EXISTENCE AND STABILITY OF TRAVELLING WAVES SOLUTIONS IN A KINETIC MODEL OF TWO-PHASE FLOWS

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1 Introduction

The model that we study here is the simplest kinetic model for two-phase fluids and it reads as follows:

$$\left\{\begin{array}{l} \partial_t u + u\partial_x u - \nu\partial_{xx} u = -\mathcal{C}\dfrac{\rho_p}{\rho_g}(u-u_p) \\ \partial_t f + v\partial_x f + \partial_v(f\beta) = 0 \end{array}\right. \qquad \left\{\begin{array}{l} u(x,0) = u_0(x) \\ f(x,v,0) = f_0(x,v) \end{array}\right. \tag{1.1}$$

where

$$\left\{\begin{array}{l} \beta = -\mathcal{C}(r)(v-u) \\ \rho_p(x,t) = \dfrac{4}{3}\pi\rho_l r^3 \displaystyle\int_v f(x,v,t)dv \\ (\rho_p u_p)(x,t) = \dfrac{4}{3}\pi\rho_l r^3 \displaystyle\int_v f(x,v,t)v\,dv \end{array}\right. \tag{1.2}$$

Here, we noted ρ_g the density of the gas, ρ_l the mass density of the liquid, u the velocity of the gas and f the probability density function representing the spray. Hence, ρ_p and $\rho_p u_p$ represent respectively the mass density and the momentum of the spray. Finally, ν is the viscosity of the gas and $\mathcal{C}(r)$ is the drag coefficient, depending on the radius of a droplet. Indeed, in order to model two-phase fluid flows, one can essentially use two approaches: a fluid model or a kinetic model. In the fluid model, one assumes that all the particles have the same radius r and that the velocity distribution function has a known shape such as a Dirac distribution or a Gaussian function (see [8, 9]). Since we want to consider more general velocity distribution functions, we use here a kinetic model (see [7, 12]). Moreover, we study the one-dimensional problem.

Hence, we may consider a model in which the droplets are characterized by their position x, their velocity v and their radius r. The probability density function (pdf) describing the droplets is then a function $f = f(x, v, r, t)$, and the spray equation writes (see [12]):

$$\partial_t f + v\partial_x f + \partial_v(f\beta) + \partial_r(fR) = Q(f, f)$$

where $\beta = \beta(x, v, r, t)$ is the acceleration of a droplet of radius r located at (x, v) at time t. It is written in terms of the Stokes drag force:

$$\beta = \frac{9\mu_g}{2r^2\rho_l}(v - u(x,t)) = -\mathcal{C}(v - u(x,t)) \tag{1.3}$$

where μ_g is the kinematic viscosity of the gas and u is the velocity of the gas flow. Then, the term R is the regression velocity $R = \frac{dr}{dt}$ of a droplet. We will consider non-burning droplets, so that $R = 0$. Moreover, $Q(f, f)$ is the collision term, that is a nonlinear expression accounting for the coalescence or break-up of the droplets. Here, we shall consider regimes where break-up no longer occurs and sprays that are dilute enough so that we may omit the collision term $Q(f, f)$. For this, it suffices that the volume fraction of the spray be of the order of a few percents (see [7]). Moreover, since we want to consider the simplest kinetic modelling for two-phase flow, we shall consider non-evaporating sprays. Otherwise, due to the exchanges of mass between the two phases, we would have to consider the density $\rho_g(x, t)$ of the gas as an unknown and therefore deal also with the continuity equation for the gas. Finally, since we assume that the evaporating rate R equals zero, the spray equation for the unknown $f(x, v, r, t)$ reduces to the transport equation:

$$\partial_t f + v\partial_x f + \partial_v(f\beta) = 0 \tag{1.4}$$

and we may assume without loss of generality that the particles have the same size, therefore $f(x, v, r, t) \equiv f(x, v, t)$.

Now, we consider an incompressible gas flow having a uniform constant mass density ρ_g. Moreover we want to take to consider the simplest modelling for the momentum equation of the gas. Therefore, we model it by using the Burgers' equation:

$$\partial_t u + u\partial_x u - \nu\partial_{xx} u = M. \tag{1.5}$$

Finally, the term M accounts for the exchanges of impulse between the two phases.

Indeed, assuming that the spray is dilute implies that the volume fraction occupied by the dispersed phase is small compared to that of the gas, which allows to consider that ρ_g is not modified by the presence of the spray. However, the mass density ρ_l of the liquid is typically one thousand times the mass density ρ_g of the gas. Hence, though dilute, the dispersed phase exhibits a total impulse comparable to that of the gas so that we may not neglect the action of the droplets on the gas flow and we have to consider the exchange term M in the equation of the gas which writes:

$$M(x,t) = -\frac{1}{\rho_g}\int_r \mathcal{C}\frac{4}{3}\pi\rho_l r^3 \int_v f(x,v,r,t)(v - u(x,t))\,dvdr$$

To obtain this expression, one considers the impulse $(\rho_p u_p)(x,t)$ of the spray then integrates the kinetic equation (1.4) against $\frac{4}{3}\pi\rho_l r^3 v$, getting:

$$\partial_t(\rho_p u_p) + \partial_x\left(\int_v\int_r \frac{4}{3}\pi\rho_l r^3 f(x,v,r,t)v^2\right) = \int_v\int_r \frac{4}{3}\pi\rho_l r^3 f(x,v,r,t)\,\beta dvdr \tag{1.6}$$

Writing the conservation of the total impulse $(\rho_g u + \rho_p u_p)$ of the system means that M is indeed the opposite of the rhs of Eq. (1.6). Recalling the expression (1.3) of the acceleration β yields the expression of the exchange term M.

Finally, the model (1.4)-(1.5) that we obtain is the simplest kinetic modelling for non-evaporating dilute two-phase flow accounting only for the exchanges of momentum between the two-phases. The assumptions made in order to derive the equation for the spray are usual. The main restriction of the model is the description of the motion of the gas by the Burgers' equation.

In the following, we consider initial conditions for the system where the spray function has a compact support, which is a natural assumption. The study of the global existence of a solution for the system uses classical techniques. In particular, we take advantage of the fact that the transport equation for the spray is solved explicitly thanks to the method of characteristics as soon as the gas velocity field is known. We then prove the existence of singular travelling waves for the present system and study their stability. Here, we call travelling wave for the system a couple $[\Phi(x-ct), F(x-ct,v)]$ solution of the system (1.4)-(1.5) where c is the velocity of the travelling wave. Setting to simplify the notations $\xi = x - ct$, we shall prove that the gas component $\Phi(\xi)$ and the spray component $F(\xi, v)$ of the travelling wave write:

$$\left\{ \begin{array}{l} \Phi(\xi) \equiv c\left[1 - th\left(\dfrac{c\xi}{2\nu}\right)\right] \\ \\ F(\xi, v) = m_T\ \delta(\xi) \otimes \delta(v - c) \end{array} \right.$$

where m_T is the total mass of spray, Also, one recognizes here the usual Burgers' travelling wave for the gas component. The travelling waves for the system are therefore singular in the sense that the two components Φ and F are uncoupled in this regime because the exchange term cancells. Moreover, one has to consider here solutions for the spray equation in the space of bounded measures of $IR_x \times IR_v$.

The linear and nonlinear stability for the Burgers' equation has already been studied in many details. We recall here the general procedure for the study of the stability of this equation and refer the reader to the paper of Sattinger for further details (see [10]). It is well known that the linear stability as well as the study of the asymptotic behaviour of the linearized problem are two natural steps towards the nonlinear stability under small perturbations.

An important particularity of the study of the linearized problem for the system is the choice of the perturbations for the spray component. First introduce as usually the perturbation $\tilde{w}$ of the gas component Φ of the travelling wave for the system such that the solution u of the problem writes $u(x,t) = \Phi(x - ct) + \epsilon\tilde{w}(x - ct, t)$, and where ϵ is the scaling factor. Doing so, it is classical that the functional space for the perturbation must contain the space $Vect\{\Phi'\}$. Now, if one uses the same type of perturbation for the spray as above, that is considers the initial datas for the system:

$$\left\{ \begin{array}{l} u(x,0) = \Phi(x) + \epsilon w_0(x) \\ f(x,v,0) = F(x,v) + \epsilon g_0(x,v), \end{array} \right.$$

then, he looks for solutions of the problem in the form:

$$\left\{ \begin{array}{l} u(x,t) = \Phi(x - ct) + \epsilon w(x,t) \\ f(x,v,t) = F(x - ct, v) + \epsilon g(x,v,t). \end{array} \right.$$

This procedure has two drawbacks. First, it would oblige us to consider solutions of the linearized problem for the spray component belonging to the space $Vect\{\partial_\xi \delta(\xi) \otimes \delta(v)\}$, therefore introducing derivatives of measures. Second, it would mean that the mass of the spray varies with the scaling factor ϵ and we would rather prefer that the impulse of the spray alone varies with ϵ. For those reasons, we introduce the following perturbation $\tilde{g}$ and scaling for the spray unknown f:

$$f(x,v,t) = \frac{1}{\epsilon^2}\tilde{g}\left(\frac{x-ct}{\epsilon}, \frac{v-c}{\epsilon}, t\right).$$

In the limit where ϵ tends to 0, the function f indeed tends to the travelling wave component for the spray. Finally, doing so, we shall see that it is possible to write the nonlinear problem for the system in terms of perturbations in the form:

$$\partial_t[\tilde{w},\tilde{g}] + L[\tilde{w},\tilde{g}] = \epsilon R(\tilde{w},\tilde{g},\epsilon)$$

where L is the linearized operator for the problem and R accounts for the nonlinear terms and where $\tilde{w}$ and $\tilde{g}$ belong respectively to the natural spaces $H^1(I\!R_\xi)$ and $\mathcal{M}_0^b(I\!R_x \times I\!R_v)$, the space of bounded measures of $I\!R_x \times I\!R_v$ with compact support. The linear stability consists in the study of the equation above with $\epsilon = 0$, which is the start-point towards the study of the nonlinear stability. Note finally that, compared to the Burgers' problem, the system studied here has one more degree of freedom consisting in the spray initial data.

The study is organized as follows. In Section **2**, we study the global existence and uniqueness of a regular solution for the system. Section **3** is a short section in which we identify the travelling waves for the system. In Section **4** and Section **5**, we study and solve the linearized problem. Finally in Section **6**, following the classical work of Sattinger, we study of the nonlinear stability under small perturbations.

2 Global existence and uniqueness of smooth solutions

In the following, we denote by $\mathcal{C}^p(I\!R_x)$ (resp. $\mathcal{C}^p(I\!R_v)$) the space of p-times continuously differentiable functions of $I\!R_x$ (resp. $I\!R_v$). The subscript in $\mathcal{C}_0^p$ denotes the subspace of functions having compact support. Moreover we will note $u_t = u(\cdot,t)$ and $f_t = f(\cdot,\cdot,t)$. Our aim is to prove:

Theorem 2.1 *Let $u_0 \in \mathcal{C}^2(I\!R_x)$ and $f_0 \in \mathcal{C}_0^1(I\!R_x \times I\!R_v)$, $f_0 \geq 0$ be given. Then, the problem (1.1)-(1.2) has a unique global solution $[u,f]$ with $u \in \mathcal{C}([0,+\infty[;\, \mathcal{C}^2\ (I\!R_x))$ and $f \in \mathcal{C}([0,+\infty[; \mathcal{C}_0^1(I\!R_x \times I\!R_v))$.*

Since f_t is uniquely determined by f_0 and $u_0^t = (u_s)_{0\leq s\leq t}$, we have indeed $f = f(u_0^t)$ and the system (1.1) may be seen as an equation with only the unknown u. Doing so, we may rewrite the gas equation:

$$\partial_t u + \mathbf{A}u = g(u_0^t) \tag{2.1}$$

where $\mathbf{A}$ is the operator defined by $\mathbf{A}u = -\nu\partial_{xx}u$ and $g(u_0^t) = -u\partial_x u - \mathcal{C}\frac{\rho_p(u_0^t)}{\rho_g}(u - u_p(u_0^t))$. The function g, like f, depends on all the story of u. Denoting then by S_t the heat semigroup, we have:

$$u_t = S_t u_0 + \int_0^t S_{t-s} g_s ds \tag{2.2}$$

and the study of Eq. (2.1) will consist in justifying and solving Eq. (2.2). In the sequel we will denote by $\mathcal{A}_t$ the operator $g \longrightarrow \mathcal{A}_t g = \int_0^t S_{t-s} g_s ds$.

2.1 Preliminary results

The homogeneous heat equation writes:

$$\begin{cases} \partial_t u - \nu\partial_{xx}u = 0 \\ u(x,0) = u_0(x), \quad x \in I\!R_x. \end{cases} \tag{2.3}$$

There holds:

$$u(x,t) = S_t u_0 = G(.,t) \overset{x}{*} u_0 = \int_y G(x-y,t)u_0(y)dy \tag{2.4}$$

where

$$G(x,t) = \begin{cases} \delta(x) & , \quad t = 0 \\ \dfrac{1}{2\sqrt{\pi\nu t}} exp\left(-\dfrac{x^2}{4\nu t}\right) & , \quad t > 0. \end{cases} \tag{2.5}$$

We have the following classical estimates on S_t and $\mathcal{A}_t$:

Proposition 2.2 *Let $u_0 \in \mathcal{C}^2(I\!R_x)$ and $g \in L^\infty([0,T]; L^\infty(I\!R_x))$ be given. Then, for all t, the operator S_t is defined from $\mathcal{C}^1(I\!R_x)$ into $\mathcal{C}^1(I\!R_x)$ and the operator $\mathcal{A}_t$ from $L^\infty([0,T]; L^\infty(I\!R_x))$ into $\mathcal{C}^1(I\!R_x)$. Moreover, for any $(t,t_1) \in [0,T]^2$ we have the estimates:*

$$\begin{aligned} &(i) \quad \|S_t u_0\|_{\mathcal{C}^1(I\!R_x)} \leq \|u_0\|_{\mathcal{C}^1(I\!R_x)} \\ &(ii) \quad \|S_t u_0 - S_{t_1} u_0\|_{\mathcal{C}^1(I\!R_x)} = \epsilon(|t-t_1|) \\ &(iii) \quad \|\mathcal{A}_t g - \mathcal{A}_{t_1} g\|_{\mathcal{C}^1(I\!R_x)} = \epsilon(|t-t_1|)\ \|g\|_{L^\infty([0,T];L^\infty(I\!R_x))} \end{aligned} \tag{2.6}$$

where $\epsilon : t \in [0,T] \longrightarrow I\!R^+$ denotes a function that tends to 0 when t tends to 0.

Proof: We refer the reader to [5] and [6]. •

Now, we introduce the characteristics. Let $u \in \mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))$ be given. Let $(s,t) \in [0,T]^2$ be given. The characteristics $(X^u, V^u)(t;x,v,s)$ are the solutions of the system of ODEs:

$$\left\{ \begin{aligned} &\frac{dX^u}{dt} = V^u \\ &\frac{dV^u}{dt} = -\mathcal{C}(V^u - u(X^u,t)) \end{aligned} \right. \qquad \left\{ \begin{aligned} &X^u(s;x,v,s) = x \\ &V^u(s;x,v,s) = v \end{aligned} \right. \tag{2.7}$$

We have the classical results:

Lemma 2.3 *The characteristics are well-defined for $t \in [0,T]$, unique, and are $\mathcal{C}^1$ functions of (x,v). Moreover, for any $(t,s) \in [0,T]^2$ the flow function $(x,v) \to (X,V)(t;x,v,s)$ is a $\mathcal{C}^1$-diffeomorphism of $I\!R_x \times I\!R_v$.*

Lemma 2.4 (Solution of the kinetic equation)
Let $u \in \mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))$ be given. Let $f_0 \in \mathcal{C}^1_0(I\!R_x \times I\!R_v)$, $f_0 \geq 0$ be given. The solution f of the kinetic equation (1.4) writes:

$$f(x,v,t) = e^{\mathcal{C}t} f_0(X^u(0;x,v,t), V^u(0;x,v,t)) \tag{2.8}$$

where (X^u, V^u) are the characteristics defined in Eq. (2.7). Moreover, for all t, $f_t \in \mathcal{C}^1_0(I\!R_x \times I\!R_v)$.

Note, that the solution f writes in terms of the reverse characteristics, that is the characteristics solution of the Cauchy problem (2.7) with final conditions. Moreover, the characteristics (X^u, V^u) defined in Eqs. (2.7) have of course no explicit solution. However, it is straightforward that we have the implicit representation:

Proposition 2.5
Let $u \in \mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))$ and $(s,t) \in [0,T]^2$ be given. Let (X^u, V^u) be the characteristics defined in Eq. (2.7). They may be written implicitly:

$$\left\{ \begin{aligned} &X^u(t;x,v,s) = x + \int_s^t V^u(\tau;x,v,s)d\tau \\ &V^u(t;x,v,s) = e^{-\mathcal{C}(t-s)}v + c\int_s^t e^{-\mathcal{C}(t-\tau)}u(X^u(\tau;x,v,s),\tau)d\tau \end{aligned} \right. \tag{2.9}$$

This yields the estimates on the characteristics:

Lemma 2.6 (Estimates on the characteristics)
Let $u \in \mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))$ and $(s,t) \in [0,T]^2$ be given. Let (X^u, V^u) be the characteristics defined in Eqs. (2.7). We have the estimates:

$$\left\{ \begin{aligned} &|X^u(t;x,v,s)| \leq |x| + \left[(t-s) - \frac{1-e^{-\mathcal{C}(t-s)}}{c}\right] \; \|u\|_{\mathcal{C}([s,t];L^\infty(I\!R_x))} \\ &|V^u(t;x,v,s)| \leq e^{-\mathcal{C}(t-s)}|v| + \left[1 - e^{-\mathcal{C}(t-s)}\right] \; \|u\|_{\mathcal{C}([s,t];L^\infty(I\!R_x))} \end{aligned} \right. \tag{2.10}$$

and

$$\left\{\begin{array}{l} |X^{u_1}(t;x,v,s) - X^{u_2}(t;x,v,s)| \\ \qquad\qquad\qquad \leq C\left(\|u_1\|_{\mathcal{C}([s,t];\mathcal{C}^1(I\!R_x))}, T, c\right) \ \|u_1 - u_2\|_{\mathcal{C}([s,t];\mathcal{C}^1(I\!R_x))} \\ \\ |V^{u_1}(t;x,v,s) - V^{u_2}(t;x,v,s)| \\ \qquad\qquad\qquad \leq C\left(\|u_1\|_{\mathcal{C}([s,t];\mathcal{C}^1(I\!R_x))}, T, c\right) \ \|u_1 - u_2\|_{\mathcal{C}([s,t];\mathcal{C}^1(I\!R_x))} \end{array}\right. \tag{2.11}$$

Note that the factors $[(t-s) - \frac{1-e^{-\mathcal{C}(t-s)}}{c}]$ and $[1 - e^{-\mathcal{C}(t-s)}]$ are positive, increasing functions of $(t-s)$.

Proof: The proof of inequality (2.10) is straightforward thanks to Eq. (2.9). The second inequalities are a consequence of the Gronwall Lemma. •

As a consequence, we can estimate the measure of the support of f_t. Define first:

$$\left\{\begin{array}{l} X_M(t) = \sup_{\substack{(x,v) \in I\!R_x \times I\!R_v \\ f(x,v,t) > 0}} |x| \\ \\ V_M(t) = \sup_{\substack{(x,v) \in I\!R_x \times I\!R_v \\ f(x,v,t) > 0}} |v|, \end{array}\right.$$

then we have:

Corollary 2.7 *For all t, the support of f_t is bounded and we have the estimates:*

$$supp(f_t) \in [-X_M(t), X_M(t)] \times [-V_M(t), V_M(t)]$$

where:

$$\left\{\begin{array}{l} X_M(t) \leq X_M(0) + \left(t - \dfrac{1-e^{-\mathcal{C}t}}{c}\right) \ \|u\|_{\mathcal{C}([0,t];L^\infty(I\!R_x))} \\ \\ V_M(t) \leq e^{-\mathcal{C}t} V_M(0) + (1 - e^{-\mathcal{C}t}) \ \|u\|_{\mathcal{C}([0,t];L^\infty(I\!R_x))} \end{array}\right. \tag{2.12}$$

Proof: This is a consequence of the expression (2.8) of f and the estimate (2.10). •

2.2 Existence and Uniqueness

We first prove the local existence and uniqueness for problem (1.1)-(1.2). This problem is very similar to a Cauchy-Lipschitz problem:

Theorem 2.8 (Local existence and uniqueness)
There exists $t^{**} > 0$ in $[0, T]$ such that there exists a unique solution u in $\mathcal{C}([0,t^{**}];\mathcal{C}^2(I\!R_x))$ and a unique f in $\mathcal{C}([0,t^{**}];C_0^1(I\!R_x \times I\!R_v))$ solution of Problem (1.1)-(1.2).

Let be given the interval of time $[0, T]$. Moreover, let $u \in \mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))$ be given; we set

$$g^u(x,t) = \left(-u\partial_x u - \mathcal{C}\frac{\rho_p(u)}{\rho_g}(u - u_p(u))\right)(x,t), \tag{2.13}$$

and we introduce the operator $\mathcal{T}$:

$$\mathcal{T} : u \longmapsto (\mathcal{T}u)_t = S_t u_0 + \int_0^t S_{t-s} g_s^u ds = S_t u_0 + \mathcal{A}_t g^u, \ \ for\ t \in [0,T],$$

which satisfies:

Proposition 2.9 *The operator $\mathcal{T}$ is a well-defined (locally) Lipschitz continuous operator between $\mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))$ and $\mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))$.*

Proof: Thanks to Lemma **2.2**, $\mathcal{T}$ is well-defined from $\mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))$ to $\mathcal{C}$ $([0,T];\ \mathcal{C}^1(I\!R_x))$. Indeed, if u is in $\mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))$, then one verifies easily that g^u is in $L^\infty([0,T];L^\infty$
$(I\!R_x))$. Now, since $\mathcal{T}u_1 - \mathcal{T}u_2 = \mathcal{A}_t(g^{u_1} - g^{u_2})$, and thanks to Eq. (2.6(iii)), it suffices to prove that $u \mapsto g^u$ is Lipschitz continuous from $\mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))$ into $L^\infty([0,T];L^\infty(I\!R_x))$. Finally, considering the expression (2.13) of g^u, it suffices to prove that the following functions $u \mapsto u$, $u \mapsto u\partial_x u$, $u \mapsto \rho_p(u)$, and $u \mapsto (\rho_p u_p)(u)$ are Lipschitz from $\mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))$ into $L^\infty([0,T]\ ;L^\infty(I\!R_x))$.

The first result follows from the continuity of the identity. Writing now $u_1\partial_x u_1 - u_2\partial_x u_2 = (u_1 - u_2)\partial_x u_1 + u_2(\partial_x u_1 - \partial_x u_2)$, the second result is straightforward. Finally, study for example the continuity of the application $u \mapsto \rho_p(u)$. We have:

$$[(\rho_p u_p)(u_1) - (\rho_p u_p)(u_2)]\,(t) = \int_v e^{\mathcal{C}t}\,[f_0(X^{u_1}(0;x,v,t), V^{u_1}(0;x,v,t))$$

$$-f_0(X^{u_2}(0;x,v,t), V^{u_2}(0;x,v,t))]\,vdv$$

where we used the expression (2.8) for the function f. We write then:

$$\begin{aligned}
&|(\rho_p u_p)(u_1) - (\rho_p u_p)(u_2)|\,(t) \\
&\leq e^{\mathcal{C}t} \int_{v \in supp(f_t)} \Big(\, \|\partial_x f_0\|_{L^\infty(I\!R_x)} \, |X^{u_1}(0;x,v,t) - X^{u_2}(0;x,v,t)| \\
&\qquad + \|\partial_v f_0\|_{L^\infty(I\!R_x)} \, |V^{u_1}(0;x,v,t) - V^{u_2}(0;x,v,t)| \, \Big) \; |v| \; dv.
\end{aligned}$$

Now, from the estimate (2.12) of the support of f_t, and from the estimate (2.11) on the characteristics, we have the estimate:

$$\begin{aligned}
&|(\rho_p u_p)(u_1) - (\rho_p u_p)(u_2)|\,(t) \\
&\leq e^{\mathcal{C}t} 2 V_M^2(t) \left(\|\partial_x f_0\|_{L^\infty(I\!R_x)} + \|\partial_v f_0\|_{L^\infty(I\!R_x)} \right) \\
&\quad \times \left(|X^{u_1}(0;x,v,t) - X^{u_2}(0;x,v,t)| + |V^{u_1}(0;x,v,t) - V^{u_2}(0;x,v,t)| \right) \\
&\leq C \left(\|u_1\|_{\mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))}, f_0, T, c \right) \; \|u_1 - u_2\|_{\mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))}.
\end{aligned}$$

This finishes the proof of Proposition **2.9**. •

Now, let $A > 0$ and $0 < t \leq T$ be given. We define the ball:

$$\mathbf{B}_{u_0}^{A,t} = \left\{ u \in \mathcal{C}([0,t];\mathcal{C}^1(I\!R_x)); \; u(.,0) = u_0 \textit{ and } \|u\|_{\mathcal{C}([0,t];\mathcal{C}^1(I\!R_x))} \leq A \right\}$$

Proposition 2.10 *Let $u_0 \in \mathcal{C}^1(I\!R_x)$ be given. Then, there exists $t^{**} > 0$ in $[0,T]$ such that there exists a unique $u \in \mathcal{C}([0,t^{**}];\mathcal{C}^1(I\!R_x))$ that satisfies:*

$$\forall t \in [0,t^{**}], \quad u_t = (\mathcal{T}u)_t = S_t u_0 + \int_0^t S_{t-s} g_s^u ds. \tag{2.14}$$

Proof: From Proposition **2.9**, $\mathcal{T}$ is Lipschitz continuous. Then, thanks to the contraction mapping theorem in Banach spaces, it suffices to find $A > 0$ and $t^{**} > 0$ in $[0,T]$ such that $\mathcal{T}$ be a contraction on:

$$\mathcal{T} : \mathbf{B}_{u_0}^{A,t^{**}} \longrightarrow \mathbf{B}_{u_0}^{A,t^{**}}.$$

Let $u_0 \in \mathcal{C}^1(I\!R_x)$, $A > 0$, $t^* > 0$ in $[0,T]$ and $u \in \mathbf{B}_{u_0}^{A,t^*}$ be given. Denote $v = \mathcal{T}u$. Then, we have indeed $v_{t=0} = u_0$ (that is $\lim_{t \longrightarrow 0} v_t = u_0 \in \mathcal{C}^1(I\!R_x)$). Let C denote a generic constant. From the expression (2.13) of g^u, we have immediately:

$$\begin{aligned}
&\|g^u\|_{L^\infty([0,t^*];L^\infty(I\!R_x))} \\
&\quad \leq C_1(f_0,T,c) \left(\|u\|^2_{\mathcal{C}([0,t^*];\mathcal{C}^1(I\!R_x))} + C_2(f_0,T,c) \right) \\
&\quad \leq C_1(f_0,T,c) \left(A^2 + C_2(f_0,T,c) \right)
\end{aligned}$$

so that, from (2.6(iii)), we may estimate:

$$\begin{aligned}&\|v\|_{\mathcal{C}([0,t^*];\mathcal{C}^1(I\!R_x))}\\ &\quad\leq \epsilon(t^*)\times\|u_0\|_{\mathcal{C}^1(I\!R_x)}\ +C_3\left(f_0,T,c\right)\left(A^2+C_1\left(f_0,T,c\right)\right).\end{aligned}$$

where $\epsilon(t)$ denotes a function that tend to 0 when t tends to 0. Now, let $A>\|u_0\|_{\mathcal{C}^1(I\!R_x)}$ be given. Then it is possible to chose $t^*>0$ such that

$$\epsilon(t^*)\times C_3\left(f_0,T,c\right)\Big(A^2+C_1\left(f_0,T,c\right)\Big)\leq A-\|u_0\|_{\mathcal{C}^1(I\!R_x)}.$$

Finally, thanks to (2.6(i)), for any $t^{**}\leq t^*$, we will have

$$\|v\|_{\mathcal{C}([0,t^{**}];\mathcal{C}^1(I\!R_x))}\leq A,$$

so that $\mathbf{B}_{u_0}^{A,t^{**}}$ is stable under $\mathcal{T}$.

Now, we assert that for t^{**} small enough, $\mathcal{T}$ is a contraction on $\mathbf{B}_{u_0}^{A,t^{**}}$. Using the Lipschitz continuity of $u\longrightarrow g^u$ and the estimate (2.6(iii)) for the linear operator $\mathcal{A}_t$, the existence of such a t^{**} is straightforward. Finally, there exists a unique fixed point, solution of (2.14). •

We finally prove the local existence and uniqueness:

Proof of Theorem 2.8

Let u be the function defined in Proposition **2.10**. Then u is indeed the unique solution of Problem (1.1)-(1.2) on $]0,t^{**}]$. Moreover, since $\mathcal{C}^1(I\!R_x)$ injects continuously in $\mathcal{C}^0(I\!R_x)$, this solution u is continuous and bounded in $\mathcal{C}(]0,t^{**}];\mathcal{C}^0(I\!R_x))$. The same holds for the exchange term $M(x,t)$. Then, since the parabolic equation for the gas has continuous coefficients, it is clear that u belongs to $\mathcal{C}(]0,t^{**}];\mathcal{C}^2(I\!R_x))$ (see [6]). Finally, since $u_0\in\mathcal{C}^2(I\!R_x)$, we have immediately from (2.6(ii)):

$$\lim_{t\longrightarrow 0}u_t=u_0,\quad in\ \mathcal{C}^2(I\!R_x)$$

so that this solution u belongs to $\mathcal{C}([0,t^{**}];\mathcal{C}^2(I\!R_x))$. The existence, uniqueness and regularity of $f\in\mathcal{C}([0,t^{**}];C_0^1(I\!R_x\times I\!R_v))$ follows. This finishes the proof of Theorem **2.8** •

Let us now deal with global existence. It will be provided by the following *a priori* estimate:

Proposition 2.11
Let $T>0$ be given. Let $u \in \mathcal{C}([0,T];\mathcal{C}^2(I\!R_x))$ be a solution of Problem (1.1)-(1.2). Then:

$$\|u\|_{\mathcal{C}([0,T];L^\infty(I\!R_x))} \leq \max\Big(V_M(0), \|u_0\|_{\mathcal{C}^0(I\!R_x)}\Big) \tag{2.15}$$

where $V_M(t)$ is defined in Corollary **2.7**.

Proof of Proposition 2.11:
We will need here the following Lemma:

Lemma 2.12 *Let be given $L(t) = -\partial_{xx} + a(t,x)\,\partial_x + b(t,x)$, where*

- *$a, b \in L^\infty([0,T]\times I\!R)$.*
- *$\exists\ \epsilon > 0$ such that $b(t,x) \geq \epsilon$ for all $(t,x) \in [0,T]\times I\!R$,*

  and $b - \epsilon$ has a compact support.

Let be given $u_0 \in \mathcal{C}^0(I\!R)$ such that $\sup\limits_{x\in I\!R} u_0 > 0$. Let be given $u(t,x)$ such that:

$$\begin{cases} \partial_t u + L(t)u \ \leq\ (b(t,x)-\epsilon) \displaystyle\sup_{\substack{0\leq s\leq t \\ y \in I\!R}} u(s,y) \\ \\ \partial_t u + L(t)u \text{ has a compact support in } I\!R \text{ for all } t. \\ \\ u(0,x) = u_0(x). \end{cases}$$

Then, for all (t,x), we have $u(t,x) \leq \sup\limits_{x\in I\!R} u_0(x)$.

Proof:
Assume the contrary, that is there exists $t_1 > 0$ such that:

$$\sup_{x\in I\!R} u(t_1,x) > \sup_{x\in I\!R} u_0(x).$$

Since $b > \epsilon$, then the function

$$\limsup_{x\to\pm\infty} u(t,x)$$

is a decreasing function of the variable t. Hence, there exists $t_0 \in]0,t_1]$ and $x_0 \in I\!R$ such that:

$$u(t_0,x_0) = \sup_{\substack{t\leq t_0 \\ x\in I\!R}} u(t,x).$$

At this point, we have $\partial_t u \geq 0$ and $-\partial_{xx} u \geq 0$ which yields

$$b(t_0, x_0)u(t_0, x_0) \leq (b(t_0, x_0) - \epsilon)\ \ u(t_0, x_0),$$

a contradiction. This finishes the proof of Lemma **2.12**. •

Come back now to the proof of Proposition **2.11**:. Define the function $M(t)$:

$$M(t) = \sup_{\substack{0 \leq s \leq t \\ x \in I\!R}} u(s, x),$$

and assume that there exists $t_0 > 0$ such that

$$M(t_0) > max\left(V_M(0), \sup_{x \in I\!R} u_0\right)$$

where we may assume without loss of generality that sup $u_0 > 0$. One observes here that $M(t)$ is an increasing function of t. Hence, from Corollary **2.7**, the function $V_M(t)$ lies in the intervall $[V_M(0), M(t)]$, and we have then for all t:

$$\begin{aligned} \int_v f(x, v, t)v\ dv \ &\leq \sup_{0 \leq s \leq t} \{v > 0 : f(x, v, t) > 0\} \int_v f(x, v, t)\ dv \\ &\leq \left(\int_v f\ dv\right)\ M(t) \equiv \rho_p\ M(t). \end{aligned}$$

Now, let be given $\epsilon > 0$ and set $u_\epsilon(x, t) = e^{\epsilon t}\ u(t, x)$. Introducing the functions $a(t, x) = u(t, x)$ and $b(t, x) = \frac{\rho_p}{\rho_g} + \epsilon$, we have:

$$\begin{aligned} (\partial_t + L(t))\, u_\epsilon \ &\leq (b(t, x) - \epsilon)\ \ e^{-\epsilon t} \sup_{\substack{0 \leq s \leq t \\ x \in I\!R}} e^{\epsilon s} u_\epsilon(s, x) \\ &\leq (b(t, x) - \epsilon) \sup_{\substack{0 \leq s \leq t \\ x \in I\!R}} u_\epsilon(s, x). \end{aligned}$$

so that applying Lemma **2.12**, one gets a contradiction. When ϵ tends to 0, one finally obtains the majoration of $\sup u$. In order to bound $\inf u(t, x)$, it suffices to apply the same technique as above to the function $(-u)$. This finishes the proof of Proposition **2.11**. •

Note that this is only an estimate on u but not $\partial_x u$. We need also the estimate:

Proposition 2.13
Let u be as in Proposition **2.11**. *Let $0 < \alpha < 1$ be given. Then, for all $\delta > 0$ there exists a constant $C(T; u_0, f_0, \mathcal{C}, \nu, \alpha, \delta)$ such that:*

$$\|u\|_{\mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}([\delta,T]\times I\!R_x)} \leq C(T; u_0, f_0, \mathcal{C}, \nu, \alpha, \delta)$$

Proof: The exchange term M is bounded, due to the boundedness of u and of the support of f. This enables us to apply the estimates of Chap. III of [6]. •

Finally, we prove the main result:

Proof of Theorem 2.1:
Just as in the classical Cauchy-Lipschitz theory, there exists a maximal time T^* such that the problem (1.1)-(1.2) has a solution $u \in \mathcal{C}([0, T^*[; \mathcal{C}^2(I\!R_x))$, which is unique. Let us prove that $T^* = +\infty$.

Let $T \in [0, T^*[$ be given. Then, from Proposition **2.11** and Proposition **2.13**, we know that

$$M = \|u\|_{\mathcal{C}([0,T];\mathcal{C}^1(I\!R_x))}$$

is finite. Recalling now the proof of Proposition **2.10** for the local existence, it is possible to find $\tau > 0$ such that for any $t_1 \in [0, T]$, there exists a *unique* solution of Problem (1.1)-(1.2) on $[t_1, t_1 + \tau]$. Then, the uniqueness on $[0, t^{**}]$ implies the uniqueness on $[0, T]$, then on $[0, T^*[$. Moreover, assume $T^* < +\infty$. Then, it is again possible to find a unique solution to Problem (1.1)-(1.2) on $[T^* - \tau/2, T^* + \tau/2]$, a contradiction. This finishes the proof of Theorem **2.1**. •

3 The travelling waves

We investigate here the existence of travelling waves for the system (1.1)-(1.2).

They have the form $[\Phi^0(x - ct), F(x - ct, v)]$, where we assume that Φ^0 is a Lipschitz function, F a compactly supported measure and c the velocity of the wave. In the moving frame $\xi = x - ct$, the functions $[\Phi^0(\xi, t), F(\xi, v, t)]$, are solutions of:

$$\begin{cases} [\Phi^0 - c]\, \partial_\xi \Phi^0 - \partial_{\xi\xi} \Phi^0 = -\mathcal{C}\dfrac{\rho_p}{\rho_g}(\Phi^0 - u_p) \\ \\ (v - c)\partial_\xi F + \partial_v(F\beta) = 0 \end{cases}$$

The equation for the spray is once again solved by the method of characteristics; that is we introduce the functions $(X,V)(t;\xi,v,s)$, solutions of the system of ODEs:

$$\begin{cases} \dfrac{dX}{dt} = V - c \\ \\ \dfrac{dV}{dt} = -\mathcal{C}(V - \Phi^0(X)) \end{cases} \qquad \begin{cases} X(s;\xi,v,s) = \xi \\ \\ V(s;\xi,v,s) = v \end{cases}$$

Then, thanks to the regularity of Φ^0, the characteristics are well-defined, unique and regular. Moreover for any times t and s, the characteristics define the flow

$$\begin{aligned} \mathcal{F}_s^t \;\; &: I\!R_\xi \times I\!R_v \longrightarrow I\!R_\xi \times I\!R_v \\ &: (\xi,v) \longrightarrow (X,V)(t;\xi,v,s) \end{aligned} \tag{3.1}$$

whose inverse is $\mathcal{F}_t^s$. For the sake of simplicity, we did not write the dependence of $\mathcal{F}$ on Φ^0. Finally the solution F writes in terms the characteristics as:

$$F(\xi,v,t) = e^{\mathcal{C}t} F(\mathcal{F}_t^0(\xi,v)). \tag{3.2}$$

However, of most interest will be the study of the forward trajectories $\mathcal{F}_0^t$. Let therefore be given (ξ,v); if $\mathcal{F}_0^t(\xi,v)$ is bounded let us denote by $\omega(\xi,v)$ the ω-limit set of this forward characteristic. Let us denote

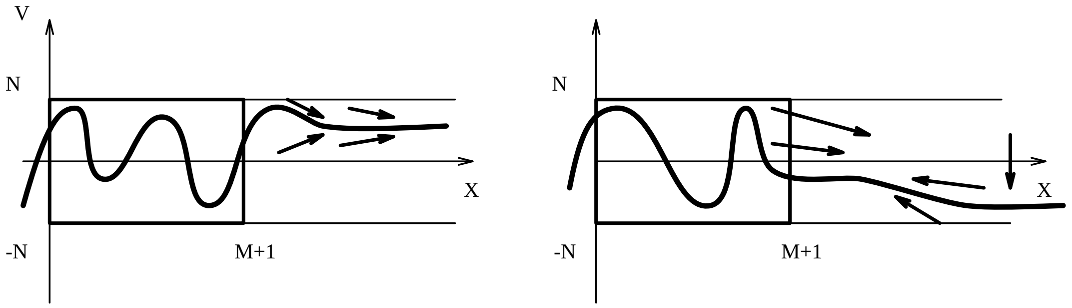
$$\mathcal{S} = \{(x,v) \in I\!R^2 : \quad v = c = \Phi^0(x)\}; \tag{3.3}$$

the set of critical points of the flow. Because F is compactly supported, the function Φ^0 is a solution of a pure steady Burgers equation for x large, and hence differs from a constant only on a compact subset, or has limits at $\pm\infty$, which are approached monotonically. This implies the following important alternative:

Proposition 3.1
Let $(\xi,v) \in (I\!R_\xi \times I\!R_v)$ be given. Then

$$\begin{cases} \textit{either} \quad \lim\limits_{t\to+\infty} |\mathcal{F}_0^t(\xi,v)| = +\infty, \\ \\ \quad \textit{or} \qquad \{\mathcal{F}_0^t(\xi,v),\ t \geq 0\} \ \textit{ is bounded and } \omega(\xi,v) \subset \mathcal{S}. \end{cases}$$

Proof: The equation for V implies that, eventually, $V(0,t,x,v)$ will be less than $M := \|\Phi^0\|_\infty$, hence less than $M+1$ in finite time. Hence it is enough to examine those initial data (x,v) that are located in the strip $\{|V| \leq M+1\}$. Let us now notice that, by setting $\Phi^0 := \Phi^0 - c$ and $V := V - c$, we may



assume $c = 0$. Let $N > 0$ be large enough so that the support of f lies in the strip $\{|X| \leq N\}$, and also such that, for $|X| \geq N$, the function $\Phi^0(X)$ is monotone and satisfies the pure steady Burgers equation. Let us set

$$\Phi^0_\pm = \lim_{X \to \pm\infty} \Phi^0(X). \tag{3.4}$$

and let us call $B_{M,N}$ the box $\{(x,v) \in [-N,N] \times [-M-1, M+1]\}$. The analysis that will follow will be carried on in the half strip $\mathcal{S}_{M,N} = \{X \geq N,\ |V| \leq M+1\}$; it is however valid in the opposite half strip $\{X \leq -N,\ |V| \leq M+1\}$. Due to the equation $\dot{X} = V$, the flow may only enter $\mathcal{S}_{M,N}$ through the gate $\{X = N,\ 0 \leq V \leq M+1\}$; let us see what happens then. Four cases are to be considered (see also figure above).

1. $\Phi^0_+ > 0$. We may choose N large enough so that $\Phi^0(X)$ is positive for $X \geq M$. The branch $\{\Phi^0(X) = V\}$ is globally attracting; therefore $X(t)$ goes to $+\infty$.

2. $\Phi^0_+ < 0$. Once again we may choose N large enough so that $\Phi^0(X)$ is negative for $X \geq N$. As long as the flow stays in the region $\{V > 0\}$, it is attracted to the X-axis at a rate at least equal to Φ^0_-; therefore it hits the X-axis in finite time - at most at the time $t_m = \dfrac{M+1}{\mathcal{C}\Phi^0_-}$. Now, the branch $\{\Phi^0(X) = V\}$ is still globally attracting; and the vector field in its vicinity decidedly points towards the box $B_{M,N}$. As a consequence, the trajectory is sent back into the box.

3. $\Phi^0_+ = 0$ and Φ^0 is compactly supported in $\mathcal{S}_{M,N}$. Bare-hand calculations show that, in this case, the flow is bounded.

4. $\Phi^0_+ = 0$ and Φ^0 is strictly monotone in $\mathcal{S}_{M,N}$. This case may be treated in the same fashion as the first two first.

As a consequence, all unbounded trajectories see their X-component go to $+\infty$. Moreover, all bounded trajectories may be considered to belong to the box $\mathcal{B}_{M,N}$. Now, the flow has the following strict Lyapounov functional:

$$\mathcal{L}(t) = \frac{1}{2}V^2 - \int_0^X \Phi^0(Y)\ dY.$$

This implies that, for all bounded trajectories, the ω-limit set lies in $\mathcal{S}$. $\bullet$

Let us now examine the consequences of this proposition. Let $\psi \in C(IR_x \times IR_v)$ be chosen; we wish to study the action of F on ψ. Since F is compactly supported, we may as well choose ψ compactly supported. Due to formula 3.2, there holds

$$\forall t \geq 0, \quad < F, \psi >=< F, \psi(\mathcal{F}_0^t) > .$$

Let us also assume that the support of ψ does not intersect $\mathcal{S}$. If we send t to $+\infty$ and use the preceding proposition, we eventually find out that $< F, \psi >= 0$. In other words, the support of F is included in $\mathcal{S}$. Going back to the expression of the exchange of momentum, we see that $\rho_p u_p - \rho u = 0$; hence the travelling wave for the gas is necessarily a travelling wave for the Burgers' equation, which writes in the moving coordinate:

$$\Phi^0(\xi) = c\left[1 - th\left(\frac{c\xi}{2\nu}\right)\right] \tag{3.5}$$

where c is the velocity whose value is imposed by the conditions at infinity:

$$c = \frac{u_0^- + u_0^+}{2}.$$

Finally, the travelling wave for the gas has the support the point $(\xi, v) = (0, c)$, so that:

$$F(\xi, v) = m_T\ \delta(\xi) \otimes \delta(v - c) \tag{3.6}$$

where m_T is the total mass of the dispersed phase and where the functions $\delta(\xi)$ and $\delta(v)$ denote the Dirac measures in the IR_ξ space and in the IR_v space respectively.

Remark 3.2 The travelling waves (3.5)-(3.6) for the system are singular solutions of the problem in the sense that they correspond to a regime where the two phases are totally uncoupled.

4 Linear stability. Spectral problem

The linearized problem means the linearized problem verified by a perturbation of the travelling wave. Let us at once say that the problem that we will eventually encounter will not look more linear than the initial one, but the important new feature will be the linearity of the characteristics.

Let $[\Phi, F]$ be a travelling wave for the system of the form discussed above. As we explained in the introduction, we consider for our problem perturbations for the spray unknown such that the total mass of the spray is independent of the scaling factor ϵ. Hence, the initial data of the perturbed problem is:

$$\begin{cases} u(x,0) = \Phi(x) + \epsilon w_0(x) \\ f(x,v,0) = \dfrac{1}{\epsilon^2} g_0\left(\dfrac{x}{\epsilon}, \dfrac{v-c}{\epsilon}\right). \end{cases}$$

Here, $g_0(\cdot,\cdot)$ is a bounded measure with compact support. Let us set

$$\eta = \frac{x-ct}{\epsilon} \quad , \quad \gamma = \frac{v-c}{\epsilon}.$$

One verifies that the total mass of the spray is independent of the value of ϵ:

$$\forall \epsilon, \quad m_T = m_0 \int_\eta \int_\gamma g_0(\eta,\gamma)\ d\eta d\gamma = m_0 \int_x \int_v \frac{1}{\epsilon^2} g_0\left(\frac{x}{\epsilon}, \frac{v-c}{\epsilon}\right)\ dxdv$$

Then, we introduce the perturbations $[\tilde{w},\tilde{g}](t)$ in the moving coordinate $(x-ct, v-c)$ such that the solution $[u,f]$ of the problem writes:

$$\begin{cases} u(x,t) = \Phi(x-ct) + \epsilon \tilde{w}(x-ct,t) \\ f(x,v,t) = \dfrac{1}{\epsilon^2} \tilde{g}\left(\dfrac{x-ct}{\epsilon}, \dfrac{v-c}{\epsilon}, t\right). \end{cases} \tag{4.1}$$

Therefore, the equation for the perturbation $\tilde{w}$ in the moving coordinate $\xi = x - ct$ may be rewritten:

$$\partial_t \tilde{w} + \partial_\xi \left[(\Phi - c)\tilde{w}\right] - \nu \partial_{\xi\xi}\tilde{w} + \epsilon \tilde{w}\partial_\xi \tilde{w} = -\frac{\mathcal{C}}{\epsilon} \frac{\rho_p^f}{\rho_g}(u - u_p^f)$$

where ρ_p^f and $(\rho_p u_p)^f$ denote respectively the density and momentum of f. Now, the rhs of the equation above is rewritten:

$$-\frac{\mathcal{C}}{\epsilon}\frac{\rho_p}{\rho_g}(u-u_p) =$$

$$= -m_0 \frac{1}{\epsilon}\frac{\mathcal{C}}{\rho_g}\ \frac{1}{\epsilon^2} \int_v \tilde{g}\left(\frac{x-ct}{\epsilon}, \frac{v-c}{\epsilon}, t\right) \left[\Phi(x-ct) + \epsilon \tilde{w}(x-ct,t) - v\right]\ dv$$

$$= -m_0 \frac{\mathcal{C}}{\rho_g} \int_\gamma \frac{1}{\epsilon} \tilde{g}\left(\frac{x-ct}{\epsilon}, \gamma, t\right) \left[\frac{\Phi(x-ct) - c}{\epsilon} + \tilde{w}(x-ct,t) - \gamma\right]\ d\gamma.$$

The bounded measure $\tilde{M}$ above is then evaluated in the moving frame $\xi = x - ct$: let $\Psi \in \mathcal{C}^1(I\!R_\xi)$ be given. We have, formally, for all t:

$$
\begin{aligned}
&\langle \tilde{M}(\cdot,t), \Psi(\cdot)\rangle_{I\!R_\xi} \\
&\quad = -m_0 \frac{\mathcal{C}}{\rho_g} \int_\eta \int_\gamma \tilde{g}(\eta,\gamma,t) \left[\frac{\Phi(\epsilon\eta) - c}{\epsilon} + \tilde{w}(\epsilon\eta, t) - \gamma \right] \times \Psi(\epsilon\eta)\, d\eta d\gamma \\
&\quad = -m_0 \frac{\mathcal{C}}{\rho_g} \int_\eta \int_\gamma g(\eta,\gamma,t) \left[\Phi'(0)\eta + \tilde{w}(0,t) - \gamma \right] \times \Psi(0) d\eta d\gamma \\
&\qquad\qquad + \|\Psi\|_{\mathcal{C}^1(I\!R_\xi)} \times O(\epsilon)
\end{aligned}
$$

Hence, at the first order in ϵ, the equation for the gas writes:

$$
\begin{aligned}
&\partial_t \tilde{w} + \partial_\xi \left[(\Phi - c)\tilde{w} \right] - \nu \partial_{\xi\xi} \tilde{w} \\
&\qquad = -m_0 \frac{\mathcal{C}}{\rho_g} \left\{ \int_\eta \int_\gamma g(\eta,\gamma,t) \left[\Phi'(0)\eta + \tilde{w}(0,t) - \gamma \right] d\eta d\gamma \right\} \delta(\xi)
\end{aligned}
$$

Now, the equation for $\tilde{g}$ writes:

$$
\partial_t \tilde{g} + \gamma \partial_\eta \tilde{g} - \mathcal{C} \partial_\gamma \left[\tilde{g} \left(\gamma - \frac{\Phi(\epsilon\eta) - c}{\epsilon} - \tilde{w}(\epsilon\eta, t) \right) \right] = 0
$$

so that at the first order in ϵ, we have:

$$
\partial_t \tilde{g} + \gamma \partial_\eta \tilde{g} - \mathcal{C} \partial_\gamma \left[\tilde{g} \left(\gamma - \Phi'(0)\eta - \tilde{w}(0,t) \right) \right] = 0
$$

Finally, the linear problem consists in the system:

$$
\left\{
\begin{aligned}
&\partial_t \tilde{w} + \partial_\xi \left[(\Phi - c)\tilde{w} \right] - \nu \partial_{\xi\xi} \tilde{w} \\
&\qquad = -m_0 \frac{\mathcal{C}}{\rho_g} \left\{ \int_\eta \int_\gamma g(\eta,\gamma,t) \left[\Phi'(0)\eta + \tilde{w}(0,t) - \gamma \right] d\eta d\gamma \right\} \delta(\xi) \\
&\partial_t \tilde{g} + \gamma \partial_\eta \tilde{g} + \partial_\gamma \left[\tilde{g}(-\mathcal{C}) \left(\gamma - \Phi'(0)\eta - \tilde{w}(0,t) \right) \right] = 0,
\end{aligned}
\right. \tag{4.2}
$$

with the corresponding initial data.

4.1 Reduced problem

In this subsection, we will see that the linearized problem (4.2) written in the moving frame amounts to studying a single equation for the gas. This result is stated in Theorem **4.1** below. Then, in Lemma **4.2**, we introduce the space of fastly decreasing functions at infinity in which we will then consider the perturbations for the gas.

The gas unknowns at any time t is now taken in the space:

$$\mathcal{B}_0 = H^1(I\!R_\xi)$$

Notice that the functions of $\mathcal{B}_0$ tend to 0 at infinity and that $\mathcal{B}_0 \subset \mathcal{C}^0(I\!R_\xi)$ where $\mathcal{C}^0(I\!R_\xi)$ denotes the space of continuous functions. Then, we consider the perturbations $\tilde{g}(\cdot,\cdot,t)$ for the spray belonging to the Banach space $\mathcal{B}_s$ defined as:

$$\mathcal{B}_s = \mathcal{M}_0^b(I\!R_x \times I\!R_v)$$

where $\mathcal{M}_0^b(I\!R_x \times I\!R_v)$ denotes the space of bounded measures with compact support on $I\!R_x \times I\!R_v$. Finally, we will note in the following $\delta_\xi = \delta(\xi)$ and $\delta_v = \delta(v)$.

Theorem 4.1 (Reduced problem)
Let the perturbations $[w_0, g_0] \in \mathcal{B}_0 \times \mathcal{B}_s$ be given. Then solving the linearized problem (4.2) is equivalent to solving the problem for the gas unknown:

$$\partial_t \tilde{w} + \partial_\xi \left[(\Phi - c)\tilde{w}\right] - \nu \partial_{\xi\xi} \tilde{w} = \frac{m_T}{\rho_g}(d_t \gamma^B)\ \delta_\xi \tag{4.3}$$

where $\gamma^B(t)$ is the solution of:

$$\begin{cases} d_{tt}\gamma^B + \mathcal{C} d_t \gamma^B - \mathcal{C}\Phi'(0)\gamma^B = \mathcal{C}(\partial_t \tilde{w})(0,\cdot) \\ \gamma^B(0) = \gamma_0^B, \quad (d_t\gamma^B)(0) = -\mathcal{C}(\gamma_0^B - \Phi'(0)\eta_0^B + \partial_t \tilde{w}(0,0)), \end{cases} \tag{4.4}$$

and where we noted

$$\begin{cases} \eta_0^B = \dfrac{m_0}{m_T} \displaystyle\int_\eta \int_\gamma g_0(\eta,\gamma)\eta\ d\eta d\gamma \\ \gamma_0^B = \dfrac{m_0}{m_T} \displaystyle\int_\eta \int_\gamma g_0(\eta,\gamma)\gamma\ d\eta d\gamma \end{cases} \tag{4.5}$$

Proof:
We first note that solving $\tilde{g}$ in the linear problem (4.2) is equivalent to solving the characteristics, that is the trajectories of the particles for the linearized problem. If the functions $(\eta,\gamma)(t;x,v,0)$ are these characteristics, then we have

$$\forall (x,v) \in Supp(g_0), medskip \\ \begin{cases} d_t\eta = \gamma \\ d_t\gamma = -\mathcal{C}\left(\gamma - \Phi'(0)\eta - \tilde{w}(0,t)\right) \end{cases} \begin{cases} \eta(0;x,v,0) = x \\ \gamma(0;x,v,0) = v \end{cases} \tag{4.6}$$

and we may now introduce the mean position and velocity (η^B, γ^B) of the spray defined as:

$$m_T\ \eta^B(t) = m_0 \int_\eta \int_\gamma g(\eta,\gamma,t)\ \eta\ d\eta d\gamma$$

$$m_T\ \gamma^B(t) = m_0 \int_\eta \int_\gamma g(\eta,\gamma,t)\ \gamma\ d\eta d\gamma.$$

It is then clear that (η^B, γ^B) are solution of the same system of ODEs as (η, γ), with the initial conditions (4.5) as stated in the Theorem. Moreover, it is now possible to rewrite the exchange term in the gas equation of the linearized problem (4.2):

$$-m_0 \frac{\mathcal{C}}{\rho_g} \left\{ \int_\eta \int_\gamma g(\eta,\gamma,t)\,[\Phi'(0)\eta + \tilde{w}(0,t) - \gamma]\, d\eta d\gamma \right\}\ \delta(\xi) =$$

$$= -\frac{\mathcal{C}}{\rho_g} m_T [\Phi'(0)\eta^B + \tilde{w}(0,t) - \gamma^B]\ \delta(\xi)$$

$$= -\frac{m_T}{\rho_g}(d_t \gamma^B)\ \delta(\xi),$$

that is the equation (4.3) of the Theorem. Finally, differentiating the equation for the velocity γ in the system of ODEs (4.6), we get:

$$d_{tt}\gamma + \mathcal{C} d_t \gamma - \mathcal{C}\Phi'(0)\gamma = \mathcal{C}\partial_t \tilde{w}(0,t)$$

with the initial condition for the derivative: $(d_t\gamma)(0) = -\mathcal{C}(\gamma(0) - \Phi'(0)\eta_0 - \partial_t \tilde{w}(0,0))$, and the same holds for γ^B. This concludes the proof of Theorem **4.1**. •

Now, following Sattinger, we restrict the linearized operator for the gas unknown to a space of fastly decreasing functions at infinity. More precisely, considering the weight function $\omega(\xi)$, a positive function such that $\lim_{|\xi|\to+\infty} \omega(\xi) = +\infty$, we introduce the subspace $\mathcal{B}_{\omega,0}$ of $\mathcal{B}_0$ of the fastly decreasing functions at infinity:

$$\mathcal{B}_{\omega,0} = \left\{ u \in \mathcal{B}_0;\ (\omega u) \in \mathcal{B}_0 \right\}.$$

Choosing properly the weight ω allows to rewrite the linearized problem by using a selfadjoint operator:

Lemma 4.2
Let $\omega(\xi) = ch(\frac{c\xi}{2\nu})$. Let $\tilde{w} \in \mathcal{C}([0,+\infty[;\mathcal{B}_{\omega,0})$ be a solution of the linearized problem (4.3) written in the reduced form. Then, the function v of $\mathcal{C}([0,+\infty[;\mathcal{B}_0)$

defined by $v(\xi,t)=\omega(\xi)\tilde{w}(\xi,t)$ is the solution of the problem:

$$\left\{\begin{array}{l} \partial_t v + Av = -\dfrac{m_T}{\rho_g}(d_t\gamma^B)\ \delta_\xi \\ \\ v(\xi,0)=\omega(\xi)w(\xi,0) \end{array}\right. \tag{4.7}$$

where $A:\mathcal{B}_0\longrightarrow H^{-1}(I\!R_x)$ denotes the linearized Burgers' operator:

$$\left\{\begin{array}{l} Av = -\nu\partial_{\xi\xi}v + a(\xi)v \\ \\ a(\xi) = \dfrac{c^2}{4\nu} - \dfrac{c^2}{2\nu}\left(1 - th^2\left(\dfrac{c\xi}{2\nu}\right)\right), \end{array}\right. \tag{4.8}$$

where γ^B is defined in (4.4) with $\tilde{w}(0,t)=v(0,t)$.

Remark that we noted here $v(\xi,t)$ the unknown for the gas though we previously used the notation v for the velocity of the particles. We think that no confusion will arise.

Proof of Lemma 4.2: The details of the proof may be found in the article of Sattinger (see [10]). •

4.2 Spectral problem. Localization of the spectrum

In this subsection, we study the spectrum of the linearized problem for the gas unknown restricted to the space $\mathcal{B}_{\omega,0}$. In Proposition **4.3**, we state the spectral problem. The main result of this subsection is Theorem **4.4** in which we prove that the spectrum (but $\{0\}$) of the linearized operator restricted to $\mathcal{B}_{\omega,0}$ is bounded away from 0.

First, we extend a function $v(t)_{t\geq 0}$ to the whole real axis $I\!R_t$ by setting $v(t)=0$ for $t<0$ so that v is a discontinuous function of the variable t. Now, let $v(\cdot,t)\in\mathcal{B}_0$ be the solution of the linearized problem (4.7). Then, the Laplace transform $\hat{v}(\xi,p)$ in the time direction of the function $v(\xi,t)$ is defined in the distribution sense as:

$$\hat{v}(\xi,p) = \langle v(\xi,t), e^{-pt}\rangle_{I\!R_t}. \tag{4.9}$$

so that the Laplace transform $\hat{v}(\xi,p)$ of $v(\xi,t)$ is the solution of the spectral problem:

Proposition 4.3 (Spectral Problem) *Let $\tilde{w}(\cdot,t)\in\mathcal{B}_{\omega,0}$ be the solution of the reduced problem (4.3). Consider $v(\cdot,t)=\omega(\cdot)\tilde{w}(\cdot,t)$, solution of problem*

(4.7). Then, the Laplace transform $\hat{v}(\xi,p)$ of $v(\xi,t)$ exists and is solution of the spectral problem:

$$(A+pI)\hat{v}+\mathcal{C}\frac{m_T}{\rho_g}\ \theta(p)\hat{v}(0,p)\ \delta_\xi = v(\xi,0) \tag{4.10}$$

where

$$\begin{cases} v(\xi,0)=\omega(\xi)w(\xi,0) \qquad \in \mathcal{B}_0 \\ \\ \theta(p)=\dfrac{p^2}{p^2+\mathcal{C}p-\mathcal{C}\Phi'(0)}. \end{cases}$$

Here, A is the operator defined in Eq. (4.8) and I denotes the identity.

Proof: The boundedness of $v(\xi,\cdot)$ allows to consider its Laplace transforms $\hat{v}(\xi,p)$. Differentiating the expression (4.9) with respect to t, and since $(\partial_t v)(\xi,0)=v(\xi,0)\times\delta(t)$, one obtains the expression of the Laplace transform of the time-derivative of v:

$$(\widehat{\partial_t v})(\xi,p)=p\hat{v}(\xi,p)-v(\xi,0).$$

Applying now the Laplace transform to the linear equation (4.7) for v yields the spectral problem:

$$(A+pI)\hat{v}+\mathcal{C}\frac{m_T}{\rho_g}(\widehat{d_t\gamma^B})\ \delta_\xi = v(\xi,0).$$

Recall now that γ^B is such that:

$$d_{tt}\gamma^B+\mathcal{C}d_t\gamma^B-\mathcal{C}\Phi'(0)\gamma=\mathcal{C}(\partial_t v)(0,t) \tag{4.11}$$

where $v(0,t)$ is discontinuous at $t=0$. Hence, so is $(d_{tt}\gamma^B)$. However, $(d_t\gamma^B)$ and γ^B are continuous functions of the variable t, and we compute the Laplace transforms:

$$\begin{cases} (\widehat{d_{tt}\gamma^B})(p)=p(\widehat{d_t\gamma^B})(p)-\mathcal{C}\ v(0,0) \\ (\widehat{\partial_t\gamma})(p)=p\hat{\gamma} \\ (\widehat{\partial_t v})(0,p)=p\hat{v}(0,p)-v(0,0) \end{cases}$$

where $\hat{\gamma}$ denotes the Laplace transform of γ^B. Hence, applying the Laplace transform to the Eq. (4.11) for γ^B gives:

$$\left(p^2+\mathcal{C}p-\mathcal{C}\Phi'(0)\right)\hat{\gamma}(p)-\mathcal{C}v(0,0)=\mathcal{C}\left(p\hat{v}(0,p)-v(0,0)\right),$$

that is exactly $(\widehat{\partial_t\gamma^B})=\theta(p)\hat{v}(0,p)$ as stated in the theorem. This concludes the proof of Proposition **4.3**. •

Now, let us investigate the spectrum of the spectral problem (4.10). That is, given any function $v(\xi,0)$ of ξ, we look for those values of p such that the spectral problem (4.10) has a unique solution in $\mathcal{B}_0$. Equivalently, setting $\lambda=-p$, $\check{v}(\lambda)=\hat{v}(-\lambda)$ and $\check{\theta}(\lambda)=\theta(-\lambda)$ we look for those λ such that the spectral problem

$$(A-\lambda I)\check{v}(\cdot,\lambda)+\mathcal{C}\frac{m_T}{\rho_g}\check{\theta}(\lambda)\check{v}(0,\lambda)=v(\cdot,0)$$

has a unique solution in $\mathcal{B}_0$. Define now the operators T_λ, S_{θ_0} and $\mathcal{S}_{\lambda,\theta_0}$ as follows:

$$\begin{cases} T_\lambda & : \mathcal{B}_0 \longrightarrow H^{-1}(I\!R_x) \\ & : v \longrightarrow (A-\lambda I)v \end{cases}$$

$$\begin{cases} S_{\theta_0} & : \mathcal{B}_0 \longrightarrow H^{-1}(I\!R_x) \\ & : v \longrightarrow \mathcal{C}\dfrac{m_T}{\rho_g}\theta_0 v(0)\ \delta_\xi \end{cases}$$

$$\begin{cases} \mathcal{S}_{\lambda,\theta_0} & : \mathcal{B}_0 \longrightarrow H^{-1}(I\!R_x) \\ & : v \longrightarrow (T_\lambda+S_{\theta_0})v \end{cases}$$

Hence, locating the spectrum of the linearized operator of the spectral problem restricted to $\mathcal{B}_{\omega,0}$ means finding those λ such that the problem:

$$\begin{cases} \mathcal{S}_{\lambda,\check{\theta}(\lambda)}\check{v}=v(\cdot,0) \\ \check{\theta}(\lambda)=\dfrac{\lambda^2}{\lambda^2-\mathcal{C}\lambda-\mathcal{C}\Phi'(0)} \end{cases} \tag{4.12}$$

has a unique solution $\check{v}(\cdot,\lambda)$ in $\mathcal{B}_0$. If, for any given $\lambda_0\in I\!R$, we denote by $P^-_{\lambda_0}$ the open left half-plane:

$$P^-_{\lambda_0}=\{\lambda\in\mathbb{C}\ :\ Re\ \lambda<\lambda_0\},$$

then our aim is to prove the main result of this subsection:

Theorem 4.4 (Resolvent set)
There exists $\lambda_0>0$ such that, for any $\lambda\in(P^-_{\lambda_0}\backslash\{0\})$, the spectral problem (4.12) has a unique solution $\check{v}(\cdot,\lambda)$ in $\mathcal{B}_0$.

This result was already studied by Sattinger for the Burgers' problem, that is for our spectral problem (4.12) where we omit the operator $S_{\check{\theta}(\lambda)}$ in $\mathcal{S}_{\lambda,\check{\theta}(\lambda)}$, and we have the classical result (see [5], Ch.V):

Proposition 4.5 *Let $\lambda_A=\dfrac{c^2}{4\nu}$. Then, for any given $\lambda\in(P^-_{\lambda_A}\backslash\{0\})$, the operator $T_\lambda=A-\lambda I$ is an isomorphism from $\mathcal{B}_0$ to $H^{-1}(I\!R_x)$ and T_λ^{-1} is a bounded operator.*

Now, we want to find the resolvent set $\rho(\mathcal{S}_{\lambda,\breve{\theta}(\lambda)})$ of the operator $\mathcal{S}_{\lambda,\breve{\theta}(\lambda)}$, that is we want to find the values of λ such that the problem (4.12) has a unique solution in $\mathcal{B}_0$. Moreover the spectrum $\sigma(\mathcal{S}_{\lambda,\breve{\theta}(\lambda)})$ is defined as $\sigma(\mathcal{S}_{\lambda,\breve{\theta}(\lambda)}) = \mathbb{C}\setminus\rho(\mathcal{S}_{\lambda,\breve{\theta}(\lambda)})$, and it would be convenient that this spectrum reduce to the set $\tilde{\sigma}(\mathcal{S}_{\lambda,\breve{\theta}(\lambda)})$ of the eigenvalues of the linearized problem. Here, by eigenvalue we mean a $\lambda \in \mathbb{C}$ such that there exists a nonzero function $\breve{v} \in \mathcal{B}_0$ satisfying $\mathcal{S}_{\lambda,\breve{\theta}(\lambda)}\breve{v} = 0$. Indeed, we will prove here that $\sigma(\mathcal{S}_{\lambda,\breve{\theta}(\lambda)}) = \tilde{\sigma}(\mathcal{S}_{\lambda,\breve{\theta}(\lambda)})$, so that the study of the set $\tilde{\sigma}$ of the eigenvalues of the linearized operator will suffice to determine the resolvent set ρ.

For our problem, we have the result:

Lemma 4.6 (Fredholm alternative) *Let λ_A be defined as in proposition* **4.5**. *Let be given $\theta_0 \in I\!R$ and $\lambda \in (P^-_{\lambda_A}\setminus\{0\})$. Then the operator $\mathcal{S}_{\lambda,\theta_0}$ is Fredholm with index $Ind(\mathcal{S}_{\lambda,\theta_0}) = 0$.*

This will ensure that as soon as the null space $N(\mathcal{S}_{\lambda,\theta_0})$ of $\mathcal{S}_{\lambda,\theta_0}$ reduces to the null space, then the operator $\mathcal{S}_{\lambda,\theta_0}$ is invertible.

Proof of Lemma 4.6: First, from Proposition **4.5**, for any given $\lambda \in (P^-_{\lambda_A}\setminus\{0\})$, we may rewrite the operator $\mathcal{S}_{\lambda,\theta_0}$ as:

$$\mathcal{S}_{\lambda,\theta_0} = (I + S_{\theta_0}T_\lambda^{-1})T_\lambda$$

where $T_\lambda : \mathcal{B}_0 \to H^{-1}(I\!R_x)$ satisfies the Fredholm alternative with index $Ind(T_\lambda) = 0$. Hence, it suffices to prove that the operator $(I + S_{\theta_0}T_\lambda^{-1})$ is Fredholm with index $Ind(I + S_{\theta_0}T_\lambda^{-1}) = 0$. It is a classical result that it suffices to prove that $\mathcal{S}_{\theta_0}T_\lambda^{-1}$ in $\mathcal{L}(\mathcal{B}_0)$ is compact (see [3]).

Indeed, let $(u_n)_{n\in I\!N}$ be a bounded sequence of $H^{-1}(I\!R_x)$. Then $v_n = T_\lambda^{-1}u_n$ is defined and bounded in $\mathcal{B}_0$. But $\mathcal{B}_0$ injects continuously in the space $\mathcal{C}^0(I\!R_\xi)$ of the continuous functions. Hence, the sequence $v_n(0)$ is defined and bounded in $I\!R$, so that the sequence $v_n(\cdot) + \theta_0 v_n(0)\delta_\xi$ is defined and bounded in $H^{-1}(I\!R_x)$. We first extract a subsequence such that $v_n(0)$ converges. Then, since $v_n(\cdot)$ tends to 0 as $|\xi| \longrightarrow +\infty$, it is possible, by a diagonal extraction, to find a subsequence of $(v_n(\cdot))$ that converges in $\mathcal{B}_0$ and the compactness of $\mathcal{S}_{\theta_0}T_\lambda^{-1}$ is proved. This concludes the proof Lemma **4.6**. •

As a consequence of Lemma **4.6** above, the operator $\mathcal{S}_{\lambda,\theta_0}$ satisfies the Fredholm alternative :

$$N(\mathcal{S}_{\lambda,\theta_0}) = \{0\} \Longleftrightarrow R(\mathcal{S}_{\lambda,\theta_0}) = H^{-1}(I\!R_x).$$

Hence, proving Theorem **4.4** is equivalent to proving the following result:

Theorem 4.7 (Localization of the spectrum)
There exists $\lambda_0 > 0$ such that $\check{\theta}$ is well-defined in $P^-_{\lambda_0}$ and such that:

$$\sigma(\mathcal{S}_{\lambda,\check{\theta}(\lambda)}) \cap P^-_{\lambda_0} = \{0\}$$

Here, we will use the decomposition of the space $H^{-1}(I\!R_x)$ and the coerciveness property (see [10]):

$$H^{-1}(I\!R_x) = N(A) \oplus R(A)$$

where $N(A)$ and $R(A)$ denote respectively the null space and the range of the Burgers' operator A defined in (4.8). Moreover, $N(A)$ is one-dimensional and the restriction to $R(A)$ of the operator A is coercive, i.e. there exists $\lambda_A > 0$ such that

$$\forall \zeta \in \mathcal{B}_0 \cap R(A), \quad \langle Aw, w\rangle \ \geq \ \lambda_A \langle \zeta, \zeta\rangle.$$

We finally prove the main result of this subsection:

Proof of Theorem 4.4 and Theorem 4.7
Let λ be in $\sigma(\mathcal{S}_{\lambda,\check{\theta}(\lambda)})$. Then there exists $v \neq 0$ in $\mathcal{B}_0$ such that:

$$(A - \lambda I)v + \mathcal{C}\frac{m_T}{\rho_g}\check{\theta}(\lambda)v(0)\ \delta_\xi = 0 \tag{4.13}$$

Now, we may decompose the function v as:

$$v = \alpha\phi + \zeta. \tag{4.14}$$

Here, ϕ denotes the positive function $\phi = \Phi'\omega$, an element of $N(A)$, α a real and w an element of the range $R(A)$ of A. Hence, since $\langle \zeta, \phi\rangle = 0$, the value of α satisfies $\langle v, \phi\rangle = \alpha\langle \phi, \phi\rangle$. Plugging the decomposition (4.14) into (4.13), and integrating against ϕ gives:

$$\lambda\alpha\langle \phi, \phi\rangle + \mathcal{C}\frac{m_T}{\rho_g}\check{\theta}(\lambda)\left(\alpha\phi(0) + \zeta(0)\right)\ \phi(0) = 0. \tag{4.15}$$

Multiplying (4.15) above by $\phi(\xi)$ and subtracting the result to (4.13) gives

$$A\zeta - \lambda\zeta + \mathcal{C}\frac{m_T}{\rho_g}\check{\theta}(\lambda)(\alpha\phi(0) + \zeta(0))(\delta_\xi - \phi(0)\phi) = 0$$

that we integrate against ζ so that we obtain finally:

$$\langle A\zeta, \zeta\rangle - \lambda\langle \zeta, \zeta\rangle + \mathcal{C}\frac{m_T}{\rho_g}\check{\theta}(\lambda)(\alpha\phi(0) + \zeta(0))\ \zeta(0) = 0. \tag{4.16}$$

Here, we used the fact that $\langle \phi, \zeta \rangle = 0$. Plugging now the value of $\check{\theta}(\lambda)(\alpha\phi(0)+\zeta(0))$ found in (4.15) into Eq. (4.16) yields the relation:

$$\langle A\zeta, \zeta \rangle - \lambda \langle \zeta, \zeta \rangle + \lambda\alpha\langle \phi, \phi \rangle \frac{\zeta(0)}{\phi(0)} = 0 \tag{4.17}$$

According to the value of α, we consider the two cases:

- <u>Case 1: $\alpha = 0$</u>.

When $\alpha = 0$, Eq. (4.15) imposes:

$$0 = \check{\theta}(\lambda)\zeta(0)\phi(0) = \frac{\lambda^2}{\lambda^2 - \mathcal{C}\lambda + \mathcal{C}\phi(0)} \zeta(0)\phi(0).$$

But denote by $\lambda_\pm$ the roots of $\lambda^2 - \mathcal{C}\lambda + \mathcal{C}\phi(0) = 0$. It is clear that $\lambda_\pm$ have positive real parts. Hence, for any $\lambda \in P^-_{min(Re\lambda_\pm)} \backslash \{0\}$ we must have here $\zeta(0) = 0$. Then Eq. (4.17) imposes that $\langle A\zeta, \zeta \rangle = \lambda\langle \zeta, \zeta \rangle$ so that finally if $\lambda \in P^-_{min(Re\lambda_\pm, \lambda_A)} \backslash \{0\}$, then the coerciveness of A on $R(A)$ imposes $\zeta \equiv 0$ and therefore $v \equiv 0$.

- <u>Case 2: $\alpha \neq 0$</u>

Normalizing v, we may chose α such that $\alpha\langle \phi, \phi \rangle = 1$. Then, taking successively the real part and the imaginary part of (4.17), we obtain the system:

$$\left\{ \begin{array}{ll} \langle A\zeta, \zeta \rangle & = Re\ \lambda \left(\langle \zeta, \zeta \rangle - \dfrac{\zeta(0)}{\phi(0)} \right) \\ \\ 0 & = Im\ \lambda \left(\langle \zeta, \zeta \rangle - \dfrac{\zeta(0)}{\phi(0)} \right) \end{array} \right.$$

We consider again two cases:

- <u>Case (a): $Im\ \lambda \neq 0$</u>

Then, we must have $\langle A\zeta, \zeta \rangle = 0$ which implies $\zeta \equiv 0$, so that thanks to (4.15), we must have:

$$\lambda = \mathcal{C}\frac{m_T}{\rho_g}\check{\theta}(\lambda)\ \frac{\phi^2(0)}{\langle \phi, \phi \rangle} \Longleftrightarrow \left\{ \begin{array}{l} \qquad \lambda = 0 \\ \\ or \quad \lambda^2 - \mathcal{C}\left(1 + \dfrac{m_T}{\rho_g}\dfrac{\phi^2(0)}{\langle \phi, \phi \rangle}\right)\lambda + \mathcal{C}\phi(0) = 0 \end{array} \right.$$

However, denote by $\tilde{\lambda}_\pm$ the roots of the second degree polynomial above. It is clear the $\tilde{\lambda}_\pm$ have positive real part. Hence, if $\lambda \in P^-_{min(Re\tilde{\lambda}_\pm)}$ is an eigenvalue, then necessarily $\lambda = 0$.

- Case (b): $Im\ \lambda = 0$

Here, $\lambda \in I\!R$. From Eq. (4.15), and since $\alpha\langle\phi,\phi\rangle = 1$, we have

$$\zeta(0) = \frac{\lambda}{\check{\theta}(\lambda)\phi(0)} - \frac{\phi(0)}{\langle\phi,\phi\rangle}$$

which expression of $\zeta(0)$ we put into (4.16). We then obtain the relation:

$$0 = -\langle A\zeta,\zeta\rangle + \lambda\langle\zeta,\zeta\rangle + \frac{\lambda}{\langle\phi,\phi\rangle} - \frac{\lambda^2}{\mathcal{C}\dfrac{m_T}{\rho_g}\check{\theta}(\lambda)\phi^2(0)},$$

and from the coerciveness of A on $R(A)$, we have the relation of dispersion:

$$0 \leq (\lambda - \lambda_A)\langle\zeta,\zeta\rangle + \frac{\lambda}{\langle\phi,\phi\rangle} - \frac{(\lambda-\lambda_-)(\lambda-\lambda_+)}{\mathcal{C}\dfrac{m_T}{\rho_g}\phi^2(0)} \tag{4.18}$$

Since $\lambda_\pm$ have positive real parts, it is clear that $\lambda < 0$ is impossible here. Then, we know that the function $\lambda \to \zeta(\cdot,\lambda)$ is analytical in the plane cut along the real axis $[\frac{c^2}{4\nu}, +\infty[$ (see [10]). Then the function in (4.18) is analytical in the same domain and is (strictly) negative at $\lambda = 0$. Hence, it stays negative in a neighborhood $[0,\lambda_d[$ of 0, so that there are no possible eigenvalue in $\lambda \in P^-_{min(Re\lambda_\pm,\lambda_d)}\backslash\{0\}$.

Finally, recalling the different cases, we set:

$$\lambda_0 = min(\lambda_A, Re\lambda_\pm, Re\tilde{\lambda}_\pm, \lambda_d).$$

Hence, $\check{\theta}$ is well-defined in $P^-_{\lambda_0}$ and we have $\sigma(\mathcal{S}_{\lambda,\check{\theta}(\lambda)}) \cap P^-_{\lambda_0} = \{0\}$. This concludes the proof of Theorem **4.4** and Theorem **4.7**. •

4.3 Solution of the spectral problem

To put an end to this Section, we solve the spectral problem (4.12). This will allow us to determine the singularities of the resolvent $(\mathcal{S}_{\lambda,\check{\theta}(\lambda)})^{-1}$ and then, in the following Section, to study the asymptotic behaviour of the linearized problem.

Theorem 4.8 (Solution of the spectral problem) *Let $\lambda \in (P^-_{\lambda_0}\backslash\{0\})$ be given. The solution $\check{v}(\xi,\lambda)$ of the spectral problem (4.12) writes on each half-line $I\!R^\pm_\xi$:*

$$\check{v}(\xi,\lambda) = \mathcal{C}\frac{m_T}{\rho_g}\check{\theta}(\lambda)\psi(0,\lambda)\frac{\phi_\mp(0,\lambda)\phi_\pm(\xi,\lambda)}{\nu W(\lambda) - \mathcal{C}\dfrac{m_T}{\rho_g}\check{\theta}(\lambda)} + \psi(\xi,\lambda) \tag{4.19}$$

where

$$\psi(\xi,\lambda) = -\frac{\langle v_0,\phi\rangle\phi(0)}{\lambda\langle\phi,\phi\rangle}\phi_0(\xi) + \left[(A-\lambda I)^{-1}\left(v_0 - \langle v_0,\phi\rangle\phi(0)\phi_0(\cdot)\right)\right](\xi) \tag{4.20}$$

Here, we introduced the three functions (ϕ_-,ϕ_+,ϕ_0) defined as the unique functions such that:

$$\begin{cases} (A-\lambda I)\phi_-(\cdot,\lambda) = 0, & \lim\limits_{\xi\to-\infty}\phi_-(\xi,\lambda) = 0 \\ (A-\lambda I)\phi_+(\cdot,\lambda) = 0, & \lim\limits_{\xi\to+\infty}\phi_+(\xi,\lambda) = 0 \\ \phi_0(\xi) = \dfrac{\phi(\xi)}{\phi(0)}, & \textit{where } \phi(\xi) = (\Phi'\omega)(\xi). \end{cases}$$

Corollary 4.9 *The solution $\check{v}(\xi,\lambda)$ of the spectral problem has the asymptotic behaviour as λ tends to 0:*

$$\check{v}(\xi,\lambda) = \frac{V(\xi)}{\lambda} + O(1)$$

where $V \in \mathcal{B}_0$ depends on the initial datas and on the physical constants of the problem.

Before this result is proved, one verifies easily that we can rewrite the spectral problem (4.12) as:

Proposition 4.10 *Finding the unique solution $\check{v} \in \mathcal{B}_0$ of the spectral problem (4.12) is equivalent to finding the unique $(\psi,\vartheta) \in (\mathcal{B}_0)^2$ verifying:*

$$\begin{cases} (A-\lambda I)\psi = v_0 \\ (A-\lambda I)\vartheta + \mathcal{C}\dfrac{m_T}{\rho_g}\check{\theta}(\lambda)(\psi(0)+\vartheta(0))\;\delta(\xi) = 0, \end{cases} \tag{4.21}$$

where we noted $v_0(\xi) = v(\xi,0)$ and the solution $\check{v}$ of the spectral problem then writes as $\check{v} = \psi + \vartheta$.

Proof of Theorem 4.8 and Corollary 4.9

Now, the solution ϑ in (4.21) equivalently writes as the unique solution of:

$$\begin{cases} (A-\lambda I)\vartheta = 0 & in\ I\!R_\xi^-\backslash\{0\} \\ (A-\lambda I)\vartheta = 0 & in\ I\!R_\xi^+\backslash\{0\} \\ \nu\left[(\partial_\xi\vartheta)(0^+)-(\partial_\xi\vartheta)(0^-)\right] = \mathcal{C}\dfrac{m_T}{\rho_g}\check{\theta}(\lambda)(\psi(0)+\vartheta(0)) & \end{cases}$$

But the first equation has a unique solution $\phi_-(\cdot,\lambda)$, bounded in $I\!R_\xi^-$, such that $\phi_-(0,\lambda)=1$ and $\phi_-(\xi,\lambda)$ tends to 0 as ξ tends to $-\infty$. In the same manner, we define the function ϕ_+, bounded in $I\!R_\xi^+$, solution of the second equation, such that $\phi_+(0,\lambda)=1$ and $\phi_+(\xi,\lambda)$ tends to 0 as ξ tends to $+\infty$. The functions $\phi_\pm$ are then analytical with respect to the variable λ in the plane cut along the real axis $[\frac{c^2}{4\nu},+\infty[$. Hence, we may look for a solution ϑ in the form:

$$\begin{cases} \vartheta(\xi,\lambda) = \alpha_-\phi_-(\xi,\lambda) & \xi\in\ I\!R^- \\ \vartheta(\xi,\lambda) = \alpha_+\phi_+(\xi,\lambda) & \xi\in\ I\!R^+ \end{cases}$$

where $\alpha_\pm$ are uniquely defined. Finally, the continuity of ϑ with respect to ϵ, together with the condition on the derivatives $(\partial_\xi\vartheta)(0^\pm)$ allow to determine $\alpha_\pm$ as the solution of the system:

$$\mathbf{M}(\lambda)\cdot\begin{bmatrix}\alpha_-(\lambda)\\ \alpha_+(\lambda)\\ \alpha_0(\lambda)\end{bmatrix} = \begin{bmatrix}0\\ \mathcal{C}\dfrac{m_T}{\rho_g}\check{\theta}(\lambda)\psi(0,\lambda)\\ 0\end{bmatrix}$$

where we introduced the matrix $\mathbf{M}$:

$$\mathbf{M}(\lambda) = \begin{bmatrix}\phi_-(0,\lambda) & -\phi_+(0,\lambda) & 0\\ -\nu\phi_-'(0,\lambda) & \nu\phi_+'(0,\lambda) & -\mathcal{C}\dfrac{m_T}{\rho_g}\check{\theta}(\lambda)\\ -\phi_-(0,\lambda) & 0 & 1\end{bmatrix}.$$

Moreover, we noted $\phi_\pm'$ the derivatives with respect to ξ of $\phi_\pm$ and we introduced the auxiliary variable

$$\alpha_0(\lambda) = \vartheta(0,\lambda) = \alpha_-(\lambda)\phi_-(0,\lambda) = \alpha_+(\lambda)\phi_+(0,\lambda).$$

Thanks to Theorem **4.4**, we know that the unknowns $(\alpha_-,\alpha_+,\alpha_0)$ are uniquely determined as soon as λ is in $P_{\lambda_0}^-\backslash\{0\}$. Hence, in $P_{\lambda_0}^-\backslash\{0\}$ the determinant $Det(\mathbf{M}(\lambda))$ of the matrix $\mathbf{M}(\lambda)$ above is nonzero and writes

$$Det(\mathbf{M}(\lambda)) = \nu W(\lambda) - \mathcal{C}\frac{m_T}{\rho_g}\check{\theta}(\lambda),$$

where we took into account the fact that $\phi_\pm(0,\lambda) = 1$. Moreover, we introduced the Wronskian of ϕ_- and ϕ_+, $W(\lambda) = (\phi_-\phi_+' - \phi_+\phi_-')(0,\lambda)$. Finally, we can write explicitly:

$$\begin{bmatrix} \alpha_-(\lambda) \\ \alpha_+(\lambda) \\ \alpha_0(\lambda) \end{bmatrix} = [\mathbf{M}(\lambda)]^{-1} \cdot \begin{bmatrix} 0 \\ \mathcal{C}\dfrac{m_T}{\rho_g}\check{\theta}(\lambda)\psi(0,\lambda) \\ 0 \end{bmatrix}$$

where $[\mathbf{M}(\lambda)]^{-1}$ is the matrix:

$$\frac{1}{\nu W(\lambda) - \mathcal{C}\dfrac{m_T}{\rho_g}\check{\theta}(\lambda)} \times$$

$$\begin{bmatrix} \nu\phi_+'(0,\lambda) & \phi_+(0,\lambda) & \mathcal{C}\dfrac{m_T}{\rho_g}\check{\theta}(\lambda)\phi_+(0,\lambda) \\ \nu\phi_-'(0,\lambda) & \phi_-(0,\lambda) & \mathcal{C}\dfrac{m_T}{\rho_g}\check{\theta}(\lambda)\phi_-(0,\lambda) \\ -\phi_-(0,\lambda)\phi_+'(0,\lambda) & -\phi_+(0,\lambda)\phi_-'(0,\lambda) & W(\lambda) \end{bmatrix}$$

and we have the expression of ϑ:

$$\vartheta(\xi,\lambda) = \mathcal{C}\frac{m_T}{\rho_g}\check{\theta}(\lambda)\psi(0,\lambda)\frac{\phi_\mp(0,\lambda)\phi_\pm(\xi,\lambda)}{\nu W(\lambda) - \mathcal{C}\dfrac{m_T}{\rho_g}\check{\theta}(\lambda)}, \quad for\ \xi \in I\!R_\xi^\pm.$$

Finally, using the same techniques as in the previous subsection, ψ decomposes as follows:

$$\psi(\xi,\lambda) = -\frac{\langle v_0,\phi\rangle\phi(0)}{\lambda\langle\phi,\phi\rangle}\phi_0(\xi) + (A-\lambda I)^{-1}(v_0 - \langle v_0,\phi\rangle\phi)$$

as stated in Eq. (4.20), so that the solution $\check{v} = \vartheta + \psi$ si known and writes as stated in Eq. (4.19). This concludes the proof of Theorem **4.8**. Finally, the result stated in the Corollary **4.9** follows from the fact the Wronskian $W(\lambda)$ introduced above is a $O(\lambda)$ when λ tends to 0 (see [1]). •

5 Linear stability. Asymptotic behaviour

In this section, we solve the linearized problem (4.2), written in the moving frame. This is equivalent to solving the reduced problem (4.3,) in the variable

$\tilde{w}$, introduced in Proposition **4.1** that is:

$$\left\{\begin{array}{l} \partial_t \tilde{w} + \partial_\xi \left[(\Phi - c)\tilde{w}\right] - \nu \partial_{\xi\xi}\tilde{w} = \mathcal{C}\dfrac{m_T}{\rho_g}(d_t \gamma^B)\ \delta_\xi \\ \\ \tilde{w}(\xi, 0) = w_0(\xi), \quad \tilde{w}(\cdot, t) \in \mathcal{B}_{\omega,0} \end{array}\right. \tag{5.1}$$

Theorem 5.1 (Asymptotic behaviour)
Let $[\tilde{w}, \tilde{g}]$ be the solutions of the linearized problem for the coupled system with initial conditions $[w_0, g_0]$. Let λ_0 be defined as in Theorem **4.4**. *Then, for all $\epsilon \in]0, \lambda_0[$, there exist a real $\eta(w_0, g_0; \mathcal{C}, \rho_g, \nu, c)$, a positive constant $C(w_0, g_0, \epsilon, \mathcal{C}, \rho_g, \nu, c)$ such that:*

$$\forall t, \quad \left\{\begin{array}{l} \|\tilde{w}(\cdot, t) - \eta \Phi'(\cdot)\|_{\mathcal{B}_{\omega,0}} \leq C\ e^{-(\lambda_0 - \epsilon)t} \\ \\ \|\tilde{g}(\cdot, t) - m_T\ \delta(\xi + \eta) \otimes \delta_v\|_{\mathcal{B}_s} \leq C \quad e^{-(\lambda_0 - \epsilon)t} \end{array}\right. \tag{5.2}$$

where m_T is the total mass of the spray.

Here, we will use the result:

Proposition 5.2 *Let $\mu \in]0, \lambda_0[$, where λ_0 is defined in Theorem* **4.4**. *Then, there exists a constant $C(w_0, g_0; \mathcal{C}, \rho_g, \nu, c)$ such that the function $v(\xi, t) = \omega(\xi)\tilde{w}(\xi, t)$ writes in terms of the well-defined integral:*

$$v(\xi, t) = C(w_0, g_0; \mathcal{C}, \rho_g, \nu, c)\ \phi(\xi) + \int_{-\mu - i\infty}^{-\mu + i\infty} \hat{v}(\xi, p)\ e^{pt}\ dp. \tag{5.3}$$

Proof of Proposition 5.2
The expression (5.3) is a consequence of the use of the inverse Laplace transform. Denote first by $P^+_{-\lambda_0}$ the open right half-plane $P^+_{-\lambda_0} = -P^-_{\lambda_0}$. Then, recalling the solution $\check{v}$ of the spectral problem in Theorem **4.8** and since $\hat{v}(p) = \check{v}(-p)$, we know that $\hat{v}$ is analytical in $P^+_{-\lambda_0} \backslash \{0\}$ and writes in $I\!R^\pm_\xi$ as:

$$\hat{v}(\xi, p) = \mathcal{C}\frac{m_T}{\rho_g}\theta(p)\psi(0, -p)\frac{\phi_\mp(0, -p)\phi_\pm(\xi, -p)}{\nu W(-p) - \mathcal{C}\dfrac{m_T}{\rho_g}\theta(p)} + \psi(\xi, -p) \tag{5.4}$$

where

$$\psi(\xi, -p) = \frac{\langle v_0, \phi \rangle \phi(0)}{p\langle \phi, \phi \rangle}\phi_0(\xi) + \left[(A + pI)^{-1}(v_0 - \langle v_0, \phi \rangle \phi(0)\phi_0(\cdot))\right](\xi) \tag{5.5}$$

and where the functions ϕ_-, ϕ_+ and ϕ_0 are defined in Theorem **4.8**.

Let us first estimate $\hat{v}(\cdot, p)$ when $p \to \infty$ in $P^+_{\lambda_0}$. Let $a > 0$ be given. Then for any p in $P^+_{\lambda_0}\backslash B(a)$, where $B(a)$ denotes the ball of radius a centered at the origin, we have the classical estimate (see [10]):

$$\forall p \in P^+_{\lambda_0}\backslash B(a), \quad \|(A+pI)^{-1}\|_{\mathcal{L}(\mathcal{B}_0)} \leq \frac{C(a)}{|p|}$$

From this, it follows that $\psi(\xi,-p) \to 0$ when $p \to \infty$ in $P^+_{-\lambda_0}$: Now, in order to estimate $\hat{v}(\cdot,p)$ of (5.4), we have to estimate as $p \to +\infty$ the term:

$$\frac{\phi_\mp(0,-p)\phi_\pm(\xi,-p)}{\nu W(-p) - \mathcal{C}\dfrac{m_T}{\rho_g}\theta(p)} \tag{5.6}$$

We know that $W(-p) = O(|\sqrt{p}|)$ while $\theta(p)$ is bounded in $P^+_{-\lambda_0}$. Moreover, we may observe that for our problem $\phi_-(\xi,-p) = \phi_+(-\xi,-p)$. Since $\phi_\pm(0,-p) = 1$, this allows us to write $W(-p) = \pm 2\phi'_\pm(0,-p)$, and then to rewrite Eq. (5.6) above as:

$$\frac{\phi_\mp(0,-p)\phi_\pm(\xi,-p)}{W(-p)} = \mp\frac{\phi_\pm(\xi,p)}{2\phi'_\pm(0,-p)}$$

But, for any given ξ, we have $|\phi_\pm(\xi,-p)| \leq C(a,\xi)\phi'_\pm(0,-p)$ as $p \to +\infty$ in $P^+_{-\lambda_0}\backslash B(a)$. Indeed, it suffices to write, for ϕ_+ for example:

$$\begin{aligned}
\phi_+(\xi,-p) &= 1 + \int_0^\xi ds \left(\phi'_+(0,-p) + \int_0^s d\zeta \; \partial_{\xi\xi}\phi_+(\zeta,-p)\right) \\
&= 1 + \int_0^\xi ds \left(\phi'_+(0,-p) - \int_0^s d\zeta \; \frac{\phi_+(\zeta,-p)}{a(\zeta)+p}\right),
\end{aligned}$$

where $a(\cdot)$ is defined in Eq. (4.8), and to use Gronwall Lemma, which yields the boundedness of (5.6). The same holds for ϕ_-.

Finally, thanks to the estimate of $\psi(\xi,\cdot)$, it is now clear that, for any ξ, $\hat{v}(\xi,\cdot)$ tends to 0 as $p \to +\infty$ in $P^+_{-\lambda_0}$. Consequently, given any $\mu > 0$, we have for any ξ (see [4, 11]):

$$v(\xi,t) = \int_{\mu-i\infty}^{\mu+i\infty} \hat{v}(\xi,p)e^{pt}\; dp,$$

independently of the value of $\mu > 0$, and we want to extend this result to the case of negative values of μ.

Hence, estimate $\hat{v}(\xi,p)$ as p tends to 0, the unique point of singularity of $\hat{v}$ in $P^+_{-\lambda_0}$. First, the second term in the expression (5.5) of $\psi(\xi,-p)$ tends to:

$$\left[(A+pI)^{-1}(v_0 - \langle v_0,\phi\rangle\phi(0)\phi_0(\cdot))\right](\xi) \to \left[A^{-1}(v_0 - \langle v_0,\phi\rangle\phi(0)\phi_0(\cdot))\right](\xi)$$

and therefore is bounded as p tends to 0. The first term in the expression (5.4) of $\psi(\xi,-p)$ is explicit and writes:

$$\frac{1}{p}\frac{\langle v_0,\phi\rangle}{\langle \phi,\phi\rangle}\phi(\xi).$$

Finally, estimate the first term in the expression (5.4) for $\hat{v}(\xi,p)$. We have the equivalences as p tends to 0 in $I\!R^{\pm}_{\xi}$:

$$\forall \xi \in I\!R^{\pm}_{\xi}, \quad \phi_{\pm}(\xi,p) \sim \phi_0(\xi) = \frac{\phi(\xi)}{\phi(0)},$$

and the functions W, θ, $\psi(0,\cdot)$ and $\phi_{\pm}(0,\cdot)$ are analytical with respect to p. Thanks to Corollary **4.9**, it follows that there exist a positive constant $C(w_0,g_0;\mathcal{C},\rho_g,\nu,c)$ *independent* of ξ, such that

$$\hat{v}(\xi,\cdot) \sim \frac{C(w_0,g_0;\mathcal{C},\rho_g,\nu,c)}{p} \times \phi(\xi) = \mathcal{Q}(p;w_0,g_0,\mathcal{C},\rho_g,\nu,c)\ \phi(\xi)$$

as p tends to 0.

Hence, since $\hat{v}(\xi,\cdot)$ tends to 0 as $p\to\infty$, since it is bounded in $P^{+}_{-\lambda_0}\backslash B(a)$ for any $a\in]0,\lambda_0[$, then for any $\mu\in]0,\lambda_0[$, the function $v(\xi,t)$ writes as:

$$v(\xi,t) = Res(\mathcal{Q})_{|p=0}\ \phi(\xi) + \int_{-\mu-i\infty}^{-\mu+i\infty} \hat{v}(\xi,p)\ e^{pt}\ dp. \tag{5.7}$$

where $Res(\mathcal{Q})_{|p=0} \equiv \left(Res(\mathcal{Q})_{|p=0}\right)(w_0,g_0;\mathcal{C},\rho_g,\nu,c)$ denotes the residue of $\mathcal{Q}$ at $p=0$ (see [4]). In the sequel, we will denote this residue by:

$$Res(\mathcal{Q})_{|p=0} = \eta(w_0,g_0;\mathcal{C},\rho_g,\nu,c) := \eta$$

This concludes the proof of Proposition **5.2**. •

Come back now to the proof of Theorem **5.1**.

Proof of Theorem 5.1: It is an easy consequence of the previous result. We recall it for the sake of completeness. Let $\epsilon\in]0,\lambda_0[$ be given. Then, set $\mu=(\lambda_0-\epsilon)+\frac{\epsilon}{2}$, so that $\mu\in]0,\lambda_0[$, and rewrite Eq. (5.7):

$$\begin{aligned} v(\xi,t) &= \eta\ \phi(\xi) + \int_{-\mu-i\infty}^{-\mu+i\infty} \hat{v}(\xi,p)\ e^{pt}\ dp \\ &= \eta\ \phi(\xi) + e^{-(\lambda_0-\epsilon)t}\int_{y=-\infty}^{y=+\infty} \hat{v}(\xi,-\lambda_0+\frac{\epsilon}{2}+iy)\ e^{-\frac{\epsilon}{2}+iy}\ dy \end{aligned}$$

where we wrote $p = -\mu + iy$. The boundedness of $\hat{v}(\xi, \cdot)$ in $P^{+}_{-\lambda_0} \backslash B(\frac{\lambda_0}{2})$ yields the exponential decay of $v(\cdot, t)$ towards an element of the null space $Vect\{\Phi'\omega\}$ of the linearized operator. Recalling that $\tilde{w}(\xi, t) = v(\xi, t)/\omega(\xi)$, the exponential decay for the gas perturbation in the moving frame is proved, as stated in Theorem **5.1**.

Finally, come back to the spray unknown. Recall that finding the solution $\tilde{g}$ solution of the linearized problem (4.2) is equivalent to studying the characteristics:

$$\forall (x, v) \in Supp(g_0),$$

$$\begin{cases} d_t \eta = \gamma \\ d_t \gamma = -\mathcal{C}(\gamma - \Phi'(0)\eta - \tilde{w}(0, t)) \end{cases} \qquad \begin{cases} \eta(0; x, v, 0) = x \\ \gamma(0; x, v, 0) = v \end{cases} \tag{5.8}$$

so that the unknowns η and γ are the solutions of the system of ODEs:

$$\partial_t \begin{bmatrix} \eta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \mathcal{C}\Phi'(0) & -\mathcal{C} \end{bmatrix} \cdot \begin{bmatrix} \eta \\ \gamma \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{C}\tilde{w}(0, t) \end{bmatrix} \tag{5.9}$$

We will use the simple result:

Proposition 5.3 *Let be given $\alpha > 0$. Let $-\lambda_{\pm}(\alpha)$ be the roots of the polynomial $p^2 + \mathcal{C}p + \mathcal{C}\alpha$, and $\mu > 0$ such that $0 < \mu < \min(Re\lambda_{\pm}(\alpha))$. Finally, let be given (η_0, γ_0) in $I\!R_x \times I\!R_v$. Consider the solutions (η, γ) of the system of ODEs:*

$$\partial_t \begin{bmatrix} \eta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\mathcal{C}\alpha & -\mathcal{C} \end{bmatrix} \cdot \begin{bmatrix} \eta \\ \gamma \end{bmatrix}$$

with the initial data $\eta(0) = \eta_0$ and $\gamma(0) = \gamma_0$. Then, (η, γ) tend to 0. Moreover, we have the estimate:

$$[\eta, \gamma] = O(e^{-\mu t}).$$

Now, come back to the study of (η, γ) defined in Eq. (5.9) above. Taking $\alpha = -\Phi'(0) > 0$, the characteristic polynomial $p^2 + \mathcal{C}p - \mathcal{C}\Phi'(0)$ of this matrix has the roots $-\lambda_{\pm}(-\Phi'(0))$, which have negative real part. Since $\tilde{w}(0, t)$ tends to a constant, then it is clear that (η, γ) tend to the steady states of (5.9), namely:

$$\begin{cases} \eta \longrightarrow -\dfrac{\tilde{w}(0, +\infty)}{\Phi'(0)} = -\eta \\ \\ \gamma \longrightarrow 0 \end{cases}$$

Now, since

$$(0, \mathcal{C}\left[\tilde{w}(0,t) - \tilde{w}(0,+\infty)\right]) = O(e^{-(\lambda_0-\epsilon)t}),$$

and since by definition of λ_0 we have $Re\lambda_{\pm}(-\Phi'(0)) \leq \lambda_0$, then

$$(\eta(t) - \eta(+\infty), \gamma(t)) = O(e^{-(\lambda_0-\epsilon)t}),$$

which yields the asymptotic behaviour for $\tilde{g}$ as stated in Theorem **5.1**. This finishes the proof of Theorem **5.1**. •

6 Nonlinear stability under small perturbations

In this part of the study, we prove the nonlinear stability of the coupled system for small perturbations Following Sattinger, the nonlinear stability under small perturbations is obtained thanks to the implicit function theorem. Recall the main features of the method for the Burgers' problem. Let be given $\epsilon > 0$ and w_0 such that the initial data for the gas writes:

$$u(x,0) = \Phi(x) + \epsilon w_0(x).$$

Then, since $u(x,t)$ is expected to converge to a translate $\Phi(x - ct - \epsilon x_\infty(\epsilon))$ of $\Phi(x - ct)$, one may look for a solution of the problem in the form:

$$u(x,t) = \Phi(x - ct - \epsilon x_\infty(\epsilon)) + \epsilon\tilde{w}(x - ct - \epsilon x_\infty(\epsilon), t)$$

where the new unknown for the gas is the perturbation $\tilde{w}$ in the moving frame $\xi = x - ct - \epsilon x_\infty(\epsilon)$. Moreover, it is important to notice that the moving frame is taken here as $\xi = x - ct - \epsilon x_\infty(\epsilon)$ where $\epsilon x_\infty(\epsilon)$ is the expected final shift. Hence, the constant $x_\infty(\epsilon)$ is also an unknown for the problem of the nonlinear stability. Finally, the nonlinear problem writes in the moving coordinate in the form:

$$\begin{cases} \partial_t \tilde{w} + A\tilde{w} = \epsilon R(\tilde{w}, x_\infty, \epsilon) \\ \tilde{w}(\cdot, 0) = w_0 + x_\infty\ \partial_\xi \Phi + \epsilon H(\cdot; x_\infty, \epsilon) \end{cases}$$

where A is the linearized operator for the Burgers' equation, R accounts for the nonlinear terms and H is a bounded function. Then, for any $\epsilon \in I\!R$ given, it is possible to write the unknowns $(\tilde{w}, x_\infty)$ of the nonlinear problem in terms of the solution of a fixed point problem. Thanks to the implicit function theorem in the Banach spaces, this allows to prove the existence of solutions $(\tilde{w}, x_\infty)$ to the nonlinear problem for small ϵ.

Now, for our problem, we have to adapt this general procedure. Indeed, our aim is to write the nonlinear problem for the coupled system in terms of the linear operator for the coupled system. Here, we shall use again the fact that the transport equation for the spray is solved explicitly by the method of characteristics. Hence, our first aim is to rewrite the nonlinear coupled problem in a reduced form involving only the gas unknown. Our second aim being to recover the linear problem in the limit when ϵ is zero, we shall see that we will have to consider a moving coordinate in which the shift is a function of the time variable which takes into account the motion of the spray, while the shift for the travelling wave of the gas is taken as a constant.

Moreover, the mass-preserving scaling introduced in Eq. (4.1) for the spray unknown allows not to work with derivatives of measures. Indeed, had we used the usual scaling, we would have had to consider perturbations for the spray belonging to the space $Vect\{\partial_\xi\delta_\xi \otimes \delta_v\} \oplus Vect\{\delta_\xi \otimes \partial_v\delta_v\}$, which is not a natural space for the problem, and would have required more attention in the treatment of the nonlinear problem.

For the sake of simplicity, we shall begin the study of the nonlinear stability in the case where the spray reduces to a single particle.

6.1 Nonlinear Stability for a one-particle spray

Let be given $\epsilon \in I\!R$, w_0 in $\mathcal{B}_{\omega,0}$, x_0 in $I\!R_\xi$ and v_0 in $I\!R_v$. Then, we look for the solutions $[u, f]$ of the nonlinear coupled problem with initial data:

$$\begin{cases} \partial_t u + u\partial_x u - \nu\partial_{xx} u = -\mathcal{C}\ \dfrac{\rho_p}{\rho_g}(u - u_p) \\ \partial_t f + v\partial_x f + \partial_v(f\beta) = 0 \end{cases} \tag{6.1}$$

with the initial data:

$$\begin{cases} u(x,0) = \Phi(x) + \epsilon w_0(x) \\ f(x,v,0) = m_T\ \delta(x - \epsilon x_0) \otimes \delta(v - \epsilon v_0), \end{cases}$$

where $\beta = -\mathcal{C}(v - u)$ and where we omit the dependence of $[u, f]$ on ϵ. Then, we introduce the new unknowns $\tilde{w}$, $X(t)$, $V(t)$ and x_∞ such that:

$$\begin{cases} u(x,t) = \Phi(x - ct - \epsilon x_\infty) + \epsilon\tilde{w}(x - ct - X(t), t) \\ f(x,v,t) = m_T\ \delta(x - ct - X(t)) \otimes \delta(v - c - V(t)). \end{cases} \tag{6.2}$$

Hence, if we want f to satisfy the transport equation of (6.1), then it suffices that the functions $(ct + X(t), c + V(t))$ are the trajectory of the particle in the fixed frame. In other words, the transport equation (6.1) for f is solved as soon as:

$$\begin{cases} \dfrac{dX}{dt} = V \\ \\ \dfrac{dV}{dt} = -\mathcal{C}\left(c + V - \Phi(X - \epsilon x_\infty) - \epsilon \tilde{w}(0,t)\right) \end{cases} \qquad \begin{cases} X(0) = \epsilon x_0 \\ \\ V(0) = \epsilon v_0 \end{cases} \tag{6.3}$$

Finally, one must notice that in Eq. (6.2) the function Φ is taken in the moving frame $x - ct - \epsilon x_\infty$ with the *constant* shift x_∞, while the perturbation $\tilde{w}$ is taken in the moving frame $x - ct - X(t)$ with the *variable* shift $X(t)$. Moreover, ϵx_∞ is the expected final shift for the coupled system in the moving frame $x - ct$.

Now, we introduce the Banach spaces of exponentially decaying functions:

$$\mathcal{E}^\mu_{\omega,0} = \left\{ v(t) \in \mathcal{C}([0,+\infty[;\mathcal{B}_{\omega,0}) : \ \sup \ e^{\mu t}\|v\|_{\mathcal{B}_{\omega,0}} < +\infty \right\}$$

$$\mathcal{E}^\mu_0 = \{ v(t) \in \mathcal{C}([0,+\infty[;\mathcal{B}_0) : \ \sup \ e^{\mu t}\|v\|_{\mathcal{B}_0} < +\infty \}$$

$$\mathcal{E}^\mu = \{ v(t) \in \mathcal{C}([0,+\infty[; I\!R) : \ \sup \ e^{\mu t}|v| < +\infty \}$$

where $\mu > 0$ will be precised later. Now, let be given (X, V) as defined in (6.3). We introduce the scaled characteristics (η, γ) such that:

$$\begin{cases} \eta = \dfrac{X - \epsilon x_\infty}{\epsilon} \\ \\ \gamma = \dfrac{V}{\epsilon} \end{cases}$$

Hence, the functions (η, γ) are equivalently defined as the solutions of the system of ODEs:

$$\begin{cases} \dfrac{d\eta}{dt} - \gamma = 0 \\ \\ \dfrac{d\gamma}{dt} + \mathcal{C}\gamma - \mathcal{C}\dfrac{\Phi(\epsilon\eta) - c}{\epsilon} - \mathcal{C}\tilde{w}(0,t) = 0 \end{cases} \qquad \begin{cases} \eta(0) = x_0 - x_\infty \\ \\ \gamma(0) = v_0. \end{cases} \tag{6.4}$$

Finally, we introduce again the function $v \in \mathcal{C}([0,+\infty[;\mathcal{B}_0)$ such that:

$$v(\xi,t) = \tilde{w}(\xi,t)\omega(\xi)$$

Then, given $\epsilon \in I\!R$, it is possible to rewrite the nonlinear problem in terms of v, x_∞, η, γ as follows:

Proposition 6.1 (Nonlinear reduced problem)
Let $\epsilon \in I\!R$ be given. Let $v \in \mathcal{E}_0^\mu$, $x_\infty \in I\!R$ $\eta \in \mathcal{E}^\mu$ and $\gamma \in \mathcal{E}^\mu$ be given. Then the functions $[u, f]$ defined in (6.2) are solutions of the coupled problem (6.1) if and only if on the one hand (η, γ) are the solutions of the system (6.4) and on the other hand, v is the solution of the problem:

$$\left\{ \begin{array}{l} \partial_t v + Av = -\mathcal{C}\dfrac{m_T}{\rho_g}\left[\dfrac{\Phi(\epsilon\eta) - c}{\epsilon} + \tilde{w}(0,t) - \gamma\right] \delta_\xi + \epsilon R(\xi, t; v, x_\infty, \eta, \gamma, \epsilon) \\ \\ v(\xi, 0) = w_0(\xi + \epsilon x_0)\omega(\xi) + x_\infty(\Phi'\omega)(\xi) + \epsilon H(\xi; x_\infty, \epsilon) \end{array} \right. \tag{6.5}$$

where A is the linearized Burgers' operator defined in (4.8). Moreover, we noted here:

$$R(\xi, t; v, x_\infty, \eta, \gamma, \epsilon) = \frac{1}{\epsilon}\partial_\xi\left[(\Phi(\xi) - \Phi(\xi + \epsilon\eta))\frac{v(\xi,t)}{\omega(\xi)}\right]\omega(\xi) + \left(\eta + \frac{v}{\omega}\right)\partial_\xi\left(\frac{v}{\omega}\right)$$

$$H(\xi; x_\infty, \epsilon) = \frac{1}{\epsilon}\left[\frac{\Phi(\xi + \epsilon x_0) - \Phi(\xi + \epsilon(x_0 - x_\infty))}{\epsilon}\omega - x_\infty(\Phi'\omega)(\xi)\right]$$

Hence, our aim is to prove the stability result:

Theorem 6.2 (Nonlinear stability) *Let λ_0 be given as in Theorem* **4.4** *of the study of the linear stability. Let $0 < \mu < \lambda_0$ be given. Then, for $\epsilon \in I\!R$ small enough, the solution $[u, f]$ of the nonlinear coupled problem converges to a travelling wave. Moreover, this convergence is exponential, of order at least $O(e^{-\mu t})$.*

Proof: Equivalently, we want to prove that, for ϵ small enough, there exist a unique $v \in \mathcal{E}_0^\mu$, a unique $x_\infty \in I\!R$, a unique $(\eta, \gamma) \in \mathcal{E}^\mu \times \mathcal{E}^\mu$ such that (η, γ) are the solutions of the system (6.4) and v is the solution of the nonlinear reduced problem (6.5). Here, the operators A and R are defined on the spaces:

$$\left\{ \begin{array}{l} A : \mathcal{B}_0 \longrightarrow H^{-1}(I\!R_\xi) \\ \\ R : \mathcal{B}_0 \longrightarrow H^{-1}(I\!R_\xi) \end{array} \right.$$

Using the decomposition $H^{-1}(I\!R_\xi) = N(A) \oplus R(A)$, we project the equation (6.5) for v onto the null space $N(A)$ of A and the range $R(A)$ of A:

$$v(\xi, t) = p(t)(\Phi'\omega)(\xi) + \zeta(\xi, t), \tag{6.6}$$

where $p(t) \in IR$ and $\zeta(t) \in \mathcal{B}_0 \cap R(A)$. Moreover, we introduce the vector $e^* = \frac{\Phi_\xi \omega}{\langle \Phi_\xi \omega, \Phi_\xi \omega \rangle}$ and the projector Q on $R(A)$ so that:

$$\left\{ \begin{array}{l} p(t) = \langle e^*, v(\cdot, t) \rangle \\ \\ \zeta(t) = Qv(\cdot, t) \end{array} \right. \qquad \left\{ \begin{array}{l} p(0) = \langle e^*, w_0 \omega \rangle + x_\infty + \epsilon \langle e^*, H(\cdot; x_\infty, \epsilon) \rangle \\ \\ \zeta(0) = Q(w_0 \omega + \epsilon H(\cdot; x_\infty, \epsilon)) \end{array} \right.$$

Finally, the equation (6.5) for v writes in terms of p and ζ as:

$$\left\{ \begin{array}{l} \partial_t p - \left\{ \mathcal{C} \dfrac{m_T}{\rho_g} (d_t \gamma) \ \langle e^*, \delta_\xi \rangle \right\} = \epsilon \langle e^*, R(p, \zeta, \eta, \gamma, x_\infty, \epsilon) \rangle \\ \\ \partial_t \zeta + \left\{ A\zeta - \mathcal{C} \dfrac{m_T}{\rho_g} (d_t \gamma) \ Q\delta_\xi \right\} = \epsilon Q R(p, \zeta, \eta, \gamma, x_\infty, \epsilon) \end{array} \right.$$

In order to make the linear problem appear, we finally rewrite the equations for the gas unknowns p and ζ as:

$$\left\{ \begin{array}{l} \partial_t p - \left\{ \mathcal{C} \dfrac{m_T}{\rho_g} (d_t \gamma^0) \ \ \langle e^*, \delta_\xi \rangle \right\} = \epsilon H^\gamma (p, \zeta, \eta, \gamma, x_\infty, \epsilon) \langle e^*, \delta_\xi \rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + \epsilon \langle e^*, R(p, \zeta, \eta, \gamma, x_\infty, \epsilon) \rangle \\ \\ \partial_t \zeta + \left\{ A\zeta - \mathcal{C} \dfrac{m_T}{\rho_g} (d_t \gamma^0) \ Q\delta_\xi \right\} = \epsilon H^\gamma (p, \zeta, \eta, \gamma, x_\infty, \epsilon) \ Q\delta_\xi \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + \epsilon Q R(p, \zeta, \eta, \gamma, x_\infty, \epsilon) \end{array} \right. \tag{6.7}$$

where we introduced the function γ^0 such that $(\eta^0, \gamma^0)(t; p, \zeta, x_\infty)$ are the solutions of the linearized system of ODEs:

$$\left\{ \begin{array}{l} \dfrac{d\eta^0}{dt} - \gamma^0 = 0 \\ \\ \dfrac{d\gamma^0}{dt} + \mathcal{C}\gamma^0 - \mathcal{C}\Phi'(0)\eta^0 - \mathcal{C}v(0, t) = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} \eta(0) = x_0 - x_\infty \\ \\ \gamma(0) = v_0. \end{array} \right.$$

Moreover, we noted:

$$H^\gamma (p, \zeta, \eta, \gamma, x_\infty, \epsilon) = \frac{1}{\epsilon} \mathcal{C} \frac{m_T}{\rho_g} \left(d_t \gamma - d_t \gamma^0 \right)$$

At this stage, we need to solve explicitly the equations for (η, γ). We will use the result:

Lemma 6.3 *Let $\epsilon \in I\!R$ be given. Let $p \in \mathcal{E}^\mu$, $\zeta \in Q\mathcal{E}_0^\mu$ and $x_\infty \in I\!R$ be given. There exists a unique $(\eta,\gamma)(t;p,\zeta,x_\infty,\epsilon)$ in $\mathcal{E}^\mu \times \mathcal{E}^\mu$ solution of the system of ODEs (6.4). Moreover, the application*

$$\begin{cases} (p,\zeta,x_\infty,\epsilon) & \longrightarrow (\eta,\gamma) \\ \mathcal{E}^\mu \times Q\mathcal{E}_0^\mu \times I\!R \times I\!R & \longrightarrow \mathcal{E}^\mu \times \mathcal{E}^\mu \end{cases}$$

is a Frechet differentiable mapping with respect to the variables (p,ζ,x_∞). Moreover, the mappings $(\eta,\gamma)(p,\zeta,x_\infty,\epsilon)$ and $(\partial_{(p,\zeta,x_\infty)}(\eta,\gamma))(p,\zeta,x_\infty,\epsilon)$ are continuous with respect to ϵ, and we have, for $|\epsilon|<1$, the estimate:

$$\sup_t \; e^{\mu t}|(\eta,\gamma)(t;p,\zeta,x_\infty,\epsilon) - (\eta^0,\gamma^0)(t;p,\zeta,x_\infty)| \le C(p,\zeta,x_\infty)\times\epsilon \qquad (6.8)$$

Proof: To prove the result, it suffices to write $(\eta,\gamma)(t;p,\zeta,x_\infty,\epsilon)$ as the solution of a fixed point:

$$\begin{cases} \eta(t) - x_0 + x_\infty - \displaystyle\int_0^t \gamma(s)\;ds = 0 \\ \gamma(t) - e^{-\mathcal{C}t}v_0 - \mathcal{C}\displaystyle\int_0^t e^{-\mathcal{C}(t-s)}\left[v(0,s) + \Phi'(0)\eta(s) + \epsilon H^\Phi(s)\right]\;ds = 0 \end{cases}$$

where $v(\xi,t)$ accounts for its expression in (6.6) and where we introduced the function

$$H^\Phi(t;\eta,\epsilon) = \frac{1}{\epsilon}\left[\frac{\Phi(\epsilon\eta)-c}{\epsilon} - \Phi'(0)\eta\right]$$

which is bounded with respect to ϵ thanks to the regularity of Φ. The regularity of η and γ as functions of $\mathcal{C}([0;+\infty[,I\!R)$ is straightforward (see [5]). Finally, since $0<\mu<\lambda_0$, the result is easily extended to the case of exponentially decreasing functions, as stated in the Lemma. This concludes the proof of Lemma **6.3**. •

It is now possible to rewrite the problem in terms of a fixed point. Denote by $\mathcal{T}$ the restriction of the semigroup operator e^{-tA} on $R(A)$, that is:

$$\begin{cases} \mathcal{T} : H^{-1} \longrightarrow \mathcal{B}_0 \\ \mathcal{T} : v \longrightarrow Qe^{-tA}v = e^{-tA}Qv. \end{cases}$$

Then, taking into account the initial data, solving the equations (6.7) is equivalent to solving the equations:

$$
\begin{aligned}
&p(t) - \left\{ \langle e^*, w_0\omega\rangle + x_\infty + \epsilon\langle e^*, H(\cdot; x_\infty, \epsilon)\rangle + \mathcal{C}\frac{m_T}{\rho_g}\langle e^*, \delta_\xi\rangle(\gamma^0(t) - v_0) \right\} \\
&\qquad - \epsilon\langle e^*, \delta_\xi\rangle \int_0^t H^\gamma(s; p, \zeta, x_\infty, \epsilon)\ ds - \epsilon \int_0^t \langle e^*, R(s; p, \zeta, x_\infty, \epsilon)\rangle\ ds = 0 \\
&\zeta(t) - \left\{ \mathcal{T}(t)\left(w_0\omega + \epsilon H(\cdot; x_\infty, \epsilon)\right) + \mathcal{C}\frac{m_T}{\rho_g}\int_0^t \mathcal{T}(t-s)(d_t\gamma^0)(s)\ \delta_\xi\ ds \right\} \\
&\qquad - \epsilon \int_0^t \mathcal{T}(t-s) H^\gamma(s; p, \zeta, x_\infty, \epsilon)\ \delta_\xi\ ds \\
&\qquad\qquad\qquad - \epsilon \int_0^t \mathcal{T}(t-s)\ R(s; p, \zeta, x_\infty, \epsilon)\ ds = 0
\end{aligned}
\tag{6.9}
$$

where (η, γ) accounts for $(\eta, \gamma)(t; p, \zeta, x_\infty, \epsilon)$, and where (η^0, γ^0) accounts for $(\eta^0, \gamma^0)\ (t;\, p,\, \zeta,\, x_\infty)$. Note also that we have put into brackets the contributions of the linearized operator. Moreover, since we want v to tend to 0 as $t \to +\infty$, then we must have $p(+\infty) = 0$, which means the condition on x_∞:

$$
\begin{aligned}
&-\langle e^*, w_0\omega\rangle - x_\infty + \epsilon\langle e^*, H(\cdot; x_\infty, \epsilon)\rangle - \mathcal{C}\frac{m_T}{\rho_g} v_0\ \langle e^*, \delta_\xi\rangle \\
&\qquad - \epsilon\langle e^*, \delta_\xi\rangle \int_0^\infty H^\gamma(s; p, \zeta, x_\infty, \epsilon) - \epsilon \int_0^\infty \langle e^*, R(p, \zeta, \eta, \gamma, x_\infty, \epsilon)\rangle\ ds = 0.
\end{aligned}
\tag{6.10}
$$

Finally the three equations (6.9)-(6.10) are put in the condensed form:

$$
\mathcal{F}(p, \zeta, x_\infty, \epsilon) = \mathbf{0} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)(p, \zeta, x_\infty, \epsilon),
$$

where

$$\mathcal{F}_1(p,\zeta,x_\infty,\epsilon) = p(t) + \left\{\mathcal{C}\frac{m_T}{\rho_g}\gamma^0(t)\langle e^*,\delta_\xi\rangle\right\} +$$

$$+\,\epsilon\langle e^*,\delta_\xi\rangle\int_t^\infty H^\gamma(s;p,\zeta,x_\infty,\epsilon)\;ds + \epsilon\int_t^\infty \langle e^*, R(s;p,\zeta,x_\infty,\epsilon)\rangle\;ds$$

$$\mathcal{F}_2(p,\zeta,x_\infty,\epsilon) = \zeta(t) -$$

$$-\left\{\mathcal{T}(t)\left(w_0\omega + \epsilon H(\cdot;x_\infty,\epsilon)\right) + \mathcal{C}\frac{m_T}{\rho_g}\int_0^t \mathcal{T}(t-s)(d_t\gamma^0)(s)\;\delta_\xi\;ds\right\}$$

$$-\,\epsilon\int_0^t \mathcal{T}(t-s)H^\gamma(s;p,\zeta,x_\infty,\epsilon)\;\delta_\xi\;ds - \epsilon\int_0^t \mathcal{T}(t-s)\;R(s;p,\zeta,x_\infty,\epsilon)\;ds$$

$$\mathcal{F}_3(p,\gamma,x_\infty,\epsilon) = x_\infty + \langle e^*,w_0\rangle + \epsilon\langle e^*, H(\cdot;x_\infty,\epsilon)\rangle - \mathcal{C}\frac{m_T}{\rho_g}v_0\langle e^*,\delta_\xi\rangle$$

$$+\,\epsilon\langle e^*,\delta_\xi\rangle\int_0^\infty H^\gamma(s;p,\zeta,x_\infty,\epsilon)\;ds + \epsilon\int_0^\infty \langle e^*, R(p,\zeta,x_\infty,\epsilon)\rangle\;ds$$

We now assert:

Proposition 6.4 *For any ϵ, the mapping $\mathcal{F}(\cdot,\cdot,\cdot,\epsilon)$ is a Frechet differentiable mapping from the Banach space*

$$\mathcal{E}^\mu \times Q\mathcal{E}_0^\mu \times I\!R$$

into itself. Moreover, the mapping $\mathcal{F}$ as well as its partial derivative $\partial_{(p,\zeta,x_\infty)}\mathcal{F}$ are continuous with respect to ϵ.

Proof: Consider for example the mapping $\mathcal{F}_2(\cdot,\cdot,\cdot,\epsilon)$ from $\mathcal{E}^\mu \times Q\mathcal{E}_0^\mu \times I\!R$ into $Q\mathcal{E}_0^\mu$. Thanks to the decaying properties of the linearized Burgers' operator, there exist a constant C such that

$$\|\mathcal{T}(t)\|_{\mathcal{L}(H^{-1}(I\!R_\xi),\mathcal{B}_0)} \le Ce^{-\lambda_0 t}.$$

Moreover, since $0<\mu<\lambda_0$, it follows that the convolution of $\mathcal{T}$ with a function of $H^{-1}(I\!R_\xi)$ of order $O(e^{-\mu t})$ is a function of $\mathcal{B}_0$ of order $O(e^{-\mu t})$. But it is quite straightforward to verify that $d_t\gamma^0$, H^γ and R are such exponentially decaying functions of order $O(e^{-\mu t})$. Indeed, it is straightforward that $(d_t\gamma^0)\in\mathcal{E}^\mu$ and $R\in\mathcal{E}_0^\mu$. Then thanks to the estimate (6.8), one verifies that $H^\gamma\in\mathcal{E}^\mu$. Hence $\mathcal{F}_2$ is indeed in $Q\mathcal{E}_0^\mu$. Finally, the regularity of $\mathcal{F}_2$ with respect to (p,ζ,x_∞) follows from the regularity of (η,γ), the regularity of Φ, and the expressions of H, H^γ and R. The continuity of $\mathcal{F}_2$ and $\partial_{(p,\zeta,x_\infty)}\mathcal{F}_2$ with respect to ϵ follows from the regularity of (η, γ), H, H^γ and R with respect to ϵ. The details are left to the reader. This concludes the proof of Proposition **6.4**. $\bullet$

Now, it is possible to apply the implicit function theorem. Indeed, when $\epsilon = 0$, the equations for $p = p^0$, $\zeta = \zeta^0$ and $x_\infty = x_\infty^0$ read:

$$\begin{cases} p^0(t) - \left\{ \langle e^*, w_0\omega\rangle + \mathcal{C}\dfrac{m_T}{\rho_g}\langle e^*, \delta_\xi\rangle(\gamma^0(t) - v_0)\right\} = 0 \\ \zeta^0(t) - \left\{ \mathcal{T}(t)(w_0\omega) + \mathcal{C}\dfrac{m_T}{\rho_g}\displaystyle\int_0^t \mathcal{T}(t-s)(d_t\gamma^0)(s)\ \delta_\xi\ ds \right\} = 0 \\ x_\infty + \langle e^*, w_0\rangle - \mathcal{C}\dfrac{m_T}{\rho_g} v_0 \langle e^*, \delta_\xi\rangle = 0 \end{cases}$$

where

$$\begin{cases} \eta^0(t) - x_0 + x_\infty - \displaystyle\int_0^t \gamma^0(s)\ ds = 0 \\ \gamma^0(t) - e^{-\mathcal{C}t} v_0 - \displaystyle\int_0^t e^{-\mathcal{C}(t-s)} \Big[\mathcal{C}v(0,s) + \mathcal{C}\Phi'(0)\eta^0(s)\Big]\ ds = 0 \end{cases}$$

Hence, the functions $(p^0, \zeta^0, \eta^0(p^0, \zeta^0, x_\infty^0), \gamma^0(p^0, \zeta^0, x_\infty^0), x_\infty^0)$ are the solutions of the linearized problem for the system and we have indeed

$$\mathcal{F}(p^0, \zeta^0, x_\infty^0, 0) = \mathbf{0}.$$

Now, we want to prove that the partial derivative $(\partial_{(p,\zeta,x_\infty)}\mathcal{F})(p^0, \zeta^0, x_\infty^0, 0)$ of $\mathcal{F}$ at the point $(p^0, \zeta^0, x_\infty^0, 0)$ is an isomorphism from $\mathcal{E}^\mu \times Q\mathcal{E}_0^\mu \times I\!R$ into itself. Here, we do not need to compute explicitly this derivative. Indeed, since $(\partial_{x_\infty}\mathcal{F})(p^0, \zeta^0, x_\infty^0, 0) = \Big(0_{\mathcal{E}^\mu}, 0_{Q\mathcal{E}_0^\mu}, 1_{I\!R}\Big)$, then it suffices to prove that $(\partial_{(p,\zeta)}\mathcal{F})(p^0, \zeta^0, x_\infty^0, 0)$ is an isomorphism from $\mathcal{E}^\mu \times Q\mathcal{E}_0^\mu$ into itself. But, when $\epsilon = 0$, finding the null space of the linear functional $(\partial_{(p,\zeta)}\mathcal{F})(\cdot, \cdot, x_\infty^0, 0)$ amounts to solving the linearized problem studied in the previous section, whose solution is unique.

Therefore, the mapping

$$(\partial_{(p,\zeta,x_\infty)}\ \mathcal{F})(p^0, \zeta^0, x_\infty^0, 0)$$

is an isomorphism from $\mathcal{E}^\mu \times Q\mathcal{E}_0^\mu \times I\!R$ into itself and an application of the implicit function theorem yields, for small enough ϵ, a solution $[(p^\epsilon, \zeta^\epsilon, x_\infty^\epsilon); (\eta^\epsilon, \gamma^\epsilon)]$ for the nonlinear coupled problem. This finishes the proof of Theorem **6.2**. $\bullet$

6.2 Nonlinear Stability

We now consider a spray not reduced to a single particle. Let be given $g_0 \in \mathcal{M}_0^b(I\!R_x \times I\!R_v)$. The initial data for the spray now reads:

$$f(x,v,0) = \frac{1}{\epsilon^2} g_0 \left(\frac{x}{\epsilon}, \frac{v}{\epsilon} \right)$$

and we look for the function $\tilde{g}$ such that the solution f for the spray writes:

$$f(x,v,t) = \frac{1}{\epsilon^2} \tilde{g} \left(\frac{x - ct - \epsilon x_\infty}{\epsilon}, \frac{v-c}{\epsilon}, t \right)$$

where ϵx_∞ is the expected final shift for the system. Hence, our aim is to prove again the stability result:

Theorem 6.5 (Nonlinear stability) *Let λ_0 be given as in Theorem* **4.4** *of the study of the linear stability. Let $0 < \mu < \lambda_0$ be given. Then, for ϵ small enough, the solution $[u, f]$ of the nonlinear coupled problem converges to a travelling wave. Moreover, this convergence is exponential, of order at least $O(e^{-\mu t})$.*

Proof:
Recall that in the previous subsection, we had looked for a solution u of the form:

$$u(x,t) = \Phi(x - ct - \epsilon x_\infty) + \epsilon \tilde{w}(x - ct - \epsilon \eta(t), t),$$

that is we had considered the fixed shift ϵx_∞ for Φ and the variable shift $\epsilon\eta(t)$ for $\tilde{w}$, where $\epsilon\eta(t)$ was the position of *the* particle in the moving frame. Now, we consider the variable shift $\epsilon\eta^B(t)$ for $\tilde{w}$, where η^B is the mean position of the spray in the moving frame. Recall first that solving the transport equation for $\tilde{g}$ amounts to solving the characteristics:

$$\forall (x,v) \in Supp(g_0),$$

$$\begin{cases} \dfrac{d\eta}{dt} = \gamma \\ \dfrac{d\gamma}{dt} = -\mathcal{C}\left(\gamma - \dfrac{\Phi(\epsilon\eta) - c}{\epsilon} - \tilde{w}\left(\epsilon(\eta - \eta_B), t\right) \right) \end{cases} \tag{6.11}$$

$$\begin{cases} \eta(0;x,v,0) = x - x_\infty \\ \gamma(0;x,v,0) = v, \end{cases}$$

so that the solution $\tilde{g}$ writes:

$$\tilde{g}(x,v,t) = e^{\mathcal{C}t} g_0\left(\eta(0;x,v,t), \gamma(0;x,v,t)\right)$$

Then, define $(\epsilon\eta^B, \epsilon\gamma^B)$, the mean position and velocity of the spray in the moving frame:

$$\left\{\begin{array}{l} m_T\eta_B(t) = m_0 \displaystyle\int_\eta\int_\gamma g(\eta,\gamma,t)\ \eta\ d\eta d\gamma = m_0 \int_x\int_v g_0(x,v)\ \eta(t;x,v,0)\ dxdv \\[2ex] m_T\gamma^B(t) = m_0 \displaystyle\int_\eta\int_\gamma g(\eta,\gamma,t)\ \gamma\ d\eta d\gamma = m_0 \int_x\int_v g_0(x,v)\ \gamma(t;x,v,0)\ dxdv \end{array}\right.$$

where

$$m_T = m_0 \int_\eta\int_\gamma g(\eta,\gamma,t)\ d\eta d\gamma = m_0 \int_x\int_v g_0(x,v)\ dxdv$$

is the total mass of the spray. Finally, we shall look for a solution u of the form:

$$\left\{\begin{array}{l} u(x,0) = \Phi(x) + \epsilon w_0(x) \\ u(x,t) = \Phi(x-ct-\epsilon x_\infty) + \epsilon\tilde{w}(x-ct-\epsilon\eta^B(t),t) \end{array}\right.$$

so that, in the equation of $v = \tilde{w}\omega$, the exchange term writes in the fixed frame:

$$\tilde{M}(x,t) = \omega(x-ct-\epsilon\eta^B(t)) \times \frac{1}{\epsilon} m_0 \frac{\mathcal{C}}{\rho_g}\int_v f(x,v,t)\times$$
$$\Big[\Phi(x-ct-\epsilon x_\infty) + \epsilon\tilde{w}(x-ct-\epsilon\eta^B(t),t) - v\Big]\ dv.$$

Hence, in the moving frame $\xi = x - ct - \epsilon\eta^B(t)$, this exchange term is evaluated in the sense of the bounded measures against the function $\Psi \in \mathcal{C}^0(I\!R_\xi)$:

$$\langle\tilde{M}(x,t), \Psi(x-ct-\epsilon\eta^B(t))\rangle_{I\!R_\xi}$$
$$= -\frac{\mathcal{C}}{\rho_g} m_0 \int_\eta\int_\gamma g(\eta,\gamma,t)\left[\frac{\Phi(\epsilon\eta)-c}{\epsilon} + \tilde{w}(\epsilon(\eta-\eta^B)) - \gamma\right]$$
$$(\Psi\omega)\Big(\epsilon(\eta-\eta^B)\Big)\ d\eta d\gamma$$
$$= -m_T\frac{\mathcal{C}}{\rho_g}\ (d_t\gamma^0)\ \langle\delta_\xi, \Psi\rangle_{I\!R_\xi} + \sqrt{\epsilon}\ \langle\tilde{H}^\gamma, \Psi\rangle_{I\!R_\xi}$$

where we introduced the bounded measure $\tilde{H}^\gamma(v, x_\infty, \eta, \gamma, \epsilon)$ such that

$$\langle H^\gamma, \Psi\rangle_{I\!R_\xi} = -\frac{1}{\sqrt{\epsilon}}\frac{\mathcal{C}}{\rho_g}\int_\eta\int_\gamma g(\eta,\gamma,t)\{\left[\frac{\Phi(\epsilon\eta)-c}{\epsilon} + \tilde{w}(\epsilon(\eta-\eta^B),t) - \gamma\right]\times$$
$$\times\ (\Psi\omega)\Big(\epsilon(\eta-\eta^B)\Big) - (d_t\gamma^0)\Psi(0)\}\ d\eta d\gamma$$

where $\tilde{w}$ accounts for $\tilde{w} = v/\omega$. Finally, the equation for the gas unknown v is rewritten in terms of the problem with initial data:

$$\begin{cases} \partial_t v + Av = -\mathcal{C}\dfrac{m_T}{\rho_g}\,(d_t\gamma^0)\,\delta_\xi + \sqrt{\epsilon}\,\tilde{H}^\gamma(v, x_\infty, \eta, \gamma, \epsilon) + \epsilon R(v, x_\infty, \eta, \gamma, \epsilon) \\ v(\xi, 0) = w_0(\xi + \epsilon x_0)\omega(\xi) + x_\infty(\Phi'\omega)(\xi) + \epsilon H(\xi; x_\infty, \epsilon) \end{cases}$$

where A is the linearized Burgers' operator defined in (4.8), where $\tilde{H}^\gamma$ is the measure defined above, and where we noted:

$$R(\xi, t; v, x_\infty, \eta^B, \epsilon) = \frac{1}{\epsilon}\partial_\xi\left[\left(\Phi(\xi) - \Phi(\xi + \epsilon\eta^B)\right)\frac{v(\xi,t)}{\omega(\xi)}\right]\omega(\xi) \\ + \left(\eta^B + \frac{v}{\omega}\right)\partial_\xi\left(\frac{v}{\omega}\right)$$

$$H(\xi; x_\infty, \epsilon) = \frac{1}{\epsilon}\left[\frac{\Phi(\xi + \epsilon\eta^B(0)) - \Phi(\xi + \epsilon(\eta^B(0) - x_\infty))}{\epsilon}\omega - x_\infty(\Phi'\omega)(\xi)\right]$$

so that using the decomposition $v(\xi, t) = \langle e^*, v(\xi, t)\rangle(\Phi\omega)(\xi) + \zeta(\xi, t)$, one obtains the equations for p, ζ and x_∞:

$$\begin{aligned} &p(t) + \left\{\mathcal{C}\frac{m_T}{\rho_g}\gamma^0(t)\langle e^*, \delta_\xi\rangle\right\} \\ &\qquad + \sqrt{\epsilon}\langle e^*, \delta_\xi\rangle\int_t^\infty \tilde{H}^\gamma(s; p, \zeta, x_\infty, \epsilon)\,ds + \epsilon\int_t^\infty \langle e^*, R(s; p, \zeta, x_\infty, \epsilon)\rangle\,ds \\ &\zeta(t) - \left\{\mathcal{T}(t)\left(w_0\omega + \epsilon H(\cdot; x_\infty, \epsilon)\right) + \mathcal{C}\frac{m_T}{\rho_g}\int_0^t \mathcal{T}(t-s)(d_t\gamma^0)(s)\,\delta_\xi\,ds\right\} \\ &- \sqrt{\epsilon}\int_0^t \mathcal{T}(t-s)\tilde{H}^\gamma(s; p, \zeta, x_\infty, \epsilon)\,\delta_\xi\,ds - \epsilon\int_0^t \mathcal{T}(t-s)\,R(s; p, \zeta, x_\infty, \epsilon)\,ds \\ &x_\infty + \langle e^*, w_0\rangle + \epsilon\langle e^*, H(\cdot; x_\infty, \epsilon)\rangle - \mathcal{C}\frac{m_T}{\rho_g}v_0\langle e^*, \delta_\xi\rangle \\ &\qquad + \sqrt{\epsilon}\langle e^*, \delta_\xi\rangle\int_0^\infty \tilde{H}^\gamma(s; p, \zeta, x_\infty, \epsilon)\,ds + \epsilon\int_0^\infty \langle e^*, R(p, \zeta, x_\infty, \epsilon)\rangle\,ds \end{aligned} \tag{6.12}$$

which is very similar to that obtained in the case of a one-particle spray. The main difference is that there appears a factor $\sqrt{\epsilon}$ instead of ϵ in the nonlinear terms. This explains when we solve the characteristics:

Lemma 6.6 *Let ϵ be given. Let $p \in \mathcal{E}^\mu$, $\zeta \in Q\mathcal{E}_0^\mu$ and $x_\infty \in I\!R$ be given. Then, for any $(x, v) \in Supp(g_0)$, there exists a unique $(\eta, \gamma)(t, x, v, 0; p, \zeta, x_\infty, \epsilon)$ in*

$\mathcal{E}^\mu \times \mathcal{E}^\mu$ solution of the system of ODEs (6.11). Moreover, the application

$$\begin{cases} (x,v;p,\zeta,x_\infty,\epsilon) & \longrightarrow (\eta,\gamma) \\ I\!R \times I\!R \times \mathcal{E}^\mu \times Q\mathcal{E}_0^\mu \times I\!R \times I\!R & \longrightarrow \mathcal{E}^\mu \times \mathcal{E}^\mu \end{cases}$$

is a Frechet differentiable mapping with respect to the variables (p,ζ,x_∞), uniformly with respect to $(x,v)\in Supp(g_0)$. Moreover, the mappings $(\eta,\gamma)(p,\zeta,x_\infty,\epsilon)$ and $(\partial_{(p,\zeta,x_\infty)}(\eta,\gamma))(p,\zeta,x_\infty,\epsilon)$ are continuous with respect to the variable ϵ, uniformly with respect to $(x,v)\in Supp(g_0)$ and we have, for $|\epsilon|<1$:

$$\sup_t \sup_{(x,v)\in Supp(g_0)} |(\eta,\gamma)(t,x,v,0;p,\zeta,x_\infty,\epsilon) - (\eta^0,\gamma^0)(t,x,v,0;p,\zeta,x_\infty)| \\ \leq C(g_0,p,\zeta)\times\sqrt{\epsilon} \tag{6.13}$$

Proof: To prove the result, it suffices to write $(\eta,\gamma)(t,x,v,0;p,\zeta,x_\infty,\epsilon)$ as the solution of a fixed point:

$$\begin{cases} \eta(t,x,v,0) - x + x_\infty - \displaystyle\int_0^t \gamma(s;x,v,0)\,ds = 0 \\ \gamma(t,x,v,0) - e^{-\mathcal{C}t}v - \mathcal{C}\displaystyle\int_0^t e^{-\mathcal{C}(t-s)}\Big[v(0,s)+\sqrt{\epsilon}H^v(s)+ \\ \qquad\qquad +\Phi'(0)\eta(s)+\epsilon H^\Phi(s)\Big]\,ds = 0 \end{cases}$$

and where we introduced the two functions

$$\begin{cases} H^\Phi(t;\eta,\epsilon) = \dfrac{1}{\epsilon}\left[\dfrac{\Phi(\epsilon\eta)-c}{\epsilon} - \Phi'(0)\eta\right] \\ H^v(t,x,v,0;v,\eta,\epsilon) = \dfrac{1}{\sqrt{\epsilon}}\mathcal{C}\left[\tilde{w}\left(\epsilon(\eta(t,x,v,0)-\eta^B(t)),t\right) - \tilde{w}(0,t)\right] \end{cases}$$

which are bounded functions of $\mathcal{E}^\mu$ with respect to ϵ, uniformly with respect to $(x,v)\in Supp(g_0)$. Indeed, one first observes that there exists a bounded set Ω $(g_0,\ p,\ \zeta,\ x_\infty)$ of $I\!R_\xi \times I\!R_v$ such that, for $|\epsilon|<1$:

$$\forall (x,v)\in Supp(g_0),\quad \forall t>0,\quad (\eta,\gamma)(t,x,v,0;p,\zeta,x_\infty,\epsilon)\in\Omega(g_0,p,\zeta,x_\infty)$$

Hence, the boundedness of H^Φ is quite straightforward. Then, since $\tilde{w}(t)$ is in $\mathcal{B}_0 = H^1(I\!R_\xi)$, and since $H^1(I\!R_\xi)$ injects continuously in $\mathcal{C}^{1,1/2}(\overline{\Omega})$, we have immediately the estimate, for $|\epsilon|<1$, $(x,v)\in Supp(g_0)$ and $t>0$:

$$\begin{aligned} e^{\mu t}&\left|\tilde{w}\left(\epsilon(\eta(t,x,v,0)-\eta^B(t)),t\right)-\tilde{w}(0,t)\right| \\ &\leq C(g_0,p,\zeta,x_\infty)\sqrt{|\eta(t,x,v,0)-\eta^B(t)|}\times\sqrt{\epsilon} \\ &\leq C(g_0,p,\zeta,x_\infty)\times\sqrt{\epsilon} \end{aligned}$$

which yields the desired boundedness of H^v. The results of the Lemma then follow. This concludes the proof of Lemma **6.6**. •

Come back now to the proof of the nonlinear stability. Since, (η, γ) are Frechet differentiable functions of (p, ζ, x_∞), then the equations (6.12) define a mapping $\mathcal{F}$ of

$$(p, \zeta, x_\infty) \in \mathcal{E}^\mu \times Q\mathcal{E}_0^\mu \times I\!R$$

into itself. The Frechet differentiability with respect to (p, ζ, x_∞) of this mapping F is quite straightforward. The continuity with respect to ϵ of the mapping $\mathcal{F}$ and of its partial derivative $\partial_{(p,\zeta,x_\infty)}\mathcal{F}$ follow now from the expression of $\tilde{H}^\gamma$ and the estimate (6.13).

Finally, when $\epsilon = 0$, one obtains the linear reduced problem, as in the case of a one-particle spray. Hence, we invoke again the implicit function theorem in order to prove the nonlinear stability for small enough ϵ. This finishes the proof of Theorem **6.5**. •

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