The Yoccoz Combinatorial Analytic Invariant

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15th February 2008

Abstract

In this paper we develop a combinatorial analytic encoding of the Mandelbrot set $\mathbf{M}$. The encoding is implicit in Yoccoz' proof of local connectivity of $\mathbf{M}$ at any Yoccoz parameter, i.e. any at most finitely renormalizable parameter for which all periodic orbits are repelling. Using this encoding we define an explicit combinatorial analytic modelspace, which is sufficiently abstract that it can serve as a go-between for proving that other sets such as the parabolic[1] Mandelbrot set $\mathbf{M_1}$ has the same combinatorial structure as $\mathbf{M}$. As an immediate application we use here the combinatorial-analytic model to reprove that the dyadic veins of $\mathbf{M}$ are arcs and that more generally any two Yoccoz parameters are joined by a unique ruled (in the sense of Douady-Hubbard) arc in $\mathbf{M}$.

1 The Douady-Hubbard description of M

The Mandelbrot set $\mathbf{M}$ equals the set of parameters $c \in \mathbb{C}$ for which the critical point 0 of $Q_c(z) = z^2 + c$ does not iterate to ∞:

$$\mathbf{M} = \{c \in \mathbb{C} \mid Q_c^n(c) \underset{n\to\infty}{\not\to} \infty\}$$

In the following we shall briefly outline the basic properties of $\mathbf{M}$ proved by Douady and Hubbard in [DH1] and [DH2]. Let ϕ_c denote the Böttcher coordinate at infinity with the radially maximally extended co-domain U_c so that $U_c \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is starshaped around infinity and for each ray $\mathcal{R}_\theta = \mathrm{e}^{i2\pi\theta}\cdot]1, \infty]$ we have $U_c \cap \mathcal{R}_\theta = \mathrm{e}^{i2\pi\theta}\cdot]r_c(\theta), \infty]$. We let $\psi_c : U_c \longrightarrow B_c = \{z \mid Q_c^n(z) \underset{n\to\infty}{\longrightarrow} \infty\}$ denote the inverse of ϕ_c. The map ψ_c is surjective if and only if $U_c = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

[1] see Definition 2.1

The Green's function $g_c(z) = \log|\phi_c(z)|$ is the harmonic function on B_c satisfying $g_c(Q_c(z)) = 2g_c(z)$ and $g_c(z) = \log|\phi_c(z)|$ on $\psi_c(U_c)$. It extends as a subharmonic function to all of $\mathbb{C}$, taking the value 0 on the filled-in Julia set $K_c = \mathbb{C}\backslash B_c$. The (external) ray $\mathcal{R}^c_\theta$ of K_c with argument $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the arc $\mathcal{R}^c_\theta = \psi_c(\mathcal{R}_\theta \cap U_c)$. The ray $\mathcal{R}^c_\theta$ is also the unbounded field line of g_c making the angle θ with the positive real axis at infinity. Notice that the straight ray $\mathcal{R}_\theta$ coincides with the external ray of argument θ for $Q_0(z) = z^2$. Notice also that Q_c acts on external rays by doubling argument and potential. (Pre)-periodicity of θ under angle doubling thus let us write it as $2^k(2^l\theta) \equiv 2^l\theta \mod 1$, where $l \geq 0$ is the pre-period and $k \geq 1$ is the period. Hence θ is (pre)-periodic if and only if it is rational and periodic if and only if its reduced form has odd denominator. We also say that $\mathcal{R}^c_\theta$ is rational, when θ is rational.

The ray $\mathcal{R}^c_\theta$ is said to land at $z \in J_c$ if $r_c(\theta) = 1$ and $\overline{\mathcal{R}^c_\theta} = \{z\} \cup \mathcal{R}^c_\theta$. If $r_c(\theta) > 1$ then the ray $\mathcal{R}^c_\theta$ is said to bump onto $z \in B_c$, where $\overline{\mathcal{R}^c_\theta} = \{z\} \cup \mathcal{R}^c_\theta$. The point z is then a (pre)-critical point for Q_c and is also the bumping point of another ray $\mathcal{R}^c_{\theta'}$ meeting $\mathcal{R}^c_\theta$ head on at z. From z and orthogonally to these emanate two bounded field lines of g_c. They may either bump again on another pre-critical point, which happens precisely if θ is periodic under angle doubling, or converge to J_c and possibly land on a point in J_c. Bumping of rays occurs only if $c \notin \mathbf{M}$.

The following was proved first by Sullivan, Douady and Hubbard, see also [P, Th. A and Prop. 2.1] for a more general result.

Theorem 1.1. *For every $c \in \mathbb{C}$ every non bumbing rational ray $\mathcal{R}^c_\theta$ lands on a (pre)-periodic point. More precisely suppose $2^{k+l}\theta \equiv 2^l\theta \mod 1$ with $k > 0$ and $l \geq 0$ minimal.*

1. *If $l = 0$ then $\mathcal{R}^c_\theta$ lands on either a repelling or a parabolic periodic point z of period k' dividing k. Moreover if parabolic, then its multiplier equals some root of unity $\mathrm{e}^{i2\pi p/q}$ with $(p,q) = 1$ and $k'q$ dividing k.*

2. *If $l > 0$ then $\mathcal{R}^c_\theta$ lands on a pre-periodic point z of pre-period l and $Q^l_c(\mathcal{R}_\theta) = \mathcal{R}^c_{2^l\theta \mod 1}$ lands on the periodic point $Q^l_c(z)$, and has the properties above.*

The landing of the (pre)-periodic ray $\mathcal{R}^c_\theta$ at the (pre)-periodic point $z(c)$ is locally stable at c when $Q^l_c(z)$ is repelling and 0 is not in the orbit of z. In particular the landing is globally stable in any connected open set for which $Q^l_c(z)$ remains repelling and 0 does not enter the forward orbit of $\overline{\mathcal{R}^c_\theta}$.

Definition 1.2. *A q-cycle under Q^k_c of external rays $\mathcal{R}^c_0, \ldots, \mathcal{R}^c_{q-1}$ co-landing on a common k-periodic point z and numbered in the counter clockwise order around z defines* combinatorial rotation number *p/q, $(p,q) = 1$ iff $Q^k_c(R_j) = R_{(j+p) \mod q}$.*

We also have conversely to the theorem above:

Theorem 1.3 (Douady-Yoccoz landing theorem). *If $z \in J_c$ is any repelling or parabolic (pre)-periodic point. Then there is at least one (pre)-periodic ray landing at z. In particular z has a combinatorial rotation number.*

For a proof of an even more general case see e.g. [P, Th. B, for the case of rational maps, the general case is identical].

Douady and Hubbard continued to define a holomorphic map $\Phi : \overline{\mathbb{C}}\backslash\mathbf{M} \longrightarrow \overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$, tangent to the identity at ∞ by the formula $\Phi(c) = \phi_c(c)$, $c \in \mathbb{C} \setminus \mathbf{M}$ and $\Phi(\infty) = \infty$. They proved that Φ is biholomorphic. We let $\Psi = \Phi^{-1}$ denote its inverse. For $\theta \in \mathbb{T}$ the parameter ray of argument θ is defined by $\mathcal{R}_\theta^{\mathbf{M}} := \Psi(\mathcal{R}_\theta)$. (We shall in the following use the term dynamical ray whenever we need to distinguish a ray in a dynamical plane from a ray in the parameter space.)

Theorem 1.4 (Douady and Hubbard). *Every pre-periodic (i.e. rational ray) $\mathcal{R}_\theta^{\mathbf{M}}$ of $\mathbf{M}$ lands. More precisely suppose $2^{k+l}\theta \equiv 2^l\theta \mod 1$ with $k > 0$ and $l \geq 0$ minimal.*

1. *If $l = 0$ then $\mathcal{R}_\theta^{\mathbf{M}}$ lands on a parameter c for which the corresponding dynamical ray $\mathcal{R}_\theta^c$ lands at a parabolic periodic point $z(c)$, with exact period $k' | \, k$ and with multiplier $\lambda = (Q_c^{k'})'(z(c))$ a primitive k/k'-th root of unity. Moreover there exists a second argument θ' with $2^k\theta' \equiv \theta' \mod 1$ such that also $\mathcal{R}_{\theta'}^{\mathbf{M}}$ lands at c and such that the Jordan curve in the dynamcial plane of Q_c, $\mathcal{R}_\theta^c \cup \mathcal{R}_{\theta'}^c \cup \{z(c)\}$ separates the parabolic petal of $z(c)$ containing c from any other parabolic petal of the orbit of $z(c)$. We also say that the two rays $\mathcal{R}_\theta^c \cup \mathcal{R}_{\theta'}^c$ are adjacent to c.*
2. *If $l > 0$ then $\mathcal{R}_\theta^{\mathbf{M}}$ lands at a parameter c for which the corresponding dynamical ray $\mathcal{R}_\theta^c$ also lands on c and for which $Q_c^{k+l}(c) = Q_c^l(c)$ is a repelling periodic point of exact period k. Moreover for any dynamical ray $\mathcal{R}_{\theta'}^c$ landing on c, the corresponding parameter ray $\mathcal{R}_{\theta'}^{\mathbf{M}}$ lands at $c \in \mathbf{M}$.*

The map Q_c has (counting multiplicity) two fixed points. The beta-fixed point $\beta(c)$ is by definition the landing point of the unique fixed ray, $\mathcal{R}_0^c$. The other fixed point $\alpha(c)$ is attracting in the main hyperbolic component H_0 bounded by the main cardioid $\heartsuit$, neutral on $\heartsuit$ and repelling outside. Moreover $\lambda(\alpha(c))$ defines a bi-holomorphic map of $\mathbb{C}\backslash[\frac{1}{4}, \infty]$ onto $\mathbb{C}\backslash[1, \infty]$, which sends zero to zero with derivative 2.

For each p/q, $(p, q) = 1$ there is a unique parameter $c = c_{p/q} \in \heartsuit$ for which $\lambda(\alpha(c)) = \mathrm{e}^{i2\pi p/q}$. For this parameter the unique q-cycle of rays $\mathcal{R}_{\theta_0}^c, \mathcal{R}_{\theta_1}^c, \ldots, \mathcal{R}_{\theta_{q-1}}^c$, $0 < \theta_0 < \theta_1 < \ldots < \theta_{q-1} < 1$ of combinatorial rotation number p/q, i.e. $2\theta_j \equiv \theta_{(j+p) \mod q} \mod 1$ lands on $\alpha(c)$. Moreover the minimal distance in $\mathbb{T}$ between neighbouring θ's in the cycle is realized by θ_{p-1} and θ_p and the corresponding parameter rays $\mathcal{R}_{\theta_{p-1}}^{\mathbf{M}}$, $\mathcal{R}_{\theta_p}^{\mathbf{M}}$ land together at $c_{p/q}$ tangentially to $\heartsuit$. The wake $W(c_{p/q})$ of $c_{p/q}$ is the Jordan domain bounded by

$\mathcal{R}^{\mathbf{M}}_{\theta_{p-1}} \cup \mathcal{R}^{\mathbf{M}}_{\theta_p} \cup \{c_{p/q}, \infty\}$ and not containing 0. It consists precisely of those parameters c for which $\alpha(c)$ is repelling and has combinatorial rotation number p/q. Define the limb $L_{p/q} := \{c_{p/q}\} \cup W(c_{p/q}) \cap \mathbf{M}$ and the uprooted limb $L^\bigstar_{p/q} := W(c_{p/q}) \cap \mathbf{M}$. The Douady-Yoccoz ray landing Theorem implies that for any point $c \in \mathbf{M}\backslash\overline{H_0}$ the fixed point $\alpha(c)$ is the landing point of at least one periodic ray and thus has a rational combinatorial rotation number. Hence the Mandelbrot set naturally decomposes into a disjoint union

$$\mathbf{M} = \overline{H_0} \cup \bigcup_{\frac{p}{q} \neq \frac{0}{1}} L^\bigstar_{p/q}.$$

Any periodic point $z(c_0)$ with multiplier $\lambda(z(c_0)) \neq 1$ can be followed holomorphically (according to the implicit function theorem applied to $Q_c^k(z) = z$) in a neighbourhood of c_0 and the corresponding multiplier function varies also holomorphically. In the particular case $\lambda(z(c_0)) \in \mathbb{D}$ there exists a topological disk $H \subset \mathbf{M}$ containing c_0 such that $z(c)$ extends to a holomorphic function on H and the map $\lambda(z(c))$ extends to homeomorphism from $\overline{H}$ onto $\overline{\mathbb{D}}$ which is analytic except possibly above 1. Such a disk H is called a hyperbolic component of $\mathbf{M}$. The unique point $c^* = c^*_H$ of H with $\lambda(z(c^*)) = 0$ is called the center of H. It satisfies $Q^k_{c^*}(0) = 0$, so that both 0 and hence c^* belongs to the orbit of $z(c^*)$. The period $k \geq 1$ of the attracting orbit is called the period of the hyperbolic component. Moreover the unique point $c_H \in \partial H$ with $\lambda(c_H) = 1$ is called the root of H. At the root the k periodic orbit coincides with another orbit of period k' a divisor of k.

For $\frac{p}{q} \neq \frac{0}{1}$, $(p,q) = 1$ the parameter $c_{p/q}$ is the root of the unique hyperbolic component $H_{p/q} \subset L_{p/q}$ of period q. The q-periodic orbit for $c_{p/q}$ coincides with the fixed point $\alpha(c_{p/q})$ of Q_c. The center $c^* = c^\bigstar_{p/q}$ of $H_{p/q}$ we shall call it the center of $L^\bigstar_{p/q}$.

Douady and Hubbard developed this theory further in [DH3], where they prove that for any hyperbolic component H of $\mathbf{M}$ with period k and center c^*, there exists a compact connected subset $\mathbf{M}_H \supset H$ of $\mathbf{M}$ and a homeomorphism $\chi_H : \mathbf{M}_H \longrightarrow \mathbf{M}$ such that: For each $c \in \mathbf{M}_H$ the map Q_c^k has a quadratic like restriction with connected Julia set, i.e. Q_c is k-renormalizable. And such that the k-periodic orbit $z_0 = 0, z_1 = c^*, \ldots, Q^{k-1}_{c^*}(z_0)$ can be followed continuously (holomorphically except at the root) in $\mathbf{M}_H$ and $z_0(c)$ is the α-fixed point of the renormalization containing 0.

2 The Yoccoz Puzzle $\mathcal{Y}$

For the rest of this section we fix a non zero rational $p/q \neq 0/1$, $(p,q) = 1$. All the introduced quantities will depend on p/q, but we shall only occasionally make reference to p/q. Also we shall fix an arbitrary choice of equipotential level say 1. For $c \in L_{\frac{p}{q}}$ let $\mathcal{GY}^0_c$ denote the graph consisting of the equipotential of level 1 for the Green's function g_c

of K_c and the segments of the external rays in the unique p/q cycle $\mathcal{R}^c_{\theta_j}$ together with their preimages by Q_c inside the equipotential chosen above along with $\alpha(c)$ and its preimage $-\alpha(c)$. The level 0 or base (p/q-Yoccoz) puzzle pieces are the closures of each of the bounded connected components of $\mathbb{C}\backslash\mathcal{G}\mathcal{Y}^0_c$. The level 0 or base ($p/q$-Yoccoz) puzzle $\mathcal{Y}^0_c$ is the collection of the $2q-1$ level 0 puzzle pieces. The level $n \in \mathbb{N}$ (p/q-Yoccoz) puzzle $\mathcal{Y}^n_c$ is the collection of closures of connected components of $Q_c^{-n}(\mathring{Y})$, where Y ranges over all of the level 0 puzzle pieces. The (p/q-Yoccoz) Puzzle for Q_c is the union $\mathcal{Y}_c = \cup_{n\geq 0}\mathcal{Y}^n_c$ of the puzzles at all levels.

Any two puzzle pieces $Y \in \mathcal{Y}^n_c$ and $Y' \in \mathcal{Y}^m_c$, $m \leq n$ are either interiorly disjoint or nested with $Y \subseteq Y'$, because potential is multiplied by two under the dynamics and the set of rays in the construction of $\mathcal{Y}^0_c$ is forward invariant

A *nest* i.e. a sequence $\mathcal{N} = \{Y^n\}_n$, $Y^n \in \mathcal{Y}^n_c$, $Y^{n+1} \subseteq Y^n$ is called *convergent* iff $\mathrm{End}(\mathcal{N}) := \bigcap_{n\in\mathbb{N}} Y^n$ is a singleton set and is called *divergent* otherwise. A nest $\mathcal{N}$ is called *critical* iff $0 \in \mathrm{End}(\mathcal{N})$ and called a *critical value nest* iff $c \in \mathrm{End}(\mathcal{N})$.

Yoccoz convergence theorems (see Theorem 3.20 for a parameter space statement) states that for any parameter c such that Q_c is non-renormalizable and all its periodic points are repelling the Julia set J_c is locally connected and Q_c is uniquely given by its Yoccoz puzzle. The latter also being phrased as "Q_c is combinatorially rigid". Moreover if Q_c is renormalizable one may iterate the Yoccoz-puzzle construction on the renormalizations and obtain local connectivity of the Julia set J_c and rigidity of Q_c for any Yocoz parameter. Rigidity means that Q_c is "determined by its combinatorics". To make this phrase more precise let us introduce:

2.1 The Universal Yoccoz Puzzle $\mathcal{U}$.

When $c \in \mathbf{M}$ the Böttcher coordinate, which conjugates Q_c to Q_0 at infinity extends to a bi-holomorphic map $\phi_c : B_c(\infty) \longrightarrow \overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$ tangent to the identity at ∞. When viewed in the Böttcher coordinates at ∞ any two Yoccoz puzzles $\mathcal{Y}_c$ and $\mathcal{Y}_{c'}$ with $c, c' \in L_{p/q}$ are "identical". In fact the graph $\mathcal{GU}^0 := \phi_c(\mathcal{GY}^0_c\backslash\{\alpha, -\alpha\}) \cup \mathbb{S}^1$ is the union of the unit circle, the circle of radius e^1 and the segments of the straight rays $\mathcal{R}_{\theta_j}, \mathcal{R}_{\theta_{j+\frac{1}{2}}}$, $0 \leq j < q$ between the two circles. The base universal (p/q-Yoccoz) puzzle $\mathcal{U}^0$ consists of the closures of the $2q$ connected components of the complement of $\mathcal{GU}^0$ between the two circles. Notice that ϕ_c is defined only on the parts of each puzzle piece outside K_c, and that the intersection with $B_c(\infty)$ of the critical puzzle piece, i.e. the puzzle piece containing the critical point 0, has two connected components. Hence this puzzle piece is 'split' in two by ϕ_c, where as the other puzzle pieces have connected intersection with $B_c(\infty)$ and so are mapped to one puzzle piece each. This is why the base universal puzzle has $2q$ pieces, where as the base puzzle of Q_c only has $2q-1$ pieces. Moreover as ϕ_c is a conjugacy, the image $\mathcal{U}^n$

of the level n puzzle $\mathcal{Y}_c^n$ consists of the $2^n 2q$ pieces $Q_0^{-n}(U)$, where U ranges over the $2q$ pieces of the base puzzle $\mathcal{U}^0$. Define $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{U}^n$. We call $\mathcal{U}$ *the universal p/q-Yoccoz puzzle.*

Let $\mathcal{Z}^0$ denote the set consisting of the unique q cycle for Q_0 of combinatorial rotation number p/q in $\mathbb{S}^1$ and its preimage (corresponding to the arguments θ_j, $\theta_j + \frac{1}{2}$ above). Then $\mathcal{Z}^0$ consists of the end points on $\mathbb{S}^1$ of the rays in the universal p/q-base puzzle $\mathcal{U}^0$. Define $\mathcal{Z}^n := Q_0^{-n}(\mathcal{Z}^0)$ so that $\mathcal{Z}^n$ consists of the end points on $\mathbb{S}^1$ of the rays in the level n universal p/q puzzle $\mathcal{U}^n$. Notice that $\mathcal{Z}^n \subset \mathcal{Z}^{n+1}$ for all n. Let $\mathcal{Z}$ denote the increasing union $\mathcal{Z} := \cup_{n\geq 0}\mathcal{Z}^n$.

Yoccoz' theorem on combinatorial rigidity can be interpreted as follows: Though the common model $\mathcal{U} = \phi_c(\mathcal{Y}_c) = \phi_{c'}(\mathcal{Y}_{c'})$ does not detect directly any differences between c and c' from the same limb. A natural sequence or tower of equivalence relations $\{\sim_n^c\}_{n\geq 0}$ on $\mathcal{Z}$ does. The equivalence relation $\sim_n^c$ is obtained by declaring τ and τ' in $\mathcal{Z}^n$ equivalent if and only if the corresponding (rational) rays for Q_c land at a common point. The notion of combinatorics of the quadratic polynomial then becomes its associated tower of equivalence relations $\sim_n^c$ on $\mathcal{Z}^n$. And combinatorial rigidity amounts to the statement that c is uniquely determined by the infinite tower $\{\sim_n^c\}_{n\geq 0}$ of equivalence relations.

This motivates the following study of which such towers are potentially possible and which are actually realized as well as the concluding combinatorial analytic model $\mathcal{C}$ of $\mathbf{M}$. The model has the virtue of being sufficiently abstract that it can be used for other sets. It is for instance also a faithfull model of the parabolic Mandelbrot set $\mathbf{M_1}$ and of the limit sets obtained when limbs of the Mandelbrot set degenerate during the holomorphic motion of $\mathbf{M}$ induced by moving the multiplier at infinity.

Definition 2.1. *In the space of quadratic rational maps modulo Möbius conjugacy, let $Per_1(1)$ denote the complex line of conjugacy classes of maps with a fixed point of multiplier 1. Then, the parabolic Mandelbrot set, $\mathbf{M_1}$, is the connectdness locus, i.e. the set of rational maps in $Per_1(1)$ with connected Julia set.*

3 The Yoccoz Combinatorial-Analytic invariant

There are many combinatorial models for the Mandelbrot set $\mathbf{M}$, e.g. kneading sequences and their generalizations, Hubbard trees etc, see e.g. [BrSl] for a comprehensive discussion and comparison. In this section we shall introduce yet another way of modelling $\mathbf{M}$. We call it the Yoccoz Combinatorial-Analytic Invariant. The model consists of combinatorial objects called p/q-equivalences and for the subset of renormalizable p/q-equivalences (see Definition 3.13) also a complex parameter in $\mathbf{M}$ containing analytical information obtained by renormalization. The set $\mathcal{F}_{p/q}$ of p/q-equivalences have the virtue of being in

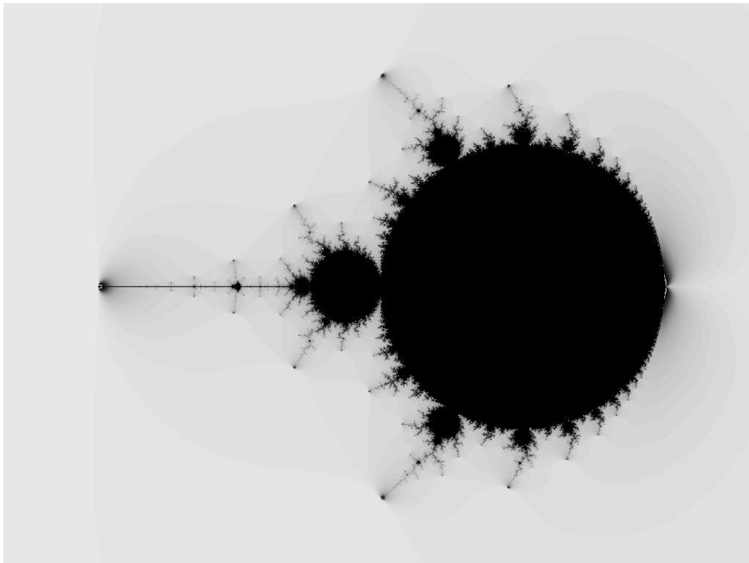

Figure 1: The parabolic Mandelbrot set $\mathbf{M_1}$.

exact one to one correspondence with the quadratic polynomials in the $L_{p/q}$-limb modulo renormalization, no more no less. In other words "if it exists combinatorially it is also realized by an essentially unique quadratic polynomial". That is $\mathcal{F}_{p/q}$ is a model for $L_{p/q}$ in which every first renormalization copy of $\mathbf{M}$ is reduced to a point. This model/notion was initially created in order to pass smoothly from quadratic polynomials to e.g. $Per_1(1)$, the space of Möbius conjugacy classes of quadratic rational maps having a fixed point of multiplier 1. Here we shall also show however that with a few extra ingredients we can reprove that e.g. veins are arcs.

3.1 Towers of equivalence relations

We consider finite and infinite tuples $(\sim_n)_{0\leq n\leq N}$, where $N \in \mathbb{N}\cup\{\infty\}$ and where for each n the object $\sim_n$ is an equivalence relation on the set $\mathcal{Z}^n$ defined in the previous section. However first a few preparations. Let $\mathcal{Z}^{-1} = \mathcal{Z}^{-2} = \ldots = \mathcal{Z}^{-q+1}$ denote the set consisting of the unique q cycle for Q_0 of combinatorial rotation number p/q in $\mathbb{S}^1$. So that $Q_0(\mathcal{Z}^j) = \mathcal{Z}^{j-1}$ when $j-1 > -q$ and $\mathcal{Z}^0 := Q_0^{-1}(\mathcal{Z}^{-1})$. Recall that $\mathcal{Z}^n := Q_0^{-n}(\mathcal{Z}^0)$ so that $\mathcal{Z}^n$ consists of the end points on $\mathbb{S}^1$ of the rays in the level n universal p/q puzzle $\mathcal{U}^n$. Moreover $\mathcal{Z}^n \subset \mathcal{Z}^{n+1}$ for all n by the forward invariance of $\mathcal{Z}^0$ and $\mathcal{Z} := \cup_{n\geq 0}\mathcal{Z}^n$.

For $E \subset \mathbb{S}^1$ we let $H(E)$ denote E union its hyperbolic convex hull in $\mathbb{D}$. Note that the notion of hyperbolic convexity of subsets of $\overline{\mathbb{D}}$ is well defined. Because any two distinct points on $\mathbb{S}^1$ are joined by a unique hyperbolic geodesic in $\mathbb{D}$. For $U \subset \overline{\mathbb{D}}$ we let δU

denote the set $U \cap \mathbb{S}^1$. We shall say that a subset $U \subset \overline{\mathbb{D}}$ is *ideally convex* if and only if $U = H(\delta U)$. That is U is hyperbolically convex with extremal boundary δU.

Define universal and trivial equivalence relations $\sim_j$ on $\mathcal{Z}^j$, $-q < j < 0$ with just one equivalence class, all of $\mathcal{Z}^j$.

Definition 3.1. *A possibly infinite tuple of equivalence relations* $(\sim_n)_{0 \leq n \leq N}$ *is called a tower if it satisfies the following admissibility conditions, (see also Kiwi [Ki] for a more complete discussion):*

i) (Proper) *For any finite* n, $0 \leq n \leq N$ *and any (equivalence) class* E *of* $\sim_n$ *the set* $Q_0(E)$ *is a class of* $\sim_{(n-1)}$.

ii) (Conservative) *For any* m, n, $-1 \leq m < n \leq N$: $\sim_n |_{\mathcal{Z}^m} = \sim_m$.

iii) (Union) *If* $N = \infty$ *then* $\sim_\infty = \displaystyle\bigcup_{n=0}^{\infty} \sim_n$.

iv) (Un-crossed) *For any* n, $0 \leq n \leq N$ *and any two classes* E *and* E' *of* $\sim_n$:

$$H(E) \cap H(E') = \emptyset.$$

Note that $\sim_N = \displaystyle\bigcup_{n=0}^{N} \sim_n$ when N is finite by the Conservative property. Also reversely $\sim_n = \sim_N |_{\mathcal{Z}_n}$. Henceforth we shall abbreviate the notation $(\sim_n)_{0 \leq n \leq N}$ for a tower of height N to just $\sim_N$.

Remark 3.2. *Though finite and infinite towers share the same admissibility properties, we consider them conceptually different. The finite towers* $\sim_N$ *are the nodes of a tree with root* $\sim_0$ *and with a branch connecting* $\sim_N$ *back to* $\sim_{N-1}$. *We denote this tree by* $\mathcal{T}$ *or* $\mathcal{T}_{p/q}$ *if we need to emphasise the dependence of* p/q. *The infinite towers* $\sim_\infty$ *on the other hand are the infinite branches of this tree starting at* $\sim_0$. *We denote the set or space of all infinite branches* $\mathcal{T}^\infty$ *or* $\mathcal{T}^\infty_{p/q}$ *if necessary. In order to obtain the right space we will eventually have to pass to a quotient* $\mathcal{F}$ *of* $\mathcal{T}^\infty$. *And we shall equip the space* $\mathcal{F}$ *with a Hausdorff topology.*

In view of the Conservative property we define the level of a class E as the minimal $n \geq 0$ for which $E \subset \mathcal{Z}^n$.

For any un-crossed equivalence relation $\sim$ on $\mathbb{S}^1$ we define the graph $\mathcal{G}_\sim$ of $\sim$ as

$$\mathcal{G}_\sim = \bigcup_{E \text{ a class of } \sim} H(E) \subset \overline{\mathbb{D}}.$$

Definition 3.3. *A* gap G *of a finite tower* $\sim_n$ *is any connected component of* $\overline{\mathbb{D}}\backslash\mathcal{G}_{\sim_n}$. *The set* $\delta G = G \cap \mathbb{S}^1$ *is called the* essential boundary *of* G.

Notice for reference that the term Gap was used earlier by Thurston to denote also the interior of the convex hull of a class.

Notice also that gaps are relatively open ideally convex subsets of $\overline{\mathbb{D}}$.

A gap G of an uncrossed equivalence relation $\sim_n$ is critical iff $0 \in G$ and similarly a class E of $\sim_n$ is critical iff $0 \in H(E)$. Clearly an un-crossed equivalence relation $\sim$ either has precisely one critical gap or precisely one critical class and not both. We shall denote the critical gap/class of $\sim_n$ by G_n^*/E_n^* or just G^*/E^* if the level is clear or not essential.

The reader shall easily supply a proof of the following lemma

Lemma 3.4. *Suppose* $\sim_N$ *is a tower. Then for any finite* n, $0 \leq n \leq N$ *and any gap* G_n *of* $\sim_n$ *there exists a gap* G_{n-1} *of* $\sim_{(n-1)}$ *such that* $Q_0(\delta G_n) = \delta G_{n-1}$. *Moreover* Q_0 *is injective on* δG_n *if* G_n *is not critical and* $2:1$ *if* G_n *is critical.*

Definition 3.5. *We define* $Q_0(G_n) := G_{n-1}$, *when* $Q_0(\delta G_n) = \delta G_{n-1}$.

It is easy to see that $Q_0(G_n) = G_{n-1}$ (in the usual sense) up to a homotopy fixing $\mathbb{S}^1$ pointwise. Note that for $2 - q < j < 0$ and any non critical gap G of $\mathcal{Z}^j$ we can define similarly the image of G by Q_0. Note also that for any non critical gap G of $\sim_0$ there is a minimal $0 < j < q$ such that $Q_0^j(G)$ is the critical gap of $\mathcal{Z}^{-j}$.

Definition 3.6. *For a finite tower* $\sim_N$ *with critical gap* G_N^* *define the* critical period $k \geq 1$ *of* $\sim_N$ *as the minimal* $k \geq 1$ *for which* $Q_0^k(G_N^*)$ *is again a critical gap (of* $\sim_{N-k}$*).*

Note that in fact $k \geq q$ always.

Let $\sim_N$ be a finite tower and let $0 < n \leq N$. If $\sim_n$ has a critical gap G_n^* then the $\sim_{(n-1)}$ gap $G_n' = Q_0(G_n^*)$ (or just G') is called *the critical value gap for* $\sim_n$. Similarly if $\sim_n$ has a critical class E_n^* then the $\sim_{(n-1)}$ class $E_n' = Q_0(E_n^*)$ (or just E') is called *the critical value class for* $\sim_n$.

Remark that the critical value gap G'/*class* E' *for* $\sim_n$ *is a gap/class of the equivalence relation* $\sim_{(n-1)}$ *one level down.*

3.2 Basic tower properties

Lemma 3.7. *Let* $\sim_N$ *be a tower, and let* E *be a class of* $\sim_N$. *Then* Q_0 *is injective on* E *if* E *is not critical and* $2:1$ *if* E *is critical.*

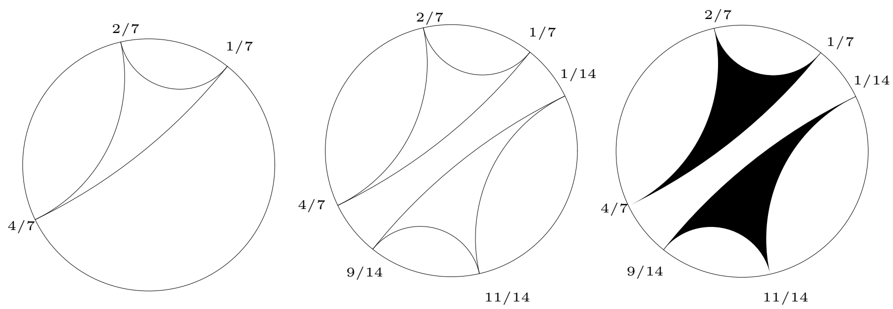

Figure 2: $\frac{p}{q} = \frac{1}{3}$, $\mathcal{Z}^{-1} = \mathcal{Z}^{-2} = \{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$, $\mathcal{Z}^{0} = \{\frac{1}{14}, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}, \frac{9}{14}, \frac{11}{14}\}$. On the left $\sim_{-1}=\sim_{-2}$, in the middle $\sim_0$, on the right the graph $\mathcal{G}_{\sim_0}$, $H(E)$ is in black for non trival classes.

Proof. The map Q_0 is injective on any open half circle and maps any two opposite points z and $-z$ to the same point. If E is not critical, then E is contained in an open half circle of $\mathbb{S}^1$, hence Q_0 is injective on E. If E is critical, then so is $-E$ as $Q_0(E) = Q_0(-E)$. Hence $E = -E$, i.e. $E = Q_0^{-1}(Q_0(E))$, that is Q_0 is $2:1$ on E. □

Let $\sim_N$ be a tower and let $0 \leq n \leq N$ be finite. A gap G_n of $\sim_n$ is a *descendent* of a gap G_m of $\sim_m$ with $m < n$, if $Q_0^{n-m}(G_n) = G_m$. Similarly a class E_n of $\sim_n$ is a *descendent* of a class E_m of $\sim_m$ with $m < n$, if $Q_0^{n-m}(E_n) = E_m$.

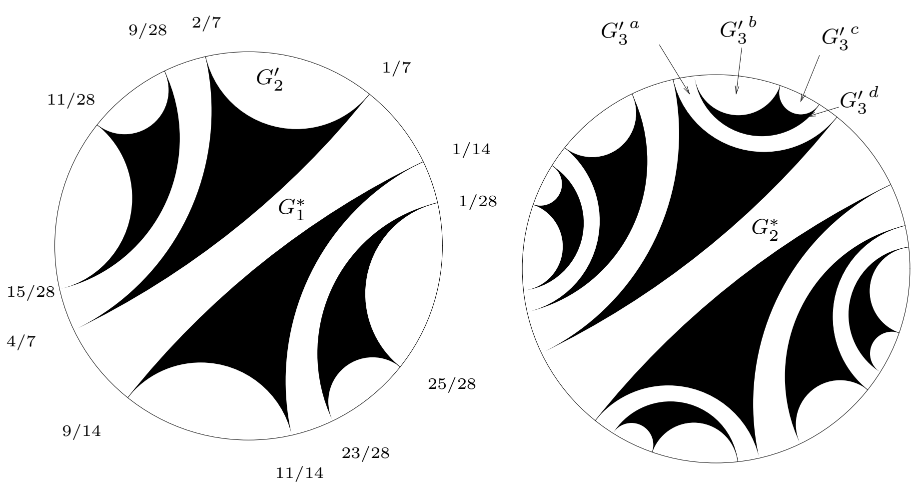

Figure 3: On the left $\sim_1$, on the right $\sim_2$ with the three possibilities for the critical value class for $\sim_3$.

Lemma 3.8. *Let $\sim_N$ be a finite tower. Then either $\sim_N$ has a critical gap G^*_N and all equivalence classes of $\sim_N$ have q elements. Or it has a critical class E^*_n of level n, $q \leq n \leq N$, E^*_n and any of its descendent has $2q$ elements and any other class has q*

elements. Moreover in the latter case, $\sim_N$ has a unique extension $\sim_n$ for every $n \geq N$ and thus a unique infinite height extension.

Proof. If $\sim_N$ has a critical gap G^*_N, then Q_0 is injective hence bijective on any class E of $\sim_N$. Moreover any class is a descendent of the class $\mathcal{Z}^{-1}$, which has q elements.

Suppose next that $\sim_N$ has a critical class E^*_N with critical value class $E'_N = Q_0(E^*_N)$. Then for any class $E \neq E'_N$ of $\sim_N$ the preimage $Q_0^{-1}(E)$ is separated into two non empty sets by E^*_N, each one of which has to be a class of any extension $\sim_{N+1}$ of $\sim_N$. As also the critical class $E^*_N = Q_0^{-1}(E')$ of $\sim_N$ and thus of $\sim_{N+1}$ is determined the extension $\sim_{N+1}$ is completely determined by $\sim_N$. Hence by induction any extension $\sim_n$, $n \geq N$ of $\sim_N$ is uniquely determined by $\sim_N$, including $n = \infty$. Let N be the level of the critical class E^*_N. Then $\sim_{N-1}$ has a critical gap so that by the first case any class of $\sim_{N-1}$ has precisely q elements. Hence E^*_N has $2q$ elements and any other class of $\sim_N$ has q elements. By the lemma above Q_0 is injective on any non critical class and since E^*_N is already a critical class and remains the unique so for any extension, Q_0 is injective hence bijective on any class of level $n > N$. Thus the lemma follows. $\square$

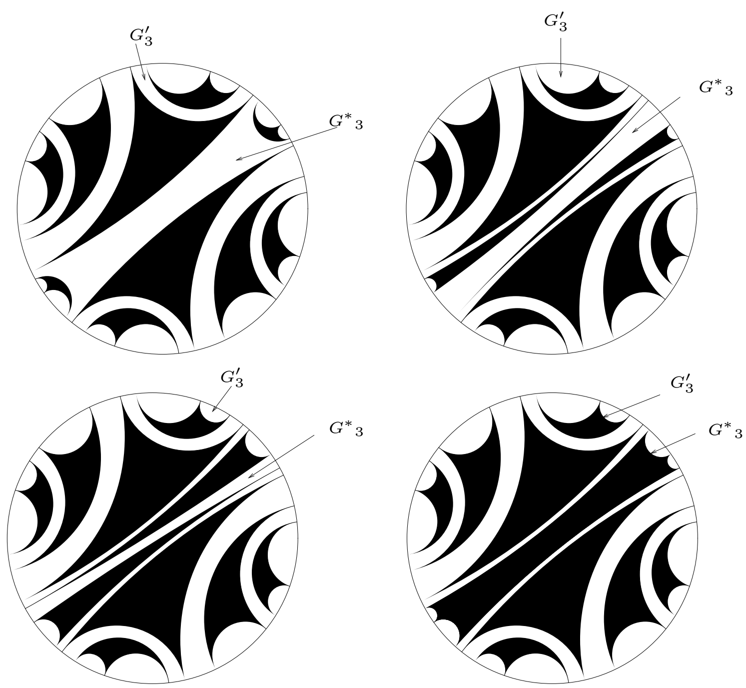

Figure 4: Description of the Gaps included in $G^*{}_2$ for the three fertile towers $\sim_3$, the last one is the terminal one. The critical period is 3 above left, 4 above right, 5 below left.

The reader shall easily verify that

Lemma 3.9. *For any finite height tower $\sim_N$ with critical value gap G'_N and for any gap $G \subseteq G'_N$ or class $E \subseteq G'_N$ of $\sim_N$ there exists a unique extension $\sim^G_{N+1}$ respectively $\sim^E_{N+1}$ with critical value gap $G'_{N+1} = G$/class $E'_{N+1} = E$.*

In view of the above a finite tower $\sim_N$ is *fertile* if it has a critical gap, as suggested by Lemma 3.9. Moreover a possibly infinite tower $\sim_N$ with critical class of level $n \leq N$ is called *terminal* with combinatorics $\sim_n = \sim_N |_{\mathcal{Z}^n}$ as suggested by Lemma 3.8.

3.3 Yoccoz parameter puzzle pieces.

To shorten notation we shall write $\mathcal{R}^{\mathbf{M}}(\tau) := \mathcal{R}^{\mathbf{M}}_\theta$ and $\mathcal{R}^c(\tau) := \mathcal{R}^c_\theta$, where $\tau = \mathrm{e}^{i2\pi\theta}$. Let us recall the connection with quadratic polynomials. For $c \in L_{p/q}$ the coordinate ϕ_c induces an un-crossed equivalence relation $\sim^c_n$ on each $\mathcal{Z}^n$ as follows: for $\tau_1, \tau_2 \in \mathcal{Z}^n$: $\tau_1 \sim^c_n \tau_2$ iff ϕ_c^{-1} has the same radial limit at τ_1 and τ_2, i.e. the two rays $\mathcal{R}^c(\tau_1)$ and $\mathcal{R}^c(\tau_2)$ co-land on the same point of $Q_c^{-(n+1)}(\alpha)$. Defining $\sim^c_\infty = \cup_{n\geq 0} \sim^c_n$ it is easy to see that $\sim^c_\infty$ is an infinite tower. We shall see that this tower essentially determines c. For c in the open wake $W(c_{p/q})$, but not in $L_{p/q}$ we have to reinterpret the notion of co-landing slightly. If a ray $\mathcal{R}^c(\tau)$, $\tau \in \mathcal{Z}^n$ bumps into the critical point 0 then also the ray $\mathcal{R}^c(-\tau)$ bumps into 0. At 0 the two rays bifurcate and land on two distinct points of $Q_c^{-n}(-\alpha(c))$. One speaks of broken rays in this case. Similarly any iterated preimage of τ and $-\tau$ gives rise to broken rays passing through a precritical point. We define the class of τ as the maximal subset $E \subset \mathcal{Z}^n$ containing τ and such that the set $\cup_{\tau' \in E}\overline{\mathcal{R}^c(\tau')}$ is connected in $\mathbb{C}$. Notice that this definition coincides with the previous definition, when $c \in L_{p/q}$.

Notice in parsing that for $p/q \neq p'/q'$ the sets $\mathcal{Z}(p/q)$ and $\mathcal{Z}(p'/q')$ are disjoint and hence so are any two equivalence relations on the two sets.

Definition 3.10. *For a finite tower $\sim_N$ define the* Yoccoz parameter puzzle piece $\mathcal{YP}(\sim_N)$ *of $\sim_N$ as the set*

$$\mathcal{YP}(\sim_N) := \{c \in W(c_{p/q}) | \sim^c_N = \sim_N \quad \textit{and } g_c(c) < 2^{-N+1}\}.$$

If $\sim_N$ is fertile with period $k \geq q$, then a parameter $c_0 \in \mathcal{YP}(\sim_N)$ is called a center of $\mathcal{YP}(\sim_N)$ *iff $Q^k_{c_0}(0) = 0$.*

Note that by construction $\mathcal{YP}(\sim_{N+1}) \subseteq \mathcal{YP}(\sim_N)$, when $\sim_N = \sim_{N+1} |_{\mathcal{Z}^N}$. Note also that $\mathcal{YP}(\sim_N)$ is open, when $\sim_N$ is fertile and neither open nor closed when $\sim_N$ is terminal. Finally we remark that we have excluded the parabolic parameter $c_{p/q}$ with q-renormalizable combinatorics $\sim^\bigstar_\infty = \sim^{c^\bigstar_{p/q}}_\infty$ from $\mathcal{YP}(\sim^\bigstar_N)$ by allowing only parameters $c \in L^\bigstar_{p/q}$. The parameter $c_{p/q}$ is somewhat special, because it is both a boundary point of

$\mathcal{YP}(\sim_N^{\bigstar})$ and has the right combinatorics. (See also the comment after the statement of Theorem 3.15)

We refer to $\mathcal{YP}(\sim_N)$ as a *level N parameter puzzle piece.*

A priori however such parameter puzzle pieces could be empty. But the following slight reformulation of Thurston's existence and uniqueness theorem for (quadratic) branched coverings of the sphere shows that none are empty.

Theorem 3.11 (Thurston realization theorem). *Any finite height tower $\sim_N$ is realized by some quadratic polynomial $Q_c(z) = z^2 + c$. Moreover*

i) If $\sim_N$ has a critical class E_n^* of level $n \leq N$, then there is a unique c such that $\sim_N^c = \sim_N$, $Q_c^n(c) = -\alpha(c)$, and hence $Q_c^{n+1}(c) = \alpha(c)$.

ii) If $\sim_N$ has a critical gap with critical period k, then there exists a unique c for which $\sim_N^c = \sim_N$ and $Q_c^k(0) = 0$, but $Q_c^j(0) \neq 0$ for $0 < j < k$.

The above Thurston realization theorem can be proved in several ways. For instance given an admissible tower $\sim_N$ one may construct an un-obstructed Hubbard tree, which in turn yields a branched degree 2 covering to which Thurston's theorem applies. Or alternatively one may apply the spider theorem to the Hubbard tree. However there is no known algorithm which produces a Hubbard tree from a finite admissible tower. Here we shall give a very short different proof using Shishikura's idea (see [R]) to prove that the Mandelbrot set is locally connected at Yoccoz parameters. It has the virtue of simultanuously proving that parameter puzzle pieces for fertile towers are Jordan disks and in particular connected. Recall that

$$\mathcal{GY}_c^n = \bigcup_{k=0}^{n} \bigcup_{Y \in \mathcal{Y}_c^k} \partial Y.$$

We extend the notion of *periods of gaps* to gaps G of $\sim_N$ contained in the critical value gap G'_N for $\sim_N$. The period k of such a gap G is $k = 1 + l$, where l is the minimal integer for which $Q_0^l(G)$ is critical, i.e. $Q_0^l(G) = G^*_{N-l}$, the critical gap of $\sim_{N-l}$. Thus the period k is the critical period of the unique extension $\sim_{N+1}$ of $\sim_N$ having critical value gap $G'_{N+1} = G$.

Lemma 3.12. *Let $\sim_N$ be any fertile tower of height N and critical period k. Then there exists a unique extension $\sim_{N+1}$ of $\sim_N$ of the same critical period k. Any other extension has strictly higher period. In particular there exists a unique infinite tower $\sim_\infty$, which extends $\sim_N$ and for which the finite towers $\sim_n = \sim_\infty |_{\mathcal{Z}^n}$ with $n \geq N$ also have critical period k.*

Proof. Let G^*_N denote the critical gap of $\sim_N$ and let $G'_N = Q_0(G^*_N)$ denote the critical value gap for $\sim_N$. Then $G^*_{N-k} = Q_0^k(G^*_N) = Q_0^{k-1}(G'_N)$ is the critical gap of $\sim_{N-k}$. Let $G^*_{N+1-k} \subset G^*_{N-k}$ denote the critical gap of $\sim_{N+1-k}$. Then G'_N contains a unique gap G of $\sim_N$ with $Q_0^{k-1}(G) = G^*_{N+1-k}$ and hence of period k, because $Q_0^{k-1} : \delta G'_N \longrightarrow \delta G^*_{N-k}$ is a homeomorphism. Consequently $\sim_N$ has a unique fertile extension $\sim_{N+1}$ with critical period k, the one with critical value gap $G'_{N+1} = G$. Moreover $Q_0^j(G_1)$ is not critical for $0 \le j < k$ for any other gap G_1 of $\sim_N$ contained in G'_N.

By induction for each $n \ge N$ there exists a unique tower $\sim_n$ extending $\sim_N$ and having critical period k. □

Definition 3.13. *Possibly reducing the height N above we can suppose that N is the minimal height n for which $\sim_n$ has critical period k. In this case the infinite tower $\sim_\infty$ is said to be* renormalizable with combinatorics $\sim_N$ *and renormalization period k.*

For $c \in L^{\bigstar}_{p/q}$, $n \in \mathbb{N}$ and G a gap of $\sim^c_n$ we define $Y^n_c(G) \in \mathcal{Y}^n_c$ to be the unique level n puzzle piece for Q_c corresponding to the gap G, i.e. $\mathcal{R}^c(\tau) \cap Y^n_c(G) \neq \emptyset$ iff $\tau \in \delta G$. We shall abuse this notation and use the generic term $Y^n_c(G^*)$ for the critical puzzle piece at level n for Q_c, as the level n is already indicated.

Lemma 3.14. *Let $\sim_\infty$ be renormalizable with combinatorics $\sim_N$. If $\mathcal{YP}(\sim_N)$ has a center c_0, then $\sim_\infty = \sim^{c_0}_\infty$.*

Proof. Let k be the renormalization period of the tower. Proof by induction on $n \ge N$. By definition $\sim^{c_0}_N = \sim_N$ and $Q^k_{c_0}(0) = 0$. Thus suppose $n \ge N$ and that $\sim^{c_0}_n = \sim_n$. Let $G'_n \supseteq G'_{n+1}$ denote the critical value gaps for $\sim_n$ and $\sim_{n+1}$ respectively and let $G^*_{n-k} = Q_0^{k-1}(G'_n) \supseteq Q_0^{k-1}(G'_{n+1}) = G^*_{n+1-k}$ denote the critical gaps of levels $n-k$ and $n+1-k$ respectively. Then

$$\begin{array}{ccc}
Y^n_{c_0}(G'_{n+1}) & \xrightarrow{\hookrightarrow} & Y^{n-1}_{c_0}(G'_n) \ni c_0 \\
{\scriptstyle Q^{k-1}_{c_0}}\big\downarrow & & \big\downarrow{\scriptstyle Q^{k-1}_{c_0}} \\
0 \in Y^{n+1-k}_{c_0}(G^*_{n+1-k}) & \xrightarrow{\hookrightarrow} & Y^{n-k}_{c_0}(G^*_{n-k}) \ni 0,
\end{array}$$

where $\hookrightarrow$ denotes the inclusion and where $Q^{k-1}_{c_0} : Y^{n-1}_{c_0}(G'_n) \longrightarrow Y^{n-k}_{c_0}(G^*_{n-k})$ is an isomorphism with $Q^{k-1}_{c_0}(c_0) = 0$. It follows that $c_0 \in Y^n_{c_0}(G'_{n+1})$ and hence that $\sim^{c_0}_{n+1} = \sim_{n+1}$. □

Theorem 3.15 (Shishikura realization theorem). *For every $N \ge 0$ and for every fertile tower $\sim_N$, let k denote its critical period and G'_N denote its critical value gap. Then*

i) the Yoccoz parameter puzzle piece $\mathcal{YP}(\sim_N)$ is a Jordan domain in particular it is not empty. Moreover for every $c \in \mathcal{YP}(\sim_N)$ there is a canonical homeomorphism $h_c : \partial Y^{N-1}_c(G'_N) \longrightarrow \partial\mathcal{YP}(\sim_N)$ with $h_c = \Psi \circ \phi_c$ where both sides are defined, *ii) the Yoccoz parameter puzzle piece $\mathcal{YP}(\sim_N)$ has a unique center c_0, i.e. $Q_{c_0}^k(0) = 0$ and $\sim_N^{c_0} = \sim_N$,*

iii) there exists a canonical holomorphic motion with base point $c_0 \in \mathcal{YP}(\sim_N)$

$$H = H_{\sim_N} : \mathcal{YP}(\sim_N) \times (\mathcal{GY}_{c_0}^N \cup \{z | g_{c_0}(z) \geq 2^{-N}\}) \longrightarrow \mathbb{C},$$

such that $H(c, z) = \psi_c \circ \phi_{c_0}(z)$ where both sides are defined,

iv) the homeomorphisms h_c defined in i) extends to homeomorphisms of $Y_c^{N-1}(G'_N) \cap \mathcal{GY}_c^N$ onto their common image, such that $h_c = \Psi \circ \phi_c$, where both sides are defined and such that $h_{c_0} = h_c \circ H_c$,

v) for any terminal extension $\sim_{N+1}$ of $\sim_N$ there is a unique parameter $c' \in \mathcal{YP}(\sim_N)$ such that $\sim_{N+1}^{c'} = \sim_{N+1}$ and $Q_{c'}^{N+1}(c') = -\alpha(c')$, in particular $c' \in \mathcal{YP}(\sim_N) \cap \mathbf{M}$

Note that the holomorphic motion statement iii) above does not apply to the special parameter $c_{p/q} \in \partial \mathcal{YP}(\sim_N^\bigstar)$, though this parameter does have the combinatorics $c_{p/q}$. As usual in the literature we shall write H_c for the quasi-conformal homeomorphism $z \mapsto H(c, z)$. Also for $c \in \mathcal{YP}(\sim_N)$ the map $(c', z) \mapsto H(c', H_c^{-1}(z))$ is essentially the same holomorphic motion, but with base point c instead of c_0.

For E a class of $\sim_N^c$ we denote by E^c the union of ray closures $E^c = \cup_{\tau \in E} \overline{\mathcal{R}^c(\tau)}$. Moreover if none of the rays $\mathcal{R}^c(\tau)$, $\tau \in E$ are broken rays, then the common landing point of these rays is denoted by $w(c, E)$.

Proof. The proof is by induction on N. The induction basis consists of the towers $\sim_N$ with $N < q$ and $k = q$. These have common critical value gap $G' = G'_N$ with essential boundary $\delta G'$ bounded by the two points τ_{p-1} and τ_p of the p/q orbit for Q_0. Hence it follows from the introductory section on the Mandelbrot set $\mathbf{M}$ that

$$\mathcal{YP}(\sim_N) = W(c_{p/q}) \cap \{c | g_c(c) < 2^{-(N-1)}\}$$

and that $c_0 = c_{p/q}^*$ is a center for $\mathcal{YP}(\sim_N)$. For $c \in \mathcal{YP}(\sim_N)$ define a homeomorphism $h_c : \partial Y_c^{N-1}(G'_N) \longrightarrow \partial \mathcal{YP}(\sim_N)$ by

$$h_c(z) = \begin{cases} \Psi \circ \phi_c(z) & \text{if } z \in \partial Y_c^{N-1}(G'_N) \setminus J_c, \\ c_{p/q} & \text{if } z = \alpha(c) \end{cases}$$

This fulfils i) and ii). For iii) recall that a ray $\mathcal{R}^c(\tau)$ moves holomorphically with the parameter c, except when 0 is either in the ray or in a forward iterate of the ray. And moreover for a (pre)periodic ray $\mathcal{R}^c(\tau)$ landing at a (pre)periodic repelling point z, the closed ray $\{z\} \cup \mathcal{R}^c(\tau)$ moves holomorphically with c, provided the critical point 0 is not in the forward orbit of $\{z\}\cup\mathcal{R}^c(\tau)$. For $c\in\mathcal{YP}(\sim_N)$ we have $c\notin\mathcal{GY}_c^{N-1}$ and $g_c(0)<2^{-N}$ so that $0\notin\mathcal{GY}_c^N\cup\{z|g_{c_0}(z)\geq 2^{-N}\}$. Hence $(c,z)\mapsto\psi_c\circ\phi_{c_0}(z)$ induces a holomorphic motion

$$H:\mathcal{YP}(\sim_N)\times(\mathcal{GY}_{c_0}^N\cup\{z|g_{c_0}(z)\geq 2^{-N}\})\longrightarrow\mathbb{C},$$

such that $H(c,z)=\psi_c\circ\phi_{c_0}(z)$ where both sides are defined. Because a holomorphic motion of a set X automatically induces a holomorphic motion of its closure.

For iv) there is only something to prove when $\delta G'_N\cap\mathcal{Z}^N\neq\emptyset$. For this to occur we need at least $N\geq q$. Suppose $E\subset\delta G'_N\cap\mathcal{Z}^N$ is a class of level N and let $\sim_{N+1}$ denote the unique extension of $\sim_N$ with critical value class E. Then for each $\tau\in E$ the parameter ray $\mathcal{R}^{\mathbf{M}}(\tau)$ lands at a parameter $c\in\mathcal{YP}(\sim_N)$ for which $\mathcal{R}^c(\tau)$ lands on c and $Q_c^k(c)=-\alpha(c)$. Then since $c\in\mathcal{YP}(\sim_N)$ any of the other dynamical rays $\mathcal{R}^c(\tau')$, $\tau'\in E$ also lands on the same c. Hence by Theorem 1.4 any of the parameter rays $\mathcal{R}^{\mathbf{M}}(\tau')$, $\tau'\in E$ lands on that same c. It follows that $\Psi\circ\phi_c$ extends as a homeomorphism from E^c onto $X:=\{c\}\cup\cup_{\tau\in E}\mathcal{R}^{\mathbf{M}}(\tau)$ and hence that $\Psi\circ\phi_{c'}$ extends as a homeomorphism of $E^{c'}$ onto X for any $c'\in\mathcal{YP}(\sim_N)$. This proves iv) and we have also proved v) for the same price, since any terminal extension $\sim_{N+1}$ of $\sim_N$ has a level N critical value class $E\subset\delta G'_N$.

For the inductive step, suppose that the theorem holds for any fertile tower of height N and let $\sim_{N+1}$ be a fertile tower of height $N+1$, critical period k and critical value gap G'_{N+1}. Write $\sim_N:=\sim_{N+1}|_{\mathcal{Z}^N}$ and let $k_0\leq k$ denote the critical period and $G'_N\supseteq G'_{N+1}$ the critical value gap of $\sim_N$. By the induction hypothesis $\mathcal{YP}(\sim_N)$ has a center c_0 and for every $c\in\mathcal{YP}(\sim_N)$ their exists a homeomorphism

$$h_c:Y_c^{N-1}(G'_N)\cap\mathcal{GY}_c^N\longrightarrow h_c(Y_c^{N-1}(G'_N)\cap\mathcal{GY}_c^N)$$

with $h_c=\Psi\circ\phi_c$, where both are defined and with $h_c\circ H_c=h_{c_0}$. Let $D=D(\sim_{N+1})$ denote the bounded connected component of the complement of the Jordan curve $h_{c_0}(\partial Y_{c_0}^N(G'_{N+1}))$. We claim that $D\subseteq\mathcal{YP}(\sim_{N+1})$: Notice at first that $c\in D$ implies that $c\in Y_c^N(G'_{N+1})$, because $h_c\circ H_c=h_{c_0}$ for any $c\in\mathcal{YP}(\sim_N)$ $(\supseteq D)$ so that $h_c(\partial Y_c^N(G'_{N+1}))=\partial D$ and because $\Phi(c)=\phi_c(c)$ for $c\in D\backslash\mathbf{M}$ so that $c\in Y_c^N(G'_{N+1})$ at least for these c. But then $c\in Y_c^N(G'_{N+1})$ for all $c\in D$ by continuity. From this it follows that $D\subseteq\mathcal{YP}(\sim_{N+1})$ because generally we have $\sim_{N+1}^c=\sim_{N+1}$ if and only if $\sim_N^c=\sim_N$ and $c\in Y_c^N(G'_{N+1})$.

Finally $D=\mathcal{YP}(\sim_{N+1})$, by the pigeon hole principle, since any connected component of

$$(\mathcal{YP}(\sim_N)\cap\{c\mid g_c(c)<2^{-N}\})\backslash h_{c_0}(Y_{c_0}^{N-1}(G'_N)\cap\mathcal{GY}_{c_0}^N)$$

corresponds in the above way to a unique height $N+1$ extension of $\sim_N$. Taking for h_c the restriction of h_c above to $\partial Y_c^N(G'_{N+1})$ completes the proof of i).

To prove ii) we need to prove that the map $c\mapsto Q_c^{k-1}(c)=Q_c^k(0)$ has a unique zero in $D=\mathcal{YP}(\sim_{N+1})$. For $k=k_0$ we have $c_0\in D$ so that c_0 is also a center for $D=\mathcal{YP}(\sim_{N+1})$. For the other case $q\leq k_0<k$ notice that for any $c\in\overline{\mathcal{YP}(\sim_N)}$ the map

$z \mapsto Q_c^{k-1}(z) : Y_c^N(G'_{N+1}) \to Y_c^{N+1-k}(G^*)$ is an isomorphism and thus has a unique zero in the interior of $Y_c^N(G'_{N+1})$. Hence the curve $z \mapsto Q_c^{k-1}(z) : \partial Y_c^N(G'_{N+1}) \to \mathbb{C}^*$ has winding number 1 around the origin. Let $\eta : \overline{\mathbb{D}} \longrightarrow \overline{\mathcal{YP}(\sim_N)}$ be a homeomorphic extension of the Riemann map with say $\eta(0) = c_0$.

Using iv) we can define a continuous map $K : [0,1] \times \partial D \longrightarrow \mathbb{C}^*$ by

$$K(s,c) = H(\eta(s \cdot \eta^{-1}(c)), Q_{c_0}^{k-1}(h_{c_0}^{-1}(c))),$$

whenever $(s,c) \notin \{1\} \times \partial\mathcal{YP}(\sim_N)$ and extend it continuously to the remaining part. That is for any $c' \in \partial D \cap \partial\mathcal{YP}(\sim_N)$ there exists a class E of level $n < N$ such that $c' \in E^{c'}$ and $c' \in \mathcal{YP}(\sim_n)$. Moreover by the induction hypothesis the map $(c,z) \mapsto \psi_c \circ \phi_{c_0}(z)$ defines a holomorphic motion of $Q_{c_0}^{k-1}(E^{c_0})$ on $\mathcal{YP}(\sim_n)$. (For parameters c' with $g_{c'}(c') = 2^{-N}$ the point $Q_{c_0}^{k-1}(h_{c_0}^{-1}(c'))$ belongs to the equipotential at level 2^{-N-1+k} and the map $(c,z) \mapsto \psi_c \circ \phi_{c_0}(z)$ defines a holomorphic motion of this equipotential in a large neighbourhood of D.) Then for any pair (s,c) we have $K(s,c) \in \partial\mathcal{Y}_{c'}^{N+1-k}(G^*)$ for some $c' \in \overline{\mathcal{YP}(\sim_N)}$ and hence $K(s,c)$ is never vanishing.

For any $c \notin \mathbf{M}$:

$$\begin{aligned} K(s,c) &= \psi_{\eta(s\cdot\eta^{-1}(c))} \circ \phi_{c_0} \circ Q_{c_0}^{k-1}(h_{c_0}^{-1}(c)) \\ &= Q_{\eta(s\cdot\eta^{-1}(c))}^{k-1} \circ \psi_{\eta(s\cdot\eta^{-1}(c))} \circ \phi_{c_0}(h_{c_0}^{-1}(c)) \\ &= Q_{\eta(s\cdot\eta^{-1}(c))}^{k-1} \circ \psi_{\eta(s\cdot\eta^{-1}(c))} \circ \Phi(c). \end{aligned}$$

It follows by continuity that for all $c \in \partial D$, $K(0,c) = Q_{c_0}^{k-1}(h_{c_0}^{-1}(c))$ and $K(1,c) = Q_c^{k-1}(c)$. That is K is the desired homotopy in $\mathbb{C}^*$ from $c \mapsto Q_{c_0}^{k-1}(h_{c_0}^{-1}(c))$ to $c \mapsto Q_c^{k-1}(c)$. Hence the function $c \mapsto Q_c^{k-1}(c) = Q_c^k(0)$ has a unique zero $c \in D$, that is a unique center in $\mathcal{YP}(\sim_{N+1}) = D$.

The proofs of iii), iv) and v) are similar to the proofs for the induction base and are left to the reader. □

3.4 The p/q-equivalences.

Not all infinite towers are realized, but almost. Generally any tower "neighbouring" an infinite terminal tower is not realized with the unique renormalizable tower of period q and combinatorics $\sim_0$ as the only exception. We shall obtain a complete set of realized combinatorial invariants by defining an equivalence relation on infinite towers.

To be more precise and to fix the ideas, suppose $\sim_\infty^T$ is an infinite terminal tower with critical value class E'_n. That is $\sim_n = \sim_\infty^T |_{\mathcal{Z}_n}$ has a critical class with image E'_n and $\sim_{n-1} = \sim_\infty^T |_{\mathcal{Z}_{n-1}}$ has a critical gap with critical value gap G'_{n-1} containing $H(E'_n)$. Then

G'_{n-1} contains exactly q gaps $G^1, \ldots, G^q$ of $\sim_{n-1}$, which are adjacent to E'_n, i.e. with $H(E'_n) \cap \partial G^j \neq \emptyset$, because $H(E'_n)$ is a q-gon. Let $\sim^j_n$ denote the unique extension of $\sim_{n-1}$ with critical value gap G^j, for $j = 1, \ldots, q$. Furthermore by induction on m for each $j = 1, \ldots, q$ there exists a unique fertile extension $\sim^j_m$ of $\sim^j_n$ with critical value gap the unique gap of $\sim^j_{m-1}$ contained in G^j and adjacent to $H(E'_m)(= H(E'_n))$. Denote by $\sim^j_\infty$ the corresponding infinite tower.

We shall say that $\sim^T_\infty$ is *adjacent* to any of the q towers $\sim^1_\infty, \ldots, \sim^q_\infty$ and vice versa.

Definition 3.16. *We consider the smallest equivalence relation on the set $\mathcal{T}^\infty = \mathcal{T}^\infty_{p/q}$ of infinite towers such that any two adjacent towers are equivalent. The equivalence classes will be called p/q-equivalences and will generically be denoted by F. The space of all p/q-equivalences will be called $\mathcal{F}_{p/q}$ and we write*

$$\mathcal{F} = \bigcup_{p/q \neq 0/1, (p,q)=1} \mathcal{F}_{p/q}.$$

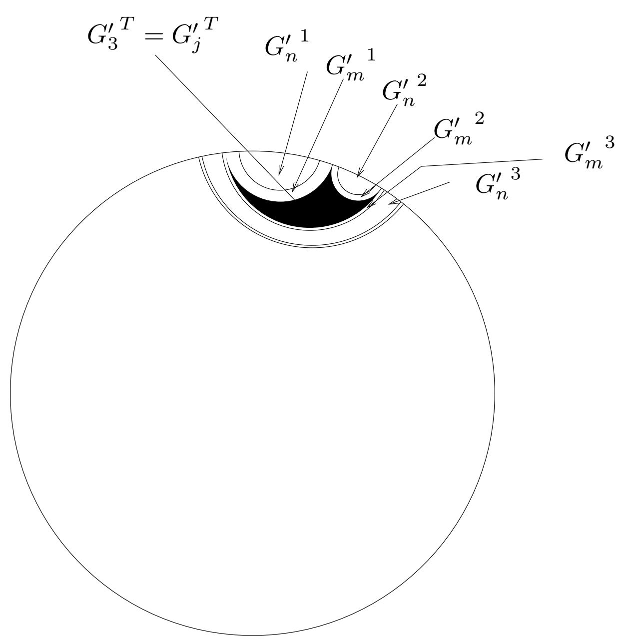

Figure 5: Illustration in a sector of the equivalence $\sim^1$, $\sim^2$, $\sim^3$, adjacent to the terminal one $\sim^T$ with critical class in black.

Definition 3.17. *We define the combinatorial projection $\Xi_C : L_{p/q} \longrightarrow \mathcal{F}_{p/q}$ as the natural map given by the Yoccoz combinatorics:*

$$\Xi_C(c) = F(\sim^c_\infty).$$

Proposition 3.18. *There are three types of p/q-equivalences $F \in \mathcal{F}_{p/q}$:*

1. *There is one p/q-equivalence $F_\bigstar = F_\bigstar(p/q)$ with countably many elements. It consists of the unique q-renormalizable tower $\sim_\infty^\bigstar = \sim_\infty^\bigstar (p/q)$ (with combinatorics $\sim_0 = \sim_0 (p/q)$), countably infinitely many terminal towers $\sim_\infty^T$ adjacent to $\sim_\infty^\bigstar$ and for each such terminal tower $\sim_\infty^T$ the $q-1$ other infinite towers adjacent to it.*
2. *There are countably many p/q-equivalences F consisisting of just one terminal tower $\sim_\infty^T$ and its q adjacent infinite towers $\sim_\infty^1, \ldots, \sim_\infty^q$.*
3. *There are uncountably many p/q-equivalences F consisting of just one infinite tower.*

Proof. There are uncountably many infinite towers $\sim_\infty$ and countably many of these are terminal towers $\sim_\infty^T$ (each one uniquely specified by its critical value class). Moreover any fertile tower has uncountably many extensions of which infinitely many are terminal extensions. We thus need to prove that any two terminal towers $\sim_\infty^{T_1}$ and $\sim_\infty^{T_2}$ belongs to the same p/q-equivalence if and only if they are both adjacent to $\sim_\infty^\bigstar$ the unique q-renormalizable tower, and that $\sim_\infty^\bigstar$ is adjacent to infinitely many terminal towers.

For $n \geq 0$ let G'_n denote the critical value gap for $\sim_n^\bigstar = \sim_\infty^\bigstar |_{\mathcal{Z}^n}$. Recall that $E_0 = \{\tau_0, \ldots, \tau_{q-1}\}$ is the Q_0 invariant equivalence class consisting of the unique p/q-orbit.

Then $Q_0^q : \delta\overline{G'_{n+q}} \longrightarrow \delta\overline{G'_n}$ is a degree 2 covering. Moreover $G'_i = G'_0$ as sets for $0 \leq i < q$ and E_0 is the only class adjacent to these gaps. It hence follows by induction on $m \geq 0$ that

i) The initial class E_0 is adjacent to any of the critical value gaps G'_m.

ii) $G'_{i+qm} = G'_{qm}$ as sets for $0 \leq i < q$ and $m \geq 0$. In particular any class $E \neq E_0$ adjacent to G'_{i+qm} is also adjacent to G'_{qm}. Moreover any such class E has level $qm' - 1$ with $1 \leq m' \leq m$ and is already adjacent to $G'_{qm'}$.

iii) any class E adjacent to G'_{qm} is adjacent to G'_n for any $n \geq mq$ and is the critical value class of a terminal tower. In particular the class $Q_0^q(E)$ is also adjacent to G'_{qm} and $\sim_\infty^\bigstar$ is adjacent to the terminal tower with critical value class E.

iv) The number of classes $E \subset \delta G'_{mq-1}$ of level $mq-1$ equals 2^{m-1}.

By iii) any class E adjacent to some G'_n is the critical value class of a terminal tower $\sim_\infty^T$ adjacent to the tower $\sim_\infty^\bigstar$. And by iv) $\sim_\infty^\bigstar$ is adjacent to (countably) infinitely many such terminal towers. To complete the proof we only need to prove that any infinite tower $\sim_\infty \neq \sim_\infty^\bigstar$ is adjacent to at most one terminal tower. For this we set up a bit of ad hoc notation. Denote by G^q the critical gap of $\sim_0$ and for $0 < j < q$ write $G^j = Q_0^{q-j}(G^q)$.

So that the gaps of $\sim_0$ are $\pm G^j$, $1 \leq j \leq q$, with $G^q = -G^q$. Let $I, -I \subset \mathbb{S}^1$ denote the intervals with $\delta G^q = I \cup -I$. Finally for this proof it is convenient to let " mod q" take its values in $\{1, \ldots, q\}$.

Let $\sim_\infty^T$ be any terminal tower with critical value class E'_{n+1} of level $n+1 \geq q-1$. For $m \in \mathbb{N}$ write $\sim_m^T = \sim_\infty^T |_{\mathcal{Z}^m}$. Let $G'_n \supset E'_{n+1}$ denote the critical value gap for $\sim_n^T$ and let $k \geq q$ denote the critical period of G'_n and thus of $\sim_n^T$.

Case 1: If $k = q$ we saw above that $n = mq - 1$, $\sim_\infty^\bigstar$ is one of the q towers adjacent to $\sim_\infty^T$ and that the critical value gap G'_n for $\sim_n^\bigstar = \sim_n^T$ is adjacent to both E_0 and E'_{n+1}. We need to prove that the $q-1$ other towers adjacent to $\sim_\infty^T$ are not adjacent to any other terminal tower. For $m > n$ and $1 \leq j \leq q$ let G_m^j denote the unique gap of $\sim_m^T$ adjacent to E_0 and contained in G^j. Then similarly $-G_m^j \subset -G^j$ is adjacent to $-E_0$ and the sets $\pm K_m = H(\pm E_0) \cup \cup_{j=1}^q \pm \overline{G}_m^j$ converge in the Hausdorff topology on compact sets in the plane to $\pm H(E_0) = H(\pm E_0)$, because $Q_0(\delta K_m) = \delta K_{m-1}$, so that δK_m converge to E_0.

Let $\widehat{G}_n^1, \ldots, \widehat{G}_n^{(q-1)} \subset G'_n$ denote the $q-1$ gaps adjacent to E'_{n+1} and of periods strictly larger than q. Possibly exchanging indices we can suppose that Q_0^n maps $\delta \widehat{G}_n^j$ $1:1$ onto $\delta(-G^j)$. Let $\sim_{n+1}^1, \ldots, \sim_{n+1}^{(q-1)}$ denote the $q-1$ extensions of $\sim_n^T$ with critical value gaps $\widehat{G}_n^1, \ldots, \widehat{G}_n^{q-1}$. Moreover let $J \subset I$ be the minimal interval such that $J \cup -J$ contains the critical class $Q_0^{-1}(E'_{n+1})$ for $\sim_{n+1}^T$. Then the critical gaps of those $q-1$ fertile towers and any of their descendents are contained in $H(J \cup -J)$. It follows that for any $m \geq n+n$, for any $1 \leq j < q$ and any extension $\sim_m$ of $\sim_{n+1}^j$, the gap $\widehat{G}_m^j \subseteq \widehat{G}_n^j$ adjacent to E'_{n+1} maps $1:1$ onto $-G_{m-n}^j$ by Q_0^n. In particular this holds for the $q-1$ towers $\sim_\infty^1, \ldots, \sim_\infty^{(q-1)}$ adjacent to $\sim_\infty^T$ and with critical value gaps $\widehat{G}_m^j$ for every level $m+1$. It follows that the sets $H(E'_{n+1}) \cup \cup_{j=1}^{q-1} \overline{\widehat{G}_m^j}$ converge to $H(E'_{n+1})$ in the Hausdorff topology and thus these $q-1$ adjacent towers of $\sim_\infty^T$ can not be adjacent to any other terminal tower, because any two distinct classes are strongly separated.

To recapitulate, this completes the proof that $F_\bigstar = F(\sim_\infty^\bigstar)$ consists of $\sim_\infty^\bigstar$, countably many terminal towers $\sim_\infty^T$ adjacent to $\sim_\infty^\bigstar$ and for each such terminal tower $q-1$ towers adjacent to it, but not adjacent to any other terminal tower.

Case 2: If $k > q$ there is a minimal $N = mq \leq n$, $1 \leq m$ such that $\sim_N^T$ has critical value gap G'_N of period $k' > q$ and the critcal value gap G'_{N-1} for $\sim_{N-1}^T$ has period q. Hence G'_{N-1} contains a unique level $(N-1)$-class E_{N-1} separating G'_N from E_0. Let $\sim_\infty^{T_1}$ denote the terminal tower with critical value class $E_{N-1} \subset G'_{N-1}$. Then E_{N-1} and $\sim_\infty^{T_1}$ are as in the first case treated above, where $k = q$. We shall use the ad hoc terminology of Case 1, but relative to E_{N-1}. With this terminology the gap G'_N equals one of the gaps $\widehat{G}_{N-1}^j$, $1 \leq j < q$ which are adjacent to E_{N-1}. Morever $\sim_\infty^T$ and the q towers $\sim_\infty^1, \ldots, \sim_\infty^q$ adjacent to $\sim_\infty^T$ are all descendents of the fertile tower $\sim_N^T$ for which E_{N-1} is a class. Hence the gaps $\pm G_m^j$ with $m \geq N$ persists also for the appropriate restrictions of these towers. Enumerating the q towers $\sim_\infty^1, \dots, \sim_\infty^q$ adjacent to $\sim_\infty^T$ such that the critical value gaps $\widetilde{G}_m^j$ for the level $m+1 > n$ restrictions $\sim_{m+1}^j = \sim_\infty^j |_{\mathcal{Z}^{m+1}}$ are mapped by Q_0^n into $-G^j$ we obtain for $m \geq N+n$ that $Q_0^n(\widetilde{G}_m^j) = -G_{m-n}^j$. As the map $Q_0^n : E'_{n+1} \longrightarrow -E_0$ is bijective the map Q_0^n is also injective on $\widetilde{G}_m^j$ for m sufficiently large, since $-K_l$ converge to $-E_0$ in the Hausdorff topology. Consequently the towers $\sim_\infty^j$ can not be adjacent to any terminal tower other than $\sim_\infty^T$. This completes the proof. $\square$

Definition 3.19. *For every p/q equivalence F define the Yoccoz parameter nest*

$$\mathcal{YP}(F) = \bigcap_{n \in \mathbb{N}} \bigcup_{\sim_\infty \in F} \mathcal{YP}(\sim_\infty |_{\mathcal{Z}^n}).$$

Note that $\mathcal{YP}(F) = \Xi_C^{-1}(F)$ except when $F = F_\bigstar$, where $\mathcal{YP}(F) = \Xi_C^{-1}(F) \setminus \{c_{p/q}\}$.

Theorem 3.20 (Yoccoz' Parameter nests Theorem).
For any non-zero, irreducible rational p/q and for any $c \in L_{p/q}^\bigstar$ let $F = \Xi_C(c)$.

Then either

Q_c is not renormalizable, $\mathcal{YP}(F) = \{c\}$ and the sets $\cup_{\sim_\infty \in F} \mathcal{YP}(\sim_\infty |_{\mathcal{Z}^n})$, $n \in \mathbb{N}$ form a fundamental system of open neighbourhoods of c in $L_{p/q}^\bigstar$,

or

Q_c is $k \geq q$-renormalizable and:
If $k > q$ then $F = \{\sim_\infty^c\}$, $\sim_\infty^c$ is k-renormalizable with combinatorics $\sim_N = \sim_\infty^c |_{\mathcal{Z}^N}$ for some $N \geq q$ and for every $m \geq N$ and for every $c \in \mathcal{YP}(F)$ the restriction $Q_c^k : \mathcal{Y}_m^c(G^*) \longrightarrow \mathcal{Y}_{m-k}^c(G^*)$ is quadratic-like with connected filled-in Julia set. Moreover the Douady-Hubbard straightening map $\chi_F : \mathcal{YP}(F) \longrightarrow \mathbf{M}$ is a homeomorphism and the sequence of puzzle pieces $\mathcal{YP}(\sim_\infty^c |_{\mathcal{Z}^n})$, $n \in \mathbb{N}$ form a fundamental system of neighbourhoods of $\mathcal{YP}(F)$ in $L_{p/q}^\bigstar$.

If $k = q$ then $F = F_\bigstar = F(\sim_\infty^\bigstar)$, where $\sim_\infty^\bigstar$ is the unique q-renormalizable tower (with combinatorics $\sim_0$) and Q_c is q-renormalizable with connected filled-in Julia set. Moreover the Douady-Hubbard straightening map $\chi_{F_\bigstar} : \mathcal{YP}(F_\bigstar) \longrightarrow \mathbf{M} \backslash \{\frac{1}{4}\}$ is a homeomorphism and $\displaystyle\bigcup_{\sim_\infty \in F_\bigstar} \mathcal{YP}(\sim_\infty |_{\mathcal{Z}^n})$ is a fundamental system of open neigbourhoods of $\mathcal{YP}(F_\bigstar)$.

For a proof see e.g. Hubbard, [H] and Roesch, [R]. In particular, for the proof of the existence of the homeomorphism χ_F (the Douady-Hubbard straightening map) see the proof of Theorem 14.6 of [H].

Corollary 3.21. *For every p/q-equivalence $F \in \mathcal{F}_{p/q}$*

$$\mathcal{YP}(F) \neq \emptyset.$$

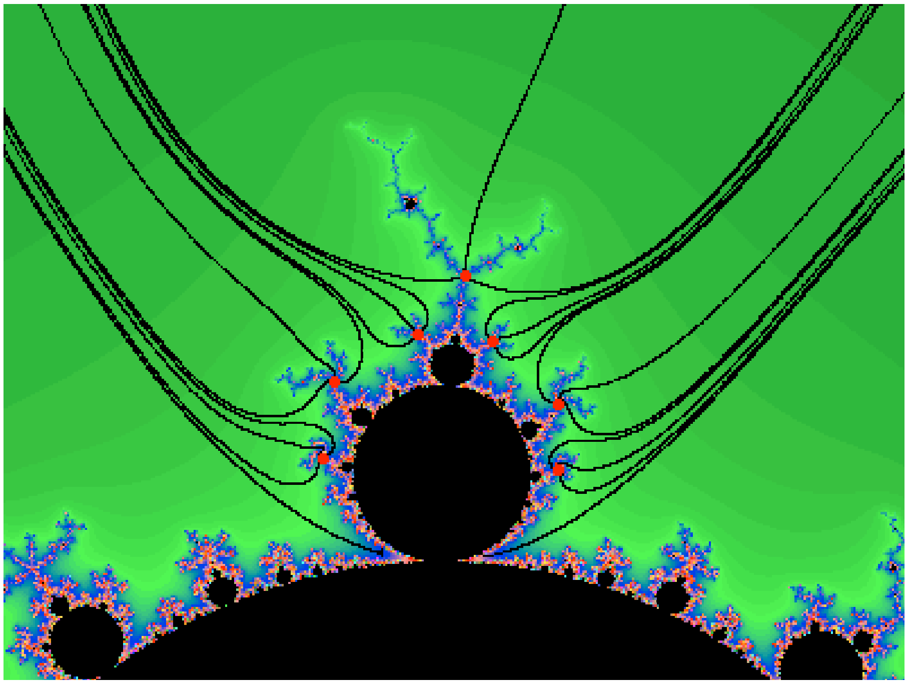

Figure 6: For $p/q = 1/3$, a representation of $\mathcal{YP}(F(\sim_\infty^\bigstar |_{\mathcal{Z}^8}))$.

In particular the combinatorial projection $\Xi_C : L_{p/q} \longrightarrow \mathcal{F}_{p/q}$ *is surjective.*

Proof. If $F = F_\bigstar$ then $\mathcal{YP}(F)$ is homeomorphic to $\mathbf{M}\backslash\{\frac{1}{4}\}$ and in particular not empty. If F is a class with $q+1$ elements one of which is a terminal tower $\sim_\infty^T$ with a critical class of level N; then by Theorem 3.15 there exists a unique parameter $c \in L_{p/q}^\bigstar$ such that $Q_c^N(0) = -\alpha(c)$ and hence $\sim_\infty = \sim_\infty^c$. Thus $\mathcal{YP}(F) \neq \emptyset$ also in this case. Finally suppose $F = \{\sim_\infty\}$ and let c_n be the unique center of the parameter puzzle piece $\mathcal{YP}(\sim_\infty |_{\mathcal{Z}^n})$ for each $n \geq 0$. Let c be any accumulation point of the sequence $\{c_n\}_n$. We claim that $\sim_\infty = \sim_\infty^c$. First note that $c \in L_{p/q}^\bigstar$ and let F_1 denote the p/q equivalence class of $\sim_\infty^c$. Secondly by the Yoccoz Parameter Nest theorem the sequence of sets $\cup_{\sim_\infty \in F}\mathcal{YP}(\sim_\infty^c |_{\mathcal{Z}^n})$, $n \in \mathbb{N}$ form a fundamental system of neighbourhoods of $\mathcal{YP}(F_1)$. If $F_1 = \{\sim_\infty^c\}$ then for every M there exists an N such that $c_n \in \mathcal{YP}(\sim_\infty^c |_{\mathcal{Z}^M})$ for every $n \geq N$, (by nestedness of puzzle pieces, then $N = M$) that is $\sim_\infty |_{\mathcal{Z}^M} = \sim_\infty^c |_{\mathcal{Z}^M}$ for every M. Thus $\sim_\infty = \sim_\infty^c$. If F_1 has $q+1$ elements then by nestedness of parameter puzzle pieces and arguments similar to the above there exists a unique non-terminal tower $\sim_\infty^j \in F_1$ such that $\sim_\infty = \sim_\infty^j$ and hence $F = F_1$, which contradicts the assumption that $F = \{\sim_\infty\}$. Finally a completely analogous argument applies in the case $F_1 = F_\bigstar$. $\square$

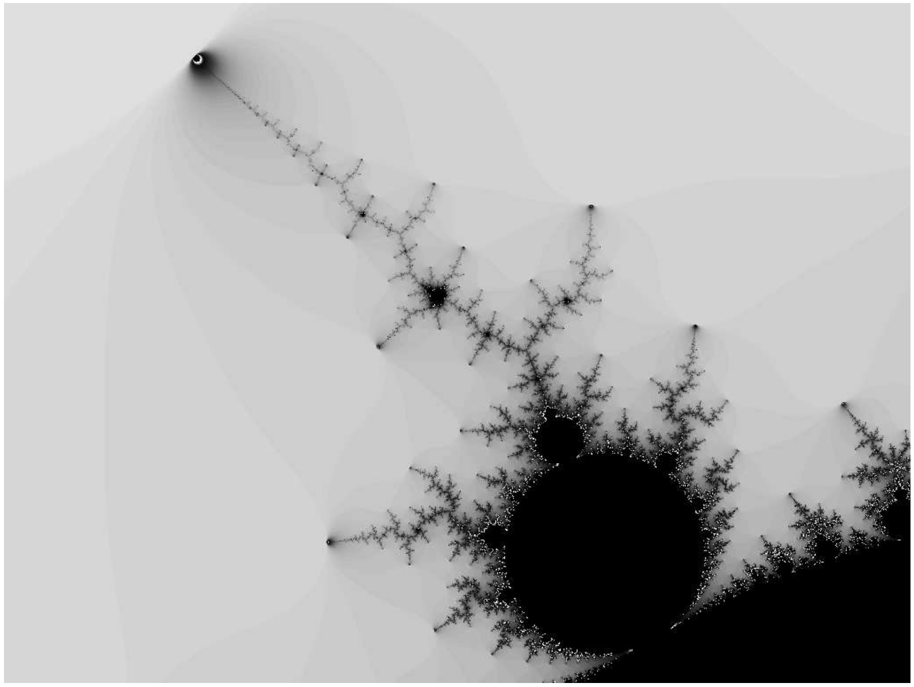

Figure 7: The "1/3" member of $\mathbf{M_1}$.

The following construction and Corollary is instrumental in showing that there is a dynamics preserving bijection between the Mandelbrot set $\mathbf{M}$ and the parabolic Mandelbrot set $\mathbf{M_1}$. Also it is used in a proof that q-renormalization of $L_{p/q}$ converge to $\mathbf{M_1}$ under the holomorphic motion parametrized by the multiplier λ of the fixed point at ∞, when $\lambda \in \mathbb{D}$ converge subtangentially to $\mathrm{e}^{-i2\pi p/q}$.

Definition 3.22. *For $p/q \neq 0/1$ with $(p,q) = 1$ define the space $\mathcal{C}_{p/q}$ of all p/q (quadratic) combinatorial analytic invariants as*

$$\begin{aligned}\mathcal{C}_{p/q} \quad &= \quad \{F | F \textit{ is a non-renormalizable } p/q \textit{ equivalence}\} \\ &\cup \{(F,c) | F \textit{ is a } k > q \textit{ renormalizable } p/q \textit{ equivalence and } c \in \mathbf{M}\} \\ &\cup \{(F_\bigstar, c) | F_\bigstar = F_\bigstar(p/q) \textit{ and } c \in \mathbf{M} \backslash \{\frac{1}{4}\}\}\end{aligned}$$

Moreover define the space $\mathcal{C}$ of all quadratic combinatorial analytic invariants as

$$\mathcal{C} = \overline{\mathbb{D}} \cup \bigcup_{p/q \neq 0/1} \mathcal{C}_{p/q}.$$

Recall that $\mathcal{Z}^\infty_{p/q} \cap \mathcal{Z}^\infty_{p'/q'} = \emptyset$ so that $\mathcal{C}_{p/q} \cap \mathcal{C}_{p'/q'} = \emptyset$, whenever $p/q \neq p'/q'$.

Combination of the above leads immediately to the following combinatorial analytic description of the Mandelbrot set

Corollary 3.23. *There exists a dynamically natural bijective mapping $\Xi : \mathbf{M} \longrightarrow \mathcal{C}$ given by:*
$\Xi(c) = \lambda$ *if* Q_c *has a non-repelling fixed point of multiplier* λ,
$\Xi(c) = \Xi_C(c) \in \mathcal{C}_{p/q}$ *if* $c \in L^{\bigstar}_{p/q}$ *is non-renormalizable and*
$\Xi(c) = (\Xi_C(c), \chi_{\Xi_C(c)}(c))$ *if* Q_c *is renormalizable.*

That is a parameter $c \in \mathbf{M}$ is uniquely determined by either

- Q_c has a non-repelling fixed point, i.e. c belongs to the filled-in cardioid
- or Q_c is non-renormalizable and c is uniquely determined by the combinatorial invariant $\Xi_C(c) = F(\sim^c_\infty)$
- or Q_c is renormalizable and c is uniquely determined by the combinatorial and analytic pair $(F, \chi_F(c))$, where $F = \Xi_C(c) = F(\sim^c_\infty)$.

Notice also that when $F = F(\sim_\infty)$ where $\sim_\infty$ is renormalizable, then F is uniquely determined by its combinatorics $\sim_N$, which is a finite tower. The Yoccoz Parameter Nest Theorem shows that any non-renormalizable parameter c with both fixed points repelling is uniquely determined by $\sim^c_\infty$, i.e. is combinatorially rigid. Iterating the combinatorial analytic invariant, that is applying it iteratively to initially all the first renormalized copies of $\mathbf{M}$ obtained at renormalizable combinatorics F, then to second renormalizable copies of $\mathbf{M}$ in the first renormalizable copies, etc. associates to any parameter c a possibly infinite string of combinatorial invariants. By iteration of the Yoccoz Parameter Nest Theorem any c which is only finitely many times renormalizable, is uniquely determined by the finite string of (iterated) combinatorial invariants plus possibly the eigenvalue of a non-repelling fixed point for the quadratic like map of the last renormalization. Graczyk and Światek, [G-S], Lyubich, [L] and Kahn, [Ka2], [K-L] have shown that even many of the infinitely renormalizable parameters c are combinatorially rigid, i.e. are uniquely determined by the infinite sequence of iterated combinatorial invariants. The remaining question being if all infinitely renormalizeable parameters are combinatorially rigid.

4 Arcs in the Mandelbrot set.

In this section we shall topologize the space $\mathcal{F}_{p/q}$ and realize it as a set of subsets of $\overline{\mathbb{D}}$ in a way reminiscent of the Thurston Lamination. We shall use this to reprove that any vein in $\mathbf{M}$ is an arc and we shall prove that more generally any two Yoccoz parameters are joined by a unique ruled arc in the sense of Douady and Hubbard. Results which are yet unpublished but which are described in the ph.d. thesis of Johannes Riedl, [Ri] and for which the proof was later simplified by Jeremy Kahn, [S]. Here ruled arc means that any passage of a hyperbolic component takes place along hyperbolic geodesics to and from the center of that component.

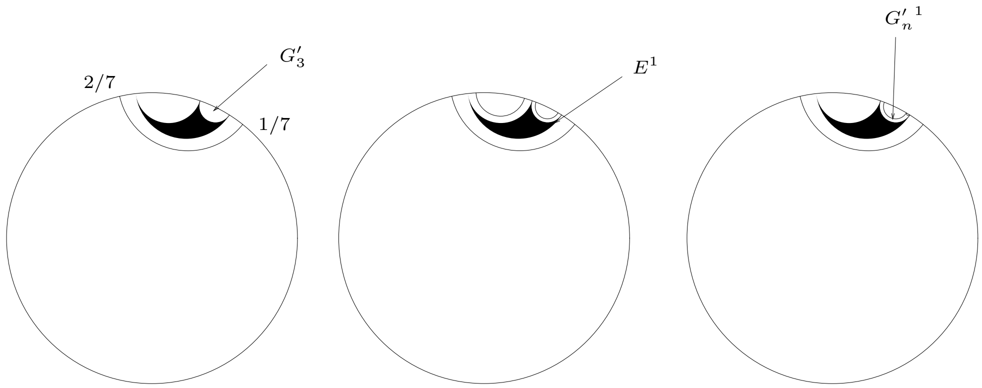

Figure 8: Determination of $\sim^1_\infty$ for the fertile tower of critical value Gap G'_3.

Let $\sim_n \in \mathcal{T}_{p/q}, \sim_n \neq \sim^\star_n$ be a fertile tower with critical value gap G'_n and let $E^1, E^2, \ldots, E^r$ be the finite number of classes of $\sim_{(n-1)}$ bounding G'_n in $\overline{\mathbb{D}}$. Denote by $\sim^j_\infty$, $1 \leq j \leq r$ the infinite towers adjacent to the terminal towers with critical value class E^j and with $\sim^j_\infty |_{\mathcal{Z}^n} = \sim_n$. Then we define

$$\mathcal{U}(\sim_n) = \{\sim_\infty \in \mathcal{T}_{p/q} | \sim_\infty |_{\mathcal{Z}^n} = \sim_n\} \setminus \{\sim^1_\infty, \sim^2_\infty, \ldots, \sim^r_\infty\}.$$

Next let $\sim_n \in \mathcal{T}_{p/q}$, be a terminal tower with infinite extension $\sim^T_\infty \notin F_\star$. Let $\sim^1_\infty, \ldots, \sim^q_\infty \in F(\sim^T_\infty)$ be the q infinite towers adjacent to $\sim^T_\infty$. Then we define

$$\mathcal{U}(\sim_n) = F(\sim^T_\infty) \cup \bigcup_{j=1}^{q} \mathcal{U}(\sim^j_\infty |_{\mathcal{Z}^n}).$$

Finally suppose that $\sim_n = \sim^\star_n$ or that $\sim_n$ is a terminal tower with infinite extension belonging to $F_\star$. Write $F'_\star = F_\star \setminus \{\sim_\infty | \sim_\infty = \sim^\star_\infty \text{ or } \sim_\infty \text{ is terminal}\}$. Then we define

$$\mathcal{U}(\sim_n) = F_\star \cup \{\sim_\infty | \sim_\infty |_{\mathcal{Z}_n} = \sim^\star_n\} \cup \bigcup_{\sim_\infty \in F'_\star} \mathcal{U}(\sim_\infty |_{\mathcal{Z}^n}).$$

Note that for any class $F \in \mathcal{F}_{p/q}$ and any $\sim_n$ with $F \cap \mathcal{U}(\sim_n) \neq \emptyset$ we have $F \subset \mathcal{U}(\sim_n)$ by construction. Furthermore for any two finite towers $\sim_n$ and $\sim_m$, say with $n < m$ the intersection $\mathcal{U}(\sim_n) \cap \mathcal{U}(\sim_m)$ is of the form $\mathcal{U}(\sim'_m)$ where $\sim_m = \sim'_m$ if and only if $\sim_m |_{\mathcal{Z}^n} = \sim_n$.

It follows that the sets $\mathcal{U}(\sim_n)$, $\sim_n \in \mathcal{T}_{p/q}$ which we shall call open sets form the basis of a topology on $\mathcal{T}^\infty$ and that the projection of any such set into $\mathcal{F}_{p/q}$ is open in the quotient topology on $\mathcal{F}_{p/q}$, since any such set is saturated under the projection. We shall abuse the notation and say that $\mathcal{U}(\sim_n)$ is also an open subset of $\mathcal{F}_{p/q}$, though technically it is its projection which is an open subset. We remark for later use that:

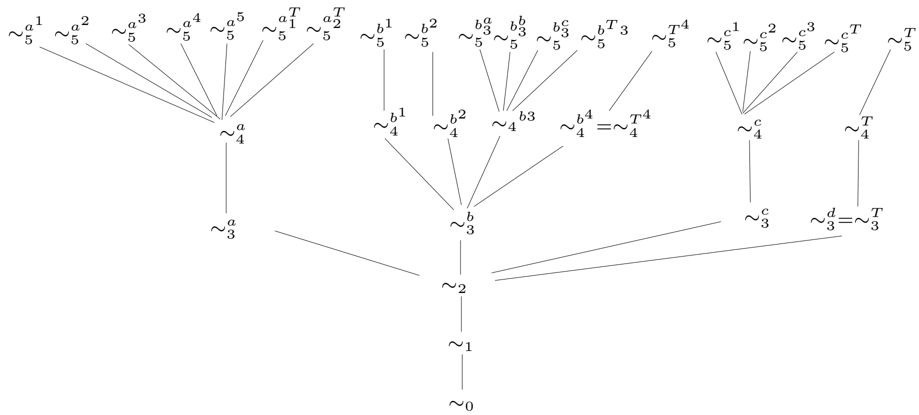

Figure 9: The tree $\mathcal{T}_{p/q}$ for $p/q = 1/3$ until level 5 .

Lemma 4.1. *The quotient topology on $\mathcal{F}_{p/q}$ is Hausdorff.*

Proof. Let $F^1, F^2 \in \mathcal{F}_{p/q}$ and suppose $F^1 \neq F^2$. We prove the lemma case by case. If $F^i = \{\sim_\infty^i\}$ for $i = 1, 2$ choose any n such that $\sim_n^1 \neq \sim_n^2$. Then the open sets $\mathcal{U}(\sim_n^i)$ are disjoint open neighbourhoods of $\sim_\infty^i$ for $i = 1, 2$. If say $F^1 = \{\sim_\infty\}$ and F^2 contains a terminal tower $\sim_\infty^T$, but $F^2 \neq F_\bigstar$, then let $\sim_\infty^j$, $1 \leq j \leq q$ denote the q infinite towers adjacent to $\sim_\infty^T$ and for each n let $G_n'^j$ denote the critical value gap of $\sim_n^j = \sim_\infty^j|_{\mathcal{Z}^n}$. Since $F^1 \neq F^2$ there exists an n such that $\sim_\infty |_{\mathcal{Z}^n} =: \sim_n \neq \sim_n^j$ for each value of $j \in \{1, \ldots, q\}$. Hence $\mathcal{U}(\sim_n)$ and $\mathcal{U}(\sim_n^T)$ are disjoint neighbourhoods of F^1 and F^2 respectively. Next if $F^1 = \{\sim_\infty\}$ and $F^2 = F_\bigstar$, then let m be maximal such that $\sim_{(m-1)}$ has critical period q. Let $\sim_\infty^T$ be the terminal tower adjacent to $\sim_\infty^\bigstar$ and such that its critical value class E'_m separates the critical value gap G'_m for $\sim_m$ from the period q critical value gap for $\sim_m^\bigstar$. From here proceed as in the case above.

Next suppose that F^1 contains a terminal tower $\sim_\infty^T$ and $F^1 \neq F^2 = F_\bigstar$. Let m be maximal such that $\sim_{(m-1)}^T$ is fertile with critical period q. Denote by $\sim_\infty^{T2}$ the terminal tower whose critical value class $E_m'^2$ separates the critical value gap for $\sim_m^\bigstar$ from the critical value gap G'_m for $\sim_m^T$ if $\sim_m^T$ is fertile or its critical value class E'_m if $\sim_m^T$ is terminal. In any of the two cases $\sim_n^T$ has a critical value class E'_N for some $N \geq m$ and E'_N is separated from the critical value gap for $\sim_N^\bigstar$ by $E_m'^2$. As only the infinite tower $\sim_\infty^\bigstar \in F_\bigstar$ is adjacent to more than one inifinte terminal tower, there exists $n \geq N$ such that $\mathcal{U}(\sim_n^T)$ and $\mathcal{U}(\sim_n^{T2})$ are disjoint neighbourhoods of F^1 and $F^2 = F_\bigstar$. Finally we have the case where $F^1 \neq F_\bigstar \neq F^2$ and F^i each contains a terminal tower $\sim_\infty^{Ti}$, with critical value class E'^i,$i = 1, 2$. Taking n sufficiently large both $\sim_n^{Ti}$ are terminal, and the two critical value classes are separated by the critical value class of at least one other terminal tower, since only the renormalizable tower $\sim_\infty^\bigstar$ is adjacent to more than one terminal tower. For such n the open sets $\mathcal{U}(\sim_n^{Ti})$ are disjoint neighbourhoods of $\sim_\infty^{Ti}$. $\square$

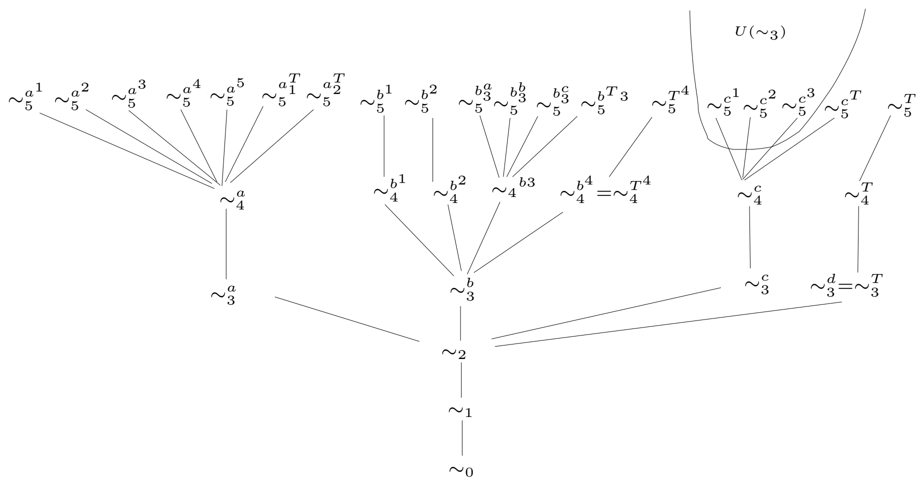

Figure 10: The elements at level 5 of the neighborhood $\mathcal{U}(\sim_3)$.

Corollary 4.2. *The combinatorial projection $\Xi_C : L_{p/q} \longrightarrow \mathcal{F}_{p/q}$ has the following properties:*

1. *The fiber above every point is compact and connected.*
2. *The fiber above any non-renormalizable class is a singleton, i.e. Ξ_C is injective above the non-renormalizable classes.*
3. *For any renormalizable class F (including $F_\bigstar$) the fiber $\Xi_C^{-1}(F)$ is homeomorphic to $\mathbf{M}$ by a canocical homeomorphism χ_F.*
4. *For each $c \in L_{p/q}^\bigstar$ and each $n \in \mathbb{N}$ the open neighbourhood of c in $L_{p/q}^\bigstar$,*

$$\bigcup_{\sim_\infty \in \Xi_C(c)} \mathcal{YP}(\sim_\infty |_{\mathcal{Z}^n})$$

is mapped onto the open neighbourhood $\mathcal{U}(\sim_n^c)$ of $\Xi_C(c)$ in $\mathcal{F}_{p/q}$ by Ξ_C.

4.1 Impressions of p/q-equivalences in $\overline{\mathbb{D}}$.

Let $G' = G'(p/q) = G'_0(p/q)$ denote the critical value gap of $\sim_0 = \sim_0 (p/q)$ and let $Comp(\overline{G'})$ denote the set of compact subsets of $\overline{G'}$.

Definition 4.3. *We define the impression map $\mathcal{I} : \mathcal{T}_{p/q} \longrightarrow Comp(\overline{G'})$ by the following: If $\sim_\infty$ is terminal with critical value class E' we define $I(\sim_\infty) = E'$. If $\sim_\infty$ is not terminal let G'_n denote the critical value gap of $\sim_\infty |_{\mathcal{Z}^n}$, $n \in \mathbb{N}$. Then we define*

$$\mathcal{I}(\sim_\infty) = \bigcap_{n \geq 0} \overline{G'_n}$$

Moreover we define the impression of a p/q-equivalence F as

$$\mathcal{I}(F) = \bigcup_{\sim_\infty \in F} \mathcal{I}(\sim_\infty)$$

similarly the impression of a subset $\mathcal{V} \subseteq \mathcal{F}_{p/q}$ is defined as $\mathcal{I}(\mathcal{V}) = \bigcup_{F \in \mathcal{V}} \mathcal{I}(F)$.

Recall that a subset $U \subseteq \overline{\mathbb{D}}$ is ideally convex if and only if it is hyperbolically convex with extremal boundary $\delta U = U \cap \mathbb{S}^1$.

As any gap or class is ideally convex, the reader shall easily verify the following Lemma:

Lemma 4.4. *For any $\sim_\infty \in \mathcal{T}^\infty_{p/q}$ and for any $F \in \mathcal{F}_{p/q}$ the impressions $\mathcal{I}(\sim_\infty)$ and $\mathcal{I}(F)$ are compact and ideally convex, in particular they are connected.*

Note that an impression $\mathcal{I}(\mathcal{V})$ for a subset $\mathcal{V} \subseteq \mathcal{F}_{p/q}$ is not necessarily compact.

Proposition 4.5. *We have $\mathcal{I}(\mathcal{F}_{p/q}) = \overline{G'}$, and moreover:*

1. *If $\sim_n \in \mathcal{T}_{p/q}$ is fertile with critical value gap G'_n and critical period $k > q$. Then*
$$\mathcal{I}(\mathcal{U}(\sim_n)) = G'_n.$$

2. *If $\sim^T_\infty \notin F_\bigstar$ is terminal with critical value class E'_m and $\sim^j_\infty$, $1 \leq j \leq q$ are the adjacent towers with critical value gaps G'^j_n for $\sim^j_n$, when $n \geq m$, then*
$$\mathcal{I}(\mathcal{U}(\sim^T_n)) = E' \cup \bigcup_{j=1}^{q} G'^j_n.$$

3. *Finally let $\sim^{Tj}_\infty$, $1 \leq j \leq r$ denote the terminal towers adjacent to $\sim^\bigstar_\infty$ and with $\sim^{Tj}_n \neq \sim^\bigstar_n$. Similarly let $\sim^i_n$, $1 \leq i \leq s$ denote the fertile towers in $F_\bigstar$ with $\sim^i_n \neq \sim^\bigstar_n$. Denote by E'^j_n and G''^i_n the respective critical value classes and gaps. Then*
$$\mathcal{I}(\sim^\bigstar_n) = \mathcal{I}(\sim^{Tj}_n) = \mathcal{I}(\sim^i_n) = G'^\bigstar_n \cup \bigcup_{j=1}^{r} E'^j_n \cup \bigcup_{i=1}^{s} G''^i_n.$$

Proof. Note first that for any fertile tower $\sim_n$, $n \geq 1$ with critical value gap G'_n, with terminal children $\sim^{Tj}_{(n+1)}$, $j = 1, \ldots, r$ having critical value classes E'^{j}_{n+1} and with fertile children $\sim^{i}_{(n+1)}$, $i = 1, \ldots, s$ having critical value gaps G''^{i}_{n+1} we have

$$G'_n = \bigcup_{j=1}^{r} E'^{j}_{n+1} \cup \bigcup_{i=1}^{s} G''^{i}_{n+1}.$$

Hence no point in $z \in G'_n$ is lost under the subdivision into critical value gaps and classes at the next level. And moreover once a point is trapped in some critical value class E'_m it remains there. It then follows by induction that $\mathcal{I}(\mathcal{F}_{p/q}) = \overline{G'}$. Moreover it also follows that for any fertile tower $\sim_n \in \mathcal{T}_{p/q}$, $\sim_n \neq \sim^{\bigstar}_n$ with critical value gap G'_n that $G'_n \subseteq \mathcal{I}(\mathcal{U}(\sim_n)) \subseteq \overline{G'_n}$. We shall prove that we have equality to the left. Let $E^1, E^2, \ldots, E^r$ be the finite number of classes of $\sim_{(n-1)}$ bounding G'_n in $\overline{\mathbb{D}}$. Denote by $\sim^j_\infty$, $1 \leq j \leq r$ the infinite towers adjacent to the terminal towers with critical value class E^j and with $\sim^j_\infty |_{\mathcal{Z}^n} = \sim_n$. For any $m > n$ write $B_m = \{\sim_m \mid \sim_m |_{\mathcal{Z}^n} = \sim_n\} \setminus \{\sim^1_m, \ldots, \sim^r_m\}$ then

$$\bigcup_{\sim_m \in B_m} \mathcal{U}(\sim_m) \subset \mathcal{U}(\sim_n) = \bigcup_{m \geq n} \bigcup_{\sim_m \in B_m} \mathcal{U}(\sim_m).$$

As

$$\mathcal{I}(\bigcup_{\sim_m \in B_m} \mathcal{U}(\sim_m)) \subset \mathcal{U}(\sim_n) \subset G'_n$$

for all $m \geq n$ we have $\mathcal{I}(\mathcal{U}(\sim_n)) \subseteq G'_n$ and thus equality. This proves 1. From this 2. and 3. also easily follows, once we note that for $\sim_\infty$ adjacent to $\sim^T_\infty$ the impression $\mathcal{I}(\sim_\infty)$ equals to the adjacent side of the class and impression $\mathcal{I}(\sim^T_\infty)$. □

Corollary 4.6. *If $F_1 \in \mathcal{F}_{p/q}, F_2 \in \mathcal{F}_{p'/q'}$ and $\mathcal{I}(F_1) \cap \mathcal{I}(F_2) \neq \emptyset$ then $p/q = p'/q'$ and $F_1 = F_2$.*

Proof. We have $p/q = p'/q'$, because $G'(p/q) \cap G'(p'/q') = \emptyset$, when $p/q \neq p'/q'$, simply because the sets $\mathcal{Z}(p/q)$ and $\mathcal{Z}(p'/q')$ are disjoint and hence so are any two equivalence relations on the two sets. Next let $\mathcal{U}(\sim^1_n)$ and $\mathcal{U}(\sim^2_n)$ be disjoint neighbourhoods of F^1 and F^2 respectively (for existence of such neighbourhoods see the proof of Lemma 4.1). By Proposition 4.5 these two sets have disjoint relatively open impressions. Disjointness of $\mathcal{I}(F^1)$ and $\mathcal{I}(F^2)$ then follows, as these sets are compact. □

Definition 4.7. *For any $z \in \overline{G'}$, there exists (by Proposition 4.5) a class $F \in \mathcal{F}_{p/q}$ such that $z \in \mathcal{I}(F)$. Moreover, Corollary 4.6 implies that F is unique, so that we can define a map $\mathcal{J} : \overline{G'} \longrightarrow \mathcal{F}_{p/q}$ by (a "left inverse to $\mathcal{I}$")*

$$\mathcal{J}(z) = F.$$

Corollary 4.8. *The map $\mathcal{J} : \overline{G'} \longrightarrow \mathcal{F}_{p/q}$ is continous with compact and ideally convex fibers $\mathcal{J}^{-1}(F) = \mathcal{I}(F)$.*

Proof. Let $F \in \mathcal{F}_{p/q}$ be arbitrary and represented by $\sim_\infty$ which is either the sole tower in F, or the sole terminal tower in F or $\sim_\infty^\bigstar$ if $F = F_\bigstar$. Then the sets $\mathcal{U}(\sim_n)$ form a neighbourhood basis at F and by Proposition 4.5 these are mapped by $\mathcal{I}$ to open neighbourhoods of $\mathcal{I}(F) = \mathcal{J}^{-1}(F)$. □

Proposition 4.9. *The topological space $\mathcal{F}_{p/q}$ is uniquely arcwise connected. More precisely for $F^0 \neq F^1$ let $z^i \in \mathcal{I}(F^i)$ for $i = 0, 1$ and let $R \subset \overline{G'}$ be the finite, infinite or bi-infinite geodesic segment for the hyperbolic metric on $\mathbb{D}$ joining z^0 to z^1. Then the image of the restriction $\Gamma_R = \mathcal{J}_R : R \longrightarrow \mathcal{F}_{p/q}$ is an arc from $F^0 = \mathcal{J}(z^0)$ to $F^1 = \mathcal{J}(z_1)$ and is the unique such arc.*

Proof. The arc R is a hyperbolic geodesic and the fibers of $\mathcal{J}$ are compact and ideally convex. It follows that the continuous curve Γ_R from $F^0 = \mathcal{J}(z_0)$ to $F^1 = \mathcal{J}(z_1)$ can not have loops. Thus its image $\gamma = \mathcal{J}(R)$ is an arc. (The curve Γ_R will have infinitely many stationary points though.) Also a point $F \in \mathcal{F}_{p/q}$ belongs to γ if and only if the fiber $\mathcal{I}(F) = \mathcal{J}^{-1}(F)$ separates z^0 and z^1 in $\overline{\mathbb{D}}$, because hyperbolic geodesics cross at most once. But then the impression $\mathcal{I}(\sigma)$ of any curve $\sigma \subset \mathcal{F}_{p/q}$ joining F^0 to F^1 contains $\mathcal{I}(\gamma)$ and thus $\gamma \subseteq \sigma$. If $\sigma \neq \gamma$ then there is a point $F \in \sigma \setminus \gamma$. By compactness of γ there exists a neighbourhood $\mathcal{U}(\sim_n)$ of F and a finite collection of open sets $\mathcal{U}(\sim_{n_j}^j)$, $1 \leq j \leq t$ covering γ such that $\mathcal{U}(\sim_n)$ and $\mathcal{U}(\sim_{n_j}^j)$ are disjoint for every j. The impression $\mathcal{I}(\mathcal{U}(\sim_n))$ is separated from R by the impression of a terminal class $F^2 \notin \gamma$ as $\mathcal{I}(\mathcal{U}(\sim_n)) \cap \mathcal{I}(\mathcal{U}(\sim_{n_j}^j)) = \emptyset$ for all j and $\mathcal{I}(\gamma) \subset \cup_{j=1}^t \mathcal{I}(\mathcal{U}(\sim_{n_j}^j))$. But then σ can not be injective, because it must visit F^2 both on the way from F^0 to F and on the way further from F to F^1. This proves that the arc γ connecting F^0 to F^1 is the unique such arc and in particular does not depend on the choice of points $z^i \in \mathcal{I}(F^i)$, $i = 0, 1$. □

Corollary 4.10. *The topological space $\mathcal{F}_{p/q}$ is a tree.*

Proof. Let $F \in \mathcal{F}_{p/q}$ be arbitrary and represented by $\sim_\infty$ which is either the sole tower in F, or the sole terminal tower in F or $\sim_\infty^\bigstar$ if $F = F_\bigstar$. Then the sets $\mathcal{U}(\sim_n)$ form a neighbourhood basis for F. A rerun of the proof above shows that these sets are arcwise connected, because $\mathcal{I}(\mathcal{U}(\sim_n))$ is ideally convex. A uniquely arcwise connected and locally arcwise connected space is a tree. □

To complete the proof that veins are arcs let $w = w_{m/2^n} = \mathrm{e}^{i2\pi m/2^n}$ be a dyadic point. And let R be the radial segment between w and its first intersection z_w towards the origin with E_0. We shall prove by induction on n that $\Xi_C^{-1}(\mathcal{J}(R))$ contains an arc connecting the root $c_{p/q}$ of $L_{p/q}$ to the dyadic tip $\Xi_C^{-1}(\mathcal{J}(w))$, where $G'(p/q) \ni w$. First we need some preparatory statements.

For $z_0, z_1 \in \mathbb{S}^1$ we shall use the notation $[z_0, z_1]_{\mathbb{S}^1}$ to denote the closed arc of $\mathbb{S}^1$ from z_0 to z_1 in the counter clockwise direction.

Proposition 4.11 (The Douady Tuning Algorithm). *Let $\sim_\infty$ be $k \geq q > 1$ renormalizable. Then $\delta(\mathcal{I}(\sim_\infty))$ is a Q_0^k-invariant Cantor-set on which Q_0^k is conjugate to the one-sided shift on two symbols. More precisely let $J = J(\sim_\infty) = [z_0, z_1]_{\mathbb{S}^1}$ denote the minimal interval with $\delta(\mathcal{I}(\sim_\infty)) \subset J$. Then z_0 and z_1 are fixed points of Q_0^k and J contains two disjoint arcs $J_0 = [z_0, z_1']_{\mathbb{S}^1}$ and $J_1 = [z_0', z_1]_{\mathbb{S}^1}$ such that $f = Q_0^k : J_i \longrightarrow J$ is an orientation preserving diffeomorphism, affine in the angular coordinate, for $i = 0, 1$ ($f(z) = z_i \cdot Q_0^k(z/z_i)$). The corresponding f-invariant Cantor set equals $\delta(\mathcal{I}(\sim_\infty))$.*

Proof. For each n let G_n' denote the critical value gap for $\sim_n = \sim_\infty|_{\mathcal{Z}^n}$ and choose N_0 such that $\sim_n$ has critical period k for each $n \geq N_0$. Then $Q_0^k : \delta\overline{G_n'} \longrightarrow \delta\overline{G_{n-k}'}$ is a $2:1$ covering for any $n \geq N_0$. It follows that the restriction $Q_0^k : I^1 := \delta(\mathcal{I}(\sim_\infty)) \longrightarrow I^1$ is a $2:1$ covering map. Because the sets $\delta\overline{G_n'}$ are nested with non-emty intersection $I^1 = \cap_{n\geq 0}\delta\overline{G_n'}$.

For $1 \leq j < k$ let $I^{(j+1) \mod k} = Q_0^j(I^1)$. Then $Q_0 : I^0 \longrightarrow I^1$ is a $2:1$ covering which is locally the restriction of a diffeomorphism. And for $1 \leq j < k$ $Q_0 : I^j \longrightarrow I^{(j+1) \mod k}$ is a diffeomorphism (i.e. homeomorphism which is locally the restriction of a diffeomorphism). For such j let J^j denote the minimal subarc of $\mathbb{S}^1$ containing I^j. Then $H(J^j) \cap H(I^0) = \emptyset$, because $H(I^j)$ is a connected set disjoint from $H(I^0)$ and $\mathbb{D}\backslash H(I^0)$ is a collection of disjoint hyperbolic half-spaces. It follows that Q_0 is injective on each arc J^j as $0 \in H(I^0)$. Write $J = J^1 = [z_0, z_1]_{\mathbb{S}^1}$, then $z_0, z_1 \in I^1$ and $Q_0^{-1}(J^1)$ has two connected components J', J'' with end points in I^0 and with $I^0 \subset J' \cup J''$. As Q_0 is injective on J^{k-1}, this arc contains one diffeomorphic preimage under Q_0 of each of the arcs J', J''. These preimages have end points in I^{k-1} and their union contains I^{k-1}. And recursively each J^{k-j}, $0 < j < k$ contains one diffeomorphic preimage under Q^j of each of the arcs J', J''. These preimages have end points in I^{k-j} and their union contains I^{k-j}. As the end points z_0, z_1 of $J = J^1$ belongs to I^1 we obtain in the final case $j = k-1$ above that J contains two disjoint subarcs $J_0 = [z_0, z_0']_{\mathbb{S}^1}$ and $J_1 = [z_0', z_1]_{\mathbb{S}^1}$ such that $I^0 \subset J_0 \cup J_1$ and $f = Q_0^k : J_0 \cup J_1 \longrightarrow J$ is a $2:1$ covering. As f is orientation preserving the end points z_0 and z_1 are f-fixed points. Moreover as f is expanding the invariant subset $C(\sim_\infty) = \cap_{n\geq 0} f^{-n}(J)$ is a Cantor set and there is a unique homeomorphism $\eta = \eta_{\sim_\infty} : \Sigma_2 = \{0,1\}^{\mathbb{N}} \longrightarrow C(\sim_\infty)$ conjugating the shift on Σ_2 to f and with $\eta((i)_n) = z_i$, for $i = 0, 1$. As $\delta(\mathcal{I}(\sim_\infty)) = I^1 \subset J^1$ is $Q_0^k = f$-invariant we have $I^0 \subset C(\sim_\infty)$. On the other hand it follows from the conjugacy above that the set of backwards orbits of z_0 and z_1 are dense in $C(\sim_\infty)$. As f is $2:1$ on I^0 any such backwards orbit is also in the closed set I^0. Hence the other inclusion and thus equality of the two sets follows. $\square$

Remark 4.12. *In the following we shall be working with renormalizable towers $\sim_\infty$. However in order to shorten/simplify notation we shall often use the index $F = F(\sim_\infty)$ instead of $\sim_\infty$. The only case where this is not always appropriate is the case $F = F_\bigstar(p/q) = F(\sim_\infty^\bigstar (p/q))$, where the impression $\mathcal{I}(F)$ and its essential boundary $\delta\mathcal{I}(F)$ contains but do not equal those of $\sim_\infty^\bigstar (p/q)$ for $q \geq 3$.*

Definition 4.13. *For $\sim_\infty$ renormalizable with period $k > 1$ let*

$$\eta_F : \Sigma_2 = \{0,1\}^{\mathbb{N}} \longrightarrow \delta\mathcal{I}(\sim_\infty)$$

as in the proof of Proposition 4.11 denote the unique topological conjugacy of the shift map on Σ_2 to Q_0^k on $\delta\mathcal{I}(\sim_\infty)$ mapping $(i)_n$ to z_i for $i = 0, 1$.

We remark that the actual tunning algorithm is based on the proposition above in a slightly different phrasing. The combinatorial part of the algorithm states that if we write the arguments $\theta^0, \theta^1 \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ of z_0, z_1 in base two, $\theta^i = .\overline{\epsilon_1^i \ldots \epsilon_k^i}$ for $i = 0, 1$, then the argument $\theta \in \mathbb{T}$ of the point $\eta_F((\sigma_j)_j)$ is given by:

$$\theta(\eta_F((\sigma_j)_j)) = \sum_{j=1}^{\infty} \frac{\epsilon_1^{\sigma_j} \ldots \epsilon_k^{\sigma_j}}{2^{kj}} = .\epsilon_1^{\sigma_1} \ldots \epsilon_k^{\sigma_1} \epsilon_1^{\sigma_2} \ldots \epsilon_k^{\sigma_2} \ldots \epsilon_1^{\sigma_j} \ldots \epsilon_k^{\sigma_j} \ldots .$$

Or equivalently writing $p_i = \epsilon_1^i \ldots \epsilon_k^i \in \{0, 1, \ldots, 2^k - 1\}$ for $i = 0, 1$ we have

$$\theta(\eta_F((\sigma_j)_j)) = \sum_{j=1}^{\infty} \frac{p_{\sigma_j}}{2^{kj}}.$$

This formula follows from $\theta(z_i') = \epsilon_1^{1-i} \ldots \epsilon_k^{1-i} \overline{\epsilon_1^i \ldots \epsilon_k^i}$ for $i = 0, 1$, which in turn follows from the restrictions $Q_0^k : J^i \longrightarrow J$ being diffeomorphisms.

The open complementary arcs

$$D_{r/2^s} = D_{r/2^s}(\sim_\infty) =]\eta_F(\epsilon_1 \ldots \epsilon_{s-1} 0\overline{1}), \eta_F(\epsilon_1 \ldots \epsilon_{s-1} 1\overline{0})[_{\mathbb{S}^1}$$

of the Cantor set $\delta\mathcal{I}(\sim_\infty)$ are naturally labelled by the dyadic fractions $r/2^s$, where $r = .\epsilon_1 \ldots \epsilon_{s-1} 1$. Because both of the binary fractions $.\epsilon_1 \ldots \epsilon_{s-1} 1\overline{0} = .\epsilon_1 \ldots \epsilon_{s-1} 0\overline{1}$ represents $r/2^s$. We shall moreover denote by $D_0 = D_0(\sim_\infty)$ the arc $]z_1, z_0[_{\mathbb{S}^1}$.

Notice that if $\sim_\infty$ is renormalizable then $\delta\mathcal{I}(\sim_\infty)$ can not contain dyadic points $w_{m/2^n} = \mathrm{e}^{i2\pi m/2^n}$, because $\delta\mathcal{I}(\sim_\infty)$ is Q_0^k-invariant and does not contain the fixed point $1 = w_0$ of Q_0. The complementary intervals $D_{r/2^s} = D_{r/2^s}(\sim_\infty)$ will however contain infinitely many dyadic points each. The dyadic point $w_{m/2^n}$ in $D_{r/2^s} = D_{r/2^s}(\sim_\infty)$ with the smallest denominator or equivalently smallest n will be called the *leading* dyadic point of $D_{r/2^s}$.

Lemma 4.14. *Suppose that $\sim_\infty$ is k-renormalizable and that the complementary arc $D_{1/2}$ of $\delta\mathcal{I}(\sim_\infty)$ has leading dyadic $w_{r/2^s}$. Let $w_{m/2^n}$ be any dyadic point contained in some complementary interval $D_{m'/2^{n'}}$ of $\delta\mathcal{I}(\sim_\infty)$. Then*

$$k(n'-1) \le n - s.$$

Proof. The map $Q_0^{k(n'-1)}$ maps $D_{m'/2^{n'}}$ diffeomorphically onto $D_{1/2}$ and thus maps $w_{m/2^n}$ onto $w_{m/2^{n-k(n'-1)}} \in D_{1/2}$. Hence $n - k(n'-1) \ge s$. $\square$

Definition 4.15. *Let $R = R_{m/2^n} = [0, w_{m/2^n}]$ denote the closure of the hyperbolic geodesic in $\mathbb{D}$ connecting the origin to $w_{m/2^n} \in G'(p/q)$. For $\sim_\infty$ renormalizable with $\mathcal{I}(\sim_\infty) \cap R \neq \emptyset$ define $K_F = [s_F, t_F]$, $0 < s_F < t_F < 1$ by $[s_F \cdot w_{m/2^n}, t_F \cdot w_{m/2^n}] = \mathcal{I}(\sim_\infty) \cap R$.*

Note that $[0, s_F \cdot w_{m/2^n}[\subset H(D_0(\sim_\infty))$ and that there is a unique dyadic $m(F)/2^{n(F)}$ such that $]t_F \cdot w_{m/2^n}, w_{m/2^n}] \subset H(D_{m(F)/2^{n(F)}}(F))$, because $\mathcal{I}(\sim_\infty) = H(\delta\mathcal{I}(\sim_\infty))$. Note also that $m(F)/2^{n(F)}$ depends on both $m/2^n$ and $\sim_\infty$, but that either $n(F) = n = 1$ or $n(F) < n$, according to Lemma 4.14.

Definition 4.16. *For a dyadic fraction $m/2^n$ the corresponding* dyadic tip *of $\mathbf{M}$ is the parameter $c = d_{m/2^n} = \Xi_C^{-1}(w_{m/2^n})$ equivalently it is the landing point of the external parameter ray $\mathcal{R}^{\mathbf{M}}_{m/2^n}$. For a homeomorphic copy $\mathbf{M}'$ of $\mathbf{M}$ with homeomorphism $\chi : \mathbf{M}' \longrightarrow \mathbf{M}$ the $m/2^n$* tip *of $\mathbf{M}'$ is defined as $\chi^{-1}(d_{m/2^n})$.*

Lemma 4.17. *Let $F = F(\sim_\infty)$, where $\sim_\infty$ is a k-renormalizable tower and let $\mathbf{M}_F = \Xi_C^{-1}(F)$. Then the prime end impression in $\mathbf{M}$ under Ψ of a point $w \in \mathbb{S}^1$ intersects the copy $\mathbf{M}_F$ if and only if $w \in \delta\mathcal{I}(F)$. Moreover the parameter rays $\mathcal{R}^{\mathbf{M}}_{z_i}$, corresponding to the Q_0^k fixed points $z_i \in \delta(\mathcal{I}(\sim_\infty))$, $i = \{0, 1\}$ co-land at the root of $\mathbf{M}_F$ and for any dyadic $r/2^s$, r odd the parameter rays corresponding to the two end points of $D_{r/2^s}(\sim_\infty)$, co-land at the $r/2^s$ dyadic tip of $\mathbf{M}_F$. Furthermore in the particular case $F = F_\bigstar(p/q)$ and $\sim_\infty^T \in F_\bigstar$ the terminal tower adjacent to $\sim_\infty^\bigstar$ in $D_{r/2^s}(\sim_\infty^\bigstar)$ any of the parameter rays corresponding to the $q-2$ other points of the critical value class E' of $\sim_\infty^T$ also co-land at $\mathbf{M}_{F_\bigstar}$.*

Proof. For $n \in \mathbb{N}$ let G'_n denote the critical value gap of $\sim_n = \sim_\infty |_{\mathcal{Z}^n}$ and for $c \in \mathbf{M}_F$ let $\mathcal{Y}'_n$ denote the corresponding critical value Yoccoz puzzle piece for Q_c. Choose N_0 such that $\sim_n$ has critical period k for all $n \ge N_0$. Then the restriction $Q_0^k : \delta G'_n \longrightarrow \delta G'_{n-k}$ has degree 2 and the restriction $f = Q_c^k : \mathcal{Y}'_n \longrightarrow \mathcal{Y}'_{n-k}$ is proper of degree 2.

Suppose first that $k > q$ so that $G'_n \subset\subset G'_{n-k}$ and f is quadratic like with filled-in Julia set $K'_c = \cap_{n\ge 0} \mathcal{Y}'_n$. It follows immediately that for any $z \in \mathbb{S}^1$ the corresponding prime-end impression under ϕ_c^{-1} in the filled-in Julia set K_c of Q_c intersects K'_c if and only if $z \in \delta(\mathcal{I}(\sim_\infty))$. And since $\mathbf{M}_F = \mathcal{YP}(F) = \cap_{n\geq 0}\mathcal{YP}(\sim_n)$ the similar statement holds in parameter space.

Let $\mathcal{R}^c_{z_i}$ denote the two k-periodic rays for Q_c, where z_i, $i = 0, 1$ are the two Q_0^k fixed points in $\delta(\mathcal{I}(\sim_\infty))$. Then by the dynamical ray landing Theorem 1.1 each of these rays lands on a non-attracting $Q_c^k = f$-fixed point in K'_c and each assigns combinatorial rotation number $0/1$ to their landing points. However f being quadratic like with connected filled-in Julia set has a unique non-repelling fixed point β'_c admitting combinatorial rotation number $0/1$. Hence for any $c \in \mathbf{M}_F$ the two rays $\mathcal{R}^c_{z_i}$ co-land at β'_c and their union with β'_c separates 0 from c in $\mathbb{C}$. By the Douady-Hubbard parameter landing Theorem 1.4 the corresponding parameter rays $\mathcal{R}^{\mathbf{M}}_{z_i}$ co-land at the root or cusp of $\mathbf{M}_F$.

Moreover for any dyadic $r/2^s$, r odd the two end points w_0, w_1 of $D_{r/2^s}$ are mapped by Q_0^{ks} to z_0, z_1. Hence the corresponding dynamical rays for Q_c co-land at the $r/2^s$ dyadic tip of K'_c, which maps by f^s to β'_c. Again by Theorem 1.4 the parameter rays of the same arguments co-land at the $r/2^s$ dyadic tip of $\mathbf{M}_F$.

Finally we leave details of the case $k = q$ to the reader. It is build into the p/q-Yoccoz puzzle. To see that the prime-end impression of a point $z \in \mathbb{S}^1$ intersects $\mathbf{M}_{F_\bigstar}$ if and only if $z \in \delta(\mathcal{I}(F_\bigstar))$ recall that the sets $\Xi_C^{-1}(\sim_n^\bigstar)$ form a fundamental system of neigbourhoods of $\mathbf{M}_{F_\bigstar}$. $\square$

To obtain uniqueness of connecting arcs in $\mathbf{M}$ we need a canonical way of passing hyperbolic components. Douady and Hubbard encompassed this by the following regularization.

Definition 4.18. *Define D-H-ruled arcs of* $\mathbf{M}$ *as those whose passage of a hyperbolic component does so via hyperbolic geodesics through the center.*

Let $\psi_{H_0} : H_0 \longrightarrow \mathbb{D}$ denote the Douady-Hubbard multiplier map of the fixed point from the component H_0 of $\mathbf{M}$ bounded by the cardioid. Denote also by ψ_{H_0} the homeomorphic extension to the closures. For each rotation number p/q, $(p, q) = 1$ define the path $\gamma_{p/q} : [0, 1] \longrightarrow \mathbf{M}$ by

$$\gamma_{p/q}(t) = \begin{cases} \psi_{H_0}^{-1}(1 - 2t), & 0 \leq t \leq \frac{1}{2} \\ \psi_{H_0}^{-1}(2(t-1)\mathrm{e}^{i2\pi p/q}), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Theorem 4.19. *Any dyadic tip* $d_{m/2^n}$ *is connected to the root* $d_0 := \frac{1}{4}$ *of* $\mathbf{M}$ *by a unique D-H-ruled arc* $\Gamma_{m/2^n}$ *in* $\mathbf{M}$*. In particular* $d_{m/2^n} \in L_{p/q}$ *is connected to the root* $c_{p/q}$ *of* $L_{p/q}$ *by a unique D-H-ruled arc.*

Proof. The proof goes by induction on n the exponent of the dyadic numerator. For $n = 1$ the vein from the cusp $d_0 = \frac{1}{4} \in \mathbf{M}$ to the dyadic tip $d_{1/2^1} = -2$ is the real interval $[\frac{1}{4}, -2]$

and is thus an arc. Choose any parametrization $\Gamma_{1/2} : [0,1] \longrightarrow [\frac{1}{4}, -2]$ with $\Gamma_{1/2}(0) = \frac{1}{4}$. Let $m/2^n$, $n > 1$ be arbitrary and suppose that for any $m'/2^{n'}$ with $n' < n$, there is a unique ruled arc $\Gamma_{m'/2^{n'}} : [0,1] \longrightarrow \mathbf{M}$, with $\Gamma_{m'/2^{n'}}(0) = d_0$ and $\Gamma_{m'/2^{n'}}(1) = d_{m'/2^{n'}}$.

Let p/q be given by $d_{m/2^n} \in L_{p/q}$, let $R = R_{m/2^n}$ be as in Definition 4.15 and define

$$Ren = Ren_R = \{F \in \mathcal{J}(R) | F \text{ is renormalizable}\}.$$

For each $F \in Ren$ define K_F and $m(F)/2^{n(F)}$ as in Definition 4.15 above. Furthermore define

$$K = K_R = \bigcup_{F \in Ren} K_F.$$

For each renormalizable p/q-equivalence F recall that $\mathbf{M}_F = \Xi_C^{-1}(F) \supseteq \mathcal{YP}(F)$ and the map $\chi_F : \mathbf{M}_F \longrightarrow \mathbf{M}$ is the straightening homeomorphism. Define

$$\widehat{\Gamma} = \widehat{\Gamma}_{m/2^n}(t) = \begin{cases} \gamma_{p/q}(\frac{t}{s_{F_\bigstar}}), & 0 \leq t < s_{F_\bigstar}, \\ \Xi_C^{-1}(\mathcal{J}(t)) & t \notin K, \\ \chi_F^{-1} \circ \Gamma_{m(F)/2^{n(F)}}(\frac{t - t_F}{s_F - t_F}), & F = \mathcal{J}(t), t \in K. \end{cases}$$

We shall prove that $\widehat{\Gamma}$ is a curve with connected fibers and thus its image contains an arc $\Gamma_{m/2^n} : [0,1] \longrightarrow \mathbf{M}$ with $\Gamma_{m/2^n}(0) = d_0$ and $\Gamma_{m/2^n}(1) = d_{m/2^n}$.

For $F = \mathcal{J}(t)$ non-renormalizable we either have $F = \{\sim_\infty\}$ or we can choose to represent F by a unique terminal tower $\sim_\infty = \sim_\infty^T \in F$. With this convention we have

$$\Xi_C(\bigcup_{\sim_\infty \in F} \mathcal{YP}(\sim_\infty |_{\mathcal{Z}^n})) = \mathcal{U}(\sim_n).$$

Continuity at t thus follows from the continuity of $\mathcal{J}$. Continuity at $s_{F_\bigstar}$ is by construction and at $t_{F_\bigstar}$ the argument is the same as above. For any other $F \in Ren$ note first that $\widehat{\Gamma}(t) \to \Xi_C^{-1}(F)$, when $t \to K_F$ from outside by the same argument as above. Hence for continuity at s_F we need to prove that $\widehat{\Gamma}(t)$ can approach only the root of $\Xi_C^{-1}(F)$, when t approaches s_F from below. And for continuity at t_F we need to prove that $\widehat{\Gamma}(t)$ can approach only the dyadic tip $\chi_F^{-1}(d_{m(F)/2^{n(F)}})$ of $\Xi_C^{-1}(F)$, when t converges to t_F from above. Continuity at s_F now follows because $[0, s_F \cdot w_{m/2^n}[$ is on one side of the hyperbolic geodesic connecting z_0 and z_1 and the rest of $\mathcal{I}(F)$ is on the other side, and because the two external rays of period k corresponding to z_0 and z_1 co-land at the root of $\Xi_C^{-1}(F)$. Similarly continuity at t_F follows because $]t_F \cdot w_{m/2^n}, w_{m/2^n}[$ is on one side of the hyperbolic geodesic connecting the two endpoints y_0 and y_1 of $D_{m(F)/2^{n(F)}}$, 0 and the rest of $\mathcal{I}(F)$ are on the other side, and because the two rays corresponding to y_0 and y_1 co-land at the dyadic tip $\chi_F^{-1}(d_{m(F)/2^{n(F)}})$ of $\Xi_C^{-1}(F)$. $\square$

Definition 4.20. *A* Yoccoz parameter $c \in \mathbf{M}$ *is a at most finitely many times renormalizable parameter for which every periodic cycle is repelling.*

Theorem 4.21. *Any two Yoccoz parameters c^1 and c^2 are connected in $\mathbf{M}$ by a unique D-H-ruled arc.*

Proof. We shall suppose the two parameters belong to the same limb $L_{p/q}$, and leave to the reader the easier case, where they are in different limbs. Write $\sim_\infty^i = \sim_\infty^{c^i}$ and $F^i = F(\sim_\infty^i)$ for $i = 0, 1$. Choose points $z^i \in \mathcal{I}(\sim_\infty^i)$ and let $R = R_{z^1 z^2}$ denote the hyperbolic geodesic connecting z^1 and z^2. Then by Proposition 4.9 the image $\mathcal{J}(R)$ is the unique arc connecting F^1 to F^2 in $\mathcal{F}_{p/q}$. Moreover arguing as in the proof of Theorem 4.19 the set $\Xi_C^{-1}(\mathcal{J}(R))$ contains a unique arc $\widehat{\Gamma}$ connecting $\mathcal{YP}(F^1)$ to $\mathcal{YP}(F^2)$. (Note that when $\sim_\infty$ is renormalizable with $F(\sim_\infty) \neq F^i$, $i = 1, 2$, and with $\mathcal{I}(\sim_\infty) \cap R \neq \emptyset$ as in Definition 4.15 then $\widehat{\Gamma} \cap \mathbf{M}_F$ is a dyadic vein of $\mathbf{M}_F$ except possibly for one renormalizable tower $\sim_\infty$, for which the intersection is contained in the union of two dyadic veins corresponding to entry and exit of $\widehat{\Gamma}$ into/out of $\mathbf{M}_F$.) If c^i is not renormalizable then $\mathcal{YP}(F^i) = \{c^i\}$ for $i = 1$ and/or $i = 2$. Thus to complete the argument we shall prove that if say c^1 is renormalizable, (we leave the similar case c^2 renormalizable to the reader) then $\Xi_C^{-1}(\mathcal{J}(R) \setminus F^1)$, which converges to some dyadic tip $\chi_{F^1}(d_{m/2^n})$ (possibly d_0) of $\mathbf{M}_{F^1}$, can be extended by an arc in $\mathbf{M}_{F^1}$ from $\chi_{F^1}(d_{m/2^n})$ to c^1. This is however equivalent to the initial problem applied to the renormalized (from $\mathbf{M}_{F^1}$) parameters $\chi_{F^1}(c^1)$ and $d_{m/2^n}$. But c^1 was assumed to be at most finitely many times renormalizable, and $\chi_{F^1}^{-1}(c^1)$ is one time less renormalizable than c^1, so we can handle the problem recursively in a finite number of steps. $\square$

References

[BrSl] H. Bruin and D. Schleicher, *Symbolic Dynamics of Quadratic Polynomials*, Reports Institut Mittag-Leffler, Preprint series: Probability and Conformal Mappings - 2001/2002, No. 07

[BuSe] S. Bullett and P. Sentenac *Ordered orbits of the shift, square roots, and the devil's staircase*, Math. Proc. Cambridge Philos. Soc. 115 (1994), no. 3, 451–481.

[DH1] A. Douady & J.H. Hubbard, *Etude dynamique des polynômes complexes, I*, publications mathématiques d'Orsay, 84-02, 1984.

[DH2] A. Douady & J.H. Hubbard, *Etude dynamique des polynômes complexes, II*, publications mathématiques d'Orsay, 85-01, 1985.

[DH3] A. Douady & J.H. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Scient. Ec. Norm. Sup., t.18, p. 287-343, 1985.

[G-S] J. Graczyk and G. Światek, *Induced expansion for quadratic polynomials,* Ann. Sci. École Norm. Sup. t. 29 (1996), p. 399–482

[H] J. H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J. C. Yoccoz*, In *Topological Methods in Modern Mathematics*, 467-511, Goldberg and Phillips eds, Publish or Perish 1993.

[L-V] O. Lehto and K. J. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag New-York, 1973.

[Ka2] J. Kahn, *A priori bounds for some infinitely renormalizable quadratics: I. Bounded primitive combinatorics* IMS06-05, preprint from Inst. of Math. Sciences, SUNY StonyBrook, USA.

[K-L] J. Kahn and M. Lyubich, *A priori bounds for some infinitely renormalizable quadratics: II. Decorations* IMS06-06, preprint from Inst. of Math. Sciences, SUNY StonyBrook, USA.

[Ki] I. Kiwi, *Rational laminations of complex Polynomials* in 'Laminations and foliations in dynamics, geometry and topology', Stony Brook M.Y. 1998, 111-254, Contemp Math, 269 AMS Providence R.I.

[L] M. Lyubich, *Dynamics of quadratic polynomials, I-II*, Acta Math. 178 (1997), p. 185–297.

[P] C.L. Petersen, *On the Pommerenke-Levin-Yoccoz inequality* Ergod. Th. & Dynam. Syst. **13** 1993 785–806.

[Ri] J. Riedl, *Arcs in Multibrot sets, Locally connected Julia sets, and Their Construction by Quasiconformal Surgery* ph. D thesis Fakultät fur ür Mathematik der Technichen Universität München. 14 September 2000.

[R] P. Roesch, *Holomorphic motions and puzzles*, in "The Mandelbrot set, Theme and Variations", Ed. Tan Lei, LMS Lect. Note Ser. 274, Cambridge Univ. Press, 2000, p.117–131.

[S] D. Schleicher, On Fibers and Renormalization of Julia sets and Multibrot sets. arXiv:math9902156v1

Acknowledgement:

The first author would like to thank The Fields Institute, Toronto and Université Paul Sabatier, Toulouse for their hospitality during part of the writing of this paper. The authors thank A. Chéritat for his help with the pictures.

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