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~~The strategy~~

IV). Proof of the de Giorgi theorem.

The strategy of the proof is an "improvement of flatness" that was known to de Giorgi and possibly before. If the graph of u is in a very flat cylinder, it will be in an even flatter cylinder inside $B_{1/2}$ — possibly by tilting the coordinates. This argument will be explained in due course, but we need some additional elements on solutions of linear equations.

1°). Viscosity harmonic functions are C^2 harmonic.

It is obvious that, if $\Delta u = \Delta v = 0$, then $\Delta(u+v) = 0$ [illegible] in the classical sense. It seems logical that this property will hold true at the viscosity solutions level, but this is far from obvious! Here is the result.

<u>Th.</u> The sum of two harmonic functions in the viscosity sense is harmonic in the viscosity sense.

The proof relies on a variant of the Alexandrov-Bakelman-Pucci argument (here it is again!).

<u>Th.</u> (~~Jensen~~) Let φ be semiconcave in B_1, and have a strict minimum at 0. Then, for all $a > 0$ let E_a be the set of contact points with paraboloids $P_{c,a,y}$. Then $|E_a \cap B_a| > 0$.

The proof will be postponed after the proof of the 1^{st} theorem.

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$$u_\varepsilon(x) = \inf_y \left(u(x) + \frac{1}{\varepsilon} |y - x|^2 \right),$$

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Both u_ε and v_ε are $\frac{1}{\varepsilon}$ semi-concave, therefore they satisfy (because they are a.e. twice differentiable)

$$-\Delta u^\varepsilon \geq 0, \quad -\Delta v^\varepsilon \geq 0 \text{ a.e}$$

Let now φ be a test-function such that $u+v-\varphi$ has a minimum at x_0.

We may assume $x_0 = 0$ and the min to be strict. Hence we may apply the theorem on semiconcave functions: for every n, we may touch the graph of $u+v-\varphi$ by paraboloids of aperture $\frac{1}{n}$ inside $B_{\frac{2}{n}}$, and the contact set has nonzero measure. Therefore we may choose x_n such that

(i). $-\Delta u(x_n) \geq 0$, $-\Delta v(x_n) \geq 0$.

(ii). $|x_n| \leq \frac{2}{n}$.

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~~Because~~ Hence we have:

$$-\Delta\varphi(x_n) \geq -\frac{N}{n} - \Delta(u+v)(x_n)$$
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Thus $-\Delta\varphi(0) \geq 0$. Letting $\varepsilon \to 0$ yields the viscosity relation for $u+v$.

Proof of the Alexandrov - Bakelman - Pucci

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Moreover the contact point is inside $B_a(y)$, thus inside B_{2a}.

Because ψ fills B_a, we have

$$|B_a| \leq \int_{\text{contact set}} \det |D\psi| \leq \frac{K^N}{a^N} |\text{contact set}|.$$

If now φ is only semi-concave, consider $\varphi * \rho_\varepsilon$ where ρ_ε is the classical approximation of identity. Because x_0 is a strict minimum, the sequen. of minima of φ, x_ε, converge to x_0. Apply the preceding considerations to $\varphi * \rho_\varepsilon$: the measure of the contact set is $O(\alpha^N)$, independent of ε. $\boxtimes$

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<u>Proof.</u> For e on the unit sphere: apply the "Harnack $\Rightarrow$ Hölder" on $\frac{u(x+he) - u(x)}{|h|^\beta}$ where β is the initial Hölder exponent. Then $\alpha = (n_\beta \beta) - 1$, n_β is the 1^{st} n such that $n\beta > 1$. $\boxtimes$

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It is equivalent to prove the

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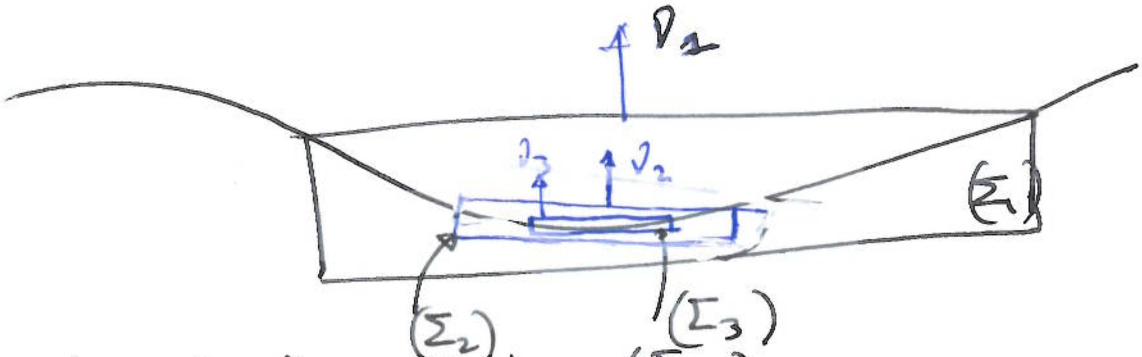
Then there is a new system of coordinates $(x', x_{N+1}) \in \mathbb{R}^{N+1}$ such that:

$$\partial E \cap B_{1/2} = \{ (x', v(x')),\ -(1-\nu)\delta \le v(x) \le (1-\nu)\delta \}$$

The reason why this implies $C^{1,\alpha}$ regularity is seen on a simple picture. Iterate the theorem:



$$\text{Angle}(\nu_n, \nu_{n+1}) = \text{Flatness}(\Sigma_n)$$
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In the end, $(\nu_n)_n$ converges geometrically to some $\nu_\infty \in \mathbb{R}^{N+1}$ the graph of u, rescaled to the length of Σ_n, converges to a plane in B_1.

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$$v_\varepsilon = \frac{u_\varepsilon}{\varepsilon}.$$

By the geometric Harnack inequality, v is

Hölder uniformly in ϵ. Thus, v_ε converges, up to a subsequence, to some v_∞.

We have $-\operatorname{div} \dfrac{Dv_\varepsilon}{\sqrt{1+\epsilon^2|Dv_\varepsilon|^2}} = 0$. By the stability property of viscosity solutions, ~~we~~ we have $-\Delta v_\infty = 0$. But then $v_\infty \in C^{1,\alpha}(B_{1/2})$. Thus its graph ~~is~~ can be put in a very flat cylinder in a neighbourhood of 0. By the convergence of v_ε to v, so can be the graph of v_ε. This is a contradiction. ∎

We have now finished the discussion about the regularity of minimal surfaces. In the next paragraphs we give some applications.

3°). Rigidity.

Th. Let $(E_n)_n$ be a sequence of minimal sets such that $\mathbb{1}_{E_n} \longrightarrow \mathbb{1}_{E_\infty}$ in $L^1(B_1)$. Assume E_∞ is a $C^{1,\alpha}$ set. Then there is $n_0 \in \mathbb{N}$ such that E_n is $C^{1,\alpha}$ for all $n \geq n_0$.

Proof. If E_0 is $C^{1,\alpha}$, ~~it is includ~~ we may rescale the picture such that, around O:

$$\left\{x_N \leq -\frac{\delta}{2}\right\} \subset E_\infty \subset \left\{x_N \leq \frac{\delta}{2}\right\}.$$

We claim the existence of $n_0 \in \mathbb{N}$ such that these inclusions also hold for E_n, with $\frac{\delta}{2}$ possibly replaced by δ. Indeed, by assumption:

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$$d(x_n, E_\infty) \geq \frac{3\delta}{2}.$$

By the positive density theorem:

$$|E_n \cap B_{\frac{\delta}{10}}(x_n)| \geq C\delta^N.$$

This contradicts the convergence in measure. $\boxtimes$

4º). Phase transitions, Allen-Cahn, de Giorgi conjecture.

In 1979, de Giorgi asked the following question.

Let $u(x)$ be a solution of

$$\begin{cases} -\Delta u = u - u^3 \quad (\mathbb{R}^N). \\ |u| \le 1. \\ \partial_{x_N} u \ge 0 \end{cases}$$

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It is important that the convergence is assumed to be pointwise in x'. If we strengthen it into uniform convergence, then the result is true in all space dimensions.

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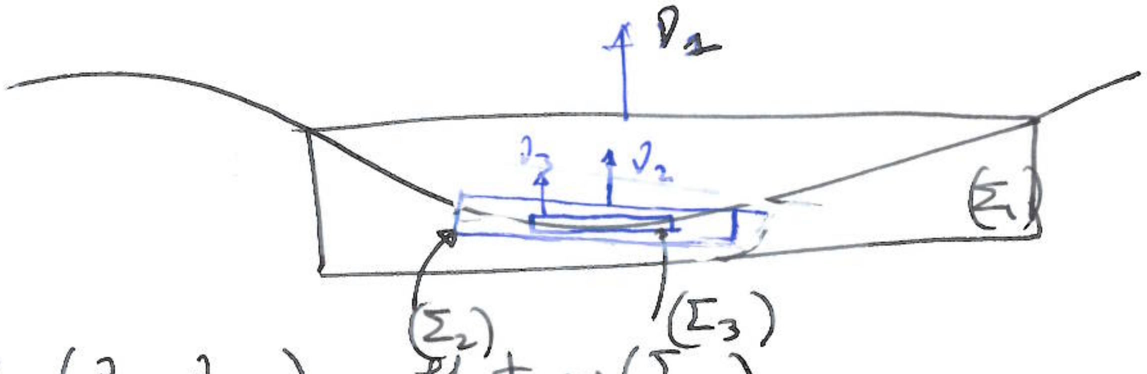

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By the positive density theorem:

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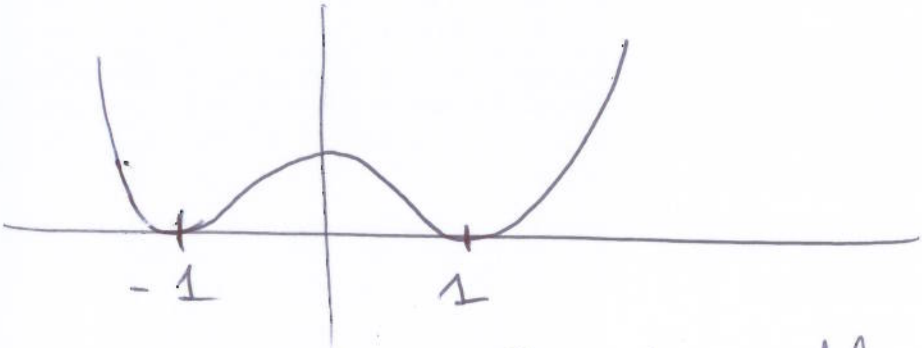
$$E_B[u] = \int_B \left(\frac{1}{2}|\nabla u|^2 - F(u)\right)dx,$$

more precisely: if B is a bounded open subset of $\mathbb{R}^N$ we wish

$$E_B[u] \le E_B[v] \quad \text{if } v = u \text{ on } \partial B.$$

F is a double-well potential



with equal depths at ± 1. (Unequal depth is a totally different problem), supposed to be analytic. And the Euler-Lagrange equation is $-\Delta u = F'(u)$.

In the de Giorgi example, $F(u) = -\frac{u^2}{2} + \frac{u^4}{4} + \frac{1}{4} = \frac{1}{4}(1-u)^4$.

We are looking at things from far away, i.e. we change x into $\frac{x}{\varepsilon}$, $\varepsilon \ll 1$.

And the energy becomes

$$E_B^\varepsilon[u] = \varepsilon^{N+1}\int_B \left(\varepsilon|\nabla u|^2 - \frac{1}{\varepsilon}F(u)\right)dx.$$

What happens as $\varepsilon \to 0$? Well, we have

$$\varepsilon^{N-1} E_B^{\varepsilon}[u] \geq \int_B \frac{1}{2} F(u) |\nabla u| \, dx .$$

Let u^{ε} be a minimiser, the Euler-Lagrange equation is $-\varepsilon \Delta u = \frac{1}{\varepsilon} F'(u)$. Heuristically: $u^{\varepsilon} \longrightarrow \pm 1$, and the region $u \sim 1$ is separated by that $u \sim -1$ by an interface Σ. Assume that it is true, and that the limiting function is indeed

$$u^{\infty}(x) = \mathbb{1}_E(x) - \mathbb{1}_{\mathbb{R}^N \setminus E}(x) .$$

Then: $\displaystyle\int_B \frac{1}{2} F(u) |\nabla u| = \int_B F(u_{\pm}) \, |D \mathbb{1}_E| \, dx .$

Th. (Modica - Mortola) [MM] Minimisers of E_B^{ε} converge to minimisers of the limiting functional.

In other words: E is a minimal set. Recall that, because we made a blow-up, that E is defined in $\mathbb{R}^N$. By the final theorem on minimal surfaces, E is $C^{1,\alpha}$ if $N \leq 7$.

If therefore we make a reverse blow-up of E, ~~i.e~~ at O (assume that $O \in \partial E$), $x \longmapsto \frac{x}{\varepsilon}$, then, in the limit, ∂E becomes a hyperplane.

$\underline{N=2}$: Ghoussoub - Gui.

$\underline{N=3}$: Ambrosio, Cabré.

$\underline{N=4,5}$: Ghoussoub - Gui (special cases).

$\underline{N \leq 7}$: Savin. (2003) ; "simple-looking" version

It is to be noted that neither Ghoussoub-Gui, nor Ambrosio-Cabré, used the full background of the pb. Savin was able to make this link explicit, and thus to get the full result up to $N \leq 7$.

For $N=9$, a truly multi-D solution of the de Giorgi problem was constructed in 2008 by del Pino, Kowalczyk, Wei.

A more general version of the conjecture is still open, i.e. replace $\partial_{N} u \geq 0$, $|u| \leq 1$ by "local minimizer".

Here are the known results -

Th Let u be a local minimiser of E in space dimension $N \leq 7$. Then the level sets of u are hyperplanes.

Th. Let u be a solution of

$$\begin{cases} -\Delta u = F'(u) \quad \text{in } \mathbb{R}^N, \ N \leq 8. \\ \partial_N u \geq 0, \\ \lim\limits_{x_N \to \pm\infty} u(x', x_N) = \pm 1. \end{cases} \qquad (*)$$

pointwise in $x' \in \mathbb{R}^{N-1}$. Then the level sets of u are hyperplanes -

Explanation: heuristically, the level sets of

$$u_\varepsilon(x) = u(\varepsilon x)$$

will converge to minimal surfaces. The best general dimension possible is 7: in $N=8$ dimensions, there is a nontrivial minimal cone.

In dimensions $N \leq 8$, the additional condition $\partial_N u \geq 0$ implies that the level sets of u are locally Lipschitz graphs (in fact we have $\partial_{x_N} u > 0$ by the strong maximum principle).

However, in dimensions $N \geq 9$ there are minimal singular graphs (Bombieri, de Giorgi, Giusti [BGG])

Still open: what happens to a local minimiser of E in dimension $N=8$, satisfying $\partial_{x_N} u \geq 0$ and $-1 \leq u \leq 1$? Not clear that it will converge pointwise to 1.

Th. (del Pino, Kowalczyk, Wei ∃, 2008). There are nontrivial local minimisers for $N \geq 9$.

Once again the dimension 8 is a borderline case, but much has been done to understand this de Giorgi problem.

Here are some very quick explanations.

- The theorem in space dimensions $N \leq 8$ is implied by the theorem in 7 dimensions.

Lemma. A solution of $(*)$ is a local minimiser.

Remark. Here, regularity is not an issue.

Proof of the lemma. Let B be a ball, u the solution of $(*)$ that we are studying, and v solving

$$\begin{cases} -\Delta v = F'(v) & (B). \\ v = u & (\partial B). \end{cases}$$

$$|v| \leq 1, \partial_{x_N} v \geq 0.$$

Because of the pointwise cv of u to 1, there is $t_0 > 0$ large such that:

$\forall t \geq t_0, \quad u(x', x_N + t) \geq v(x)$.

Therefore there is t_* such that

$$(i). \quad u(x', x_N + t_*) \geq v(x).$$

$$(ii). \quad \exists (x'_*, x_{N*}) \in \overline{B} \,/\, v(x'_*, x_{N*}) = u(x'_*, x_{N*}$$

- Assume $t_* > 0$. Then the strong max ple implies ~~$\Rightarrow u(x_* \; u(t_* + x',$~~

  - $u(x', t_* + x_N) > u(x', x_N) = v$ on ∂B.

  Therefore $(x'_*, x_{N*}) \in B$. This is an interior contact point, this cannot be.
- Therefore $t_* = 0$, and $u \geq v$.

Reversing the inequalities we have $v = u$. ∎

This explains why one needs the limits ± 1.

The major result is the following.

<u>Th.</u> (Savin). If the $\frac{1}{2}$-level set of u inside B_ℓ is such that:

- $u(0) = \frac{1}{2}$.
- $\exists \delta_* > 0$ such that $\{u = \frac{1}{2}\} \cap B_\ell \supset \{x_N \leq -\delta \ell\}$
  $\subset \{x_N \leq \delta \ell\}$

then there is $\delta_0 > 0$ such that: there is $\nu > 0$, universal, such that

$$\left\{ u = \frac{1}{2} \right\} \cap B_{\frac{\ell}{2}} \supset \left\{ x_N \geq -(1-\nu)\delta \right\}.$$

$$\left\{ u = \frac{1}{2} \right\} \cap B_{\frac{\ell}{2}} \subset \left\{ x_N \leq +(1-\nu)\delta \right\}.$$

~~Fro~~

From there, one can make an improvement of flatness at any scale.

let us now look at $u(\varepsilon x)$ and the rescaled functional $\int \left(\varepsilon |Du|^2 - \frac{1}{\varepsilon} F(u) \right) dx$. Then:

- the $\frac{1}{2}$ level set of u inside, say, B_1, will converge to a minimal set. However, in space dimension ≤ 7, minimal sets are smooth.
- therefore there is $\rho < 1$ such that

$$\left\{ u_\varepsilon = \frac{1}{2} \right\} \cap B_\rho \subset \left\{ x_N \leq +\delta \right\} \cap B_\rho.$$

$$\supset \left\{ x_N \geq -\delta \right\} \cap B_\rho.$$

~~Back~~ with $\frac{\delta}{\rho} <$ the δ_0 of the improvement of flatness.

Back to the original variable:

$$\left\{ u = \frac{1}{2} \right\} \cap B_{\rho/\varepsilon} \subset \left\{ x_N \leq \delta \right\}$$
$$\supset \left\{ x_N \geq -\delta \right\}.$$

If ν is the improvement of flatness quantity, set $\frac{\varepsilon}{\rho} = (1-\nu)^n$, $n = \frac{\log \rho/\varepsilon}{|\log(1-\nu)|} \longrightarrow +\infty$.

Therefore
$$\left\{ u = \frac{1}{2} \right\} \cap B_1 \subset \left\{ x_N \leq \frac{\varepsilon}{\rho} \delta \right\}$$
$$\supset \left\{ x_N \geq -\frac{\varepsilon}{\rho} \delta \right\}.$$

ε arbitrary $\Rightarrow \left\{ u = \frac{1}{2} \right\} \cap B_1$ is a hyperplane

The proof of the Harnack inequality has 2 steps:

- an "ε - viscosity relation".
- an "ε - improvement" of the contact sets.

These are truly nontrivial points that will not be developped here.

V). Nonlocal models.

Let us end this lecture by one word

In [CafRS], we ~~realised~~ realised that one could change the BV norm into a Sobolev norm. In particular, ~~compu~~ numerical computations are easy because Sobolev norms have nice interpretations in terms of Fourier transforms. On the other hand they are not local...

Here is a very brief account of what we know.

let $0 < \alpha < \frac{1}{2}$, and $\|\cdot\|_{H^\alpha}$ be the norm

$$\|u\|_{H^\alpha} = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2\alpha}} \, dx\, dy .$$

Does not really make sense if u is not compactly suported, but never mind...

~~We remark that~~

let us consider the sets E having finite H^α norm:

$$\|\mathbb{1}_E\|_{H^\alpha} = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(\mathbb{1}_E(x) - \mathbb{1}_E(y))^2}{|x-y|^{N+2\alpha}} \, dx\, dy$$

$$= 2 \int_E \int_{\mathbb{R}^N \setminus E} \frac{dx\,dy}{|x-y|^{N+2\alpha}} .$$

df. E is α-minimal in B_1 if, for any F agreeing with E outside B_1, and such that $\mathbb{1}_{F\cap B_1} \in H^\alpha$, then: $\|\mathbb{1}_E\|_{H^\alpha} \leq \|\mathbb{1}_F\|_{H^\alpha}$.

Both integrals need not converge. However they will have the same nonconvergent part, and therefore we in the end get a well-defined minimisation problem.

Th. (Caffarelli, R., Savin). Flatness $\Rightarrow C^{1,\delta}$.

From then on: dimension reduction and so on.

- Question: minimal cones? In what dimensions?

- ε-approximations of the pb?

Ex: $\displaystyle\int\left(\int \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}\,dy\right) - F(u)\Big)\,dx$.

Analogue of the Hyperbolic tangent was just constructed by Cabré and Sire.

A lot remains to be done, and it is useful to understand these things, because they are related to models in physics.

That's all!

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