No invariant line field on the boundary of bounded Fatou components

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The boundary of U does not support an invariant line field.

There can be indifferent irrational points, the polynomial can be infinitely renormalizable.

Invariant line field

A measurable line field supported on J
is the data of a real line through the origin in the tangent space at each point $z \in E$ where $E \subset J$ has positive Lebesgue measure, such that the slope is a mesurable function of z.

This line field is invariant if $f^{-1}(E) = E$ and f' maps the line at z to the line at $f(z)$.

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The line field corresponds to the big axis of the ellipse field of the tangent map.

Idea of the proof

- we construct sequences of neighborhoods of points of ∂U (puzzle pieces $P_n(z)$)
- either there exists a sequence (i_n), some $D > 0$ such that

$$\deg(f^{i_n} : P_{i_n}(x) \to P_0) \leq D \qquad (*)$$

- or we can construct enhanced nests $\widetilde{K_n} \subset K_n \subset K'_n$ around recurrent critical points

The enhanced nest has the property $K'_n \setminus K_n$ and $K_n \setminus \widetilde{K_n}$ do not intersect the post-critical set, and the moduli are bounded from below.

Sketch of the proof of non invariant line field

Let X be our set. Assume that we have a graph Γ_0 that cut X in pieces. Denote by $\Gamma_n = f^{-n}(\Gamma_0)$, by $P_n(x)$ the connected components of $\mathbb{C} \setminus \Gamma_n$.

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We say that $P_{n+k}(c)$ is a successor of $P_n(c)$ if $f^k(P_{n+k}(c)) = P_n(c)$ and each critical point appears at most twice in the sequence of pieces $\{f^i(P_n(c)) \mid 0 \leq i \leq k\}$

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- X_4 is the set of point $z \in X \setminus (X_1 \cup X_2)$ such that $\omega_{comb}(z) \neq \emptyset$ and for all $(c, c') \in \omega_{comb}(z)$, $c \in \omega_{comb}(c')$ and $c' \in \omega_{comb}(c)$, moreover any $P_n(c)$ for $c \in \omega_{comb}(z)$ has only finitely many succesors.

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Let x be a point such that $\nu(x) = 1$ and ν is almost continuous at x

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Using the nest $K'_n \supset K_N \supset \widetilde{K_n}$,
one can define maps $g_n : U_n(x) \to V_n(x)$

- that preserve ν
- of degree $d \in [2, N]$
- such that $shape(U_n(x), x) \leq M$ and $shape(V_n(x), x) \leq M$ for some M.
- $diam(U_n(x)) \to 0$ and $diam(V_n(x)) \to 0$.

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where

$$shape(P, x) = \frac{max_{z \in \partial P} d(z, x)}{d(x, \partial P)}$$

Rescalling by some linear map

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The map $h_n = \beta_n^{-1} \circ g_n \circ \alpha_n$ admits a limit h defined at least on $B(0, 1/M)$ which is holomorphic of degree in $[2, N]$.

Assume that $h'(z) \neq 0$ on $D \subset B(0, 1/M)$ then $h'(z) \geq \delta$ on D for some $\delta > 0$.

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Let $\mu_n = \alpha_n^* \nu$ and $\nu_n = \beta_n^* \nu$ then $\mu_n = h_n^* \nu$
Then

$$Leb\{z \in B(0, 1/M) \mid |\mu_n(z) - 1| \geq \varepsilon\} \to 0$$

$$Leb\{z \in B(0, 1/M) \mid |\nu_n(z) - 1| \geq \varepsilon\} \to 0$$

$$Leb(\{z \in D \mid |\nu_n(h_n(z)) - 1| \geq \varepsilon\}) \leq \frac{N}{\delta^2} Leb\{w \in h_n(D) \mid |\nu_n(w) - 1| \geq \varepsilon\}$$

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But $deg(h) \geq 2$ in $B(0, 1/M)$. Contradiction.

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The maps g_n are $f^{-t_n} \circ f^{r_n}$.

Sketch of proof

Idea of the proof that $Leb(X_2 \cup X_3) = 0$
1) We can find P_r with $\overline{P_r} \subset P_0$ such that $f^{i_n}(x) \in P_r$ with $r > 0$.

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3) Then let $B = D(y, \varepsilon)$ be a ball in $P_r \cap \mathcal{F}$, then $shape(B_{i_n+r}(y_n), y_n) \leq C'$ where $B_{i_n+r}(y_n)$ is a component of $f^{-i_n}(B)$ in $P_{i_n+r}(x)$;

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4) One can deduce that

$$\frac{Leb(P_{i_n+r}(x) \cap J)}{Leb(P_{i_n+r}(x))} < 1$$

therefore x is not a density point.

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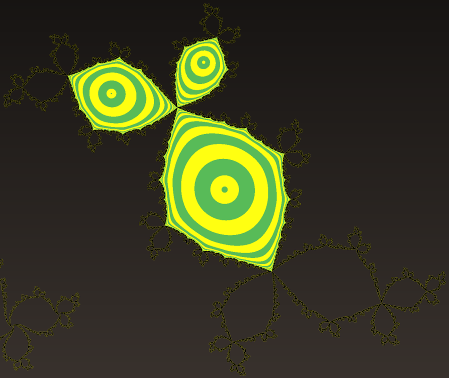
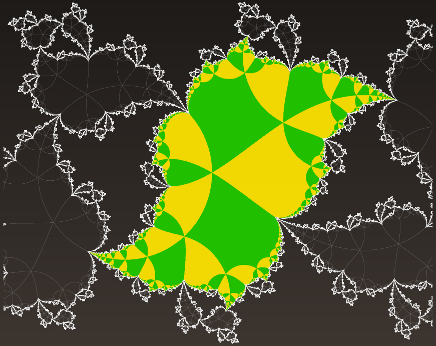


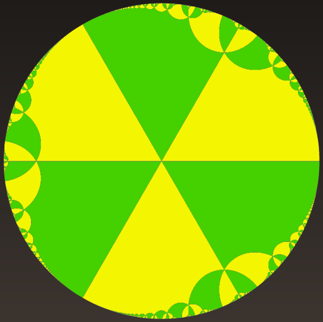
Consequence: the return map in U is conjugate

- either to $z \mapsto z^d$
- or to the Blaschke product of degree d having a parabolic point:

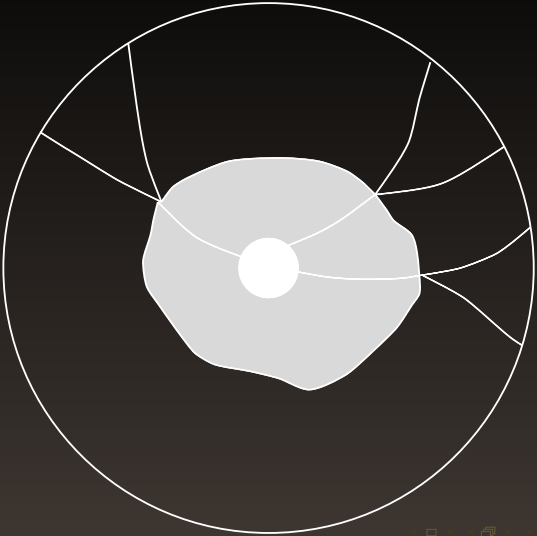
$$B(z) = \frac{z^d + v_d}{1 + v_d z^d}, \ v_d = \frac{d-1}{d+1}$$



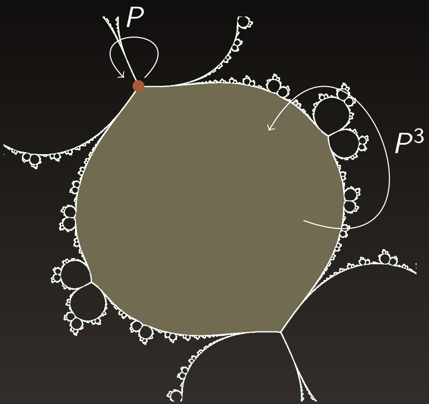
$d = 3$



Construction of Puzzles in the attracting case



In the parabolic case : Parabolic rays



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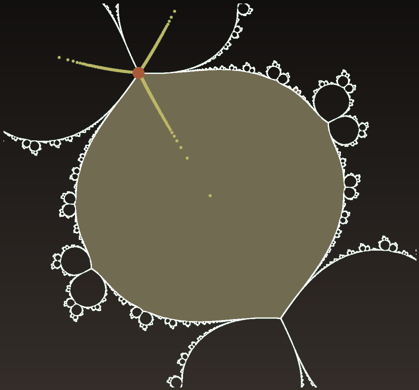

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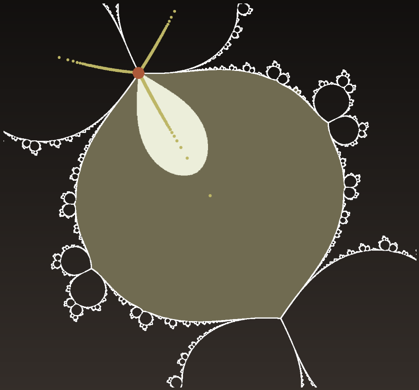

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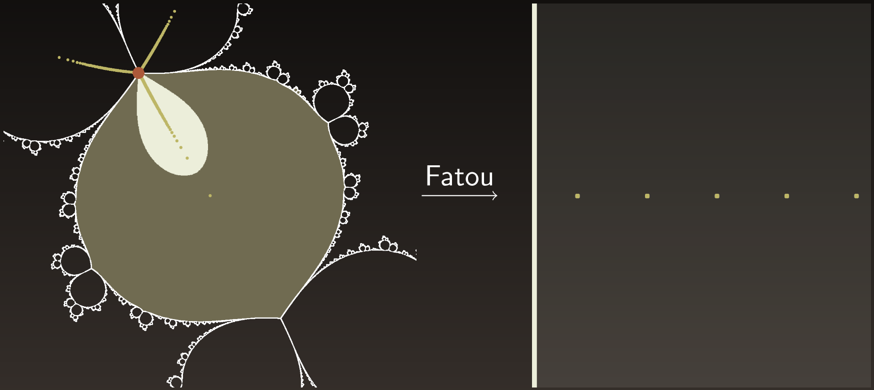

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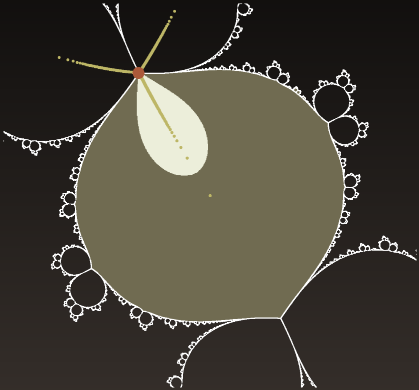
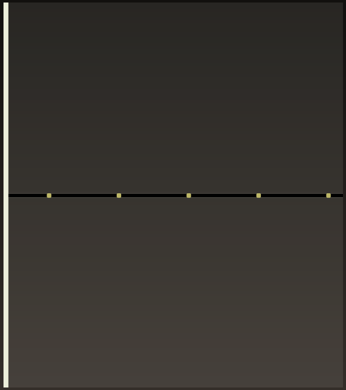


$\xrightarrow{\text{Fatou}}$



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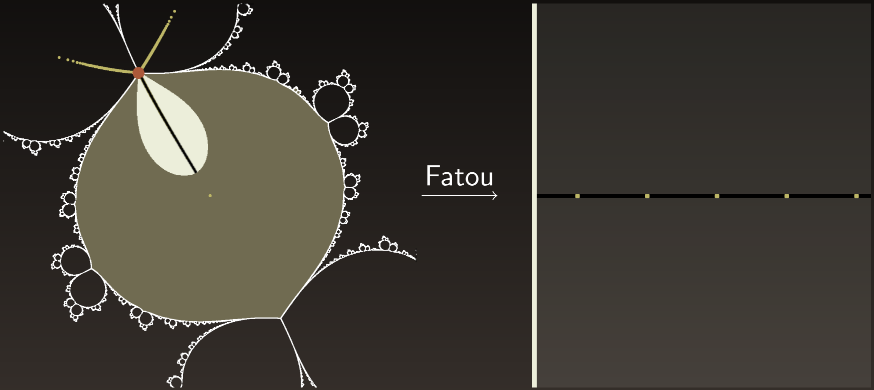

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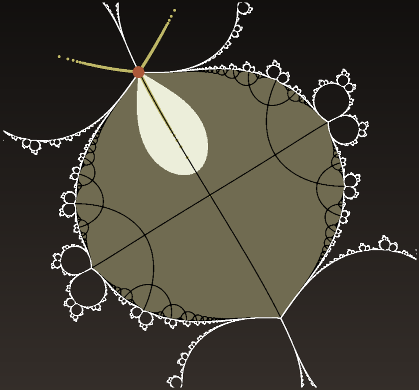

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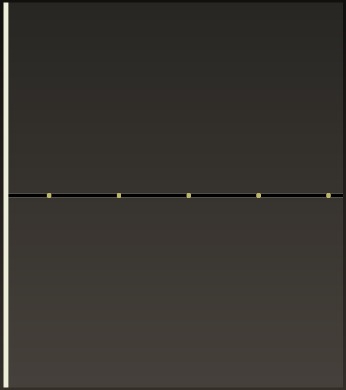

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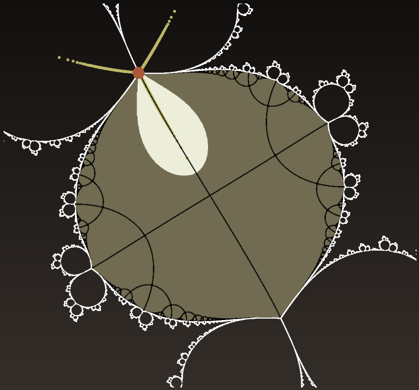

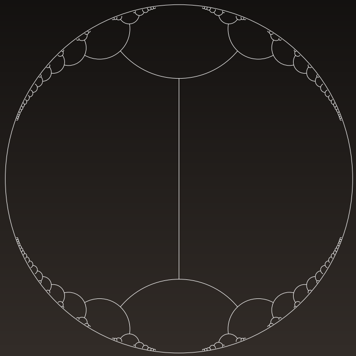

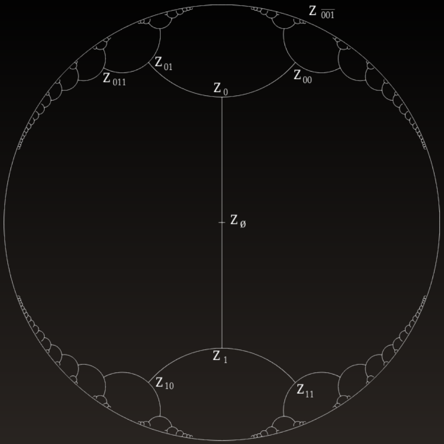
Z $\overline{001}$
Z 01
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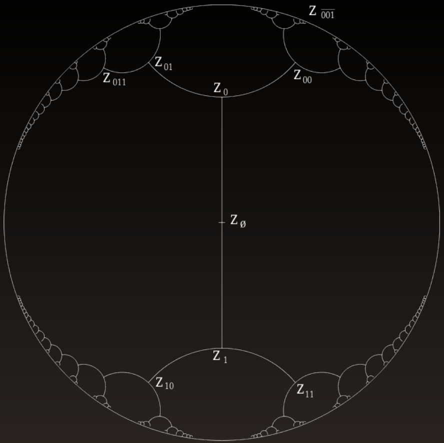
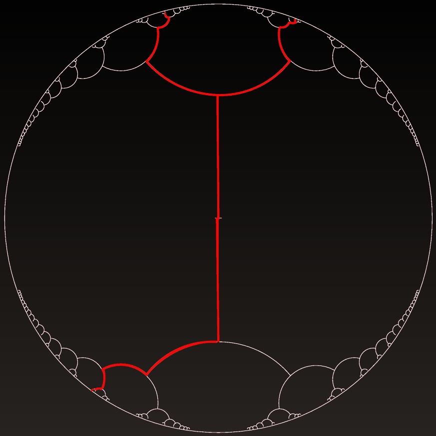
Z 11

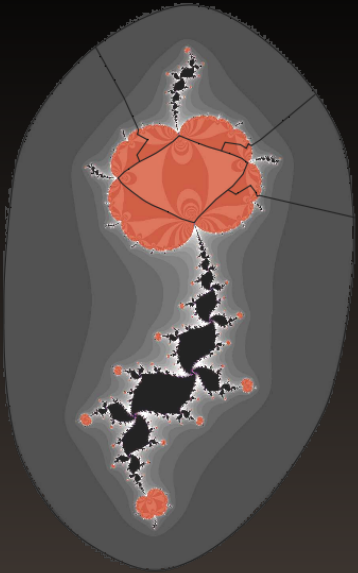




For any itinerary $\underline{\varepsilon} = \varepsilon_0 \cdots \varepsilon_n \cdots$ with $\varepsilon_i \in \{0, 1\}$ define the parabolic ray $\gamma_{\underline{\varepsilon}}$ to be the minimal arc in the tree joining the points $z_{\varepsilon_0 \cdots \varepsilon_n}$ and $z_\emptyset$.

$$B(\gamma_\varepsilon) = \gamma_{\sigma(\varepsilon)} \cup [0, \frac{1}{3}]$$

Construction of puzzles



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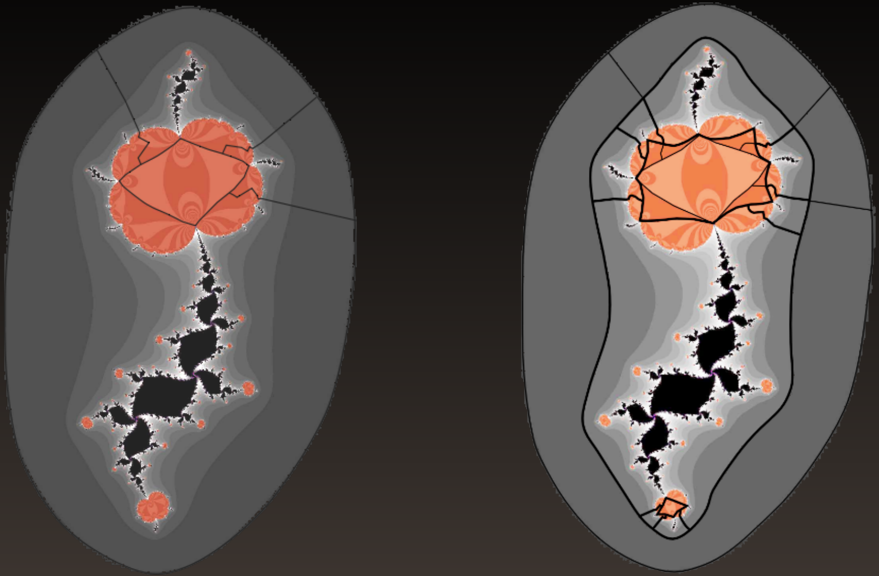
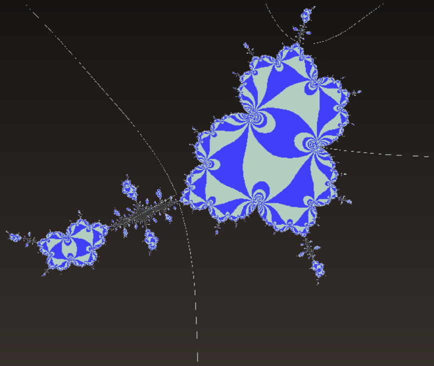


PROPOSITION: If x is eventually periodic on ∂U,

- either $E(x) := \cap \overline{P_n(x)} = \{x\}$
- or there exist external rays $R_\infty(\zeta), R_\infty(\zeta')$ landing at x and separating $\overline{U}$ from $E(x) \setminus \{x\}$.



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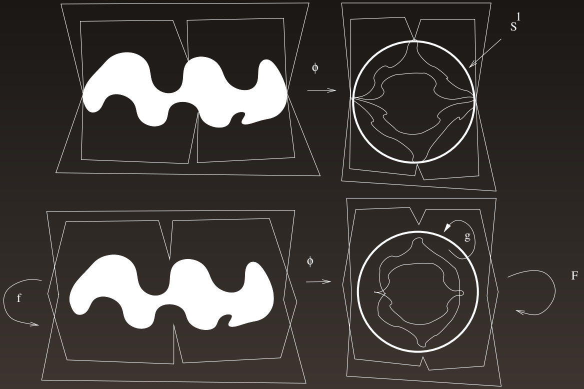

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Fixed points for f in $E(x)$ with a fixed access correspond to fixed points of $g = F_{|S^1}$ in $\mathbf{S}^1$ with a fixed access (by F).

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We have actually three accessible fixed points in $E(x)$ obtained as the limits of the rays bounding the "fixed" nest $P_n(x)$:

- the boundary rays $R_\infty(\zeta_n)$ give $\zeta_n \uparrow \zeta$ so a fixed ray $R_\infty(\zeta)$ landing at y in $E(x)$;
- the boundary rays $R_\infty(\zeta_n')$ give $\zeta_n' \downarrow \zeta'$ so a fixed ray $R_\infty(\zeta')$ landing at y' in $E(x)$;
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Denote by τ, τ' and 1 the points on S^1 corresponding to y, y' and x.

Claim :

The fixed points of g are weakly repelling, *i.e.* $\mid g(z) - p \mid_{\mathbb{S}^1} > \mid z - p \mid_{\mathbb{S}^1}$.

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Here in each interval of $S^1 \setminus \{\tau, \tau', 1\}$.

Therefore some strict inverse image of $R_\infty(\zeta'')$, say $R_\infty(\eta)$,
lands on $E(x)$ at a preimage of x.

So $R_\infty(\eta)$ lies between $R_\infty(\zeta_n)$ and $R_\infty(\zeta)$ (or $R_\infty(\zeta'_n)$ and $R_\infty(\zeta')$).

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Hence $\eta = \zeta$ or ζ'

contradiction.

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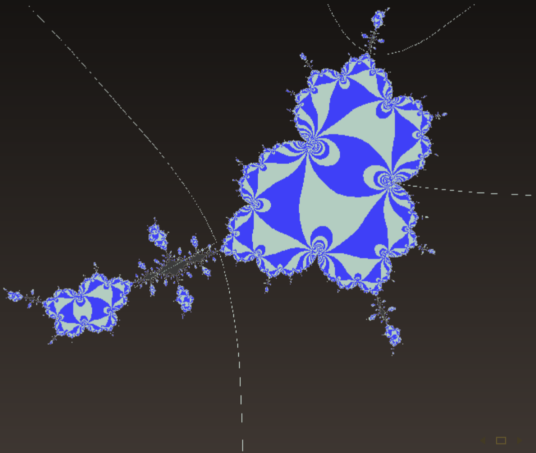
Using the same kind of argument and Denjoy Wolff's Theorem, we obtain that $E(x)$ is separated from U by these two rays.



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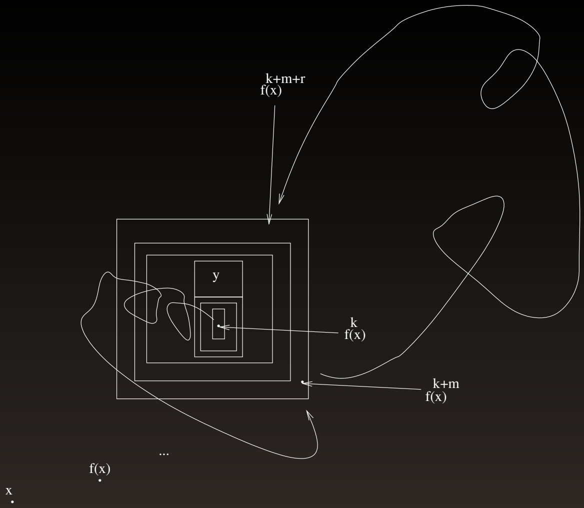
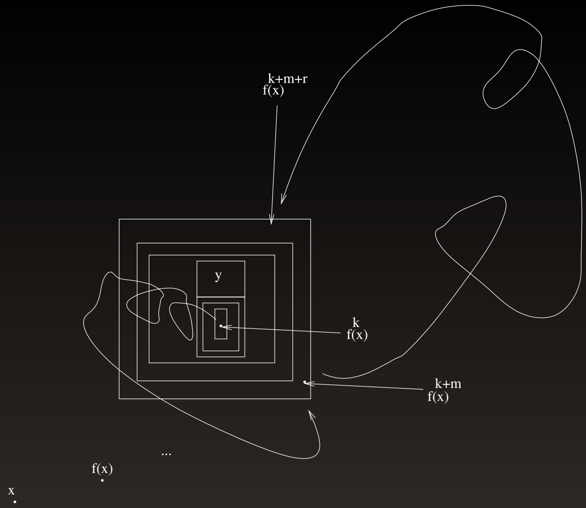

k+m+r
f(x)
y
k
f(x)
k+m
f(x)
...
f(x)
x



For every n take the first k such that $f^k(x) \in P_n(y)$, the first $m \geq n$ such that $f^k(x) \in P_m(y) \setminus \overline{P_{m+1}(y)}$ and the first r such that $f^r(f^{k+m}(x)) \in P_1(y)$.

The map $f^{k+m+r} : P_{k+m+r+1}(x) \to P_1(y)$ has bounded degree.

Lemma: For any given sequence k_n, if $\mathcal{O} := \{f^{k_n}(x),\ n \geq 0\}$ does not accumulate on a parabolic point, it is possible to find a graph Γ and some r such that $P_r \subset\subset P_0$ contains infinitely many points of $\mathcal{O}$.

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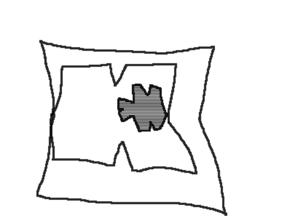

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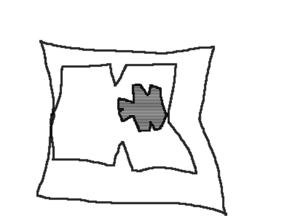

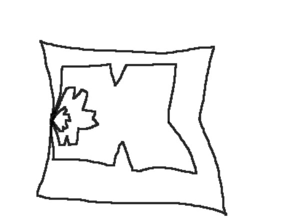

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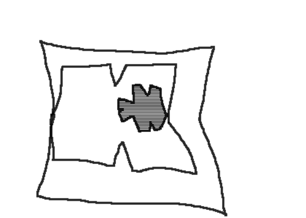

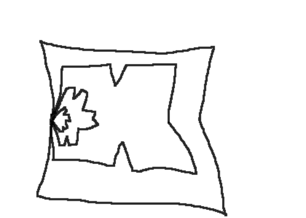

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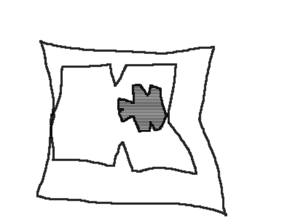

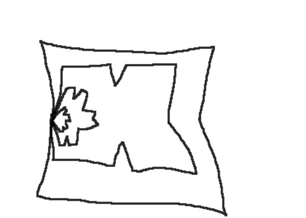

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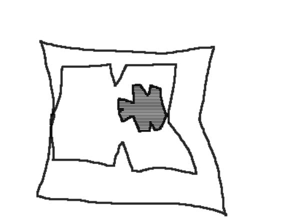

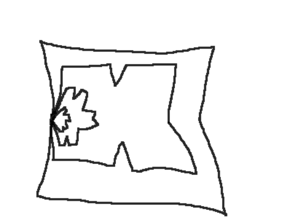

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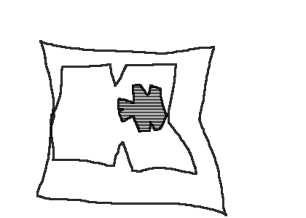

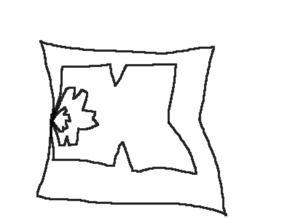

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- $f^{p_n}(K_n) = K_{n-1}$, $p_{n+1} \geq 2p_n$, $deg(f^{p_n} : K_n \to K_{n-1}) \leq C(b, \delta)$.

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- $f^{p_n}(K_n) = K_{n-1}$, $p_{n+1} \geq 2p_n$, $deg(f^{p_n} : K_n \to K_{n-1}) \leq C(b, \delta)$.
- $h(K'_n) - h(K_n) \geq r(K_{n-1}) \to \infty$

The persistently recurrent case

Every critical point accumulated by x has finitely many successors. Denote by $\Gamma(P)$ the last successor of P.

Lemma: Any puzzle piece has at least two successors.

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Using "Kahn-Lyubich covering Lemma", we can prove that $\liminf \mu_n$ is bounded from below.

Hence $E(c) = \{c\}.\square$

The Covering Lemma

Theorem

Let $f : U \to V$ be a degree D ramified covering. For any $\eta > 0$, there exists $\varepsilon = \varepsilon(\eta, D) > 0$ such that :

- if $A \subset\subset A' \subset\subset U$ and $B \subset\subset B' \subset\subset V$ are sequences of disks ;
- if f is a proper map from A to B, and from A' to B' with degree d ;
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Then

- $mod(U \setminus A) > \varepsilon$
- or $mod(U \setminus A) > \frac{\eta}{2d^2} mod(V \setminus B)$.

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