that, for* $|l| \le M$ *and* $\varepsilon \le \varepsilon_0(M)$*, uniqueness for (8)-(9) holds in the following way:*
- *the speed* c_ε *is unique,*
- *the solutions* T_ε *and* Y_ε *are unique up to the* x*-translations.*

As we will see in the course of the proof, this result also yields a construction of the waves. The last theorem of this section is the one of [36]; we restate it in a slightly different way.

THEOREM 2.3. *([36]) There is a function* $D_\varepsilon(\sigma, k, l)$*, holomorphic in* σ*, such that (2.1) has a nontrivial solution if and only if* $D_\varepsilon(\sigma, k, l) = 0$*; moreover the solution* *space is one-dimensional. Further, there holds:* $\lim_{\varepsilon\to 0} D_\varepsilon(\sigma,k,l) = D(\sigma,k,l)$ *uniformly on the sets* $\{(\sigma,k,l)\in B_M(0)\times[-M',M']\times[-M'',M'']\}$.

The proofs of Theorems 2.1, 2.2 and 2.3 will follow from a few steps, namely:
- The operator $\mathcal{T}_\varepsilon$
- upper bound for $|\sigma|+k^2$,
- uniqueness,
- uniform expansions for ϕ and ψ,
- computation of $\phi'(x_3^{\varepsilon +}) - \phi'(x_3^{\varepsilon -})$ and matching.

The limiting problem, the operator $\mathcal{T}_\varepsilon$ and more estimates on T_ε.

Many results of this subsection - but not all - were already known; see [10], [11], [19], [20], [36]. However, they have the merit of being self-contained, and to end up with a result - namely, the estimates for ${\mathcal{T}_\varepsilon}^{-1}$, from which everything in the sequel will naturally follow. The goal of this paragraph is the proof of the following proposition:

PROPOSITION 2.4. *The operators $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\varepsilon^*$ are both isomorphisms from $UC_0^2([\xi_3^\varepsilon,+\infty[)$ to $UC([\xi_3^\varepsilon,+\infty[)$. Further they satisfy*

$$(2.2)\qquad \begin{aligned} \|\mathcal{T}_\varepsilon^{-1}\|_{\mathcal{L}\left(UC([\xi_3^\varepsilon,+\infty[),UC_0^2([\xi_3^\varepsilon,+\infty[)\right)} &= O(\mathrm{Log}^4\varepsilon)\\ \|(\mathcal{T}_\varepsilon^*)^{-1}\|_{\mathcal{L}\left(UC([\xi_3^\varepsilon,+\infty[),UC_0^2([\xi_3^\varepsilon,+\infty[)\right)} &= O(\mathrm{Log}^4\varepsilon)\end{aligned}$$

Let us first write the limiting problem for p_ε, which reads

$$(2.3)\qquad -p'' + pe^p = 0 \text{ on } I\!R \quad p(+\infty) = 0.$$

Problem (2.3) has a unique strictly increasing, negative solution satisfying $p(0) = -1$, and which is given by

$$(2.4)\qquad \int_{-1}^{p(\xi)} \frac{dq}{\sqrt{2\big(1+(q-1)e^q\big)}} = \xi$$

This easily yields the

LEMMA 2.5. *The function p_ε satisfies*

$$(2.5)\qquad \|p_\varepsilon - p\|_{C^1([\xi_3^\varepsilon,+\infty[)} = O(\varepsilon\mathrm{Log}^4\varepsilon).$$

PROOF. From the *a priori* estimates for T_ε and Y_ε in [10], there holds

$$\|p_\varepsilon\|_{C^1([\xi_3^\varepsilon,+\infty[)} + \|q_\varepsilon\|_{C^1([\xi_3^\varepsilon,+\infty[)} = O(\mathrm{Log}\varepsilon);$$

moreover $\|W_\varepsilon\|_{C^1([\xi_3^\varepsilon,+\infty[)} = O(\varepsilon\mathrm{Log}\varepsilon)$. Therefore the equation for p_ε may read as

$$(2.6)\qquad -p_\varepsilon'' + p_\varepsilon e^{p_\varepsilon} = O(\varepsilon\mathrm{Log}\varepsilon) \text{ on } I\!R \quad p_\varepsilon(0) = p(0).$$

The integral formula (2.4) yields, for $\xi_3^\varepsilon < \xi < \xi_2^\varepsilon$:

$$\int_{p_\varepsilon}^{p} \frac{dq}{\sqrt{2\big(1+(q-1)e^q\big)}} = O(\varepsilon \mathrm{Log}\varepsilon) \tag{2.7}$$

which implies: $\|p_\varepsilon - p\| = O(\varepsilon \mathrm{Log}\varepsilon)$ for $\xi_3^\varepsilon < \xi \le \xi_2^\varepsilon$. Now, we also have for $\xi > \xi_2$

$$\frac{1}{2}{p'}^2 + (1-p)e^p = 1$$
$$\frac{1}{2}{p'_\varepsilon}^2 + (1-p_\varepsilon)e^{p_\varepsilon} = 1 + O(\varepsilon \mathrm{Log}^2\varepsilon)$$

which readily yields (2.5). •

We also need with some precision the asymptotic behaviour of p_ε and p'_ε as $\xi \to +\infty$. Another lemma will tell it to us.

LEMMA 2.6. *There exist constants $M > 0$, $m > 0$ independent of ε, such that, for $\xi \ge \xi_3^\varepsilon$, there holds*

$$\begin{aligned} m p_\varepsilon(\xi) \le p'_\varepsilon(\xi) \le M p_\varepsilon(\xi) \\ m q_\varepsilon(\xi) \le q'_\varepsilon(\xi) \le M q_\varepsilon(\xi) \end{aligned} \tag{2.8}$$

PROOF. From Lemma 2.5, the equation for q_ε and the mean value theorem, there exists a function $a_\varepsilon(\xi)$, bounded from below by a constant $\delta > 0$, independent of ε, such that q_ε satisfies

$$-q''_\varepsilon + \varepsilon c_\varepsilon q'_\varepsilon + a_\varepsilon(\xi) q_\varepsilon = 0. \tag{2.9}$$

Therefore there exists a constant $C > 0$ such that $|q_\varepsilon(\xi)| \le Ce^{-\delta\xi}$; this estimate, plugged in the equation for p_ε

$$-p''_\varepsilon + \varepsilon c_\varepsilon p'_\varepsilon = q_\varepsilon e^{p_\varepsilon} \quad \text{for } \xi \ge \xi_2^\varepsilon$$

yields the same result for p_ε. It is to be noted that the estimate also holds for the first and second derivatives of p_ε and q_ε. Now, let us write

$$-p''_\varepsilon + \varepsilon c_\varepsilon p'_\varepsilon + p_\varepsilon = O(e^{-\delta\xi} q_\varepsilon)$$
$$-\Lambda q''_\varepsilon + \varepsilon c_\varepsilon q'_\varepsilon + q_\varepsilon = O(e^{-\delta\xi} q_\varepsilon)$$

the "O" being independent of ε. A classical fixed point argument - see [14], Chapter 3, Sect. 8 - yields exact exponential estimates for p_ε and q_ε; hence (2.8) for $\xi \ge \xi_0$, ξ_0 being large enough. Application of Lemma 2.5 between ξ_3^ε and ξ_0 completes the proof. •

Lemmas 2.5 and 2.6 have the

COROLLARY 2.7. *There exists $q > 0$ such that $\xi_3^\varepsilon = q\mathrm{Log}\varepsilon(1 + O(\varepsilon))$.*

The proof is at this stage trivial, and is omitted.

PROOF OF PROPOSITION 2.4. We start with the following operator $\mathcal{T}_\infty$, operating between $UC_0^2([\xi_3^\varepsilon, +\infty[)$ and $UC([\xi_3^\varepsilon, +\infty[)$:

$$\mathcal{T}_\infty = -\frac{d^2}{d\xi^2} + (1+p)e^p$$

For $f(\xi) \in UC([\xi_3^\varepsilon, +\infty[)$, the equation $\mathcal{T}_\infty u = f$ has a unique solution, given by the integral formula

$$u(\xi) = p'(\xi) \int_{\xi_3^\varepsilon}^{\xi} \left(\int_{\xi'}^{+\infty} p'(\eta) f(\eta)\ d\eta \right) \frac{d\xi'}{p'^2(\xi')}; \tag{2.10}$$

Of course this holds because p' never vanishes. From (2.10) one obviously sees that ${\mathcal{T}_\infty}^{-1}$ satisfies (2.2). It only remains to see that

$$\mathcal{T}_\varepsilon = \mathcal{T}_\infty + \varepsilon \left(c_\varepsilon \frac{d}{d\xi} + \frac{1}{\varepsilon}\big((1+p_\varepsilon)e^{p_\varepsilon} - (1+p)e^p\big) - \frac{l}{\Lambda} e^{p_\varepsilon} \right),$$

which, with the aid of Lemma 2.5, easily implies the proposition. The same argument is valid for $\mathcal{T}_\varepsilon^*$. •

Upper bound for $|\sigma| + k^2$.

A few notations are first needed. Recall that Δ_ε, etc are defined by (1.21) and (1.22). Let us define the two integral operators $\mathcal{K}_1^\varepsilon(k,\sigma)$ and $\mathcal{K}_2^\varepsilon(k,\sigma)$ by

$$\begin{aligned} \mathcal{K}_1^\varepsilon(k,\sigma)\phi = -\frac{c_\varepsilon}{\Lambda\Delta_\varepsilon} \Big(r_2^\varepsilon(k,\sigma) &\int_{-\infty}^{x} e^{r_2^\varepsilon(k,\sigma)(x-y)} \phi(y)\ dy \\ + r_1^\varepsilon(k,\sigma) &\int_{x}^{+\infty} e^{r_1^\varepsilon(k,\sigma)(x-y)} \phi(y)\ dy \Big) \\ \mathcal{K}_2^\varepsilon(k,\sigma) = \frac{1}{\Lambda\Delta_\varepsilon} \Big(\int_{-\infty}^{x} e^{r_2^\varepsilon(k,\sigma)(x-y)} &\phi(y)\ dy + \int_{x}^{+\infty} e^{r_1^\varepsilon(k,\sigma)(x-y)} \phi(y)\ dy \Big) \end{aligned} \tag{2.11}$$

Thus the unknown ψ in (2.1) can be written as

$$\psi := \frac{\varepsilon l}{\Lambda} \mathcal{K}^\varepsilon(k,\sigma) = \frac{\varepsilon l}{\Lambda} \big(\phi + \mathcal{K}_1^\varepsilon(k,\sigma)\phi + \sigma \mathcal{K}_2^\varepsilon(k,\sigma)\phi \big) \tag{2.12}$$

When $\sigma = 0$, these three operators will simply be denoted by $\mathcal{K}^\varepsilon(k)$, $\mathcal{K}_1^\varepsilon(k)$ and $\mathcal{K}_2^\varepsilon(k)$.

In the remaining of this part, we assume that we are dealing with a nontrivial solution of (2.1), which additionally satisfy

$$\|\phi\|_\infty = 1. \tag{2.14}$$

This condition is the one that we are eventually going to contradict if $|\sigma|+k^2$ is assumed to be unbounded. We also set

$$\phi(x_3^\varepsilon)=\alpha(k,\sigma) \tag{2.15}$$

and we may assume that $\alpha(k,\sigma)\neq 0$. We could indeed find another point, no further from x_3^ε than ε^{100}, at which ϕ would not vanish.

Let us now start with the upper bound. A first glance at (2.1) leads to the estimate $|\sigma|+k^2=O(\frac{1}{\varepsilon^2})$. We will first prove a somewhat less rough estimate:

LEMMA 2.8. *There exists $C>0$ such that*

$$|\sigma|+k^2\leq -C\frac{\mathrm{Log}\varepsilon}{\varepsilon}.$$

PROOF. Let us set:

$$-V_\varepsilon(x)=g'(T_\varepsilon)+\frac{\varepsilon l}{\Lambda}f(T_\varepsilon)+f'(T_\varepsilon)W_\varepsilon;$$

we have, by differentiating the equation for T_ε in (8) and setting $T_\varepsilon'=\phi_0$:

$$-\phi_0''+c_\varepsilon\phi_0'+V_\varepsilon(x)\phi_0=f'(T_\varepsilon)\frac{\varepsilon l}{\Lambda}\mathcal{K}_1^\varepsilon(0)\phi_0.$$

However, from (2.11) and application of Lemmas 2.5 and 2.6, we notice that, for $x\geq x_3^\varepsilon$:

$$\begin{aligned}\mathcal{K}_1^\varepsilon(0)\phi_0&=-c_\varepsilon\int_x^{+\infty}e^{c_\varepsilon(x-y)}\phi_0(y)\ dy\\ &\geq -c_\varepsilon(1-T_\varepsilon)\\ &\geq mc_\varepsilon\varepsilon\mathrm{Log}\varepsilon\phi_0\end{aligned} \tag{2.16}$$

On the other hand, for $x\leq x_3^\varepsilon$ we have trivially

$$f'(T_\varepsilon)\frac{\varepsilon l}{\Lambda}\mathcal{K}_1^\varepsilon(0)\phi_0\geq -CT_\varepsilon' \tag{2.17}$$

because the left-handside of (2.17) has compact support, uniformly in ε. As a consequence we have, for a large constant $C>0$:

$$-\phi_0''+c_\varepsilon\phi_0'+V_\varepsilon(x)\phi_0\geq C\mathrm{Log}\varepsilon\phi_0. \tag{2.18}$$

To make the operator of the right-handside self-adjoint, it suffices to make the change of functions

$$\phi\mapsto\bar\phi=e^{-\frac{c}{2}x}\phi, \tag{2.19}$$

and we end up with the operator

$$L = -\frac{d^2}{dx^2} + (V_\varepsilon + \frac{c_\varepsilon^2}{4}).$$

Because of (2.18) we have, for every $\phi \in H^1(I\!R)$:

$$\big(L\phi|\phi\big)_{L^2} \geq C\,\mathrm{Log}\varepsilon\|\phi\|_{L^2}^2 \tag{2.20}$$

Let us now come back to Problem (2.1) and its solution (σ, ϕ, ψ). If $\bar\phi$ satisfies (2.19), we have

$$L\bar\phi + (k^2 - \sigma)\bar\phi = \frac{l}{\Lambda\varepsilon} e^{\frac{c_\varepsilon}{2}x} f(T_\varepsilon)\big(\mathcal{K}_1^\varepsilon(k,\sigma)\bar\phi + \sigma\mathcal{K}_2^\varepsilon(k,\sigma)\bar\phi\big).$$

Therefore we have

$$\begin{aligned}&\big(L\bar\phi|\bar\phi\big)_{L^2} + (k^2 - \sigma)\|\bar\phi\|_{L^2}^2\\ =&\frac{\varepsilon l}{\Lambda}\Big(-\big(e^{\frac{c_\varepsilon}{2}x} f(T_\varepsilon)\mathcal{K}_1^\varepsilon(k,\sigma)\bar\phi|\bar\phi\big)_{L^2} + \sigma\big(e^{\frac{c_\varepsilon}{2}x} f(T_\varepsilon)\mathcal{K}_2^\varepsilon(k,\sigma)\bar\phi|\bar\phi\big)_{L^2}\Big)\end{aligned} \tag{2.21}$$

and we are led to estimate the terms in the right-handside of (2.21). On the one hand we have - here we will exceptionally denote by ξ the Fourier variable -

$$\begin{aligned}|\big(\mathcal{K}_1^\varepsilon(k,\sigma)\bar\phi|\bar\phi\big)_{L^2}| &= |\big(\widehat{\mathcal{K}_1^\varepsilon(k,\sigma)\bar\phi}|\hat{\bar\phi}\big)_{L^2}\\ &= \frac{c_\varepsilon}{|\Delta_\varepsilon|}\Big|\int_{-\infty}^{+\infty} \frac{\xi}{\Lambda\xi^2 + ic_\varepsilon\xi + \Lambda k^2 - \sigma}\ |\hat\phi(\xi)|^2\ d\xi\Big|\\ &\leq C\|\bar\phi\|_{L^2}^2\end{aligned} \tag{2.22}$$

The last inequality is true because $\sigma \notin \bigcup\limits_{1/2\leq\Lambda\leq 3/2} P_\Lambda$. In the same spirit, we have

$$|\big(\mathcal{K}_2^\varepsilon(k,\sigma)\bar\phi|\bar\phi\big)_{L^2}| \leq \frac{C}{|\sigma|}\|\bar\phi\|_{L^2}^2 \tag{2.23}$$

Gathering (2.22), (2.23) and using (2.20) we obtain, by setting $\sigma = \sigma_1 + i\sigma_2$:

$$\frac{C\,\mathrm{Log}\varepsilon}{\varepsilon} \leq -\sigma_1 + k^2 \leq \frac{C}{\varepsilon} \qquad |\sigma_2| \leq \frac{C}{\varepsilon}$$

Thus $k^2 = O(\frac{1}{\varepsilon})$, and σ falls into the region $\mathcal{C}_\varepsilon$ below.

Fig. 2.1

The lemma is therefore proved. $\bullet$

Lemma 2.8 will allow us to obtain uniform estimates for ϕ on both sides of x_3^ε. The following lemma, however, is trivial and owes nothing to Lemma 2.8.

LEMMA 2.9. *On the interval* $]-\infty, x_3^\varepsilon]$, *there exists* $\delta > 0$ *and* $C > 0$ *such that there holds, uniformly in* σ*:*

$$|\phi(x) - \alpha(k,\sigma)e^{\rho_1^\varepsilon(k,\sigma)(x-x_3^\varepsilon)}| \leq C\varepsilon^{90}e^{\delta(x-x_3^\varepsilon)}.$$

PROOF. On the interval $]-\infty, x_3^\varepsilon]$ the function ϕ satisfies

$$-\phi'' + c_\varepsilon\phi' + (k^2 - \sigma)\phi = \varepsilon^{90}h_\varepsilon(x)$$

where h_ε has compact support and is uniformly bounded in ε. The fact that σ does not belong to the P_Λ implies the lemma. $\bullet$

Here is the last intermediate result before we prove our boundedness result for $\sigma + k^2$; it concerns the behaviour of ϕ for $x \geq x_3^\varepsilon$.

LEMMA 2.10. *For* $x \geq x_3^\varepsilon$ *the following estimate holds, uniformly in* σ *and* k*:*

$$\|\phi - \frac{\alpha(k,\sigma)}{T_\varepsilon'(x_3^\varepsilon)}T_\varepsilon'\|_{L^\infty([x_3^\varepsilon,+\infty[)} = O(\varepsilon\mathrm{Log}^4\varepsilon).$$

PROOF. Let us note that $\|\psi\|_\infty = O(\varepsilon\mathrm{Log}^4\varepsilon)$; therefore, in the stretched variable ξ, the function ϕ satisfies an equation of the form

$$-\phi'' + \varepsilon c_\varepsilon\phi' + h(p_\varepsilon)\phi + \varepsilon^2(k^2 - \sigma)\phi = O(\varepsilon\mathrm{Log}^4\varepsilon).$$

Therefore, from Lemma 2.8, the function

$$u(\xi) = \phi(\xi) - \frac{\alpha(k,\sigma)}{p'_\varepsilon(\xi_3^\varepsilon)} p'_\varepsilon(\xi)$$

satisfies

$$\big(\mathcal{T}_\varepsilon + O(\varepsilon \mathrm{Log}\varepsilon)\big)u = O(\varepsilon \mathrm{Log}^4\varepsilon), \qquad \text{for } \xi \geq \xi_3^\varepsilon, \qquad u(\xi_3^\varepsilon) = 0$$

and Lemma 2.10 follows from Proposition 2.4. •

PROOF OF THEOREM 2.1. In (2.1) we multiply the equation for ϕ by $e_0^*(x) := e^{-c_\varepsilon x} T'_\varepsilon$, and we integrate on $I\!R$; we note $< e_0^*, . >$ this action. Notice that all the integrals will converge. We obtain:

$$(2.24) \qquad (k^2 - \sigma) < e_0^*, \phi >= \big\langle e_0^*, \frac{\varepsilon l}{\Lambda} f(T_\varepsilon)\big(\mathcal{K}_1^\varepsilon(k,\sigma)\phi + \sigma \mathcal{K}_2^\varepsilon(k,\sigma)\phi - \mathcal{K}_1^\varepsilon(0)T'_\varepsilon\big)\big\rangle.$$

1. Estimates for $< e_0^*, f(T_\varepsilon)\mathcal{K}_1^\varepsilon(0)T'_\varepsilon >$.

From formula (2.11) we infer that $f(T_\varepsilon)\mathcal{K}_1^\varepsilon(0)T'_\varepsilon = O(\mathrm{Log}\varepsilon)$ for $x \geq x_3^\varepsilon$, and $O(\varepsilon^{90})$ for $x \geq x_3^\varepsilon$. Therefore

$$(2.25) \qquad \frac{\varepsilon l}{\Lambda} | < e_0^*, f(T_\varepsilon)\mathcal{K}_1^\varepsilon(0)T'_\varepsilon > | = O(\varepsilon \mathrm{Log}\varepsilon).$$

2. Estimates for $< e_0^*, f(T_\varepsilon)\mathcal{K}_1^\varepsilon(k,\sigma)\phi >$.

Let us first estimate $< e_0^*, f(T_\varepsilon)\mathcal{K}_1^\varepsilon(k,\sigma)T'_\varepsilon >$; the desired upper bound will follow from Lemma 2.10. The location of σ and k allow us to write, for some large constant $K > 0$:

$$(2.26) \qquad \Big|\frac{r_2^\varepsilon(k,\sigma)}{\Delta_\varepsilon}\Big| + \Big|\frac{r_1^\varepsilon(k,\sigma)}{\Delta_\varepsilon}\Big| = O(1);$$

therefore we may bound $< e_0^*, f(T_\varepsilon)\mathcal{K}_1^\varepsilon(k,\sigma)T'_\varepsilon >$ as follows:

$$\begin{aligned}
&| < e_0^*, f(T_\varepsilon)\mathcal{K}_1^\varepsilon(k,\sigma)T'_\varepsilon > | \\
\leq & \int_{x_3^\varepsilon}^{+\infty} e^{-c_\varepsilon x} T'_\varepsilon f(T_\varepsilon)\, dx \Big| \frac{r_2^\varepsilon(k,\sigma)}{\Delta_\varepsilon} \int_{-\infty}^{x} e^{r_2^\varepsilon(k,\sigma)(x-y)} T'_\varepsilon(y)\, dy \\
& \qquad + \frac{r_1^\varepsilon(k,\sigma)}{\Delta_\varepsilon} \int_{x}^{+\infty} e^{r_1^\varepsilon(k,\sigma)(x-y)} T'_\varepsilon(y)\, dy\Big| + O(e^{90}) \\
\leq & \frac{C}{\varepsilon}\left(1 + \varepsilon \int_{\xi_3^\varepsilon}^{+\infty} e^{-\varepsilon c_\varepsilon \xi} p_\varepsilon\right)
\end{aligned}$$

Taking into account Lemmas 2.9 and 2.10 we have

$$\begin{aligned}
&| < e_0^*, f(T_\varepsilon)\mathcal{K}_1^\varepsilon(k,\sigma)(\phi - \alpha(k,\sigma)T'_\varepsilon) > | \\
=& \Big|\frac{r_2^\varepsilon(k,\sigma)}{\Delta_\varepsilon} \int_{-\infty}^{x} e^{r_2^\varepsilon(k,\sigma)(x-y)} \big(T'_\varepsilon(y) - e^{\rho_1^\varepsilon(k,\sigma)(x-x_3^\varepsilon)}\, dy\Big|\ |\alpha(k,\sigma)| \\
&+ O(\mathrm{Log}^4\varepsilon)
\end{aligned}$$

This implies:

$$\frac{\varepsilon l}{\Lambda} < e_0^*, f(T_\varepsilon)\mathcal{K}_1^\varepsilon(k,\sigma)\phi >= K_\varepsilon(k,\sigma)\alpha(k,\sigma), \tag{2.27}$$

with $K_\varepsilon(k,\sigma) = O(1)$, uniformly in σ and k.

3. Estimates for $< e_0^*, f(T_\varepsilon)\mathcal{K}_2^\varepsilon(k,\sigma)\phi >$.

In the same spirit as above, we may write

$$\frac{\varepsilon l}{\Lambda}\sigma < e_0^*, f(T_\varepsilon)\mathcal{K}_2^\varepsilon(k,\sigma)\phi >= K'_\varepsilon\alpha(k,\sigma) + O(\varepsilon \mathrm{Log}^4\varepsilon), \tag{2.28}$$

with $K'_\varepsilon(k,\sigma) = O(1)$, uniformly in σ and k. Let us set

$$Q_\varepsilon(k,\sigma) = K_\varepsilon(k,\sigma) + K'_\varepsilon(k,\sigma). \tag{2.29}$$

4. Estimates for $< e_0^*, \phi >$.

From Lemmas 2.9 and 2.10, we readily obtain:

$$< e_0^*, \phi >= \left(\frac{T'_\varepsilon(x_3^\varepsilon)}{\rho_1^\varepsilon(k,\sigma)} + \int_{x_3^\varepsilon}^{+\infty} e^{-c_\varepsilon x}(T'_\varepsilon)^2\ dx\right) + O(\varepsilon \mathrm{Log}^4\varepsilon). \tag{2.30}$$

5. Conclusion.

Gathering estimates (2.25), (2.27), (2.28) and (2.30) we end up with

$$\left\{\left(\frac{T'_\varepsilon(x_3^\varepsilon)}{\rho_1^\varepsilon(k,\sigma)} + \int_{x_3^\varepsilon}^{+\infty} e^{-c_\varepsilon x}(T'_\varepsilon)^2\ dx\right)(k^2-\sigma) - Q_\varepsilon(k,\sigma)\right\}\alpha(k,\sigma) = O(\varepsilon \mathrm{Log}^4\varepsilon). \tag{2.31}$$

When $\sigma \notin \bigcup_{1/2\leq\Lambda\leq 3/2} P_\Lambda$ and when the amount $|\sigma| + k^2 \to +\infty$ there holds

$$\left|\left(\frac{T'_\varepsilon(x_3^\varepsilon)}{\rho_1^\varepsilon(k,\sigma)} + \int_{x_3^\varepsilon}^{+\infty} e^{-c_\varepsilon x}(T'_\varepsilon)^2\ dx\right)(k^2-\sigma) - Q_\varepsilon(k,\sigma)\right| \to +\infty;$$

therefore if $|\sigma| + k^2$ is larger than a constant that can be explicitly computed from (2.31) we have

$$\alpha(k,\sigma) = O(\varepsilon \mathrm{Log}^4\varepsilon).$$

From Lemmas 2.9 and 2.10, we infer

$$\|\phi\|_\infty = O(\varepsilon \mathrm{Log}^4\varepsilon). \tag{2.32}$$

This contradicts (2.14), and Theorem 2.1 is proved. •

Uniqueness for the planar front.

PROOF OF THEOREM 2.2. Let us consider two solutions $(T_{1\varepsilon}, Y_{1\varepsilon}, c_{1\varepsilon})$, $(T_{2\varepsilon}, Y_{2\varepsilon}, c_{2\varepsilon})$ of (8)-(9) such that $T_{1\varepsilon}(x_3^\varepsilon) = T_{2\varepsilon}(x_3^\varepsilon)$ and let us suppose that for a sequence $\varepsilon_n \to 0$ (also denoted ε) we have

$$(T_{1\varepsilon}, Y_{1\varepsilon}, c_{1\varepsilon}) \neq (T_{2\varepsilon}, Y_{2\varepsilon}, c_{2\varepsilon}).$$

Therefore we have $T_\varepsilon^1 \not\equiv T_\varepsilon^2$ on $[x_3^\varepsilon, +\infty[$. Indeed, if $T_{1\varepsilon} \equiv T_{2\varepsilon}$ on $[x_3^\varepsilon, +\infty[$, since

$$T_{i\varepsilon}(x) \simeq 1 - a_i e^{\lambda_i x} \quad \text{as} \quad x \to +\infty,$$

where

$$\lambda_i = \frac{c_{i\varepsilon} - \sqrt{c_{i\varepsilon}^2 + \dfrac{4\Lambda}{\varepsilon^2}}}{2\Lambda},$$

we have $\lambda_1 = \lambda_2$, $c_{1\varepsilon} = c_{2\varepsilon}$ and consequently $T_{2\varepsilon} = T_{1\varepsilon}$, $Y_{2\varepsilon} = Y_{1\varepsilon}$ on $\mathbb{R}$.

We then consider $p_{i\varepsilon}(\xi)$, $r_{i\varepsilon}(\xi)$ defined in (1.24) and we set

$$\alpha_\varepsilon = ||p_{2\varepsilon} - p_{1\varepsilon}||_{C^1([\xi_3^\varepsilon, +\infty[)} \neq 0 \quad \text{and} \quad \phi_\varepsilon = \frac{p_{2\varepsilon} - p_{1\varepsilon}}{\alpha_\varepsilon}. \tag{2.33}$$

$(p_{i\varepsilon}, r_{i\varepsilon})$ are solutions for $\xi \geq \xi_3^\varepsilon$ of equations

$$\begin{aligned} &-p'' + \varepsilon c p' + e^p(p + r) = 0,\\ &-\Lambda r'' + \varepsilon c r' = -\varepsilon l p''. \end{aligned}$$

Subtracting equations on $p_{2\varepsilon}$ and on $p_{1\varepsilon}$ and dividing by α_ε, we get

$$-\phi_\varepsilon'' + \varepsilon c_{1\varepsilon}\phi_\varepsilon' + e^{p_{1\varepsilon}}(\phi_\varepsilon + \frac{r_{2\varepsilon} - r_{1\varepsilon}}{\alpha_e}) + \frac{e^{p_{2\varepsilon}} - e^{p_{1\varepsilon}}}{\alpha_e}(p_{2\varepsilon} + r_{2\varepsilon}) + \varepsilon\frac{(c_{2\varepsilon} - c_{1\varepsilon})}{\alpha_\varepsilon}p_{2\varepsilon}' = 0.$$

We claim that

$$\frac{(c_{2\varepsilon} - c_{1\varepsilon})}{\alpha_\varepsilon} = O(1). \tag{2.34}$$

Indeed, multiplying the equation $-\Lambda W' + cW = -\varepsilon l T'$ by $e^{\frac{-cx}{\Lambda}}$ and integrating on $[x, x_3^\varepsilon]$ implies

$$e^{\frac{-cx}{\Lambda}} W(x) = e^{\frac{-cx_3^\varepsilon}{\Lambda}} W(x_3^\varepsilon) - \varepsilon l \int_x^{x_3^\varepsilon} T'(y) e^{\frac{-cy}{\Lambda}}\, dy$$

and therefore from (2.5)

$$||W_{2\varepsilon} - W_{1\varepsilon}||_{L^\infty([x_1^\varepsilon, x_3^\varepsilon])} = O(\varepsilon)|c_{2\varepsilon} - c_{1\varepsilon}| + O(\varepsilon)||T_{2\varepsilon}' - T_{1\varepsilon}'||_{L^\infty([x_1^\varepsilon, x_3^\varepsilon])}. \tag{2.35}$$

The same treatment to the equation for T yields

$$e^{-cx}T'(x) = e^{-cx_3^\varepsilon}T'(x_3^\varepsilon) + \int_x^{x_3^\varepsilon} f(T(y))(1 - T(y) + W(y))e^{-cy}\, dy.$$

We first remark that from (2.33), since $T'_{i\varepsilon}(x_3^\varepsilon) = p'_{i\varepsilon}(\xi_3^\varepsilon)$,

$$T'_{2\varepsilon}(x_3^\varepsilon) - T'_{1\varepsilon}(x_3^\varepsilon) = O(1)\alpha_\varepsilon. \tag{2.36}$$

Then, from estimates on $f(T_{i\varepsilon}(y))$ for $y \in [x_1^\varepsilon, x_3^\varepsilon]$, we deduce

$$\begin{aligned}||T'_{2\varepsilon} - T'_{1\varepsilon}||_{L^\infty([x_1^\varepsilon,x_3^\varepsilon])} =& O(1)|c_{2\varepsilon} - c_{1\varepsilon}| + O(\varepsilon)||T_{2\varepsilon} - T_{1\varepsilon}||_{L^\infty([x_1^\varepsilon,x_3^\varepsilon])} \\ &+ O(\varepsilon)||W_{2\varepsilon} - W_{1\varepsilon}||_{L^\infty([x_1^\varepsilon,x_3^\varepsilon])} + O(1)\alpha_\varepsilon,\end{aligned}$$

that is, from (2.34)-(2.36) and the fact that $T_{2\varepsilon} = T_{1\varepsilon}$ for $x = x_3^\varepsilon$

$$||T_{2\varepsilon} - T_{1\varepsilon}||_{L^\infty([x_1^\varepsilon,x_3^\varepsilon])} = O(1)|c_{2\varepsilon} - c_{1\varepsilon}| + O(1)\alpha_\varepsilon. \tag{2.37}$$

We next integrate the equation on T on $[x_1^\varepsilon, x_3^\varepsilon]$ and we find

$$-T'(x_3^\varepsilon) + cT(x_3^\varepsilon) = \int_{x_1^\varepsilon}^{x_3^\varepsilon} f(T(y))(1 - T(y) + W(y))dy$$

since $T'(x_1^\varepsilon) = cT(x_1^\varepsilon)$. Thus, from (2.35)-(2.37) and again estimates on $f(T(y))$ for $y \in [x_1^\varepsilon, x_3^\varepsilon]$ we have

$$(c_{2\varepsilon} - c_{1\varepsilon})T_{1\varepsilon}(x_3^\varepsilon) = (T'_{2\varepsilon}(x_1^\varepsilon) - T_{1\varepsilon}(x_1^\varepsilon)) + O(\varepsilon)(|c_{2\varepsilon} - c_{1\varepsilon}| + \alpha_\varepsilon)$$

and estimate (2.34) is proved.

Thus it is easy to check with the integral representation of $r_{i\varepsilon}(\xi)$ that

$$\frac{r_{2\varepsilon} - r_{1\varepsilon}}{\alpha_\varepsilon} = O(\varepsilon). \tag{2.38}$$

Hence estimates (2.34) and (2.38) imply

$$\mathcal{T}_{1\varepsilon}\phi_\varepsilon = O(\varepsilon),$$

where $\mathcal{T}_{1\varepsilon}$ is the operator defined in (1.25) associated with $p_{1\varepsilon}$. Since $\phi_\varepsilon(\xi) = 0$, we deduce from Proposition 2.4 that $||\phi_\varepsilon||_{C^1([\xi_3^\varepsilon,+\infty[)} = 0(\varepsilon\mathrm{Log}^4\varepsilon)$, which is a contradiction with (2.33). This concludes the proof of Theorem 2.2. •

Asymptotic expansions of the eigenfunctions.

Let us first point out that now, due to Theorem 2.1, all the expansions will converge uniformly with respect to σ and k, just because there are no eigenfunctions for large σ or k. This will not be recalled.

Let us denote by $(\phi^{k,\sigma}, \psi^{k,\sigma})$ a solution of (2.1) such that $\phi^{k,\sigma}(x_3^\varepsilon) = T'_\varepsilon(x_3^\varepsilon)$. An expansion of $\phi^{k,\sigma}$ for $x \le x_3^\varepsilon$ is readily given by Lemma 2.9 with all the needed precision; for $x \ge x_3^\varepsilon$ Lemma 2.10 tells us that the first term is $T'_\varepsilon(x)$. Let us set

$$\tilde{\phi}^{k,\sigma} = \begin{cases} T'_\varepsilon(x_3^\varepsilon)e^{r_1^\varepsilon(k,\sigma)(x-x_3^\varepsilon)} & \text{if } x \le x_3^\varepsilon \\ T'_\varepsilon(x) & \text{if } x \ge x_3^\varepsilon \end{cases} \tag{2.39}$$

and

$$K_1^{\varepsilon,k,\sigma} = -\mathcal{K}_1^\varepsilon(k,\sigma)\tilde{\phi}^{k,\sigma} + \sigma\mathcal{K}_2^\varepsilon(k,\sigma)\tilde{\phi}^{k,\sigma}. \tag{2.40}$$

Define $\phi_1^{k,\sigma}$ by its expression in the stretched variable ξ:

$$\mathcal{T}_\varepsilon\phi_1^{k,\sigma} = \frac{l}{\Lambda}K_1^{\varepsilon,k,\sigma}(\xi)e^{p_\varepsilon}, \qquad \phi_1^{k,\sigma}(\xi_3^\varepsilon) = 0. \tag{2.41}$$

PROPOSITION 2.11. *For $\xi \geq \xi_3^\varepsilon$, the first-order expansion of $(\phi^{k,\sigma}, \psi^{k,\sigma})$ in ε is given by*

$$\begin{aligned} \phi^{k,\sigma}(\xi) &= p_\varepsilon'(\xi) + \varepsilon\phi_1^{k,\sigma}(\xi) + O(\varepsilon^2\mathrm{Log}^4\varepsilon) \\ \psi^{k,\sigma}(\xi) &= \frac{\varepsilon l}{\Lambda}\big(\phi^{k,\sigma} + K_1^{\varepsilon,k,\sigma} + O(\varepsilon\mathrm{Log}^4\varepsilon)\big) \end{aligned} \tag{2.42}$$

PROOF. From Lemmas 2.9 and 2.10, and also in view of the definitions of $\mathcal{K}^\varepsilon$ and $\mathcal{K}_1^\varepsilon$, we infer that the second equation of (2.42) is valid. The function $\tilde{\phi}_1^{k,\sigma}(\xi) := \dfrac{\phi^{k,\sigma}(\xi) - p_\varepsilon'(\xi)}{\varepsilon}$ is a solution of

$$\mathcal{T}_\varepsilon\tilde{\phi}_1^{k,\sigma} = \frac{l}{\Lambda}K_1^{\varepsilon,k,\sigma}(\xi)e^{p_\varepsilon} + O(\varepsilon\mathrm{Log}\varepsilon), \qquad \tilde{\phi}_1^{k,\sigma}(\xi_3^\varepsilon) = 0.$$

This, together with Proposition 2.4, implies Proposition 2.11. •

The dispersion relation.

Let $\mathcal{K}^\varepsilon(\phi)$ be the linear operator defined by (2.12). Let us consider the problem $(\mathcal{P})$: find $\phi \in C_0(I\!R)$, of class C^1 on $]-\infty, x_3^\varepsilon]$ and $[x_3^\varepsilon, +\infty[$, satisfying the first equation of (2.1) on both sides of x_3^ε.

LEMMA 2.12. *Problem $(\mathcal{P})$ has a unique solution, denoted by $\phi^{k,\sigma}$, satisfying $\phi^{k,\sigma}(x_3^\varepsilon) = T_\varepsilon'(x_3^\varepsilon)$. Moreover, $\phi^{k,\sigma}$ is holomorphic in σ.*

PROOF. Its existence may easily be shown by a fixed point theorem: one may indeed express $(\mathcal{P})$ under the following form. Find ϕ such that

$$\begin{aligned} \phi''(x) + c_\varepsilon\phi'(x) + (k^2 - \sigma)\phi(x) &= \varepsilon^{90}L_{\mathrm{left}}(k,\sigma)\phi(x) \qquad (x \leq x_3^\varepsilon) \\ -\phi''(\xi) + \varepsilon c_\varepsilon\phi'(\xi) + h'(p_\varepsilon)\phi(\xi) &= \varepsilon L_{\mathrm{right}}(k,\sigma)\phi(\xi) \qquad \xi \geq \xi_3^\varepsilon \\ \phi(x_3^\varepsilon) &= T_\varepsilon(x_3^\varepsilon) \end{aligned} \tag{2.43}$$

where L_{left} and L_{right} are linear operators on $C_0(I\!R)$ whose norms do not exceed $\mathrm{Log}^4\varepsilon$. This proves the existence and uniqueness of ϕ; further the operator L_{left} (resp. L_{right}) depends analytically on σ. Because it appears with ε^{90} (resp. ε), we also get the analytic dependence of ϕ with respect to σ. the lemma. •

PROOF OF THEOREM 2.3. Lemma 2.12 tells us that there exists a solution to (2.1) if and only if the function $\phi^{k,\sigma}$ found in this lemma satisfies ${\phi^{k,\sigma}}'(x_3^{\varepsilon}-) = {\phi^{k,\sigma}}'(x_3^{\varepsilon}+)$.

This is the dispersion relation, which is, again by Lemma 2.12, analytic in σ. Let us therefore set

$$D_\varepsilon(\sigma,k,l) = \phi^{k,\sigma\prime}(x_3^{\varepsilon\,-}) - \phi^{k,\sigma\prime}(x_3^{\varepsilon\,+}) \tag{2.44}$$

To prove Theorem 2.3, it remains to find the limit of D_ε as $\varepsilon \to 0$. We already have:

$$\phi^{k,\sigma\prime}(x_3^{\varepsilon\,-}) = c_\varepsilon \rho_1^\varepsilon(k,\sigma) + O(\varepsilon \mathrm{Log}\varepsilon). \tag{2.45}$$

Turn to the right handside. Differentiating with respect to ξ the equation for $\phi^{k,\sigma}$ in (2.42), and taking the value at ξ_3^ε, we have

$$\begin{aligned}\phi^{k,\sigma\prime}(\xi_3^{\varepsilon\,+}) =& p_\varepsilon''(\xi_3^\varepsilon) + \varepsilon \phi_1^{k,\sigma\prime}(\xi_3^\varepsilon) + O(\varepsilon^2 \mathrm{Log}^4\varepsilon)\\ =& \varepsilon\big(c_\varepsilon^2 + \phi_1^{k,\sigma\prime}(\xi_3^\varepsilon) + O(\varepsilon \mathrm{Log}^4\varepsilon)\big)\end{aligned} \tag{2.46}$$

Let us now notice that, due to Proposition 2.11, $K_1^{\varepsilon,k,\sigma}(\xi)$ tends to a constant as $\varepsilon \to 0$; namely:

$$\begin{aligned}K_1^{\varepsilon,k,\sigma} =& \frac{c_\varepsilon}{\Delta_\varepsilon(k,\sigma)\big(\rho_1^\varepsilon(k,\sigma) - r_2^\varepsilon(k,\sigma)\big)}\big(\sigma - c_\varepsilon r_2^\varepsilon(k,\sigma)\big) + O(\varepsilon \mathrm{Log}^4\varepsilon)\\ =& \frac{c_\varepsilon}{\Delta_\varepsilon^2(k,\sigma)}\big(\sigma - c_\varepsilon r_2^\varepsilon(k,\sigma)\big) + O(\varepsilon \mathrm{Log}^4\varepsilon)\end{aligned} \tag{2.47}$$

Multiplying (2.41) by p_ε' and integrating by parts, then using (2.47), we find out that

$$c_\varepsilon \phi^{k,\sigma\prime}(\xi_3^{\varepsilon\,+}) = \frac{l c_\varepsilon}{\Delta_\varepsilon^2(k,\sigma)}\big(\sigma - c_\varepsilon r_2^\varepsilon(k,\sigma)\big) + O(\varepsilon \mathrm{Log}^4\varepsilon). \tag{2.48}$$

We gather (2.45) and (2.46), then we use (2.48); this implies Theorem 2.3. •

3. EXISTENCE OF THE BIFURCATION

The results of the preceding section allow us to operate near the case $\varepsilon = 0$, for which everything is explicit. Coming back to the notations of Section 2, we will set

$$\begin{aligned} \phi_1(x) &= \phi^{k_1,0}(x), \qquad & \psi_1(x) &= \psi^{k_1,0}(x) \\ \Phi_1(x,y) &= \phi_1(x)\cos k_1 y, \qquad & \Psi_1(x,y) &= \psi_1(x)\cos k_1 y \end{aligned} \tag{3.1}$$

We will prove the following result:

THEOREM 3.1. *For $\varepsilon > 0$ small enough, there exist two positive reals l_ε^0, $s_0^\varepsilon > 0$ and a C^2 curve $(l^\varepsilon(s), T(s), W(s))$, defined on $[-s_0^\varepsilon, s_0^\varepsilon]$, such that $l^\varepsilon(0) = l_0^\varepsilon$, and satisfying (1.12). Further, the functions $T(s), W(s)$ admit the following expansion, near $s = 0$:*

$$\begin{aligned} T(s,x,y) &= T_\varepsilon(x) + s\phi_1(x)\cos k_1 y + o(s) \\ W(s,x,y) &= W_\varepsilon(x) + s\psi_1(x)\cos k_1 y + o(s). \end{aligned}$$

Recall that the function $\mathcal{F}_\varepsilon$ is a C^2 mapping from the space X to the space Y, these two spaces being respectively defined by (1.5) and (1.7). To apply Theorem 1.7 of [15], it will be sufficient to find some $l_\varepsilon^0 > 0$, such that:

1. the null space of $\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0, T_\varepsilon, W_\varepsilon)$ is one-dimensional, spanned by (Φ_1, Ψ_1),
2. the range of $\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0, T_\varepsilon, W_\varepsilon)^*$ has codimension 1,
3. $\big\langle(\Phi_1^*, \Psi_1^*), \partial^2_{l,(T,W)}\mathcal{F}_\varepsilon(l_\varepsilon^0, T_\varepsilon, W_\varepsilon)(\Phi_1, \Psi_1)\big\rangle \neq 0$.

The ingredients needed to check Points 1-3 are standard: we first remark that $\partial_{T,W}\mathcal{F}_\varepsilon(l, T, W)$ is Fredholm, for a solution (l, T, W) of (1.12); then we prove that the null space of $\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0, T_\varepsilon, W_\varepsilon)$ at l_ε^0 - to be defined yet - is of geometric and algebraic dimension one. The latter fact could be checked with the aid of the approximate Evans function and Lemma 6.2 of [2]; nevertheless we will again follow another way by computing, to suitable order, the product $\big\langle(\Phi_1^*, \Psi_1^*), (\Phi_1, \Psi_1)\big\rangle$. This is by no means useless extra work: we still need to know its sign for the next section. Therefore it is reasonable to use all the help this computation can give.

The following questions will be addressed to:
- The operator $\partial_{T,W}\mathcal{F}_\varepsilon(l, T, W)$ is Fredholm,
- the adjoint equation for (2.1),
- Existence of the bifurcation.

The operator $\partial_{T,W}\mathcal{F}_\varepsilon(l, T, W)$ is Fredholm for small ε.

We first need a general lemma; let $H^2_{\partial y}(\Sigma)$ denote the space of all H^2 functions of Σ having zero y derivatives at $I\!R \times \{0, 1\}$. Let, as before, $H(x)$ be the Heaviside function. In the sequel, we will often use the notations

$$\Sigma_- = I\!R_- \times [0,1], \qquad \Sigma_+ = I\!R_+ \times [0,1]. \tag{3.2}$$

LEMMA 3.2. *Let α be a nonnegative constant. Let $\mathcal{M} : H^1(\Sigma) \mapsto H^2_{\partial y}(\Sigma)$ be a continuous operator, and choose $c > \frac{c_\varepsilon}{2}$. Let $T_\varepsilon : X_r \mapsto L^2(\Sigma) \cap B_{\mu_r}$ be defined as:*

$$T_\varepsilon = -\Delta + c\partial_x + \frac{H(x)}{\varepsilon^2}\big(I - \varepsilon(\alpha I + \mathcal{M})\big). \tag{3.3}$$

Then, for $\varepsilon > 0$ small enough, T_ε is an isomorphism.

PROOF. For each $v \in L^\infty(\Sigma)$, the linear problem

$$
\begin{aligned}
-\Delta u + cu_x + \frac{H(x)}{\varepsilon^2}\big((1-\varepsilon\alpha)u - \varepsilon v\big) &= 0 \qquad (\Sigma) \\
u_y(x,0) = u_y(x,1) &= 0
\end{aligned}
\tag{3.4}
$$

has a unique solution $u := S_\varepsilon v$ as soon as ε is small enough. To see that $u \in B_{\mu_r}$, one only has to make the change of unknown $u \mapsto \mu_r v$. The solution u belongs to X_r if $v \in H^1(\Sigma)$. To prove the lemma, it suffices to prove that the operator S_ε is such that

$$
\|S_\varepsilon \mathcal{M}\|_{\mathcal{L}(H^1)} = O(\varepsilon) \tag{3.5}
$$

For $v \in H^1(\Sigma)$, let us set $\bar v = \|\mathcal{M}v\|_\infty$, and let us set $\bar u := S_\varepsilon \bar v$. We note that $|u| \le \bar u$; moreover

$$
\bar u(x) = \begin{cases} \dfrac{r_\varepsilon}{r_\varepsilon + c}\dfrac{\varepsilon}{1-\varepsilon\alpha}\bar v e^{cx} & \text{if } x \le 0 \\ \dfrac{-c}{r_\varepsilon + c}\dfrac{\varepsilon}{1-\varepsilon\alpha}\bar v e^{r_\varepsilon x} + \dfrac{\varepsilon}{1-\varepsilon\alpha}\bar v H(x) & \text{if } x \ge 0 \end{cases}
$$

We have set

$$
r_\varepsilon = \frac12\left(-c + \sqrt{c^2 + \frac{4}{\varepsilon^2}(1-\varepsilon\alpha)}\right)
$$

As a consequence:

$$
\|u\|_\infty + \|u\|_{L^2(\Sigma_-)} \le C\varepsilon \|v\|_{H^1(\Sigma)}; \tag{3.6}
$$

multiplying equation (3.4) and using (3.6) yields (3.5). •.

In order to handle equations with continuous coefficients, let H_δ be a C^∞, non-decreasing function, agreeing with H outside $[-\frac{1}{\delta}, \frac{1}{\delta}]$. Let $T_{\varepsilon,\delta}$ be defined as in (3.3), except that H is replaced by H_δ.

COROLLARY 3.3. *The operator $T_{\varepsilon,\delta}$ is an isomorphism from $H^2_{\partial_y}(\Sigma)$ to $L^2(\Sigma)$ when ε and δ are small enough.*

PROOF. Let $S_{\varepsilon,\delta}$ be defined as in (3.4) and let us estimate $(S_\varepsilon - S_{\varepsilon,\delta})v$. Set $u := S_\varepsilon v$; $u^\delta := S_{\varepsilon,\delta}$; subtracting the equation for u to the equation for u^δ, and noticing that, for $\delta = O(1)$:

$$
\|u^\delta\|_{L^\infty(\Sigma)\cap L^2(\Sigma)} \le \frac{C}{\varepsilon^2}, \tag{3.7}
$$

we find out, after the usual multiplication by $u - u^\delta$ and integration by parts:

$$
\begin{aligned}
\int_\Sigma |\nabla(u-u^\delta)|^2 \; + \frac{1-\varepsilon\alpha}{\varepsilon^2}\int_{\Sigma_+} |u-u^\delta|^2 \\
\le C\big(\|u^\delta\|_{L^2(\Sigma)} + \|v\|_{H^1(\Sigma)}\big)\|H - H_\delta\|_{L^2(\Sigma)}
\end{aligned}
\tag{3.8}
$$

Taking $\delta = \varepsilon^4$ implies, together with elliptic regularity, the corollary. •

We are now ready to state the main proposition of this section. Recall that $X^\#$ and $Y^\#$ are defined by (1.5) and (1.7).

PROPOSITION 3.4. *Let (l, c, T, W) be a solution of (1.12). For $\varepsilon > 0$ small enough, the operator $\partial_{T,W}\mathcal{F}_\varepsilon(l, T, W)$ has a decomposition of the form*

$$\partial_{T,W}\mathcal{F}_\varepsilon(l, T, W) = \mathcal{A} + \mathcal{B} \tag{3.9}$$

where $\mathcal{A}$ is an isomorphism from X to Y (resp. from $X^\#$ to $Y^\#$) and $\mathcal{B}\mathcal{A}^{-1}$ is a compact operator of Y (resp. $Y^\#$).

PROOF. With the notations of Corollary 3.3, let us set

$$\mathcal{A}(u, w) = \begin{pmatrix} -\Delta u + cu_x + \dfrac{H_{\varepsilon^4}(x)}{\varepsilon^2}u - \dfrac{H_{\varepsilon^4}(x)}{\varepsilon^2}w \\ -\Lambda\Delta w + cw_x - (1-\Lambda)\Delta u \end{pmatrix} \tag{3.10}$$

$$\mathcal{B}(u, w) = \begin{pmatrix} \mu_\varepsilon(u)T_x - \big(g'(T) + f'(T)W + \dfrac{H_{\varepsilon^4}(x)}{\varepsilon^2}\big)u + \big(\dfrac{H_{\varepsilon^4}(x)}{\varepsilon^2} - f(T)\big)w \\ \mu_\varepsilon(u)W_x \end{pmatrix} \tag{3.11}$$

Obviously, we have $\partial_{T,W}\mathcal{F}_\varepsilon(l, T, W) = \mathcal{A} + \mathcal{B}$; let us prove that $\mathcal{A}$ is an isomorphism from X to Y. Take $(f_1, f_2) \in Y$; let us try and solve $\mathcal{A}(u, w) = (f_1, f_2)$.

1. Consider first the equation for w, and let us define the operator $\mathcal{M}$.

• Due to the definition of Y_2 the equation

$$\begin{gathered} -\Lambda\Delta w_1 + c\partial_x w_1 = f_2 \quad (\Sigma) \\ \partial_y w_1(x, 0) = \partial_y w_1(x, 1) = 0 \end{gathered} \tag{3.12}$$

is uniquely solvable in $H^2_{\partial_y}(\Sigma)$: to see this, it suffices to notice that, by taking the pointwise mean of (3.12) over $[0, 1]$ we get

$$< w_1 > (x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{< \hat{f}_2 > (\xi)}{\Lambda\xi^2 + ic\xi} e^{ix\xi}\, d\xi; \tag{3.14}$$

then the problem (3.12), with f_2 replaced by $\tilde{f}_2$ is uniquely solvable in $\tilde{H}^2_{\partial_y}(\Sigma)$, due to the Poincaré inequality.

• Because Δu is the sum of an x-derivative and $\partial^2_{yy}u$, which has a zero mean, the problem

$$\begin{gathered} -\Lambda\Delta w_1 + c\partial_x w_1 = (1-\Lambda)\Delta u \quad (\Sigma) \\ \partial_y w_1(x, 0) = \partial_y w_1(x, 1) = 0 \end{gathered} \tag{3.15}$$

is uniquely solvable in H^2, for any given u in X_r. Let us write $w = \dfrac{\varepsilon l}{\Lambda}u + z$; we have

$$\begin{gathered} -\Lambda\Delta z + c\partial_x z = \frac{\varepsilon lc}{\Lambda}u_x \quad (\Sigma) \\ \partial_y w_1(x, 0) = \partial_y w_1(x, 1) = 0 \end{gathered} \tag{3.16}$$

This implies, taking the y-mean of (3.16): $\| < z > \|_{H^2} = O(\varepsilon)\|u\|_{H^1}$; then Poincaré's inequality allows a similar estimate for $\tilde{z}$. We define the operator $\mathcal{M}$ as $u \mapsto \frac{1}{\varepsilon}z$.

2. Let us now solve the equation in u. We have to consider the problem

$$T_{\varepsilon,\varepsilon^4}u = \frac{H_{\varepsilon^4}w_1}{\varepsilon^2},$$

the right-handside depending only on f_2. In the definition of $T_{\varepsilon,\varepsilon^4}$ (3.3), we have taken $\alpha = \frac{lc}{\Lambda}$. Application of Corollary 3.3 tells us that we indeed have a solution u in $H^2_{\partial_y}(\Sigma)$.

3. It remains to prove that the solution (u, w) is in X. The equation for $w^\# := \mu_r^\# w$ reads:

$$-\Delta w^\# + (c - 2r)\partial_x w^\# + r(c - r)w^\# = (1 - \Lambda)e^{rx}\Delta u^\# \tag{3.17}$$

which proves that a B_{μ_r} estimate holds for w, in terms of the $H^1 \cap L^\infty$ norm of u. The same argument for u holds; hence $\mathcal{A}$ is an isomorphism. form X to Y. In view of the above considerations, and in particular (3.17), the fact that $\mathcal{A}$ is an isomorphism from $X^\#$to $Y^\#$ is obvious.

The compactness of $\mathcal{B}\mathcal{A}^{-1}$ is standard; see [21]. The key ingredient is that the functions that are involved tend to 0 uniformly in y as $|x| \to +\infty$. The integral terms are obviously compact, and this ends the proof of Proposition 3.4. •

The adjoint equation for (2.1).

In this subsection, we are interested in solving the adjoint system for (2.1), with $\sigma = 0$ and $k \neq 0$, i.e

$$\begin{cases} -\phi^{*\prime\prime} - c_\varepsilon\phi^{*\prime} + k^2\phi^* + (1-\Lambda)(-\psi^{*\prime\prime} + k^2\psi^*) = \big(g'(T_\varepsilon) + f'(T_\varepsilon)W_\varepsilon\big)\phi^* \\ \qquad\qquad -\Lambda\psi^{*\prime\prime} - c_\varepsilon\psi^{*\prime} + \Lambda k^2\psi^* = f(T_\varepsilon)\phi^* \end{cases} \tag{3.18}$$

A solution of (3.18) such that $\phi^*(x_3^\varepsilon) = T_\varepsilon'(x_3^\varepsilon)$ will be denoted by $(\phi^{k,*}, \psi^{k,*})$. When $k = 0$, we furthermore ask $(\phi^{k,*}, \psi^{k,*}) \in L^2(I\!R)^2$. Let us notice that it is always possible to make such an assumption; see the remark at the beginning of Section 2. We are going to prove the follwing proposition:

PROPOSITION 3.5. *Let $(\phi^{k,*}, \psi^{k,*})$ a solution of (3.18). For $k \in I\!R$, there exists $\rho(k) > 0$ such that the following expansion holds:*

1. *For $x \geq x_3^\varepsilon$,,*

$$\begin{aligned} \phi^{k,*}(x) &= e^{-c_\varepsilon\varepsilon(x-x_3^\varepsilon)}T_\varepsilon'(x) + O(\varepsilon^2\mathrm{Log}^8\varepsilon)e^{-\rho(k)(x-x_3^\varepsilon)} \\ \varepsilon\psi^{k,*} &= e^{\varepsilon r_2^\varepsilon(k)(x-x_3^\varepsilon)} + O(\varepsilon^2\mathrm{Log}^8\varepsilon)e^{-\rho(k)(x-x_3^\varepsilon)} \end{aligned} \tag{3.19}$$

2. *For $x \leq x_3^\varepsilon$,*

$$\begin{aligned} \phi^{k,*}(x) &= \big(1 - \frac{l\rho_1^\varepsilon(k)}{\Delta_\varepsilon(k)}(x - x_3^\varepsilon)\big)T'(x_3^\varepsilon)e^{\rho_1^\varepsilon(k)(x-x_3^\varepsilon)} + O(\varepsilon^{90}) \\ \varepsilon\psi^{k,*}(x) &= \frac{1}{\Delta_\varepsilon(k)}e^{\rho_1^\varepsilon(k)(x-x_3^\varepsilon)} + O(\varepsilon\mathrm{Log}^8\varepsilon) \end{aligned} \tag{3.20}$$

When $k = 0$, we may take a weight of the form $\dfrac{\rho(k)}{\varepsilon^2}$, *with* $\rho(k) > 0$.

PROOF. The second equation of (3.18) yields:

$$\begin{aligned}\psi^{k,*}(x) = \frac{1}{\Delta_\varepsilon(k)}\Big(&\int_{-\infty}^{x} e^{r_2^\varepsilon(k)(x-y)} f(T_\varepsilon)\phi^{k,*}(y)\ dy \\ &+ \int_x^{+\infty} e^{r_1^\varepsilon(k)(x-y)} f(T_\varepsilon)\phi^{k,*}(y)\ dy\Big)\end{aligned} \tag{3.21}$$

Therefore we have

$$\|\psi^{k,*}\|_{C^2(I\!R)} \le \frac{C}{\varepsilon}\|\phi^{k,*}\|_\infty. \tag{3.22}$$

The equation for $\phi^{k,*}$, read on the domain $]-\infty, x_3^\varepsilon]$, yields:

$$\|\phi^{k,*}\|_{L^\infty(]-\infty,x_3^\varepsilon])} = O(1) + O(e^{90})\|\phi^{k,*}\|_\infty. \tag{3.23}$$

Stretching the x-variable on the domain $[x_3^\varepsilon, +\infty[$ and using the equation for $\psi^{k,*}$ yields

$$(\mathcal{T}_\varepsilon^* + O(\varepsilon \mathrm{Log}^4\varepsilon))\phi^{k,*} = O(\varepsilon\mathrm{Log}^4\varepsilon), \quad \phi^{k,*}(\xi_3^\varepsilon) = p_\varepsilon'(\xi_3^\varepsilon); \tag{3.24}$$

therefore, combining (3.22), (3.23) and (3.24) to Proposition 2.4 we get: $\|\phi^{k,*}\|_\infty = O(1)$; therefore $\|\psi^{k,*}\|_\infty = O(\frac{1}{\varepsilon})$. The rest of the proof is now easy. Indeed, the function $\tilde\phi^{k,*} := \phi^{k,*} - e^{-c_\varepsilon\varepsilon(\xi-\xi_3^\varepsilon)}p_\varepsilon'$ is a solution of

$$\mathcal{T}_\varepsilon^*\tilde\phi^{k,*} = O(\varepsilon^2\mathrm{Log}^4\varepsilon); \qquad \tilde\phi^{k,*}(\xi_3^\varepsilon) = 0.$$

This, combined to Proposition 2.4, proves the first part of (3.19), with $\rho(k) = 0$. Then we use (3.21) to get, for $x \ge x_3^\varepsilon$:

$$\begin{aligned}&\psi^{k,*}(x)\\ =&\frac{1}{\Delta_\varepsilon(k)}\Big(\int_{x_3^\varepsilon}^{x} e^{r_2^\varepsilon(k)(x-y)} f(T_\varepsilon)T_\varepsilon'(y)\ dy + \int_x^{+\infty} e^{r_1^\varepsilon(k)(x-y)} f(T_\varepsilon)T_\varepsilon'(y)\ dy\Big) + O(1)\\ =&\frac{1}{\varepsilon\Delta_\varepsilon(k)}e^{\varepsilon r_2^\varepsilon(k)(\xi-\xi_3^\varepsilon)} + O(1)\end{aligned}$$

This yields the second part of (3.19), with $\rho(k) = 0$. Once this is proved, we get $\psi^{k,*}(x_3^\varepsilon) = \dfrac{1}{\Delta_\varepsilon}$, and Part 2 of the proposition follows.

It now remains to prove that we may add a weight function. The characteristic exponents of (3.18) at $+\infty$ are the eigenvalues of the matrix

$$A_\infty^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k^2 & -c_\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{1}{\Lambda\varepsilon^2} & 0 & -k^2 - \dfrac{1}{\Lambda\varepsilon^2} & -c_\varepsilon \end{pmatrix} \tag{3.25}$$

Its characteristic roots are

$$
(3.26) \qquad
\begin{aligned}
\lambda_1^*(k) &= \frac{-c_\varepsilon - \Delta_\varepsilon(k)}{2}\\
\lambda_2^*(k) &= \frac{-c_\varepsilon + \Delta_\varepsilon(k)}{2}\\
\lambda_3^*(k) &= \frac{-c_\varepsilon - \sqrt{c_\varepsilon^2 + 4(k^2 + \frac{1}{\Lambda\varepsilon^2})}}{2}\\
\lambda_4^*(k) &= \frac{-c_\varepsilon + \sqrt{c_\varepsilon^2 + 4(k^2 + \frac{1}{\Lambda\varepsilon^2})}}{2}
\end{aligned}
$$

Therefore we take $\rho(k) = \dfrac{\Gamma(k)}{4}$, and the proof of of Proposition 3.5 is over. •

Finally, in order to investigate the product $\big\langle(\Phi_1^*, \Psi_1^*), (\Phi_1, \Psi_1)\big\rangle$, we will use the following proposition.

LEMMA 3.6. *Let $(\phi^{k,*}, \psi^{k,*})$ be a solution of (3.18). There is a positive function $\beta(k)$ such that*

$$
(3.27) \qquad \lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} \left(\phi^{k,*}\phi^{k,\sigma} + \psi^{k,*}\psi^{k,\sigma}\right)\, dx = \beta(k).
$$

PROOF. Let us first notice that, in view of Proposition 2.11, we have

$$
\int_{x_3^\varepsilon}^{+\infty} \left(\phi^{k,*}\phi^{k,\sigma} + \psi^{k,*}\psi^{k,\sigma}\right)\, dx = O(\varepsilon \mathrm{Log}^4 \varepsilon);
$$

therefore

$$
\lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} \left(\phi^{k,*}\phi^{k,\sigma} + \psi^{k,*}\psi^{k,\sigma}\right)\, dx = \int_{-\infty}^{0} \left(\phi_\infty^{k,*}\phi_\infty^{k,\sigma} + \psi_\infty^{k,*}\psi_\infty^{k,\sigma}\right)\, dx,
$$

where $\phi_\infty^{k,*}$, $\phi_\infty^{k,\sigma}$, $\psi_\infty^{k,*}$, $\psi_\infty^{k,\sigma}$ satisfy the following linear differential systems, posed on $I\!R_-$:

$$
(3.28) \qquad
\begin{cases}
-\phi_\infty^{k,\sigma\,''} + \sqrt{2}\phi_\infty^{k,\sigma\,'} + k^2\phi_\infty^{k,\sigma} = 0\\
-\psi_\infty^{k,\sigma\,''} + \sqrt{2}\psi_\infty^{k,\sigma\,'} + k^2\psi_\infty^{k,\sigma} = l(-\phi_\infty^{k,\sigma\,''} + k^2\phi_\infty^{k,\sigma})
\end{cases}
\begin{cases}
\phi_\infty^{k,\sigma}(0) = \sqrt{2}, \qquad \psi_\infty^{k,\sigma}(0) = l\sqrt{2}(1 - \sqrt{2}\dfrac{r_2(k)}{\Gamma^2(k)})\\
\phi_\infty^{k,\sigma}(-\infty) = 0, \qquad \psi_\infty^{k,\sigma}(-\infty) = 0
\end{cases}
$$

$$
(3.29)\qquad
\begin{cases}
-\phi_\infty^{k,*''} - \sqrt{2}\phi_\infty^{k,*'} + k^2\phi_\infty^{k,*} = l(-\psi_\infty^{k,*''} + k^2\psi_\infty^{k,*}) \\
-\psi_\infty^{k,*''} - \sqrt{2}\psi_\infty^{k,*'} + k^2\psi_\infty^{k,*} = 0
\end{cases}
\quad
\begin{cases}
\phi_\infty^{k,*}(0) = \sqrt{2}, \qquad \psi_\infty^{k,*}(0) = \dfrac{1}{\Gamma(k)} \\
\phi_\infty^{k,*}(-\infty) = 0, \qquad \psi_\infty^{k,*}(-\infty) = 0
\end{cases}
$$

We find:

$$
\beta(k) = 2\Big(\frac{1}{\Gamma(k)} + \frac{l\rho_1(k)}{\Gamma^4(k)} + \frac{l}{\sqrt{2}\Gamma^2(k)}(1 - \frac{2}{\Gamma^2(k)})\Big) > 0.
$$

which concludes the proof of the lemma. •

Notice that we have said nothing yet about the actual existence of solutions to (3.18). This is, however, not really troublesome, and will be addressed to in the next section.

Existence of the bifurcation

For fixed $k \in I\!R$, there exists a real C^∞ function $\sigma(k, l)$, which is positive for $l < l_0 := 2(1 + 2k^2)$, negative for $l > l_0$, and which corresponds to the root of $\sigma \mapsto D(\sigma, k, l)$ with minimal real part. Furthermore, with the notations of (1.22) we have:

$$
\frac{dD}{d\sigma} = 2(3\Gamma - 2)\Gamma\frac{d\Gamma}{d\sigma} - l(1 + \frac{d\Gamma}{d\sigma});
$$

therefore

$$
(3.30)\qquad \frac{dD}{d\sigma}(0, k, l_0(k)) = -4 + 4\Gamma(0, k, l_0(k)) - 2\Gamma^2(0, k, l_0(k)) < 0.
$$

This implies:

$$
(3.31)\qquad \frac{d\sigma}{dl}(k, l_0(k)) < 0.
$$

By (3.30),(3.31), Rouché's Theorem and the implicit function Theorem, there exist a real, C^∞ function, denoted by $\sigma_\varepsilon(k, l)$, and a real, C^∞ function $l_\varepsilon^0(k)$, such that

$$
(3.32)\qquad
\begin{aligned}
&\sigma_\varepsilon(k, l_\varepsilon^0(k)) = 0, \qquad \frac{d\sigma_\varepsilon}{dl}(k, l_\varepsilon^0(k)) < 0, \\
&\lim_{\varepsilon\to 0} l_\varepsilon^0(k) = l_0(k), \qquad \lim_{\varepsilon\to 0} \sigma_\varepsilon(k, l) = \sigma(k, l),
\end{aligned}
$$

In the particular case of our cylinder, the relevant value for k is $k_1 = \pi$. We set:

$$
(3.33)\qquad l_\varepsilon^0 := l_\varepsilon^0(k_1), \quad \sigma_\varepsilon(l) := \sigma_\varepsilon(k_1, l),
$$

and we will stick to this notation until the end of the paper. From Theorem 2.3, the problem $(1.18)_n$, with $n = 1$ and $\sigma = 0$, has a unique solution (ϕ_1, ψ_1), such that $\phi_1(x_3^\varepsilon) = T_\varepsilon'(x_3^\varepsilon)$. The functions Φ_1 and Ψ_1, given by (3.1), are unambiguously defined. The operator $\mathcal{L}_1$ being Fredholm from $H^2(I\!R)$ to $L^2(I\!R)$, there exists a solution (ϕ_1^*, ψ_1^*) to (3.18) with $k = k_1$, such that $\phi_1^*(x_3^\varepsilon) = T_\varepsilon'(x_3^\varepsilon)$. Let us set

$$
(3.34)\qquad \Phi_1^*(x, y) = \phi_1^*(x)\cos k_1 y, \qquad \Psi_1^*(x, y) = \psi_1^*(x)\cos k_1 y.
$$

We notice that $\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0, T_\varepsilon, W_\varepsilon)(\Phi_1, \Psi_1) = 0$, $\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0, T_\varepsilon, W_\varepsilon)^*(\Phi_1, \Psi_1)^* = 0$. Lemma 3.6 has the following corollary

COROLLARY 3.7. *There exists $\beta_1 > 0$ such that*

$$\lim_{\varepsilon\to 0}\big\langle(\Phi_1^*,\Psi_1^*),(\Phi_1,\Psi_1)\big\rangle=\beta_1.$$

PROOF. Obvious: we have

$$\big\langle(\Phi_1^*,\Psi_1^*),(\Phi_1,\Psi_1)\big\rangle=\int_0^1 \cos^2\pi y\ dy\ \int_{-\infty}^{+\infty}\big(\phi^{k,*}\phi^{k,\sigma}+\psi^{k,*}\psi^{k,\sigma}\big)\ dx,$$

and we apply Lemma 3.6. •

We are now ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. We check Points **1** to **3** of the beginning of this section.
1. In Section 1, we saw that solving the equation $\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0,T_\varepsilon,W_\varepsilon)(\Phi,\Psi)=0$ was equivalent to solving $(1.18)_0$ and $(1.18)_n$, with $\sigma=0$. From Theorem 2.1, it is sufficient to solve $(1.18)_n$ for a finite number of values for n. Then Theorem 2.3 and Rouché's Theorem - applied to a finite number of functions $D_\varepsilon(\sigma,k_n,l_\varepsilon^0)$ - tell us that, for small enough ε the only problem in the $(1.18)_n$ having a nontrivial solution is the first one, with (ϕ_1,ψ_1) as a solution. Its solution space is one-dimensional.

Then it remains to investigate $(1.18)_0$. We take the scalar product of $(1.18)_0$ with (ϕ_0^*,ψ_0^*); then from Lemma 3.6 we infer that $\mu_\varepsilon(\phi_0)=0$. This implies, by the uniqueness part of Theorem 2.3: $(\phi_0,\psi_0)=\alpha(T_\varepsilon',W_\varepsilon')$; because $\mu_\varepsilon(T_\varepsilon')=\sqrt{2}(1+o(1))$ we have: $\alpha=0$ and $(\phi_0,\psi_0)=0$.

This is sufficient to prove that $N\big(\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0,T_\varepsilon,W_\varepsilon)\big)$ is spanned by (Φ_1,Ψ_1).
2. Because of Proposition 3.4, the null space of $\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0,T_\varepsilon,W_\varepsilon)^*$ has dimension 1, and is spanned by (Φ_1^*,Ψ_1^*). Now, because of Corollary 3.7, we have:

$$Y=N\big(\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0,T_\varepsilon,W_\varepsilon)\big)\oplus R\big(\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0,T_\varepsilon,W_\varepsilon)\big),$$

which proves Point **2.**
3. From Formula (1.27) in [16] we have, with our notations:

$$\begin{aligned}&\big\langle(\Phi_1^*,\Psi_1^*),\partial^2_{l,(T,W)}\mathcal{F}_\varepsilon(l_\varepsilon^0,T_\varepsilon,W_\varepsilon)(\Phi_1,\Psi_1)\big\rangle\\&\qquad=\sigma_\varepsilon'(l_\varepsilon^0)\big\langle(\Phi_1^*,\Psi_1^*),(\Phi_1,\Psi_1)\big\rangle,\end{aligned}$$

which is nonzero from Corollary 3.7 and (3.32). The proof of Theorem 3.1 is over. •

To end this section, let us notice that we have in fact proved more than Theorem 3.1. Indeed, we may still follow the planar branch with respect to l, and find more primary bifurcations. Of course, all the bifurcating solutions from these higher values of l will be unstable, and will not have much physical meaning. However, they give an idea of the complexity of the model. Here is the precise result:

THEOREM 3.8. *Let $n\in I\!N^*$ be given. There exist $\varepsilon_0(n)>0$, $s_0^\varepsilon(n)>0$, $(n+1)$ real numbers l_ε^0, l_ε^1, ..., l_ε^n and $(n+1)$ C^2 curves $(l_i^\varepsilon(s),T_i(s),W_i(s))$, $k\in[0,n]$, defined on $[-s_0^\varepsilon(n),s_0^\varepsilon(n)]$, such that $\lim_{\varepsilon\to 0} l_i=2(1+2k_i^2)$, and*

• $l_n^\varepsilon(0)=l_n^\varepsilon$,

• $(l_i^\varepsilon(s), T_i(s), W_i(s))$ *satisfies (1.12). Further, the functions* $T_i(s), W_i(s)$ *admit the following expansion, near* $s = 0$*:*

$$T_i(s,x,y) = T_\varepsilon(x) + s\phi_i(x)\cos k_i y + o(s)$$
$$W_i(s,x,y) = W_\varepsilon(x) + s\psi_i(x)\cos k_i y + o(s).$$

Its proof is quite similar to the one of Theorem 3.1. The reason why we cannot follow the planar branch up to $l = +\infty$ is the fact that we have no uniform convergence result for $D(\sigma, k, l)$ for unbounded l.

4. COMPUTATION OF $l''_\varepsilon(0)$.

Let us recall that the knowledge of the sign of $sl'_\varepsilon(s)$ around 0 is vital if we wish to get any information about the stability of the curved flame.

There is an obvious reason why $l'_\varepsilon(0) = 0$. In view of (3), (4), $(T(s,x,1-y), W(s,x,1-y))$ is a solution of (3), (4), which is different, for small $s > 0$, from $(T(s,x,y), W(s,x,y))$. From the uniqueness part of [15] we have $l'_\varepsilon(s) = l'_\varepsilon(-s)$, which yields $l'_\varepsilon(0) = 0$. Therefore, to our despite - and also to the reader's - it is necessary to compute $l''_\varepsilon(0)$. In doing so, we prove rigorously that the speed of the curved flame is greater than the speed of the planar flame, a fact that had already been observed by Benkhaldoun and Larrouturou [5]. The results that will be proved in this section are the following.

THEOREM 4.1. *There exists $\alpha > 0$ such that, for small s and for small $\varepsilon > 0$:*

$$\mu_\varepsilon(T(s)) = c_\varepsilon(1 + \alpha s^2 + o(s^2)). \tag{4.1}$$

We will see that - besides its own independent interest - Theorem 4.1 is a necessary step in the computation of $l''_\varepsilon(0)$.

THEOREM 4.2. *There exists $\mu > 0$ such that*

$$\lim_{\varepsilon\to 0} \left(\varepsilon l''_\varepsilon(0)\right) = \mu \tag{4.2}$$

From Theorem 1.7 of [15], the functions $T(s)$ and $W(s)$ may be written under the more accurate form

$$\begin{aligned} T(s,x,y) &= T_\varepsilon(x) + s\Phi_1(x,y) + s^2 u(x,y) + o(s^2) \\ W(s,x,y) &= W_\varepsilon(x) + s\Psi_1(x,y) + s^2 w(x,y) + o(s^2) \end{aligned} \tag{4.3}$$

The couple (u, w) is orthogonal to (Φ_1, Ψ_1); it satisfies the equation

$$\begin{aligned} &\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon(0), T_\varepsilon, W_\varepsilon).(u,w) \\ &= -\frac{1}{2}\partial^2_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon(0), T_\varepsilon, W_\varepsilon).(\Phi_1, \Psi_1)^{(2)} \end{aligned} \tag{4.4}$$

Of course, both u and v depend on ε, but, we drop here the subscript in order to make the notations a little clearer. The Fourier expansions of u and w may therefore be written as

$$\begin{aligned} u(x,y) &= u_0(x) + u_1(x)\cos k_1 y + u_2(x)\cos 2k_1 y \\ w(x,y) &= w_0(x) + w_1(x)\cos k_1 y + w_2(x)\cos 2k_1 y \end{aligned} \tag{4.5}$$

Let us immediately prove that $(u_1, w_1) \equiv 0$. Since $\cos^2 k_1 y = \dfrac{1+\cos 2k_1 y}{2}$, projecting (4.4) onto $(\phi_1 \cos k_1 y, \psi_1 \cos k_1 y)$ yields: $\mathcal{L}_1(u_1, w_1) = 0$. As a consequence, (u_1, w_1) is proportional to (ϕ_1, ψ_1); therefore it has to be 0.

The equations for $(u_i, w_i), i \in \{0, 2\}$ read as follows.

$$\mathcal{L}_i.\begin{pmatrix} u_i \\ w_i \end{pmatrix} = \begin{pmatrix} \frac{1}{4}(g''(T_\varepsilon) + f''(T_\varepsilon)W_e)\phi_1^2 + \frac{1}{2}f'(T_\varepsilon)\phi_1\psi_1 \\ 0 \end{pmatrix} \tag{4.6}$$

We notice that (4.6) always has solutions.

The expression of the speed of the curved flame $\mu_\varepsilon(T(s))$ is given by

$$\begin{aligned} \mu_\varepsilon(T(s)) &= d_\varepsilon \int_{-\infty}^{x_3^\varepsilon} (T_\varepsilon + s^2 u_0)\ dx + o(s^2) \\ &= c_\varepsilon + s^2 d_\varepsilon \int_{-\infty}^{x_3^\varepsilon} u_0\ dx + o(s^2) \end{aligned} \tag{4.7}$$

Expanding the function $s \mapsto s^{-1}\mathcal{F}_\varepsilon\big(l_\varepsilon(s), T(s), W(s)\big)$ to the second order in s we find, after a tedious calculation:

$$\begin{aligned} &\frac{1}{2}\big\langle(\Phi_1^*, \Psi_1^*), (\Phi_1^*, \Psi_1^*)\big\rangle |\sigma_\varepsilon'(l_\varepsilon^0)|\ l_\varepsilon''(0) \\ =&\frac{1}{3}\big\langle(\Phi_1^*, \Psi_1^*), \partial_{T,W}^3\mathcal{F}_\varepsilon(l_\varepsilon^0, T_\varepsilon, W_\varepsilon).(\Phi_1, \Psi_1)^{(3)}\big\rangle \\ &+2\big\langle(\Phi_1^*, \Psi_1^*), \partial_{T,W}^2\mathcal{F}_\varepsilon(l_\varepsilon^0, T_\varepsilon, W_\varepsilon).(\Phi_1, \Psi_1).(u, w)\big\rangle \end{aligned} \tag{4.8}$$

We have

$$\begin{aligned} &\partial_{T,W}^2\mathcal{F}_\varepsilon(l, T_\varepsilon, W_\varepsilon).(\Phi, \Psi)(\bar\Phi, \bar\Psi) = \\ &\quad -\begin{pmatrix} (\mu_\varepsilon(\Phi)\bar\Phi_x + c(\bar\Phi)\Phi_x) + (g''(T_\varepsilon) + f''(T_\varepsilon)W_\varepsilon)\Phi\bar\Phi + f'(T_\varepsilon)(\Phi\bar\Psi + \Psi\bar\Phi) \\ (\mu_\varepsilon(\Phi)\bar\Psi_x + c(\bar\Phi)\Psi_x) \end{pmatrix} \\ &\partial_{T,W}^3\mathcal{F}_\varepsilon(l, T_\varepsilon, W_\varepsilon).(\Phi_1, \Psi_1)^{(3)} = -\begin{pmatrix} (g'''(T_\varepsilon) + f'''(T_\varepsilon)W_\varepsilon)\Phi_1^3 + 3f''(T_\varepsilon)\Phi_1^2\Psi_1) \\ 0 \end{pmatrix} \end{aligned}$$

The limit $\varepsilon \to 0$ of $\mu_\varepsilon(T(s)$ and $l_\varepsilon''(0)$ is obtained by plugging into (4.7) and (4.8) expansions of ϕ_1, ψ_1, u_i, w_i at suitable order in ε.

Let us recall that it will be useful to work with the stretched variable ξ. In the sequel, more than in the preceding sections, it will be convenient to switch from the representation of a function in terms of ξ to a representation in terms of x and also in the other way round. Possible ambiguities may arise from differentiation with respect to x or ξ. Therefore we decide to denote by $A'(x)$ the derivative of the x-function A with respect to x, and $A'(\xi)$ the derivative of the ξ-function A with respect to ξ.

This section comprises 4 steps:

- Asymptotic expansions of ϕ_1 and ψ_1,
- Computation of $\int_{-\infty}^{x_3^\varepsilon} u_0\ dx$,
- Asymptotic expansions of u and w,

- Expansion of $l_\varepsilon''(0)$.

Asymptotic expansions of ϕ_1 and ψ_1

They have already been obtained in Section 2 up to the first order for $\xi > \xi_3^\varepsilon$, and to every reasonable order for $\xi < x_3^\varepsilon$. We need an additional term for $\xi > \xi_3^\varepsilon$. Let us set $K_1^\varepsilon := K^{k_1,0}$, and recall that $\mathcal{K}_1^\varepsilon s$ is defined by (2.40) in Section 2.

PROPOSITION 4.3. *The following expansion holds:*

$$\phi_1(\xi) = p_\varepsilon'(\xi) + \varepsilon\phi_{11}(\xi) + \varepsilon^2\phi_{12}(\xi) + O(\varepsilon^3 \operatorname{Log}^4 \varepsilon) \tag{4.9}$$

with

$$\mathcal{T}_\varepsilon \phi_{11} = -\frac{l_\varepsilon^0}{\Lambda} K_1^\varepsilon e^{p_\varepsilon}, \qquad \phi_{11}(\xi_3^\varepsilon) = 0 \tag{4.10}$$

$$\mathcal{T}_\varepsilon \phi_{12} = -k_1^2 p_\varepsilon', \qquad \phi_{12}(\xi_3^\varepsilon) = 0 \tag{4.11}$$

$$\psi_1 = \frac{\varepsilon l_\varepsilon^0}{\Lambda}\big(\phi_1 + K_1^\varepsilon + O(\varepsilon^2 \operatorname{Log}^4 \varepsilon)\big) \tag{4.12}$$

PROOF. All that we have to do is to prove (4.12); in fact we must prove that the expansion for ψ in Proposition 2.11 is accurate to the second order. Let us first compute the expression of $K_1^{\varepsilon\prime}$; from (2.11) we have

$$\begin{aligned} K_1^{\varepsilon\prime} =& -\left(\frac{{c_\varepsilon}^2 r_2^{\varepsilon 2}(k_1)}{\Delta_\varepsilon(\rho_1^\varepsilon(k_1) - r_2^\varepsilon(k_1))} + \frac{c_\varepsilon r_2^\varepsilon(k_1)}{\Delta_\varepsilon} p_\varepsilon'\right)\varepsilon e^{\varepsilon r_2^\varepsilon(k_1)(\xi-\xi_3^\varepsilon)} \\ &+ \frac{c_\varepsilon r_1^\varepsilon(k_1)}{\Delta_\varepsilon}\varepsilon p_\varepsilon' - \varepsilon c_\varepsilon p_\varepsilon' + O(\varepsilon^2 \operatorname{Log}^4 \varepsilon) \\ =& -\frac{{c_\varepsilon}^2 r_2^{\varepsilon 2}(k_1)}{\Delta_\varepsilon}\varepsilon + O(\varepsilon^2 \operatorname{Log}^4 \varepsilon) \end{aligned} \tag{4.14}$$

Therefore the function $\tilde\phi_{11}'$ satisfies - we imitate the proof of Proposition 2.11 -

$$\mathcal{T}_\varepsilon \tilde\phi_{11}' = \tilde f_{11}(\xi) + O(\varepsilon \operatorname{Log}^4 \varepsilon),$$

with $|f_{11}(\xi)| = O(p_\varepsilon')$.

Let us recall that H is the Heaviside function. In order to prove the proposition, we have to estimate the remainder $R_1^\varepsilon(\xi) := \mathcal{K}_1^\varepsilon(k_1)H(x - x_3^\varepsilon)\tilde\phi_{11}(x)$; we have

$$\begin{aligned} &R_1^\varepsilon(\xi) \\ =& -\varepsilon\frac{c_\varepsilon}{\Lambda\Delta_\varepsilon}\left(r_2^\varepsilon(k_1)\int_{\xi_3^\varepsilon}^{\xi} e^{\varepsilon r_2^\varepsilon(k_1)(\xi-\eta)}\tilde\phi_{11}(\eta)\, d\eta + r_1^\varepsilon(k_1)\int_x^{+\infty} ie^{\varepsilon r_1^\varepsilon(k_1)(\xi-\eta)}\tilde\phi_{11}(\eta)\, d\eta\right) \\ =& -\varepsilon\frac{c_\varepsilon}{\Lambda\Delta_\varepsilon}\left(\int_{\xi_3^\varepsilon}^{\xi} e^{\varepsilon r_2^\varepsilon(k_1)(\xi-\eta)}\tilde\phi_{11}'(\eta)\, d\eta + \int_x^{+\infty} ie^{\varepsilon r_1^\varepsilon(k_1)(\xi-\eta)}\tilde\phi_{11}(\eta)\, d\eta\right) \\ =& -\varepsilon\frac{c_\varepsilon}{\Lambda\Delta_\varepsilon}\left(\int_{\xi_3^\varepsilon}^{\xi} e^{\varepsilon r_2^\varepsilon(k_1)(\xi-\eta)}\tilde f_{11}'(\eta)\, d\eta + \int_x^{+\infty} ie^{\varepsilon r_1^\varepsilon(k_1)(\xi-\eta)}\tilde f_{11}(\eta)\, d\eta\right) + O(\varepsilon \operatorname{Log}^4 \varepsilon) \\ =& O(\varepsilon \operatorname{Log}^4 \varepsilon) \end{aligned}$$

which proves the proposition. •

Computation of $\displaystyle\int_{-\infty}^{x_3^\varepsilon} u_0 \; dx$

The functions ϕ_0^* and ψ_0^* will be normalized so that $\phi_0^*(x_3^\varepsilon) = T_\varepsilon'(x_3^\varepsilon)$; therefore, from Lemma 3.6, there exists $\beta > 0$ such that

$$\langle \phi_0^*, T_\varepsilon' \rangle + \langle \psi_0^*, W_\varepsilon' \rangle := \beta_\varepsilon = \beta\big((1 + o(1)\big).$$

PROOF OF THEOREM 4.1. We take the scalar product of (4.6) with (ϕ_0^*, ψ_0^*) and we obtain

$$\begin{aligned}
&d_\varepsilon \beta_\varepsilon \int_{-\infty}^{x_3^\varepsilon} u_0 \; dx \\
&= \int_{-\infty}^{+\infty} \phi_0^* \Big(\frac{1}{4}\big(g''(T_\varepsilon) + f''(T_\varepsilon)W_\varepsilon\big)\phi_1^2 + \frac{1}{2} f'(T_\varepsilon)\phi_1\psi_1 \Big) \; dx \\
&= \frac{1}{4} \int_{-\infty}^{+\infty} \phi_0^* \Big(\big(g''(T_\varepsilon) + f''(T_\varepsilon)W_\varepsilon + \frac{2\varepsilon l_\varepsilon^0}{\Lambda} f'(T_\varepsilon)\big)\phi_1^2 + \frac{2\varepsilon l_\varepsilon^0}{\Lambda} f'(T_\varepsilon)\phi_1 \mathcal{K}_1^\varepsilon(0)\phi_1 \Big) dx \\
&\quad + \frac{\varepsilon l_\varepsilon^0}{2\Lambda} \int_{-\infty}^{+\infty} \phi_0^* f'(T_\varepsilon)\phi_1 K_1^\varepsilon \; dx \\
&:= I_1 + I_2
\end{aligned}$$

Because T_ε and Y_ε are solutions of (8)-(9), we have

$$\begin{aligned}
I_1 =& \frac{1}{4} \int_{-\infty}^{+\infty} \phi_0^* \Big(\big(g''(T_\varepsilon) + f''(T_\varepsilon)W_\varepsilon) + \frac{2\varepsilon l_\varepsilon^0}{\Lambda} f'(T_\varepsilon)\big)(\phi_1^2 - {T_\varepsilon'}^2) \Big) \; dx \\
&+ \frac{2\varepsilon l_\varepsilon^0}{\Lambda} \int_{-\infty}^{+\infty} f'(T_\varepsilon)(\phi_1 - T_\varepsilon')\mathcal{K}_1^\varepsilon(0)\phi_1 \; dx.
\end{aligned}$$

Let us now recall that $\mathcal{K}_1^\varepsilon(0)T_\varepsilon' = O(\varepsilon^2)$; therefore, switching to the variable ξ and setting

$$h(p) = -(1 - \frac{\varepsilon l_\varepsilon^0}{\Lambda})pe^p, \tag{4.15}$$

we find that

$$\begin{aligned}
I_1 =& \frac{1}{4\varepsilon^2} \int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* \Big(h''(p_\varepsilon)(\phi_1^2 - p'^2) + \frac{2 l_\varepsilon^0}{\Lambda}(\phi_1 - p_\varepsilon')\varepsilon^2 p_\varepsilon e^{p_\varepsilon} \Big) \; d\xi + O(\varepsilon^5) \\
=& \frac{1}{4\varepsilon^2} \int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* \; h''(p_\varepsilon)(\phi_1^2 - {p_\varepsilon'}^2) \; d\xi + O(\varepsilon)
\end{aligned} \tag{4.16}$$

We take into account that

$$\phi_1^2 - p_\varepsilon^2 = 2\varepsilon\phi_{11}p_\varepsilon' + 2\varepsilon^2\phi_{12}p_\varepsilon' + \varepsilon^2\phi_{11}^2 + O(\varepsilon^3 \mathrm{Log}^8 \varepsilon)$$

we differentiate (4.10) once; this yields

$$\mathcal{T}_\varepsilon \phi_{11}' = h''(p_\varepsilon)p_\varepsilon'\phi_{11} + \frac{l_\varepsilon^0}{\Lambda}\big(e^{p_\varepsilon}K_1^{\varepsilon\prime} + p_\varepsilon' e^{p_\varepsilon}K_1^\varepsilon\big) \tag{4.17}$$

We then multiply (4.17) by ϕ_0^*, and we integrate by parts over $]\xi_3^\varepsilon, +\infty[$. In doing so, we note that

$$\begin{aligned}\phi_0^{*\prime}(\xi) &= \frac{d}{d\xi}\big(e^{\varepsilon c_\varepsilon(\xi-\xi_3^\varepsilon)}p_\varepsilon'\big) + O(\varepsilon^2 \mathrm{Log}^4 \varepsilon)\\ &= e^{\varepsilon c_\varepsilon(\xi-\xi_3^\varepsilon)}(p_\varepsilon'' - \varepsilon c_\varepsilon p_\varepsilon') + O(\varepsilon^2 \mathrm{Log}^4 \varepsilon)\end{aligned}$$

Therefore $\phi_0^{*\prime}(\xi_3^\varepsilon) = O(\varepsilon^2 \mathrm{Log}^4 \varepsilon)$. This implies that

$$\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^*\big(h''(p_\varepsilon)p_\varepsilon'\phi_{11} + \frac{l_\varepsilon^0}{\Lambda}p_\varepsilon' e^{p_\varepsilon}K_1^\varepsilon + \frac{l_\varepsilon^0}{\Lambda}e^{p_\varepsilon}K_1^{\varepsilon\prime}\big)\ d\xi = O(\varepsilon^2 \mathrm{Log}^4 \varepsilon). \tag{4.18}$$

Then we multiply (4.17) by $e^{-\varepsilon c_\varepsilon(\xi-\xi_3^\varepsilon)}\phi_{11}$. We keep in mind that

$$\phi_{11}'(\xi_3^\varepsilon) = -\frac{l_\varepsilon^0 c_\varepsilon r_2^\varepsilon(k_1)}{\Delta_\varepsilon^2} + O(\varepsilon \mathrm{Log}^4 \varepsilon),$$

and we also remember that $\phi_0^* = e^{-\varepsilon c_\varepsilon(\xi-\xi_3^\varepsilon)}p_\varepsilon' + O(\varepsilon^2 \mathrm{Log}^4 \varepsilon)$. Consequently, we obtain the following identity:

$$\int_{\xi_3^\varepsilon}^{+\infty} h''(p_\varepsilon)\phi_0^*\phi_{11}^2\ d\xi = -\frac{{l_\varepsilon^0}^2 {r_2^\varepsilon}^2(k_1)}{\Delta_\varepsilon^4} - \frac{2l_\varepsilon^0}{\Lambda}\int_{\xi_3^\varepsilon}^{+\infty} p_\varepsilon' e_\varepsilon^p K_1^\varepsilon \phi_{11}\ d\xi + O(\varepsilon \mathrm{Log}^4 \varepsilon) \tag{4.19}$$

We now deal with ϕ_{12}. We differentiate (4.11) once; this yields

$$\mathcal{T}_\varepsilon \phi_{12}' = h''(p_\varepsilon)p_\varepsilon'\phi_{12} - k_1^2 p_\varepsilon''. \tag{4.20}$$

We then multiply (4.20) by ϕ_0^*, and we integrate by parts over $]\xi_3^\varepsilon, +\infty[$. We get

$$\int_{\xi_3^\varepsilon}^{+\infty} h''(p_\varepsilon)p_\varepsilon'\phi_0^*\phi_{12}\ d\xi = \frac{c_\varepsilon^2}{2}k_1^2 + O(\varepsilon \mathrm{Log}^4 \varepsilon) = k_1^2 + O(\varepsilon \mathrm{Log}^4 \varepsilon). \tag{4.21}$$

From (4.18), (4.19) and (4.21) we infer the following identity:

$$2I_1 = -\frac{{l_\varepsilon^0}^2 {r_2^\varepsilon}^2}{2\Delta_\varepsilon^4} - \frac{l_\varepsilon^0}{\Lambda}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* e^{p_\varepsilon}\big(\phi_{11}K_1^\varepsilon - p_\varepsilon'\frac{K_1^\varepsilon}{\varepsilon} - \frac{K_1^{\varepsilon\prime}}{\varepsilon}\big)\ d\xi + k_1^2 + O(\varepsilon \mathrm{Log}^4 \varepsilon). \tag{4.22}$$

By expanding the terms in I_2 we find

$$I_2 = \frac{l_\varepsilon^0}{2\Lambda\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* e^{p_\varepsilon}p_\varepsilon' K_1^\varepsilon\ d\xi + \frac{l_\varepsilon^0}{2\Lambda}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* e^{p_\varepsilon}p_\varepsilon' K_1^\varepsilon \phi_{11}\ d\xi. \tag{4.23}$$

Let us now remember (4.14); combining (4.22) and (4.23), we end up with

$$2d_\varepsilon\beta_\varepsilon \int_{-\infty}^{x_3^\varepsilon} u_0 \ dx = k_1^2 + l_\varepsilon^0 \frac{c_\varepsilon{}^2 r_2^\varepsilon{}^2(k_1)}{\Delta_\varepsilon^2} \int_{\xi_3^\varepsilon}^{+\infty} p_\varepsilon' e^{p_\varepsilon} \ d\xi - \frac{c_\varepsilon{}^2 r_2^\varepsilon{}^2(k_1)}{2\Delta_\varepsilon{}^4} l_\varepsilon^0{}^2 + O(\varepsilon \mathrm{Log}^4 \varepsilon);$$

that is, taking into account the identity $l_\varepsilon^0 = \Delta_\varepsilon^2 + O(\varepsilon \mathrm{Log}^4 \varepsilon)$:

$$2d_\varepsilon\beta_\varepsilon \int_{-\infty}^{x_3^\varepsilon} u_0 \ dx = k_1^2 + \rho_2^2(k_1) + O(\varepsilon \mathrm{Log}^4 \varepsilon).$$

This ends the proof of Theorem 4.1. •

Asypmptotic expansions of u_i and w_i

They will be performed as before: first, a nearly exact expression is derived for u_i in the domain $]-\infty, x_3^\varepsilon]$. It then seems reasonable that the first term in the expansion of u_i for $x > x_3^\varepsilon$ is $\frac{T_\varepsilon''}{4}$; this yields a guess for w_i. Then it remains to show that this guess is accurate up to the third order, and to finish the expansion of u_i.

Let us set $\gamma_{\varepsilon 0} = u_0(x_3^\varepsilon)$, $\gamma_{\varepsilon 2} = u_2(x_3^\varepsilon)$.

PROPOSITION 4.4. *For $x \le x_3^\varepsilon$, the expansions of u_i, $i \in \{0, 2\}$ read as follows:*

$$u_0(x) = d_\varepsilon\big(\frac{c_\varepsilon}{2}(x - x_3^\varepsilon) + 1\big)\gamma_{\varepsilon 0} e^{c_\varepsilon(x - x_3^\varepsilon)} + O(\varepsilon^{90}) \tag{4.24}$$

$$u_2(x) = \gamma_{\varepsilon 2} e^{r_2^\varepsilon(k_2)(x - x_3^\varepsilon)} + O(\varepsilon^{90}) \tag{4.25}$$

PROOF. The orders of magnitude of $\gamma_{\varepsilon i}$ being $O(1)$, one only has to write down the equations satisfied for u_i:

$$\begin{aligned} -u_0'' + c_\varepsilon u_0' + d_\varepsilon \int_{-\infty}^{x_3^\varepsilon} u_0 \ dx T_\varepsilon' &= O(\varepsilon^{90}) \\ -u_2'' + c_\varepsilon u_2' &= O(\varepsilon^{90}) \end{aligned}$$

the right-handsides being 0 for $x \le x_1^\varepsilon$. One then remembers that $d_\varepsilon \int_{-\infty}^{x_3^\varepsilon} T_\varepsilon = c_\varepsilon$, and an elementary computation ends the proof. •

The expansions of u_i and w_i for $x \ge x_3^\varepsilon$ will be given in terms of the stretched variable ξ. Let us set

$$\tilde{u}_2(x) = \gamma_{\varepsilon 2} H(x_3^\varepsilon - x) e^{r_2^\varepsilon(k_2)(x - x_3^\varepsilon)} + H(x - x_3^\varepsilon)\frac{T_\varepsilon''(x)}{4} \tag{4.26}$$

$$\begin{aligned} Q_0^\varepsilon &= 0 \\ Q_2^\varepsilon &= \mathcal{K}_1^\varepsilon(k_2)\tilde{u}_2 - \mathcal{K}_1^\varepsilon(0)\frac{T_\varepsilon''}{4} \end{aligned} \tag{4.27}$$

We then introduce the functions u_{ij}, $(i,j)\in\{0,2\}\times\{1,2\}$ defined by

$$\mathcal{T}_\varepsilon u_{i1} = \frac{1}{2}\big(h''(p_\varepsilon)p'_\varepsilon\phi_{11} + \frac{l^0_\varepsilon}{\Lambda}p'_\varepsilon K_1^\varepsilon e^{p_\varepsilon}\big) \tag{4.28}$$

$$\mathcal{T}_\varepsilon u_{i2} = -k_i^2 p''_\varepsilon + \frac{l^0_\varepsilon}{\Lambda}e^{p_\varepsilon}Q_i + \frac{1}{4}\big(h''(p_\varepsilon)p'_\varepsilon\phi_{11}^2 + 2\phi_{12}p'_\varepsilon\big) + \frac{l^0_\varepsilon}{2\Lambda}\phi_{11}e^{p_\varepsilon}K_1^\varepsilon \tag{4.29}$$

$$u_{i1}(\xi_3^\varepsilon) = \gamma_{\varepsilon i} - \frac{c_\varepsilon p'_\varepsilon(\xi_3^\varepsilon)}{4}, \qquad u_{i2}(\xi_3^\varepsilon) = 0 \tag{4.30}$$

We note that

$$u_{i1}(\xi) = \big(\gamma_{\varepsilon i} - \frac{c_\varepsilon p'_\varepsilon(\xi_3^\varepsilon)}{4}\big)p'_\varepsilon(\xi) + \tilde{u}_1(\xi), \tag{4.31}$$

the function $\tilde{u}_1$ satisfying

$$\mathcal{T}_\varepsilon \tilde{u}_1 = \frac{1}{2}\big(h''(p_\varepsilon)p'_\varepsilon\phi_{11} + \frac{l^0_\varepsilon}{\Lambda}p'_\varepsilon K_1^\varepsilon e^{p_\varepsilon}\big), \qquad \tilde{u}_1(\xi_3^\varepsilon) = 0. \tag{4.32}$$

We will see in the sequel that the expansions of u_i and w_i are needed up to the second order only; they are given by the

PROPOSITION 4.5. *For $\xi \geq \xi_3^\varepsilon$, the functions u_i, w_i, $i\in\{0,2\}$ may be written as*

$$w_i(\xi) = \frac{\varepsilon l^0_\varepsilon}{\Lambda}\big(u_i(\xi) + \mathcal{K}_1^\varepsilon(0)\frac{T''_\varepsilon}{4}(\xi) + Q_i(\xi) + O(\varepsilon \mathrm{Log}^4\varepsilon)\big) \tag{4.33}$$

$$u_i(\xi) = \frac{p''_\varepsilon(\xi)}{4\varepsilon} + u_{i1} + \varepsilon u_{i2}(\xi) + O(\varepsilon^2\mathrm{Log}^4\varepsilon) \tag{4.34}$$

Moreover, these expansions hold in $C^2([\xi_3^\varepsilon,+\infty[)$.

PROOF. Let us first consider w_0. Its equation reads

$$-\Lambda w''_0 + c_\varepsilon w'_0 + d_\varepsilon\int_{-\infty}^{x_3^\varepsilon} u_0\ dx\ W'_\varepsilon = (1-\Lambda)u''_0,$$

which implies, since $r_2^\varepsilon(0)=0$:

$$\begin{aligned} w_0 &= \frac{\varepsilon l^0_\varepsilon}{\Lambda}\left(u_0 + \mathcal{K}_1^\varepsilon(0)(u_0 - d_\varepsilon(\int_{-\infty}^{x_3^\varepsilon} u_0\ dx)\ \frac{\Lambda}{\varepsilon l^0_\varepsilon}W_\varepsilon)\right) \\ &= \frac{\varepsilon l^0_\varepsilon}{\Lambda}\big(u_0 + \mathcal{K}_1^\varepsilon(0)u_0\big) + O(\varepsilon^2\mathrm{Log}^4\varepsilon) \end{aligned} \tag{4.35}$$

Let us now expand the first coordinate of the right-handside of (4.6) in terms of ε, and let us replace u_0 by $\dfrac{T''_\varepsilon}{4}$ in (4.35); we are led to solve (4.28) with $i=0$. Obviously, we get: $u_0 = \dfrac{p''_\varepsilon}{4\varepsilon} + u_{01} + O(\varepsilon\mathrm{Log}^4\varepsilon)$. As in the previous sections, we note that, u_{01}

may well be unbounded in ε - it may behave like Logε - but this will not bother us too much because, by subtracting a suitable amount of p'_ε, we can make it locally bounded. We also note that $\tilde{u}'_1(\xi)$ is integrable; therefore, by (4.35) we have

$$\begin{aligned} w_0(x) &= \frac{\varepsilon l^0_\varepsilon}{\Lambda}\left(u_0(x) - c_\varepsilon \int_x^{+\infty} e^{c_\varepsilon(x-y)}\big(\frac{T''_\varepsilon}{4} + u_{01} + O(\varepsilon \mathrm{Log}^4 \varepsilon)\big)\ dx\right) \\ &= \frac{\varepsilon l^0_\varepsilon}{\Lambda}\big(u_0 + \mathcal{K}^\varepsilon_1(0)\frac{T''_\varepsilon}{4}(x)\big) + O(\varepsilon^2 \mathrm{Log}^4 \varepsilon) \end{aligned}$$

the last equality resulting from an integration by parts. This yields the expansion for w_0. The expansion for u_0 is from then on obvious.

Let us now turn to the case $i = 2$. It does not differ much from the case $i = 0$; therefore we will only sketch the proof. From the equation

$$-\Lambda w''_2 + c_\varepsilon w'_2 + k_2 w_2 = -(1-\Lambda)(-u''_2 + k_2^2 u_2)$$

we infer the analogue of (4.35):

$$w_2 = \frac{\varepsilon l^0_\varepsilon}{\Lambda}\big(u_2 + \mathcal{K}^\varepsilon_1(k_2) u_2\big); \tag{4.36}$$

this, as above, yields in turn $u_2 = \dfrac{p''_\varepsilon}{4\varepsilon} + u_{21} + O(\varepsilon \mathrm{Log}^4 \varepsilon)$. It follows that

$$\begin{aligned} w_2(x) =& \frac{\varepsilon l^0_\varepsilon}{\Lambda}\big(u_2 + \mathcal{K}^\varepsilon_1(k_2)\tilde{u}_2\big) \\ &- \varepsilon \frac{l^0_\varepsilon c_\varepsilon}{\Delta_\varepsilon} r^\varepsilon_2(k_2) \int_{x^\varepsilon_3}^x \big(u_{21} + O(\varepsilon \mathrm{Log}^4 \varepsilon)\big) e^{r^\varepsilon_2(k_2)(x-y)}\ dy \\ &- \varepsilon \frac{l^0_\varepsilon c_\varepsilon}{\Delta_\varepsilon} r^\varepsilon_1(k_2) \int_x^{+\infty} \big(\big(u_{21} + O(\varepsilon \mathrm{Log}^4 \varepsilon)\big) e^{r^\varepsilon_1(k_2)(x-y)}\ dy \\ =& \frac{\varepsilon l^0_\varepsilon}{\Lambda}\big(u_2 + \mathcal{K}^\varepsilon_1(k_2)\frac{T''_\varepsilon(x)}{4} + Q^\varepsilon_2 + O(\varepsilon \mathrm{Log}^4 \varepsilon)\big) \end{aligned}$$

the last equality resulting from an integration by parts, and identity (4.31). This yields the expansion for w_2. The expansion for u_2 may be obtained just as easily as above, only remembering that $Q^\varepsilon_2 = O(1)$.

Finally, we note that, because $\varepsilon\dfrac{d}{dx} = \dfrac{d}{d\xi}$, all the above expansions hold in $C^2([\xi^\varepsilon_3, +\infty[)$. •

Corollary 4.6. *The function Q^ε_0 and Q^ε_2 satisfies*

$$\|Q'_i\|_{C^1([\xi^\varepsilon_3,+\infty[)} = O(\varepsilon \mathrm{Log}^4 \varepsilon).$$

PROOF. It follows from Proposition 4.5, the expansion of the integral identity (4.27), and an integration by parts just as in the computation ok K_1^ε. •

The limit of $\varepsilon l''_\varepsilon(0)$

Unhappily, we will not only have to compute $l''_\varepsilon(0)$ instead of $l'_\varepsilon(0)$, but we will also have to compute it to the order ε^{-1} in ε, whereas one could have expected it to be of order ε^{-3}. This is of course due to the translation invariance, that allows additional orthogonality relations.

We first introduce notations. Let us set

$$\frac{1}{2}\big\langle(\Phi_1^*,\Psi_1^*),(\Phi_1^*,\Psi_1^*)\big\rangle|\sigma'(l_\varepsilon^0)|\ l''_\varepsilon(0) = -J_1 - J_2 \tag{4.37}$$

where

$$-J_1 = \frac{1}{3}\big\langle(\Phi_1^*,\Psi_1^*),\partial^3_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0,T_\varepsilon,W_\varepsilon).(\Phi_1,\Psi_1)^{(3)}\big\rangle, \tag{4.38}$$

$$-J_2 = \big\langle(\Phi_1^*,\Psi_1^*),\partial^2_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0,T_\varepsilon,W_\varepsilon).(\Phi_1,\Psi_1).(u,w)\big\rangle \tag{4.39}$$

Let us set

$$A = \int_0^1 \cos^4 k_1 y \quad \text{and} \quad B = \int_0^1 \cos^2 k_1 y.$$

In fact $A = \dfrac{3}{8}$ and $B = \dfrac{1}{2}$, but it will be more convenient to keep A and B because these expressions will still be valid in a more general framework: they reflect the geometry of the domain.

From (4.38)

$$J_1 = \frac{A}{3}\int_{-\infty}^{+\infty}\phi_1^*\big((g'''(T_\varepsilon)+f'''(T_\varepsilon)W_\varepsilon)\phi_1^3 + 3f''(T_\varepsilon)\phi_1^2\psi_1\big)dx.$$

From (4.9) we have for $x \geq x_3^\varepsilon$

$$\begin{aligned}\phi_1^3 &= {T'_\varepsilon}^3 + 3\varepsilon {T'_\varepsilon}^2\phi_{11} + 3\varepsilon^2 T'_\varepsilon\phi_{11}^2 + 3\varepsilon^2 T'_\varepsilon\phi_{12}^2 + O(\varepsilon^3\operatorname{Log}^4\varepsilon)\\ \phi_1^2 &= {T'_\varepsilon}^2 + 2\varepsilon T'_\varepsilon\phi_{11} + 2\varepsilon^2 T'_\varepsilon\phi_{12} + \varepsilon^2\phi_{11}^2 + O(\varepsilon^3\operatorname{Log}^4\varepsilon).\end{aligned} \tag{4.40}$$

Consequently from (4.12)

$$\begin{aligned}\frac{1}{A}J_1 = &\ \frac{1}{3}\int_{x_3^\varepsilon}^{+\infty}\phi_1^*\big(g'''(T_\varepsilon) + f'''(T_\varepsilon)W_\varepsilon + \frac{3\varepsilon l_\varepsilon^0}{\Lambda}f''(T_\varepsilon)\big)\phi_1^3\ dx\\ &+ \frac{\varepsilon l_\varepsilon^0}{\Lambda}\int_{x_3^\varepsilon}^{+\infty}\phi_1^* f''(T_\varepsilon)\phi_1^2\mathcal{K}_1^\varepsilon(0)T'_e\ dx + \frac{\varepsilon l_\varepsilon^0}{\Lambda}\int_{x_3^\varepsilon}^{+\infty}\phi_1^*\ f''(T_\varepsilon)\phi_1^2 K_1^\varepsilon dx + O(\varepsilon^{90})\end{aligned}$$

We expect $l''_\varepsilon(0) \sim (\dfrac{\text{constant}}{\varepsilon})$; Therefore we may get rid of a few terms:

$$\frac{\varepsilon l_\varepsilon^0}{\Lambda}\int_{x_3^\varepsilon}^{+\infty}\phi_1^* f''(T_\varepsilon)(\phi_1^2 - T_\varepsilon^2)\mathcal{K}_1^\varepsilon(0)T'_e\ dx = O(1).$$

Therefore, taking in account (4.40) and (4.15) and the behaviour of W_ε, we have

$$
\begin{aligned}
\frac{1}{A}J_1 = & \frac{1}{3}\int_{x_3^\varepsilon}^{+\infty} \phi_1^* h'''(\frac{T_\varepsilon - 1}{\varepsilon})\big(T_\varepsilon'^3 + 3\varepsilon T_\varepsilon'^2\phi_{11} + 3\varepsilon^2 T_\varepsilon'\phi_{11}^2 + 3\varepsilon^2 T_\varepsilon'\phi_{12}^2\big)\ dx \\
& + \frac{\varepsilon l_\varepsilon^0}{\Lambda}\int_{x_3^\varepsilon}^{+\infty} \phi_1^* f''(T_\varepsilon) T_\varepsilon{}'^2 \mathcal{K}_1^\varepsilon(0) T_\varepsilon'\ dx \\
& + \frac{\varepsilon l_\varepsilon^0}{\Lambda}\int_{x_3^\varepsilon}^{+\infty} \phi_1^*\ f''(T_\varepsilon)(T_\varepsilon'^2 + 2\varepsilon T_\varepsilon'\phi_{11}) K_1^\varepsilon dx + O(1)
\end{aligned} \tag{4.41}
$$

The second part of $l_\varepsilon^{0\prime\prime}(0)$ (denoted by $-J_2$) is

$$
\begin{aligned}
\frac{1}{2}J_2 = & - \big\langle(\Phi_1^*,\Psi_1^*), \partial^2_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0, T_\varepsilon, W_\varepsilon).(\Phi_1,\Psi_1).(u,w)\big\rangle \\
= & 2A\int_{-\infty}^{+\infty} \phi_1^*\big(g''(T_\varepsilon) + f''(T_\varepsilon)W_\varepsilon)u_2\phi_1 + f'(T_\varepsilon)(\phi_1 w_2 + u_2\psi_1)\big)\ dx \\
& + B\int_{-\infty}^{+\infty} \phi_1^*\big(g''(T_\varepsilon) + f''(T_\varepsilon)W_\varepsilon)u_0\phi_1 + f'(T_\varepsilon)(\phi_1 w_0 + u_0\psi_1)\big)\ dx \\
& - B\int_{-\infty}^{+\infty} \phi_1^*\big(g''(T_\varepsilon) + f''(T_\varepsilon)W_\varepsilon)u_2\phi_1 + f'(T_\varepsilon)(\phi_1 w_2 + u_2\psi_1)\big)\ dx + O(1)
\end{aligned}
$$

Estimates (4.34), (4.9) and (4.12) yield for $i \in \{0,2\}$

$$
\begin{aligned}
\phi_1 u_i = & (\frac{T_\varepsilon''}{4}(T_\varepsilon' + \varepsilon\phi_{11} + \varepsilon^2\phi_{12}) + u_{i1}T_\varepsilon' + \varepsilon(u_{i1}\phi_{11} + u_{i2}T'\varepsilon) + O(\varepsilon^2 \mathrm{Log}^4\varepsilon) \\
\psi_1 u_i = & \frac{\varepsilon l_\varepsilon^0 T_\varepsilon''}{4\Lambda}(T_\varepsilon' + \varepsilon\phi_{11} + K_1^\varepsilon + \mathcal{K}_1^\varepsilon(0)T_\varepsilon') \\
& + \frac{\varepsilon l_\varepsilon^0}{\Lambda} u_{i1}(T_\varepsilon' + K_1^\varepsilon + \mathcal{K}_1^\varepsilon(0)T_\varepsilon') + O(\varepsilon^2\mathrm{Log}^4\varepsilon)
\end{aligned}
$$

Consequently, using the last two identities, we obtain a new integral to compute. In order to reduce its size, let us notice that differentiating twice the equation for T_ε and W_ε yields an identity of the form

$$
\mathcal{L}_0\begin{pmatrix} T_\varepsilon'' \\ W_\varepsilon'' \end{pmatrix} = \begin{pmatrix} R(x) \\ 0 \end{pmatrix}.
$$

Then we stretch the variable x, and we obtain:

$$
\begin{aligned}
J_1 + J_2 = & \frac{A}{\varepsilon^2}J_3 + \frac{A}{\varepsilon^2}J_4 + \frac{2B\gamma_{\varepsilon 0} + 2(2A-B)\gamma_{\varepsilon 2} - Ac_\varepsilon p_\varepsilon'(\xi_3^\varepsilon)}{\varepsilon^2}J_5 \\
& + \frac{2(2A-B)}{\varepsilon}J_6 + \frac{2B}{\varepsilon}J_7 + \frac{A}{\varepsilon}J_8 + O(\mathrm{Log}^4\varepsilon)
\end{aligned} \tag{4.42}
$$

and the J_i have the following expressions:

$$
\begin{aligned}
J_3 &= \int_{-\infty}^{+\infty} \phi_1^*(x)R(x)\ dx \\
J_4 &= \frac{A}{\varepsilon^2}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^*\big(h'''(p_\varepsilon){p'_\varepsilon}^2\phi_{11} + h''(p_\varepsilon)p''_\varepsilon\phi_{11} + 4h''(p_\varepsilon)\tilde{u}_1 + \frac{l_\varepsilon^0}{\Lambda}K_1^\varepsilon e^{p_\varepsilon}(p''_\varepsilon + {p'_\varepsilon}^2)\ d\xi \\
J_5 &= \int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* h''(p_\varepsilon){p'_\varepsilon}^2\ d\xi
\end{aligned}
$$

$$
\begin{aligned}
J_6 &= \int_{\xi_3^\varepsilon}^{+\infty} \phi_0^*\left(h''(p_\varepsilon)\big(u_{21}\phi_{11} + u_{22}p'_\varepsilon\big) + \frac{l_\varepsilon^0}{\Lambda}e^{p_\varepsilon}\left(p'_\varepsilon Q_2^\varepsilon + u_{21}K_1^\varepsilon\right)\right)\ d\xi \\
J_7 &= \int_{\xi_3^\varepsilon}^{+\infty} \phi_0^*\left(h''(p_\varepsilon)\big(u_{01}\phi_{11} + u_{02}p'_\varepsilon\big) + \frac{l_\varepsilon^0}{\Lambda}e^{p_\varepsilon}\left(p'_\varepsilon Q_0^\varepsilon + u_{01}K_1^\varepsilon\right)\right)\ d\xi \\
J_8 &= \int_{\xi_3^\varepsilon}^{+\infty} \phi_0^*\left(h'''(p_\varepsilon)\big(p'_\varepsilon\phi_{11}^2 + {p'_\varepsilon}^2\phi_{12}\big) + h''(\varepsilon)p''_\varepsilon\phi_{12} + \frac{2l_\varepsilon^0}{\Lambda}K_1^\varepsilon p'_\varepsilon e^{p_\varepsilon}\phi_{11}\right)\ d\xi
\end{aligned}
$$

The terms J_3 to J_5 will be estimated separately; the remaining terms will be computed together.

- *Expansion of J_3*

It follows from differentiating three times the equation for T_ε and W_ε the identity

$$
\mathcal{L}_0\begin{pmatrix} T'''_\varepsilon \\ W'''_\varepsilon \end{pmatrix} + k_1^2\begin{pmatrix} T'''_\varepsilon \\ \Lambda W'''_\varepsilon \end{pmatrix} = \begin{pmatrix} R(x) \\ 0 \end{pmatrix} + k_1^2\begin{pmatrix} T'''_\varepsilon \\ \Lambda W'''_\varepsilon \end{pmatrix}.
$$

Taking the scalar product with (ϕ_1^*, ψ_1^*), we obtain

$$
J_3 = -k_1^2\int_{-\infty}^{+\infty}\left(\phi_1^* T'''_\varepsilon + (1+\varepsilon l_\varepsilon^0)\psi_1^* W'''_\varepsilon\right)\ dx \tag{4.43}
$$

We integrate (4.43) by parts, taking into account that T'''_ε and W'''_e are $O(1)$ for $x \le x_3^\varepsilon$.

$$
J_3 = \frac{k_1^2}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty}(p''_\varepsilon {\phi_1^*}' + q''_\varepsilon {\psi_1^*}')\ d\xi + O(1).
$$

Then we use the estimates for $p_\varepsilon, q_\varepsilon, \phi_1^*, \psi_1^*$, namely Lemmas 2.6 and 3.5 of the preceding sections; the final estimate for J_3 reads:

$$
J_3 = \frac{k_1^2}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} {p''_\varepsilon}^2\ d\xi + O(\mathrm{Log}^4\varepsilon). \tag{4.44}
$$

- *Expansions of J_4 and J_5.*

We differentiate twice (4.10) to obtain

$$\begin{aligned} \mathcal{T}_\varepsilon \phi_{11}'' =& h'''(p_\varepsilon){p_\varepsilon'}^2\phi_{11} + 2h''(p_\varepsilon)p_\varepsilon'\phi_{11}' + h''(p_\varepsilon)p_\varepsilon''\phi_{11} \\ &+ \frac{l_\varepsilon^0}{\Lambda}e^{p_\varepsilon}\big((p_\varepsilon'' + {p_\varepsilon'}^2)K_1^\varepsilon + 2p_\varepsilon' {K_1^\varepsilon}' + {K_1^\varepsilon}''\big) \end{aligned} \tag{4.45}$$

Multiplying (4.45) by ϕ_0^* and integrating by parts a few times, we find out that

$$\begin{aligned} &\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^*\big(-h'''(p_\varepsilon){p_\varepsilon'}^2 - 3h''(p_\varepsilon)p_\varepsilon'' + 2\varepsilon c_\varepsilon h''(p_\varepsilon)p_\varepsilon'\big)\phi_{11}\ d\xi \\ +&\frac{l_\varepsilon^0}{\Lambda}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* e^{p_\varepsilon}\big((p_\varepsilon'' + {p_\varepsilon'}^2)K_1^\varepsilon + 2p_\varepsilon' {K_1^\varepsilon}' + {K_1^\varepsilon}''\big)\ d\xi = O(\varepsilon^2 \mathrm{Log}^4\varepsilon) \end{aligned} \tag{4.46}$$

Differentiating (4.32) once, we have

$$\begin{aligned} \mathcal{T}_\varepsilon \tilde{u}_1' =& h''(p_\varepsilon)\tilde{u}_1 p_\varepsilon' + \frac{l_\varepsilon^0}{2\Lambda}e^{p_\varepsilon}\big((p_\varepsilon'' + {p_\varepsilon'}^2)K_1^\varepsilon + 2p_\varepsilon' {K_1^\varepsilon}'\big) \\ &+ \frac{1}{2}\big(h'''(p_\varepsilon){p_\varepsilon'}^2\phi_{11} + h''(p_\varepsilon)p_\varepsilon''\phi_{11} + h''(p_\varepsilon)p_\varepsilon'\phi_{11}'\big) \end{aligned} \tag{4.47}$$

We once again multiply by ϕ_0^* and integrate by parts a certain number of times, which yields

$$\begin{aligned} &\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^*\big(2h''(p_\varepsilon)\tilde{u}_1 - h''(p_\varepsilon)p_\varepsilon'' + \varepsilon c_\varepsilon h''(p_\varepsilon)p_\varepsilon'\big)\phi_{11} d\xi \\ +&\frac{l_\varepsilon^0}{\Lambda}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* e^{p_\varepsilon}\big((p_\varepsilon'' + {p_\varepsilon'}^2)K_1^\varepsilon + 2p_\varepsilon' {K_1^\varepsilon}'\big)\ d\xi = O(\varepsilon^2 \mathrm{Log}^4\varepsilon) \end{aligned} \tag{4.48}$$

Now, $\big($ (4.46) $-2\times$ (4.48) $\big)$ implies

$$J_4 = \frac{l_\varepsilon^0}{\Lambda}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* e^{p_\varepsilon}{K_1^\varepsilon}''\ d\xi + O(\varepsilon^2 \mathrm{Log}^4\varepsilon) = O(\varepsilon^2 \mathrm{Log}^4\varepsilon). \tag{4.49}$$

Multiplication of the equation

$$\mathcal{T}_\varepsilon p_\varepsilon' = O(\varepsilon^2 \mathrm{Log}^4\varepsilon),$$

by ϕ_0^* implies

$$J_5 = O(\varepsilon^2 \mathrm{Log}^4\varepsilon). \tag{4.50}$$

- *Expansions of the remaining terms*

Taking into account (4.31) we may write

$$
\begin{aligned}
J_6 =&\big(\gamma_{\varepsilon 2} - \frac{c_\varepsilon p'_\varepsilon(\xi_3^\varepsilon)}{4}\big)\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^*\big(h''(p_\varepsilon)p'_\varepsilon\phi_{11} + \frac{l_\varepsilon^0}{\Lambda}K_1^\varepsilon p'_\varepsilon e^{p_\varepsilon}\big)\ d\xi \\
&+\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^*\Big(h''(p_\varepsilon)\big(\tilde{u}_1\phi_{11} + u_{22}p'_\varepsilon\big)+\frac{l_\varepsilon^0}{\Lambda}e^{p_\varepsilon}\big(p'_\varepsilon Q_2^\varepsilon + \tilde{u}_1 K_1^\varepsilon\big)\Big)\ d\xi \\
:=&J_{61} + J_{62}
\end{aligned}
\tag{4.51}
$$

From (4.18) we know that $J_{61} = O(\varepsilon \mathrm{Log}^4 \varepsilon)$; therefore we may turn to J_{62} immediately. Let us first prove that $\tilde{u}_1$ is nothing else than ϕ'_{11}, up to a certain amount of p'_ε This amount is bound to disappear in the integration process.

We know that $\tilde{u}_1(\xi_3^\varepsilon) = 0$; therefore the function

$$
\tilde{v}_1 := \tilde{u}_1 - \frac{1}{2}\big(\phi'_{11} - \phi'_{11}(\xi_3^\varepsilon)\frac{p'_\varepsilon}{c_\varepsilon}\big)
$$

is a solution of

$$
\mathcal{T}_\varepsilon \tilde{v}_1 = O(\varepsilon \mathrm{Log}^4 \varepsilon), \qquad \tilde{v}_1(\xi_3^\varepsilon) = O(\varepsilon \mathrm{Log}^4 \varepsilon).
$$

From Proposition 2.4 we find out that $\tilde{v}_1 = O(\varepsilon \mathrm{Log}^4 \varepsilon)$;

$$
\tilde{u}_1 = \frac{1}{2}\big(\phi'_{11} - \frac{l_\varepsilon^0 r_2^\varepsilon(k_1)}{\Delta_\varepsilon^2}p'_\varepsilon\big)+O(\varepsilon \mathrm{Log}^4 \varepsilon).
\tag{4.52}
$$

It follows from (4.52) and similar considerations on J_7 - it suffices to replace u_{21} by u_{01} - that

$$
\begin{aligned}
&\frac{2(2A-B)}{\varepsilon}J_6 + \frac{2B}{\varepsilon}J_7 \\
=&\frac{2A}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^*\big(h''(p_\varepsilon)\phi_1\phi''_{11} + \frac{l_\varepsilon^0}{\Lambda}\phi_0^* e^{p_\varepsilon}K_1^\varepsilon\phi'_{11}\big)\ d\xi \\
&+\frac{2(2A-B)}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* h''(p_\varepsilon)p'_\varepsilon u_{22}\ d\xi + \frac{2B}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* h''(p_\varepsilon)p'_\varepsilon u_{02}\ d\xi \\
&+\frac{2(2A-B)}{\varepsilon}\frac{l_\varepsilon^0}{\Lambda}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* e^{p_\varepsilon}p'_\varepsilon Q_2^\varepsilon\ d\xi + \frac{2B}{\varepsilon}\frac{l_\varepsilon^0}{\Lambda}\int_{\xi_3^\varepsilon}^{+\infty} \phi_0^* e^{p_\varepsilon}p'_\varepsilon Q_0^\varepsilon\ d\xi \\
&+O(\mathrm{Log}^4 \varepsilon)
\end{aligned}
\tag{4.53}
$$

For $i \subset \{0, 2\}$ let us differentiate once the equation for u_{i2}:

$$
\begin{aligned}
\mathcal{T}_\varepsilon u'_{i2} =& -k_i^2 p'''_\varepsilon + \frac{l_\varepsilon^0}{\Lambda}p'_\varepsilon e^{p_\varepsilon}Q_i + h''(p_\varepsilon)u_{i2} \\
&+\frac{1}{4}h'''(p_\varepsilon)p'_\varepsilon\big(\phi_{11}^2 + 2\phi_{12}p'_\varepsilon\big)+\frac{1}{2}h''(p_\varepsilon)\big(\phi_{11}\phi'_{11} + p''_\varepsilon\phi_{12} + p'_\varepsilon\phi'_{12}\big) \\
&+\frac{l_\varepsilon^0}{\Lambda}\big(K_1^\varepsilon(\phi'_{11} + p'_\varepsilon\phi_{11})+Q'_i\big)e^{p_\varepsilon}
\end{aligned}
\tag{4.54}
$$

and the next identity follows from usual routine

$$
\begin{aligned}
&\frac{2(2A-B)}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} h''(p_\varepsilon)p_\varepsilon'' u_{22}\phi_0^* d\xi + \frac{2B}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} h''(p_\varepsilon)p_\varepsilon'' u_{02}\phi_0^* d\xi \\
&+\frac{2A}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} h''(p_\varepsilon)p_\varepsilon''\phi_{11}\phi_{11}'\phi_0^* d\xi \\
=&\frac{-A}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty}(h'''(p_\varepsilon)p_\varepsilon'\phi_{11}^2 - 2h''(p_\varepsilon)p_\varepsilon''\phi_{12})\phi_0^* d\xi \\
&-\frac{2l_\varepsilon^0 A}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} K_1^\varepsilon e^{p_\varepsilon}(\phi_{11}' + p_\varepsilon'\phi_{11})\phi_0^* d\xi \\
&+\frac{2(2A-B)}{\varepsilon}k_2^2\int_{\xi_3^\varepsilon}^{+\infty} p_\varepsilon'''\phi_0^* d\xi - \frac{2(2A-B)l_\varepsilon^0}{\Lambda\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} p_\varepsilon' e^{p_\varepsilon} Q_2^\varepsilon\phi_0^* d\xi \\
&-\frac{2Bl_\varepsilon^0}{\Lambda\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} p_\varepsilon' e^{p_\varepsilon} Q_0^\varepsilon\phi_0^* d\xi - \frac{2(2A-B)l_\varepsilon^0}{\Lambda\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} e^{p_\varepsilon} {Q_2^\varepsilon}'\phi_0^* d\xi \\
&-\frac{2Bl_\varepsilon^0}{\Lambda\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} e^{p_\varepsilon} {Q_0^\varepsilon}'\phi_0^* d\xi + O(\log^4\varepsilon).
\end{aligned}
\tag{4.55}
$$

It now remains to handle J_8. Let us differentiate twice equation (4.11) on ϕ_{12} :

$$
\mathcal{T}_\varepsilon\phi_{12}'' = h'''(p_\varepsilon){p_\varepsilon'}^2\phi_{12} + 2h''(p_\varepsilon)p_\varepsilon'\phi_{12}' + h''(p_\varepsilon)p_\varepsilon''\phi_{12} - k_1^2 p_\varepsilon'''. \tag{4.56}
$$

Multiplying for the last time by ϕ_0^* and doing the usual integrations by parts, we come to the identity

$$
\begin{aligned}
&\frac{A}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty}(h'''(p_\varepsilon){p_\varepsilon'}^2 + 3h''(p_\varepsilon)p_\varepsilon'')\phi_{12}\phi_0^* d\xi \\
&\qquad\qquad = \frac{-Ak_1^2}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} p_\varepsilon'''\phi_0^* d\xi + O(\varepsilon\mathrm{Log}^4\varepsilon)
\end{aligned}
\tag{4.57}
$$

We may conclude the expansion of the remaining terms in (4.42) by adding (4.55) and (4.57) and taking in account Corollary 4.6:

$$
\begin{aligned}
&\frac{2(2A-B)}{\varepsilon}J_6 + \frac{2B}{\varepsilon}J_7 + \frac{A}{\varepsilon}J_8 \\
&\qquad = \frac{-2(2A-B)k_2^2 + Ak_1^2}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} {p_\varepsilon''}^2 d\xi + O(\mathrm{Log}^4\varepsilon)
\end{aligned}
\tag{4.58}
$$

PROOF OF THEOREM 4.2. From (4.37) and (4.42), we only have to add up (4.44), (4.49), (4.50) and (4.58):

$$
\begin{aligned}
\big\langle(\Phi_1^*,\Psi_1^*),(\Phi_1^*,\Psi_1^*)\big\rangle l_\varepsilon''(0) =& \frac{4}{|\sigma_\varepsilon'(l_\varepsilon^0)|}\frac{(2A-B)k_2^2 - Ak_1^2}{\varepsilon}\int_{\xi_3^\varepsilon}^{+\infty} {p_\varepsilon''}^2 d\xi + O(\mathrm{Log}^4\varepsilon) \\
=& \frac{\pi^2}{2\varepsilon|\sigma_\varepsilon'(l_\varepsilon^0)|}\int_{\xi_3^\varepsilon}^{+\infty} {p_\varepsilon''}^2 d\xi + O(\mathrm{Log}^4\varepsilon)
\end{aligned}
.
$$

It remains to note that there holds

$$\lim_{\varepsilon\to 0}\int_{\xi_3^\varepsilon}^{+\infty} p_\varepsilon''^2\, d\xi = \int_{-\infty}^{+\infty} p_\infty^2 e^{2p_\infty}\, d\xi < +\infty,$$

and to recall Corollary 3.7, which asserts that $\big\langle (\Phi_1^*, \Psi_1^*), (\Phi_1, \Psi_1)\big\rangle$ tends to a positive constant. This ends the proof of Theorem 4.2. •

5. STABILITY OF THE WAVES

The goal of this section is to prove the multidimensional stability of the planar wave above the first critical Lewis number, and of the nonplanar wave below. There usually two steps: linearized stability, and nonlinear stability. Here, nonlinear stability is not immediate, and the difficulty holds for both the planar wave and the nonplanar wave. It comes from the fact that the linearized operator is not coercive as $x \to -\infty$. A very usual way to circumvent this difficulty [21], [32], [30] is to use weighted spaces to push the essential spectrum of the linearized operator away from zero. Let us see how it works here, and let us try and solve, for smooth compactly supported f_1 and f_2:

$$\mathcal{L}_0 \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sigma \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}; \tag{5.1}$$

recall that the oprator $\mathcal{L}_0$ is defined in (1.19). Taking the Fourier transform of the equation for ψ in (5.1) yields

$$\hat{\psi}(\xi) = \frac{\xi^2(1-\Lambda)\hat{\phi}(\xi) + \hat{f}_2(\xi)}{\Lambda\xi^2 + ic_\varepsilon\xi - \sigma}.$$

The denominator has a singularity as soon as σ belongs to the parabola $\{\Lambda\sigma_2^2 - c_\varepsilon^2\sigma_1 = 0\}$, which contains 0. Therefore operating in a weighted space is not enough to push the essential spectrum away from 0. We will retain the principle of operating in a weighted space, but we will need a more subtle argument: we will first prove that the linear semigroup has the expected dynamical properties in a space of functions exponentially decreasing as $x \to -\infty$ and exponentially increasing as $x \to +\infty$, then we will recover some uniform convergence for the nonlinear problem with barehands. In the latter framework, it will be more convenient to work with the unknowns (u, v) with

$$v = w - u \tag{5.2}$$

Let us rewrite System (1) in the coordinate of a wave of speed c - in the sequel, it will either be c_ε or $\mu_\varepsilon(T(s))$. In the unknowns T and W we get

$$\left\{ \begin{array}{l} T_t - \Delta T + cT_x - \big(g(T) + f(T)W\big) = 0 \\ W_t - \Lambda\Delta W + cW_x - (1-\Lambda)\Delta T = 0 \end{array} \right. \qquad (\Sigma) \tag{5.3}$$

$$T_y(t,x,0) = T_y(t,x,1) = W_y(t,x,0) = W_y(t,x,1) = 0 \tag{5.4}$$

Problem (5.3), (5.4) will be condensed into the notation

$$(T, W)_t + \mathcal{G}_{c,\varepsilon}(T, W) = 0, \tag{5.5}$$

Reintroducing the unknown $Y = W - T$, we will allow ourselves to write (5.3), (5.4) under the form

$$(T, Y)_t + \mathcal{G}_{c,\varepsilon}(T, Y) = 0, \tag{5.6}$$

The Cauchy Problem for (5.5), associated to initial data T_0 and W_0 which are, for instance, bounded uniformly continuous in $\overline{\Sigma}$, has a unique solution, denoted by $\mathcal{S}(t)(T_0, W_0)$, which is *a priori* only local in time. In the remaining of this section, we will denote, for a continuous function in $\overline{\Sigma}$ and $h \in I\!R$:

$$\tau_h u(x, y) = u(x + h, y). \tag{5.7}$$

We are going to prove the following result.

THEOREM 5.1. *Take $c = c_\varepsilon$, and $l < l^0_\varepsilon$. Choose (T_0, Y_0) of the form*

$$T_0 = T_\varepsilon + u_0, \qquad W_0 = W_\varepsilon + w_0, \tag{5.8}$$

with $(u_0, w_0) \in B_{\mu_r} \times B_{\mu_r}$.
1. *If $(\|(u_0, w_0)\|_{B_{\mu_r}\times B_{\mu_r}}$ is small enough there exists $x_0 \in I\!R$ such that*

$$\lim_{t\to+\infty}\Big(\|\mathcal{S}(t)(T_0, W_0) - \big(\tau_{x_0}T_\varepsilon, \tau_{x_0}W_\varepsilon\big)\|_{B_{\mu_r}\times B_{\mu_r}} \Big) = 0. \tag{5.9}$$

2. *Take $l > l^0_\varepsilon$. If T_0 or W_0 depend explicitly on y, then $(T_\varepsilon, W_\varepsilon)$ is unstable with respect to this perturbation. If neither T_0 nor W_0 depends explicitly on y, then the conclusion of Point 1 holds.*
3. *In both cases, the convergence towards the wave is exponential in time on every half-cylinder of the form $]-\infty, -a] \times [0, 1]$. The convergence of the mass fraction Y is exponential in time on the whole cylinder.*

THEOREM5.2. *Take $c = \mu_\varepsilon(T(s))$, for $s < s^\varepsilon_0$, and $l = l^\varepsilon(s) > l^0_\varepsilon$. Assume (T_0, Y_0) have the form*

$$T_0 = T(s) + u_0, \qquad W_0 = W(s) + w_0, \tag{5.10}$$

with $(u_0, w_0) \in B_{\mu_r} \times B_{\mu_r}$. If $(\|(u_0, w_0)\|_{B_{\mu_r}\times B_{\mu_r}}$ is small enough there exists $x_0 \in I\!R$ such that

$$\lim_{t\to+\infty}\Big(\|\mathcal{S}(t)(T_0, W_0) - \big(\tau_{x_0}T(s), \tau_{x_0}W(s)\big)\|_{B_{\mu_r}\times B_{\mu_r}} \Big) = 0. \tag{5.11}$$

Furthermore, the convergence towards the wave is exponential in time on every half-cylinder of the form $]-\infty, -a] \times [0, 1]$. The convergence of the mass fraction Y is exponential in time on the whole cylinder.

We believe that Theorem 5.1 still holds for $l = l^0_\varepsilon$. The section is organized as follows:
- Linear stability

- An abstract nonlinear stability result
- Nonlinear stability.
- A centre manifold theorem.

In the remainder of this section, otherwise specified, the letter c will either denote c_ε, or $\mu_\varepsilon(T(s))$, for small s. A solution of (1.12) will still be denoted by (l, T, W), and will either be $(l^0_\varepsilon, T_\varepsilon, W_\varepsilon)$ or $(l_\varepsilon(s), T(s), W(s))$.

Linear stability

The problem under consideration is

$$\partial_{T,W}\mathcal{G}_{c,\varepsilon}(l,T,W)(\Phi,\Psi) = \sigma(\Phi,\Psi) \tag{5.12}$$

The following proposition solves the stabilty problem left open in [36].

PROPOSITION 5.3. *Set $c = c_\varepsilon$. Let (σ, Φ, Ψ) be a solution of (5.12), with:*
- *either $l < l^0_\varepsilon$*
- *or $l > l^0_\varepsilon$ and $(\Phi(x,y), \Psi(x,y)) = (\Phi(x), \Psi(x))$.*

Then $\sigma = 0$ or $\mathrm{Re}\sigma > 0$.

PROOF. Obvious: for $l < l^0_\varepsilon$ and $c = c_\varepsilon$, the result readily follows from the Fourier expansion of (5.12), and application of Theorems 2.3 and 2.1. Also, if we restrict ourselves to one-dimensional solutions of (5.12), then the conclusion of Proposition 5.3 still holds. •

Therefore we may concentrate on the case $l = l^\varepsilon(s) > l^0_\varepsilon$, and $c = \mu_\varepsilon(T(s))$. We would basically like to use Theorem 1.16 in [16]; however we may not work in the spaces X and Y because in this framework the notion of simple eigenvalue has essentially no meaning. Therefore we have to work in $Y^\#$, and we will have to be careful not to introduce spurious eigenvalues. Here is the result that we are going to prove.

THEOREM 5.4. *Set $c = c_\varepsilon$ and $l < l^0_\varepsilon$, or $c = \mu_\varepsilon(T(s))$, and $l = l_\varepsilon(s)$, for small s. The operator $\partial_{T,W}\mathcal{G}_{c,\varepsilon}(l,T,W)$ is sectorial in $Y^\#$, with domain $X^\#$. Moreover,*
1. *0 is a simple eigenvalue; moreover the decompostion*

$$Y^\# = N\big(\partial_{T,W}\mathcal{G}_{c,\varepsilon}(l,T,W))\big) \oplus R\big(\partial_{T,W}\mathcal{G}_{c,\varepsilon}(l,T,W))\big) \tag{5.14}$$

holds, both in the algebraic and topological sense.
2. *The remainder of $\sigma\big(\partial_{T,W}\mathcal{G}_{c,\varepsilon}(l,T,W)\big)$ is contained in a cone of the complex plane, with a positive vertex, and aperture $< \pi$.*

The proof will be broken in four lemmas, the first one being the following elementary result in linear algebra.

LEMMA 5.5. *Let X be a vector space, and A be a linear mapping from $D(A)$ to X. Let l be a linear form on X, and $x_0 \in N(A)$ such that $< l, x_0 > \neq 0$.*
- *Let $\sigma \neq 0$ be an eigenvalue of A, such that $< l, x_0 > \neq \sigma$. Then σ is an eigenvalue of the operator B: $u \mapsto Bu = Au + < l, u > x_0$.*

• *Assume that $N(B) = \{0\}$. Then, if $N(A)$ is spanned by x_0, then $N(A^2)$ is also spanned by x_0.*

PROOF. For the first point, let v be an eigenvector for the eigenvalue σ; the vector

$$u = -\frac{< l, v >}{< l, x_0 > -\sigma} x_0 + v$$

is nonzero because $\sigma \neq 0$, and is therefore such that $Bu = \sigma u$.

As for the second point, assume there exists $\alpha \neq 0$ such that $Au = \alpha x_0$; then the vector

$$v = u - \frac{\alpha + < l, u >}{< l, x_0 >} x_0$$

belongs to $N(B)$, which is therefore at least two-dimensional.

Next, we need some information about the asymptotic behaviour of the eigenfunctions at $\pm\infty$. To do this, we merely write down (5.12) at "$x = \pm\infty$", and we plug in exponential solutions; then we use the heavy machinery of [1] and [28]. See also [9] for a quick survey of what is useful in these two papers in the case of second-order PDE's. In the sequel, we will only investigate the problem at $+\infty$, the problem at $-\infty$ being simpler. First, we need some knowledge about how a solution T and W behaves as $x \to +\infty$.

LEMMA 5.6. *There exists $\rho_1 > 0$ such that*

$$\|\big(T(x,.) - 1, W(x,.)\big)\|_{C^2([0,1])} = O(e^{-\frac{\rho_1 x}{\varepsilon}}) \tag{5.15}$$

PROOF. It is very much in the spirit of Proposition 3.4, so we are not going to dwell on it too long. Choose $r < \frac{k_1^2}{2}$; for small ε it is smaller than $\frac{1}{2\varepsilon}$. We set $U = T - 1$, and

$$\begin{gathered} T^{\#\#}(x,y) = 1 + e^{-rx}\big(T(x,y) - 1\big), \qquad W^{\#\#}(x,y) = e^{-rx} Z(x,y) \\ U^{\#\#}(x,y) = e^{-rx} U(x,y) \end{gathered} \tag{5.16}$$

The equation for W - same as in (3.16) - yields the two following facts.

1. Taking its mean in y, we have

$$< W > (x) = O(\varepsilon) \int_x^{+\infty} e^{c(x-x')} < U >' (x')\, dx',$$

hence the estimate $\| < W^{\#\#} > \|_\infty \leq C\varepsilon \|\partial_x T^{\#\#}\|_\infty$.

2. By Poincaré's inequality and elliptic regularity, we get an estimate of the type $\|W^{\#\#}\|_{H^1(\Sigma)} = O(\varepsilon)\|U^{\#\#}\|_{H^1(\Sigma)}$.

As a conclusion, if we set $W^{\#\#} = \varepsilon \mathcal{M} U^{\#\#}$, we have $\|\mathcal{M}\|_{\mathcal{L}(H^1)} = O(1)$.

We now consider the equation for $U^{\#\#}$ in Σ_+; it is of the form

$$-\Delta U^{\#\#} + (c + 2r)\partial_x U^{\#\#} + \frac{1}{\varepsilon^2}(V_\varepsilon(x) + \varepsilon \mathcal{M}) U^{\#\#} = 0;$$

for x large enough we have $V_\varepsilon(x) \geq \delta > 0$. Arguing as in Lemma 3.2 and using elliptic regularity ends the proof. •

Let now investigate the eigenvalue problem at $x = +\infty$. In the (u, v) formulation the problem reads

$$(5.17) \qquad \begin{cases} -\Delta u + cu_x = \sigma u + \dfrac{v}{\varepsilon^2} \\ -\Lambda \Delta v + cv_x = -\dfrac{v}{\varepsilon^2} + \sigma v \end{cases} \qquad (\Sigma)$$

with Neumann conditions at $y = 0$ and $y = 1$. Let us recall [1] that an exponential solution to (5.17) is a solution of the form

$$(5.18) \qquad e(x, y) = e^{\lambda x} \sum_{i=0}^{n} x^i (p_i(y), q_i(y))$$

The leading coefficient in (5.18) is a solution of the differential system

$$(5.19) \qquad \begin{cases} -p'' = \big(\lambda(\lambda - c) + \sigma\big)p + \dfrac{q}{\varepsilon^2} \\ -\Lambda q'' = \big(\lambda(\lambda - c) + \sigma - \dfrac{1}{\varepsilon^2}\big)q \end{cases}$$

with $p'(0) = p'(1) = q'(0) = q'(1) = 0$. Thus λ is a possible exponent if and only if there exists $i \in I\!N$ such that one of the two following equalities holds

$$(5.20) \qquad k_i^2 = \lambda(\lambda - c) + \sigma$$

$$(5.21) \qquad \Lambda k_i^2 = \lambda(\lambda - c) + \sigma - \frac{1}{\varepsilon^2}$$

in other words if and only if λ is a characteristic root of a problem $(1.18)_n$.

LEMMA 5.7. *Let $(\Phi, \Psi) \in X^{\#}$ be a solution of (5.12). Then there exists ρ_2 such that*

$$(5.22) \qquad \|\big(\Phi(x, .), \Psi(x, .)\big)\|_{C^2([0,1])} = O(e^{-\rho_2 x}).$$

PROOF. From the preceding lemma, we infer that, for $\varepsilon > 0$ small enough,

$$\|\big(g'(T) - g'(1) + f(T)W\big)\Phi + \big(f(T) - f(1)\big)\Psi\|_{C^2([0,1]} = O(e^{-\rho_1 x}) \quad \text{as } x \to +\infty.$$

Among the roots in (5.20) and (5.21) with positive real parts, the one with minimal real part is

$$\lambda_1 = \frac{c + \delta_\varepsilon(0, \sigma)}{2},$$

and $\mathrm{Re}\lambda_1 > r$. We conclude by using Theorem 5.6 in [1] and Lemma 5.2 in [28]. •

Therefore we have proved that the eigenfunctions for (5.12) in $X^\#$ are the same as in X.

PROOF OF THEOREM 5.4. Let us first prove the analogue of Proposition 5.3 for $c = \mu_\varepsilon(T(s))$ and $l = l_\varepsilon(s) > l_\varepsilon^0$. The operator $\partial_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0, T_\varepsilon, W_\varepsilon)$ of $Y^\#$, with domain $X^\#$ satisfies the assumptions of Lemma 1.3 of [16]. Therefore, for small enough $\delta > 0$ and $s \in [-s_0^\varepsilon, s_0^\varepsilon]$, there exists a unique real eigenvalue $\eta_\varepsilon(s)$ in the ball of the complex plane of radius δ for $\partial_{T,W}\mathcal{F}_\varepsilon\big(l_\varepsilon(s), T(s), W(s)\big)$. Moreover, $\eta_\varepsilon(0) = 0$. From Theorem 1.16 of [16] - the nonlinearity is unbounded in $Y^\#$, therefore we cannot say that we are in the framework of this result; however it only involves linear operators; therefore its proof may be copied word by word for our case - and Theorem 4.2, we have

$$\eta_\varepsilon(s) \sim -\sigma_\varepsilon'(l_\varepsilon^0)l_\varepsilon''(0)s^2 > 0.$$

Recall that $\sigma_\varepsilon(l)$ is defined by formula (3.33). From Lemma 5.5, with

$$x_0 = \begin{pmatrix} T_\varepsilon' \\ W_\varepsilon' \end{pmatrix}, \qquad < l, u >= -\mu_\varepsilon(u),$$

we infer that

$$\sigma\Big(\partial_{T,W}\mathcal{G}_{\mu_\varepsilon(T(s)),\varepsilon}(l_\varepsilon(s), T(s), W(s))\Big) \subset \sigma\Big(\partial_{T,W}\mathcal{F}_\varepsilon\big(l_\varepsilon(s), T(s), W(s)\big)\Big) \cup \{0\}.$$

We have $-\mu_\varepsilon(T_\varepsilon') < 0$. However, Theorem 2.3 and a standard argument show that there is no sequence $s_n \to 0$ such that

$$-\mu_\varepsilon(T_\varepsilon') \in \sigma\Big(\partial_{T,W}\mathcal{F}_\varepsilon\big(l_\varepsilon(s_n), T(s_n), W(s_n)\big)\Big).$$

Therefore, for small s, the conclusion of Proposition 5.3 for $c = \mu_\varepsilon(T(s))$ and $l = l_\varepsilon(s) > l_\varepsilon^0$ holds.

Let us now prove Points **1** and **2** of Theorem 5.4. For small $s \neq 0$, we have

$$N\Big(\partial_{T,W}\mathcal{F}_\varepsilon\big(l_\varepsilon(s), T(s), W(s)\big)\Big) = \{0\}.$$

Application of Lemma 5.5 tells us that 0 is an algebraically simple eigenvalue of $\partial_{T,W}\mathcal{F}_\varepsilon\big(l_\varepsilon(s), T(s), W(s)\big)$ in $Y^\#$. From Proposition 3.4 and an argument similar to the one in [30], we infer that the decomposition (5.14) holds. That (5.14) also holds for the planar wave is a consequence of Points **1** and **2** at the beginning of Section 3. This ends the proof of Point **1.**

Let us now prove Point **2** and let us come back to the notations c, l, T, W; the proof follows the same way for the planar and the nonplanar wave. We first notice that the eigenvalues of the eigenfunctions for (5.12) in $X^\#$ are the same as in X. From (a trivial extension of) Theorem 2 of [35], the operator $\partial_{T,W}\mathcal{G}_{c,\varepsilon}(l, T, W)$ is sectorial in $Y^\#$. Therefore it remains to prove that its essential spectrum lies in a half-plane of the complex plane $\{\text{Re }\sigma \geq \gamma\}$, with $\gamma > 0$. For $(u, v) \in X^{\#}$, let us write

$$\partial_{T,W}\mathcal{G}_{c,\varepsilon}(l, T, W)(u, v) := e^{rx}\mathcal{L}^{\#}(e^{-rx}u, e^{-rx}v).$$

We write $\mathcal{L}^{\#} = \mathcal{A}^{\#} + \mathcal{B}^{\#}$, with

$$\mathcal{A}^{\#}(u, w) = \begin{pmatrix} -\Delta u + (c-2r)u_x + r(c-r)u - \dfrac{H_{\varepsilon^4}}{\varepsilon^2}v \\ -\Lambda\Delta v + (c-2r)v_x + \big(r(c-r) + \dfrac{H_{\varepsilon^4}}{\varepsilon^2}\big)v \end{pmatrix} \tag{5.23}$$

The expression of $\mathcal{B}^{\#}$ is similar to the one of $\mathcal{B}$ in (3.11). Obviously, $\mathcal{A}^{\#}$ has its essential spectrum in $\{\text{Re }\sigma \geq \dfrac{1}{2r(c-r)}\}$; since $\mathcal{B}^{\#}{\mathcal{A}^{\#}}^{-1}$ is compact, we have proved Point **2** of Theorem 5.4. •

An abstract nonlinear result

The following result, althogh not directly applicable, still contains the main ingredients of the nonlinear part of he proof of Theorems 5.1 and 5.2.

Let $Z \subset Z^{\#}$ be two Banach spaces, with continuous injection from Z to $Z^{\#}$. Let A be a sectorial operator in Z and $Z^{\#}$, and f a C^2 function of Z. Assume that the equation $Au = f(u)$ admits a local C^2 manifold of solutions:

$$\lambda \mapsto \phi_\lambda, \qquad \lambda \text{ in a compact interval of } I\!R.$$

We are interested in the stability of the ϕ_λ with respect to the evolution problem

$$\frac{du}{dt} + Au = f(u), \qquad u \in Z. \tag{5.24}$$

The Cauchy Problem for (5.24), associated to an initial datum u_0, has a local solution denoted by $\mathcal{S}(t)u_0$. Let denote by $\mathcal{L}_\lambda$ the operator $A - f'(\phi_\lambda)$. Its spectrum in Z (resp. $Z^{\#}$) will be denoted by $\sigma_Z(\mathcal{L}_\lambda)$ (resp. $\sigma_{Z^\#}(\mathcal{L}_\lambda)$).

THEOREM 5.8. *Assume that the three following assumptions hold.*
1. *The function f has an expansion of the form*

$$f(u) - f(\phi_\lambda) - f'(\phi_\lambda)(u - \phi_\lambda) = Q(\lambda, u - \phi_\lambda), \tag{5.25}$$

and there exists $\gamma(\lambda)$, bounded on every compact of $I\!R$, such that

$$\|Q(\lambda, v)\|_Z \leq \gamma(\lambda)\|v\|_Z\|v\|_{Z^\#} \tag{5.26}$$

2. *For any given $\lambda \in I\!R$, $N(\mathcal{L}_\lambda)$ is spanned by ϕ_λ; moreover the decomposition*

$$Z^{\#} = N(\mathcal{L}_\lambda) \oplus R_{Z^\#}(\mathcal{L}_\lambda)$$

holds, with algebraic and topological sum. Further, $\sigma_{Z^\#}(\mathcal{L}_\lambda)\backslash\{0\}$ is contained in a cone of the complex plane with positive vertex and aperture $< \pi$.

3. *For any given λ, let e^*_λ be such that $< e^*_\lambda, \phi'_\lambda >= 1$ and $\mathcal{L}^*_\lambda e^*_\lambda = 0$. There is a positive function $\varphi(t)$, such that $\lim_{t\to+\infty} \varphi(t) = 0$, such that there holds, for all $v_0 \in Z$:*

$$\|e^{-t\mathcal{L}_\lambda} v_0 - < e^*_\lambda, v_0 > \phi'_\lambda\|_Z \leq \|v_0\|_Z \varphi(t).$$

Let $\lambda_0 \in I\!R$ and $v_0 \in Z$ be given. The Cauchy datum u_0 is taken under the form $u_0 = \phi_{\lambda_0} + v_0$, $v_0 \in Z$. If $\|v_0\|_Z$ is small enough, there exists λ^∞ such that

$$\lim_{t\to+\infty} \mathcal{S}(t)u_0 = \phi_{\lambda_\infty} \qquad \text{in } Z.$$

Moreover, there exists $\omega > 0$ such that

$$\|\mathcal{S}(t)u_0 - \phi_{\lambda_\infty}\|_{Z^\#} = O(e^{-\omega t}).$$

When $Z = Z^\#$, the above result is nothing else than Exercise 6 page 108 in [21], and the convergence of $\mathcal{S}(t)u_0$ to ϕ_{λ_∞} is exponential. The proof of the general case will therefore closely follow the one in [21].
PROOF. When v_0 is small enough we may write, at least for short times: $u(t) = \phi_{\lambda(t)} + v(t)$, with $v(t) \in R(\mathcal{L}_{\lambda_0})$. The problem is to show the validity of this formula for large times. Let Π be the projection onto $R(\mathcal{L}_{\lambda_0})$; equation (5.24) may be written under the form

$$\begin{aligned} \dot\lambda(t) &= \frac{\langle e^*_{\lambda_0}, Q\big(\lambda(t), v(t)\big)v(t) + R(t, v(t))\rangle}{\langle e^*_{\lambda_0}, \phi'_\lambda(t)\rangle} \\ \dot v + \mathcal{L}_{\lambda_0} v &= \Pi Q(\lambda, v) - \dot\lambda(t)\Pi\phi'_{\lambda(t)} + \Pi R(t, v) \end{aligned} \tag{5.27}$$

with

$$R(t, u) = \int_0^1 f''(\phi_{\lambda(t)} + \sigma\phi_{\lambda_0}).(\phi_{\lambda(t)} - \phi_{\lambda_0}).u \; d\sigma$$

Let us assume, as is usual: $|\lambda(t) - \lambda_0| + \|v(t)\|_Z \leq \eta$. We operate in $Z^\#$; writing down the integral formula for $v(t)$, using the formula for $\dot\lambda(t)$ and carrying on with the usual estimates yields the existence of $\omega > 0$ and two constants C_1 and $C_2(\eta)$, bounded for small η, such that

$$\begin{aligned} \|v(t)\|_{Z^\#} \leq& C_1 e^{-\omega t}\|v_0\|_Z + C_2(\eta)\int_0^t e^{-\omega(t-s)}\big(\|v(s)\|_Z + |\lambda(s) - \lambda_0|\big)\|v\|_{Z^\#} \; ds \\ \leq& \big(C_1 + C_2(\eta)\big)\big(e^{-\omega t}\|v_0\|_Z + \eta\int_0^t e^{-\omega(t-s)}\|v\|_{Z^\#} \; ds\big) \\ \leq& 2\big(C_1 + C_2(\eta)\big)\|v_0\|_Z e^{-\left(\omega - 2\left(C_1 + C_2(\eta)\right)\eta\right)t} \end{aligned}$$

Looking at the equation for $\lambda(t)$ in (5.27) we infer the existence of a constant $q_1 > 0$ and a function $q_2(\eta)$, with $\lim_{\eta\to 0} q_2(\eta) = 0$ such that

$$|\dot\lambda(t)| \leq \frac{q_1}{1 - q_2(\eta)}\big(\|v\|_Z + |\lambda - \lambda_0|\big)\|v\|_{Z^\#}$$

Therefore, when η is really small enough, we have $|\lambda - \lambda_0| \le \eta$, and there exists $C_3(\eta)$ such that

$$|\lambda(t) - \lambda_0| + \|v(t)\|_{Z^\#} \le C_3(\eta)\eta e^{-\frac{\omega}{2}t}. \tag{5.28}$$

It now remains to prove that $\|v(t)\|_Z$ actually remains below η. The above formulas will from then on be valid, and the theorem will be proved. We write the integral formula for v in Z, and we use (5.28); this yields

$$\|v(t)\|_Z \le C\|v_0\|_Z\big(\varphi(t) + \eta\int_0^t \varphi(t-s)e^{-\frac{\omega}{2}s}\,ds\big).$$

Therefore, when $\|v_0\|_Z$ is small enough, we have $\|v(t)\|_Z \le \eta$. •

Nonlinear stability of the waves

We treat both the cases $c = c_\varepsilon$, $c = \mu_\varepsilon(T(s))$, and we adopt the framework of Theorem 5.8. We set $Z := B_{\mu_r} \times B_{\mu_r}$ and $Z^\# = Y^\#$. The other notations are obvious.

PROOF OF THEOREMS 5.1 AND 5.2. Let us first check that Point **3** indeed holds. Let us by denote (Φ^*, Ψ^*) the vector $e^*_{\lambda_0}$ in Theorem 5.8.

Choose $(u_0, w_0) \in B_{\mu_r} \times B_{\mu_r}$, with $\|(u_0, w_0)\|_{B_{\mu_r}\times B_{\mu_r}} = 1$. Let us set

$$\begin{pmatrix} p(t) \\ r(t) \end{pmatrix} = e^{-t\partial_{T,W}\mathcal{G}_{c,\varepsilon}(l,T,W)}\begin{pmatrix} u_0 \\ w_0 \end{pmatrix} - \big\langle(\Phi^*, \Psi^*), (u_0, w_0)\big\rangle\begin{pmatrix} T_x \\ W_x \end{pmatrix} \tag{5.29}$$

From Theorem 5.4 and parabolic regularity we have $\|\big(p(t), r(t)\big)\|_{X^\#} = O(e^{-\omega t})$, for some $\omega > 0$. Let us set $q(t) = r(t) - p(t)$, and let us write the equation satified by (p, q) in Σ_+. We may write it under the form

$$\left\{\begin{array}{l} p_t - \Delta p + cp_x - \dfrac{q}{\varepsilon^2} = \varphi_1(x,y)p + \varphi_2(x,y)q \\ q_t - \Lambda\Delta q + cq_x + \dfrac{q}{\varepsilon^2} = -\big(\varphi_1(x,y)p + \varphi_2(x,y)q\big) \end{array}\right. \qquad (\Sigma_+) \tag{5.30}$$

$$p_y(t,x,0) = p_y(t,x,1) = q_y(t,x,0) = q_y(t,x,1) = 0 \qquad (I\!R_+) \tag{5.31}$$

$$p(t,0,y) = O(e^{-\omega t}), \qquad q(t,0,y) = O(e^{-\omega t}). \tag{5.32}$$

Because of Lemma 5.6, we have

$$\|\big(\varphi_1(x,.), \varphi_2(x,.)\big)\|_{C^2([0,1])} = O(e^{-\frac{\rho_1}{\varepsilon}x}).$$

Therefore the right-hadside of (5.30) decays exponentially in time. This implies, in view of the equation for q in (5.30), (5.31) and (5.32), that $\|q(t)\|_{L^\infty(\Sigma_+)}$ decays exponentially in time. This implies in turn that $\|p(t)\|_{L^\infty(\Sigma_+)}$ tends to 0 as $\to +\infty$, which proves Point **3** of Theorem 5.8.

In view of the above considerations, Point **2** of Theorem 5.8 is immediate. Point **1** does not exactly holds, due to the presence of terms of the form $f''(T)uv$, which do not satisfy (5.25). However, Problem (5.6)may be written under the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \partial_{T,Y}\mathcal{G}_{c,\varepsilon}(l,T,Y)\begin{pmatrix} u \\ v \end{pmatrix} = Q\begin{pmatrix} u \\ v \end{pmatrix} + R\begin{pmatrix} u \\ v \end{pmatrix},$$

with

$$\left\| R\begin{pmatrix} u \\ v \end{pmatrix} \right\|_Z \leq C\|u\|_Z\|v\|_Z$$

and Q satisfying (5.25). Therefore, we only have to combine the linear argument above to the nonlinear argument of Theorem 5.8 in order to get the additional information: there exists a function $\rho : \mathbb{R} \to \mathbb{R}$, with $\lim_{h\to 0} \rho(h) = 0$, such that

$$\|v(t)\|_Z \leq \rho(\|v_0\|_Z)e^{-\omega t},$$

for some $\omega > 0$. This implies Theorems 5.1 and 5.2. •

A centre manifold theorem.

In this short paragraph, we present the analogue of a well-known centre manifold result for dynamical systems with one-dimensional bifurcation, when time-exponential estimates for stability are avalilable. No detailed proofs will be given, first because the considerations of the last paragraph contain all the ingredients that we would need, and second because this would enlarge this paper beyond a reasonable limit.

In order to prove the centre manifold theorem, we introduce a second normalization, following the idea of [5]. This will allow us to obtain the wave fronts as **steady** solutions of a **common** equation. Then we will have to check that this new normalization does not induce dramatic changes to our problem; finally we will apply the abstract theorems of [21].

Let us therefore introduce the new time-dependent coordinate $X(t)$ defined by

$$\int_{-\infty}^{X(t)} \int_\omega T(t,x,y)\ dxdy = 1; \tag{5.33}$$

then we make the change of coordinates $(x,y) \mapsto (x + X(t), y)$; therefore to (1) we have to add a convection term of the form $\dot{X}(t)(T_x, Y_x)$. Moreover we have

$$\dot{X}(t) = \frac{\displaystyle\int_{\Sigma_-} f(T)Y\ dxdy + \int_\omega T_x(t,0,y)\ dy}{\displaystyle\int_\omega T(t,0,y)\ dy}; \tag{5.34}$$

this definition allows us to write system (1) under the form

$$\begin{pmatrix} T \\ Y \end{pmatrix}_t + \mathcal{H}_l \begin{pmatrix} T \\ Y \end{pmatrix} = 0 \tag{5.35}$$

with the constraint

$$\int_{\Sigma_-} u(t,x,y)\ dxdy = 1. \tag{5.36}$$

The nonlinearity $\mathcal{H}_l$ may easily be reconstructed; moreover we see that a wave solution of (1) satisfies $\mathcal{H}_l(T,Y) = 0$. Finally, if c is the speed of the (T,Y) we see that the following identity holds:

$$\partial_{T,Y}\mathcal{H}_l(T,Y) = \partial_{T,Y}\mathcal{G}_{c,\varepsilon}(l,T,Y) + < L, . > (T_x, Y_x)$$

with

$$< L, (u,v) > = \frac{\displaystyle\int_{\Sigma_-} \big(f'(T)Yu + f(T)v\big)\ dxdy + \int_\omega u_x(0,y)\ dy}{\displaystyle\int_\omega T(0,y)\ dy} - \frac{\displaystyle\int_{\Sigma_-} f(T)Y\ dxdy + \int_\omega T_x(0,y)\ dy}{\displaystyle\left(\int_\omega T(0,y)\ dy\right)^2} \int_\omega u_x(0,y)\ dy$$

For a wave solution, we have $< L, (T'_\varepsilon, Y'_\varepsilon) >= 0$. However, the linearisation of the constraint (5.36) renders the null space of $\partial_{T,Y}\mathcal{H}_l(T,Y)$ in the space $X \cap \{(u,v) : \displaystyle\int_{\Sigma_+} u(x,y)\ dxdy = 0\}$ null. The results of Chapter 6.3 of Henry [21] apply, *modulo* the modifications of the preceding section. Namely, we have the

THEOREM 5.9. *For s small enough, there exists an invariant manifold under the flow (5.35),(5.36), which we denote by*

$$\mathcal{M}_l = \{(u,v) = \alpha(\Phi_1, \Psi_1) + \mathcal{I}(\alpha, l)\}, \qquad \alpha =< (\Phi_1^*, \Psi_1^*), (u,v) > .$$

with $\mathcal{I}(0,s) = 0$, $\mathcal{I}'(0,0) = 0$. On the manifold $\mathcal{M}_s$, there exists $b_\varepsilon(l) > 0$ such that the equation of the flow has the form - up, perhaps, to some time normalization -

$$\dot\alpha(t) = \sigma_\varepsilon(l)\alpha - b_\varepsilon(l)\alpha^3 + O(\alpha^5).$$

In order to conclude this section, let us come back to our initial formulation (1), and let us state the following result.

THEOREM 5.10. *Let s be small enough, and let the Cauchy datum (T_0, Y_0) for (1) be close to $(T_\varepsilon, Y_\varepsilon)$. Then if (T_0, Y_0) does not belong to a codimension 1 manifold - i.e. the stable manifold of $(T_\varepsilon, Y_\varepsilon)$ there exists $x_0 \in \mathbb{R}$ such that*

$$\lim_{t\to+\infty} \|\mathcal{S}(t)(T_0,Y_0) - \tau_{x_0+c(s)t}(T(s), Y(x))\|_\infty = 0.$$

We have therefore provided a theoretical justification to the calculation of [5].

6. THREE-DIMENSIONAL EXTENSION

This time, we set $\Sigma = I\!R \times \omega$, where ω is a smooth bounded open subset of $I\!R^2$, such that $\bar{\omega}$ is diffeomorphic to the unit ball $\bar{B}_2$. The equations are the same as in (1); we just have to replace (2) by Neumann conditions on $\partial\omega$. The method that we have used to prove Theorem 3.1 in the two-dimensional framework may be repeated word by word, by only changing $\cos k_n y$ in the Fourier expansion (1.18) by $\zeta_n(y)$, the n^{th} eigenfunction of the Neumann problem in ω:

$$
\begin{aligned}
-\Delta\zeta_n &= k_n^2\zeta_n \qquad (\omega) \\
\frac{\partial\zeta_n}{\partial\nu} &= 0 \qquad (\partial\omega)
\end{aligned}
\tag{6.1}
$$

When it comes to proving similar results to Section 3, one has to check the equivalent of Lemma 3.2; however there is no continuous injection between $H^2(\Sigma)$ and $C^0(\overline{\Sigma})$. Therefore we have to work in $W^{2,p}(\Sigma)$, with p larger than the critical exponent, and use a standard bootstrap argument. We leave this exercise to the reader.

As for the form of the bifurcation, we saw in Section 4 that, in two space dimensions, $l'_\varepsilon(0)$ was obviously zero. This needs not be true in three space dimensions, and we are going to prove the following result:

THEOREM 6.1. *Generically with respect to ω there holds $l'_\varepsilon(0) \neq 0$.*

A look at Theorem 1.18 of [15] reveals us that the expression of $l'_\varepsilon(0)$ is given by

$$
\begin{aligned}
-\big\langle(\Phi_1^*,\Psi_1^*),(\Phi_1^*,\Psi_1^*)\big\rangle|\sigma'_\varepsilon(l_\varepsilon^0)|\; l'_\varepsilon(0) &= \frac{1}{2}\big\langle(\Phi_1^*,\Psi_1^*),\partial^2_{T,W}\mathcal{F}_\varepsilon(l_\varepsilon^0,T_\varepsilon,W_\varepsilon).(\Phi_1,\Psi_1)^{(2)}\big\rangle \\
&= I_\omega\, J_1
\end{aligned}
\tag{6.2}
$$

where

$$
\begin{aligned}
I_\omega &= \int_\omega \zeta_1^3(y)dy \\
J_1 &= \int_{-\infty}^{+\infty}\phi_1^*\Big(\big(g''(T_\varepsilon)+f''(T_\varepsilon)W_\varepsilon\big){\phi_1}^2+2f'(T_\varepsilon)\phi_1\psi_1\Big)\;dx
\end{aligned}
$$

We saw in Section 3 that $I_{[0,1]} = \displaystyle\int_0^1 \cos^3(\pi y)\;dy = 0$; we will see in this section that it is no more true. As for the integral J_1, which is *a priori* of order ε^{-2}, it will have to be computed to the order 0. This task is not more agreeable than the computation of $l''_\varepsilon(0)$, but after what we had to go through in Section 4, we can afford a last effort. This section is organized as follows:

- Computation of J_1,
- The integral I_ω is generically nonzero,
- Conclusion.

Computation of J_1

We use the fact that we already have calculated the speed of the curved wave; this will save us a few computations. Let us notice that we may write $J_1 = J_{11} + J_{12}$, with

$$
\begin{aligned}
J_{11} &= \int_{-\infty}^{+\infty} \phi_0^* \Big((g''(T_\varepsilon) + f''(T_\varepsilon)W_\varepsilon){T'_\varepsilon}^2 + 2f'(T_\varepsilon)W'_\varepsilon T'_\varepsilon \Big)\, dx \\
&= k_1^2 + \rho_2(k_1)^2 + O(\varepsilon \mathrm{Log}^4 \varepsilon)
\end{aligned}
\tag{6.3}
$$

the last equality in (6.3) resulting from Theorem 4.1. We have, just as in Section 4:

$$
\begin{aligned}
J_{12} &= \int_{-\infty}^{+\infty} \phi_1^* \Big((g''(T_\varepsilon) + f''(T_\varepsilon)W_\varepsilon){T'_\varepsilon}^2 + 2f'(T_\varepsilon)W'_\varepsilon T'_\varepsilon \Big)\, dx + O(\varepsilon) \\
&= -k_1^2 \int_{-\infty}^{+\infty} \left(\phi_1^* T''_\varepsilon + \psi_1^* W''_\varepsilon \right)\, dx + O(\varepsilon \mathrm{Log}^4 \varepsilon)
\end{aligned}
$$

This time, the contributions of the different terms for $x \leq x_3^\varepsilon$ and $x > x_3^\varepsilon$ are of the same order; therefore we have, taking into account Lemma 2.10 and Proposition 3.5:

$$
\begin{aligned}
\int_{\xi_3^\varepsilon}^{+\infty} \phi_1^* p''_\varepsilon \, d\xi &= -\frac{c_\varepsilon^2}{2} + O(\mathrm{Log}^4 \varepsilon \varepsilon) \\
\int_{-\infty}^{x_3^\varepsilon} \phi_1^* T''_\varepsilon \, dx &= \frac{c_\varepsilon^2}{{r_1^\varepsilon}^2(k_1)} \Delta_\varepsilon(k_1) + O(\mathrm{Log}^4 \varepsilon \varepsilon) \\
\int_{\xi_3^\varepsilon}^{+\infty} \psi_1^* q''_\varepsilon \, d\xi &= -\frac{l_\varepsilon^0 c_\varepsilon}{\Delta_\varepsilon(k_1)} + O(\varepsilon) \\
\int_{-\infty}^{x_3^\varepsilon} \psi_1^* W''_\varepsilon \, dx &= \frac{c_\varepsilon \Delta_\varepsilon(k_1)}{r_1^\varepsilon(k_1)} \Big(1 - \frac{c_\varepsilon}{r_1^\varepsilon(k_1)}\Big) + O(\mathrm{Log}^4 \varepsilon \varepsilon)
\end{aligned}
$$

Gathering everything we have

$$
J_1 = k_1^2 + r_2^2(k_1) + k_1^2\Big(1 + \frac{\sqrt{2}\delta(k_1)}{r_1^2(k_1)}(r_1(k_1) - 1)\Big) + O(\varepsilon \mathrm{Log}^4 \varepsilon)
\tag{6.4}
$$

and a quick glance at (6.4) tells us that J_1 is always positive.

Computation of I_ω

The result that we want to prove is possibly known, but we have not been able to find a proper reference. It is in the spirit of the studies of Micheletti [26] about the generic simplicity of the eigenvalues of a symmetric elliptic differential operator.

Let $\bar{B}_2$ be the ball of the plane with centre 0 and radius 2, and let $\mathcal{E}$ be the set of C^3 diffeomorphisms from $\bar{B}_2$ to $\mathbb{R}^2$,with the same orientation as the identity, endowed with the usual topology. We say that a property is generically true for open subsets of $\mathbb{R}^2$, whose closure is diffeomorphic to the unit ball if this property is true for a set $\{\omega = f(B_2), f \in \mathcal{E}'\}$ where $\mathcal{E}'$ is open and dense in $\mathcal{E}$.

PROPOSITION 6.2. *Generically with respect to ω, $I_\omega \neq 0$.*

In order to prove this we define

$$\mathcal{E}^* = \{f \in \mathcal{E}, \lambda(f(B_2)) \quad \text{is simple} \quad \}$$

an we consider the function I:

$$f \in \mathcal{E}^* \to I(f) = I_\omega \in I\!R \quad \text{where} \quad \omega = f(B_2).$$

We next use the following proposition whose proof is similar to [26] (In [26], only Dirichlet conditions were studied):

THEOREM 6.3. *$\mathcal{E}^*$ is open and dense in $\mathcal{E}$ and the function I is uniformly continuous on $\mathcal{E}^*$.*

Therefore I can be continuously extended on $\mathcal{E}$. We put

$$\mathcal{E}_0 = \{f \in \mathcal{E}, I(f) = 0\}.$$

$\mathcal{E}_0$ is a closed subset of $\mathcal{E}$, and we have to prove that $\mathcal{E}\backslash\mathcal{E}_0$ is dense.

LEMMA 6.4. *We have $\mathcal{E}\backslash\mathcal{E}_0 \neq \emptyset$.*

PROOF. It is based on the explicit computation of I_{ω_0} for small perturbations of the set $\omega_0 =]0, a[\times]0, b[$, $0 < b < a$. Let us consider $\phi \in C^1(\omega_0, I\!R^2)$ and let us put $\omega_\varepsilon = (Id + \varepsilon\phi)(\omega_0)$ and λ_ε the first eigenvalue > 0 of the Neuman problem on ω_ε. Since $\lambda_0 = \dfrac{\pi}{a}$ is simple, λ_ε is simple for $\varepsilon > 0$ small enough and we denote by v_ε the associated eigenvector which verifies the following variational problem

$$\forall\psi \in \omega_\varepsilon, \quad \int_{\omega_\varepsilon} \nabla v_\varepsilon(y)\nabla\psi(y)dy - \lambda_\varepsilon \int_{\omega_\varepsilon} v_\varepsilon(y)\psi(y)dy = 0. \tag{6.5}$$

We put $u_\varepsilon(x) = v_\varepsilon((I+\varepsilon\phi)(x))$ and we consider the asymptotic expansions of u_ε and λ_ε :

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x) + O(\varepsilon^2),$$
$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + O(\varepsilon^2).$$

Substitution of the previous expansions in (6.5) leads to the variational problems:

$$\forall\psi \in \omega_0, \quad \int_{\omega_0} \nabla u_o(x)\nabla\psi(x)dx - \lambda_0 \int_{\omega_0} u_o(y)\psi(x)dx = 0,$$

$$\forall\psi \in \omega_0, \quad \int_{\omega_0} \nabla u_1(x)\nabla\psi(x)dx - \lambda_0 \int_{\omega_0} u_1(y)\psi(x)dx - \lambda_1 \int_{\omega_0} u_0(y)\psi(x)dx$$
$$- a(u_0, \psi) + \int_{\omega_0} \Big(\nabla u_1(x)\nabla\psi(x) - \lambda_0 u_0(x)\psi(x)\Big)\operatorname{div}\phi(x)dx = 0$$

where $a(u,v) = \displaystyle\int \sum_{1\le i,j\le 2} (\frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i})\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}$. Since $u_0(x_1,x_2) = \cos(\frac{\pi}{a}x_1)$, u_1 is a solution of an ordinary differential equation. Therefore one can compute an asymptotic expansion of I_{ω_ε}:

$$\begin{aligned}\int_{\omega_\varepsilon} v_\varepsilon^3 &= \int_{\omega_0} u_0^3 + \varepsilon\Big(3\int_{\omega_0} u_0^2 u_1 + \int_{\omega_0} u_0^3 \mathrm{div}\phi\Big) + O(\varepsilon^2)\\ &= 4\varepsilon\lambda_0 \int_{\omega_0} (u_0^3 - u_0)\frac{\partial \phi_2}{\partial x_2} + O(\varepsilon^2).\end{aligned}$$

Thus for ϕ such that the previous integral is not equal to 0, $I_{\omega_\varepsilon} \neq 0$ for $\varepsilon > 0$ small enough. •

PROOF OF PROPOSITION 6.2. Assume that $\mathcal{E}_0$ has nonempty interior. The set $\mathcal{E}$ is path-connected; see for example Theorem 3.1, p. 185 of [22]. Therefore there exists $g_0 \in \partial(\mathrm{Int}\mathcal{E}_0)$. Let us set $\omega = g_0(B_2)$ and let us make C^3 perturbations of ω of the form $\mathrm{Id} + \varepsilon f$. From Theorem *XII* $-$ 11 of [29], there exists $\varepsilon_0(\omega) > 0$ such that, for $|\varepsilon| \le \varepsilon_0$, for all f such that $\|f\|_{C^3} \le 1$, the function $\varepsilon \mapsto I_{(\mathrm{Id}+\varepsilon f)(\omega)}$ may be expanded into a power series in ε.

Let us take $g_1 \in \mathcal{E}\backslash\mathcal{E}_0$; when $\|g_1 - \mathrm{Id}\|_{C^3}$ is small enough, the real number $\varepsilon_0(g_1(\omega))$ is greater than $\frac{\varepsilon_0(\omega)}{2}$. Let us take $g_2 \in \mathrm{Int}\mathcal{E}_0$, close enough to g_1. We have $g_2 = \mathrm{Id}_{g(\omega)} + \varepsilon f$, with ε small and $\|f\|_{C^3} \le 1$. But we also obtain that the power series of $I_{g_2(\omega)}$ is zero, a contradiction. •

We have therefore proved that, for "most" open subsets ω of the plane diffeomorphic to the unit ball, the integral I_ω is nonzero.

Conclusion

We have obtained the following theorem:

THEOREM 6.5. *For $\varepsilon > 0$ small enough, and generically with respect to ω diffeomorphic to the unit ball, there exist two positive reals l_ε^0, $m_0^\varepsilon > 0$ and a C^2 curve $(T(l), W(l))$, defined on $[l_\varepsilon^0 - m_0^\varepsilon, l_\varepsilon^0 + m_0^\varepsilon]$, and satisfying (1.12). Further, the functions $T(l), W(l)$ admit the following expansion, near $l = l_\varepsilon^0$:*

$$\begin{aligned}T(l,x,y) &= T_\varepsilon(x) + (l - l_\varepsilon^0)\phi_1(x)\zeta_1(y) + o(l - l_\varepsilon^0)\\ W(l,x,y) &= W_\varepsilon(x) + (l - l_\varepsilon^0)\psi_1(x)\zeta_1(y) + o(l - l_\varepsilon^0)\end{aligned}$$

For $l > l_\varepsilon^0$ and close to l_ε^0, the curved flame is nonlinearly stable.

Let us examine the consequences of Theorem 6.5. In particular, we will explain why, in cases, a stable three-dimensional flame occurs above the critical Lewis number.

Let us come back to the rectangle $\omega_0 =]-a,a[\times]-b,b[$, $a < b$; We claim that, for ω_1 close to ω_0, $\lambda_1(\omega_1)$ is simple. This does not come from [26] but the reader can convince himself of this fact by examining the consequences of the existence of a sequence $\omega_1^\varepsilon \to \omega_1$, for which $\lambda_1(\omega_1)$ has two eigenfunctions ζ_{11}^ε, ζ_{12}^ε : $||\zeta_{11}^\varepsilon|| = ||\zeta_{12}^\varepsilon|| = 1$ and $(\zeta_{11}^\varepsilon, \zeta_{12}^\varepsilon) = 0$.

Let ω_1 be smooth, close to ω_0 and symmetric about the y_1 axis such that λ_1 is simple. We automatically have I_ω This defines a branch $s \to (l^\varepsilon(s,\omega_1), T(s,\omega_1), Y(s,\omega_1))$ which exists on an interval of the form $[-s_0^\varepsilon(\omega_1), s_0^\varepsilon(\omega_1)]$. Now we claim that, for fixed $\varepsilon > 0$ small enough, $s_0^\varepsilon(\omega)$, for ω close to ω_1 is bounded away from 0; let $\tilde{s}_0^\varepsilon$ be a common lower bound for ω close to ω_1. Indeed this holds true because Theorem 1.16 from [15] relies upon the Implicit Function Theorem, wich turn relies on the norm of $(\partial_{T,W}\mathcal{F}_\varepsilon + \big\langle(\Phi_1^*, \Psi_1^*), \partial^2_{l,(T,W)}\mathcal{F}_\varepsilon(l_\varepsilon^0, T_\varepsilon, W_\varepsilon).\big\rangle)^{-1}$, and C^3 norms of $\mathcal{F}_\varepsilon$. These quantities are controlled when the C^2 norm of ω is controlled.

Now we know that $l^\varepsilon(s,\omega) \to l^\varepsilon(s,\omega_1)$ when $\omega \to \omega_1$ for $s \in [-\tilde{s}_0^\varepsilon, \tilde{s}_0^\varepsilon]$. By Theorem 6.1 this implies the existence of a turning point for the branch $(s, T(s,\omega), Y(s,\omega))$, at a value l_1^ε. The corresponding bifurcation diagrams are sketched on figures 6.1 and 6.2 below.

Fig. 6.1

Fig. 6.2

In particular, there appears, as announced, a stable multidimensional configuration for Λ above the critical Lewis number.

It would remain to examine the following cases:
• open subsets ω which are not simply connected;
• open subsets ω where I_ω is actually zero.

As for the first case, we believe that Proposition 6.2 remains true, even if the domain ω is not simply connected.

The second case might be more difficult: one indeed has to compute $l''_\varepsilon(0)$. In the rectangle case, because of the Moivre linearisation, one may perform the exact same computation as the one done in Section 4 and we still have $l''_\varepsilon(0) > 0$. Unfortunately, there are other cases where $I_\omega = 0$, namely the domains which have a symmetry axis. One has to deal with expressions of the form $\displaystyle\int_\omega \zeta_1^4 \; dy$ and $\displaystyle\int_\omega \zeta_1^2 \zeta_2 \; dy$, and estimating their respective sizes may not be an easy task. We postpone this task for a future paper.

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