About rigidity for rational maps

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are they Moebius conjugate?

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4. Conformal equivalence : the conjugacy is a Moebius map.

Without hypothesis $1 \not\Longrightarrow 2$

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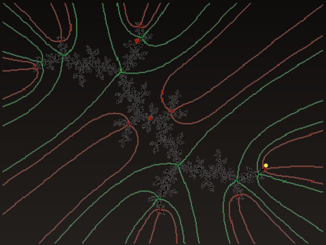

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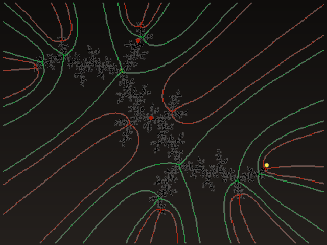

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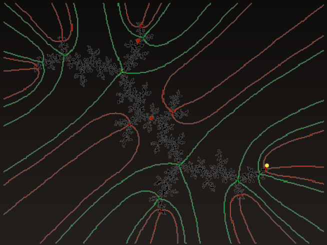

"identifying" rational angles when the external rays "co-land".

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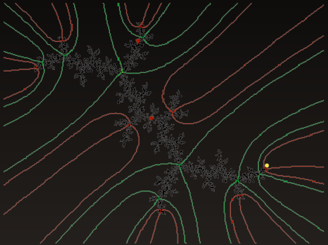

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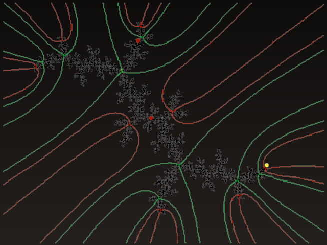

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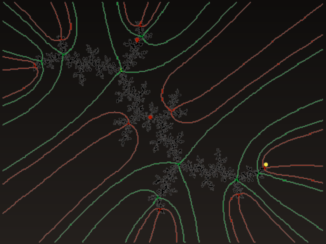

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If the Julia sets are locally connected then $1 \Longrightarrow 2$

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changing Λ gives quasi-conformally conjugate Lattès maps.

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- rational map with a fixed attracting multi connected basin such that any non trivial Julia component is a quasi-circle bounding an eventually superattracting Fatou component containing at most one postcritical point (Peng & al.)
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- cubic Newton maps in the finitely renormalizable case (R)
- ...

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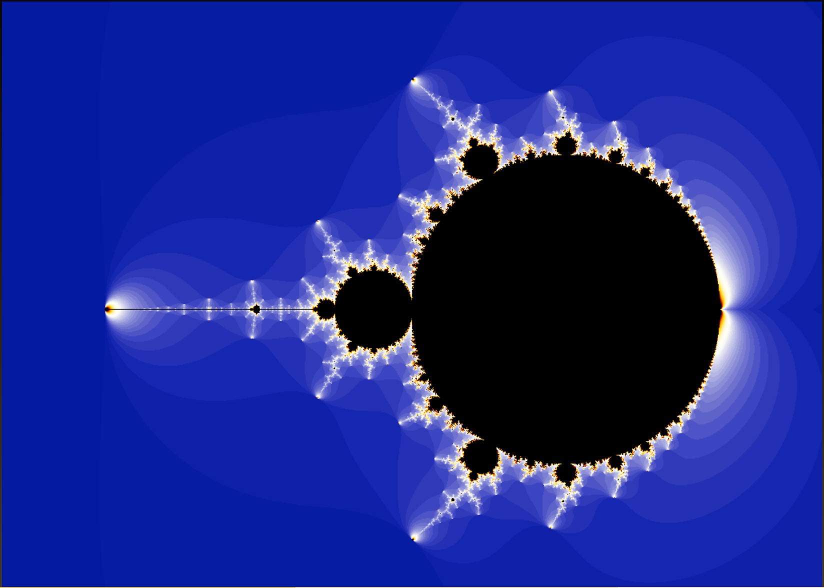
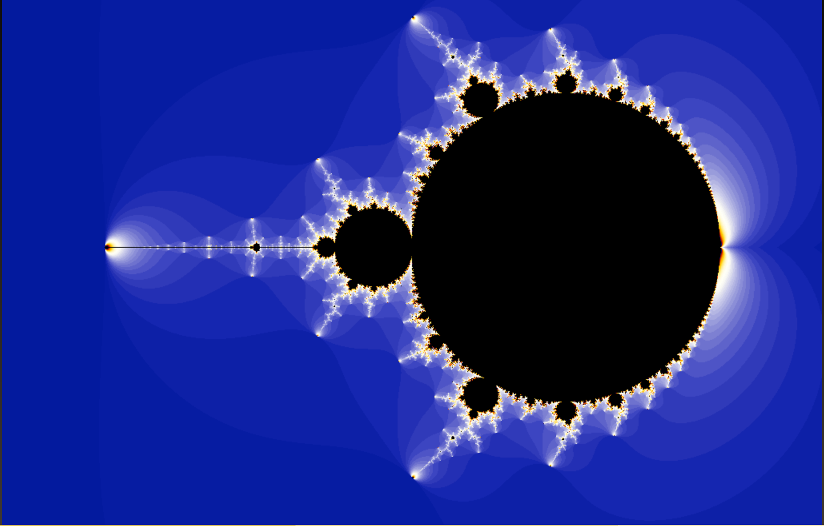


$Per_1(1) = \{f \in Rat_2 \mid \text{with a fixed point of multiplier } 1\}/PSL(2,\mathbf{C})$



The parabolic Mandelbrot set

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  The two maps have to belong to the same piece.

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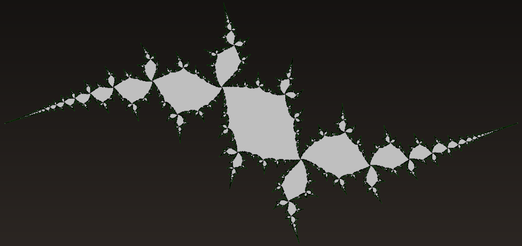
In case 2) if g and g' are combinatorially/ topologically equivalent their renormalizations are so and the points in $\mathbf{M}$ should be the same.

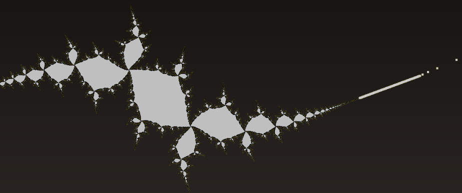
The non-renormalizable parameters of given period and given indifferent multiplier is a finite set ; these points are separated by the parameter puzzle pieces. Rigidity holds.

More precisely

The maps in $Per_1(1)$ can be represented by $g_B(z) = z + 1/z + B$ with $B \in \mathbf{C}$



Parabolic rays



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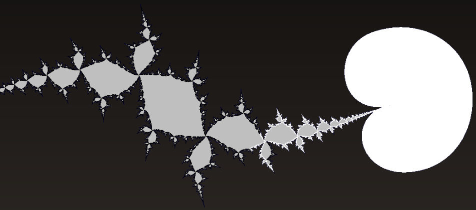

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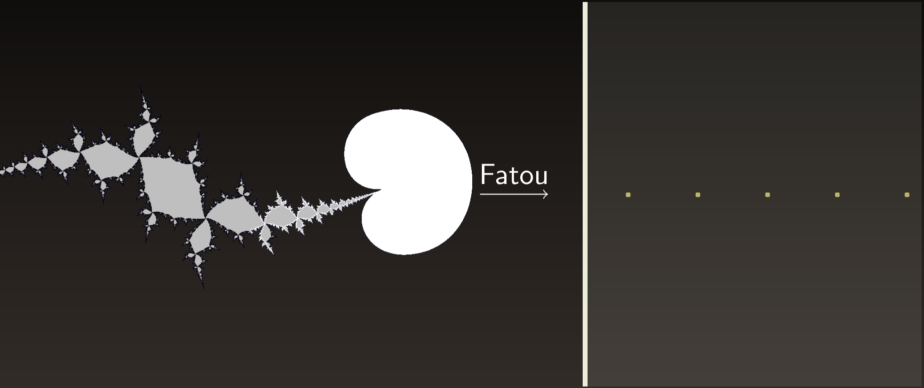

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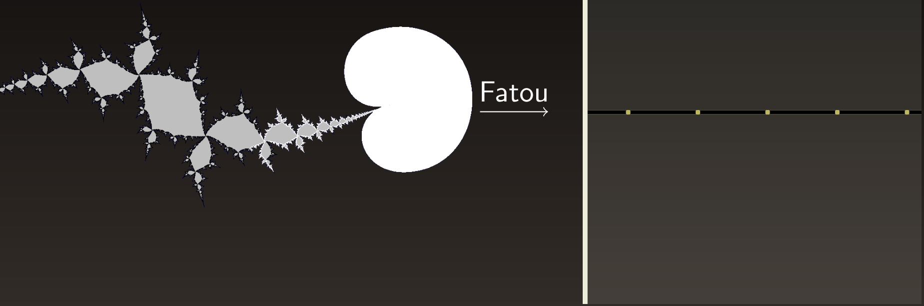

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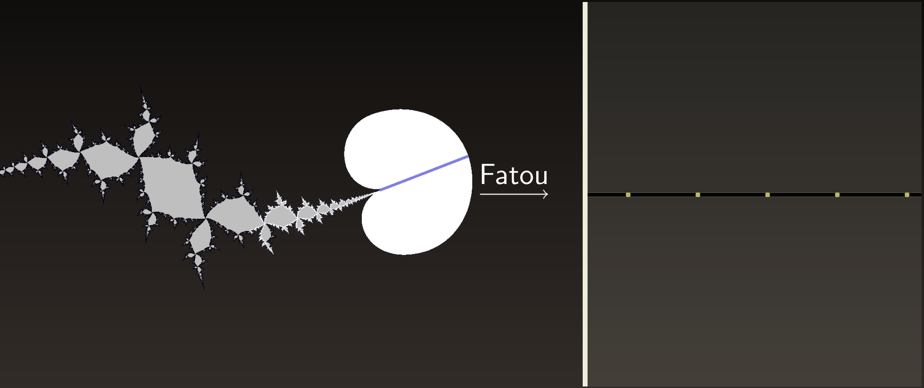

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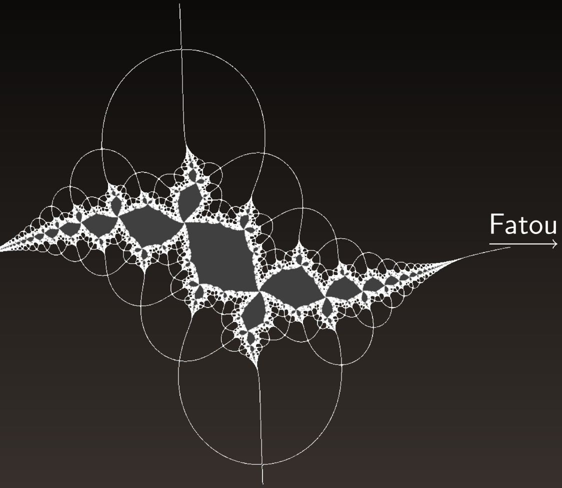
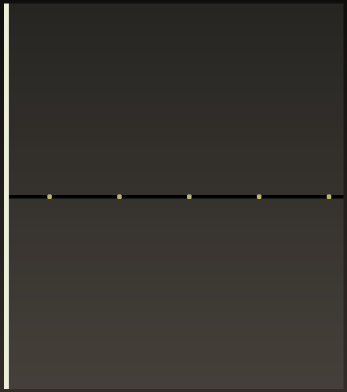
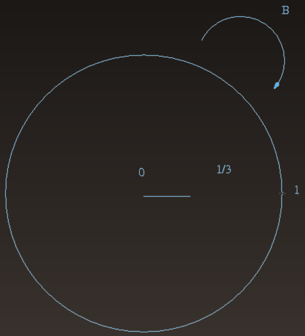




Model

In the connected case, g is conjugated in $\mathbf{C} \setminus K$ to

$$B_2(z) = \frac{z^2 + \frac{1}{3}}{1 + \frac{1}{3} z^2} \text{ on } \mathbf{D}$$



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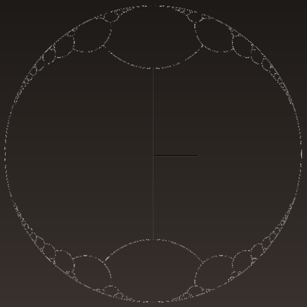

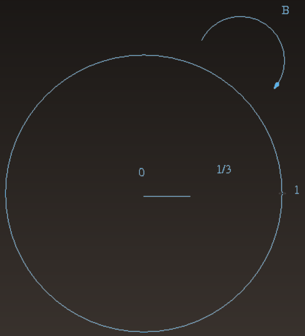

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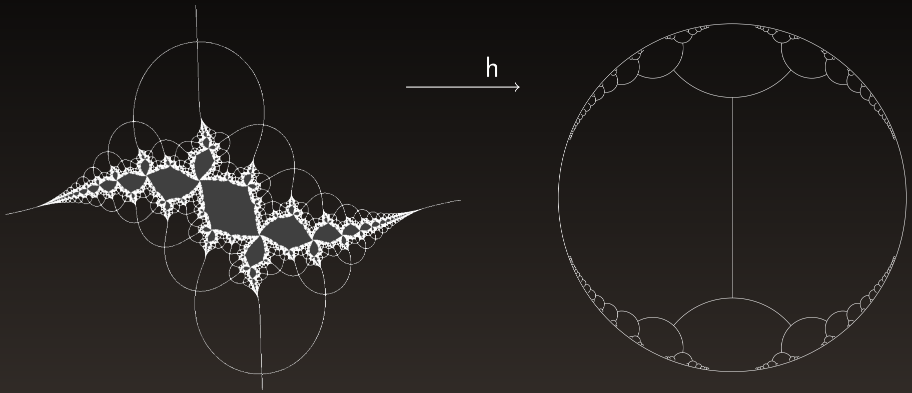

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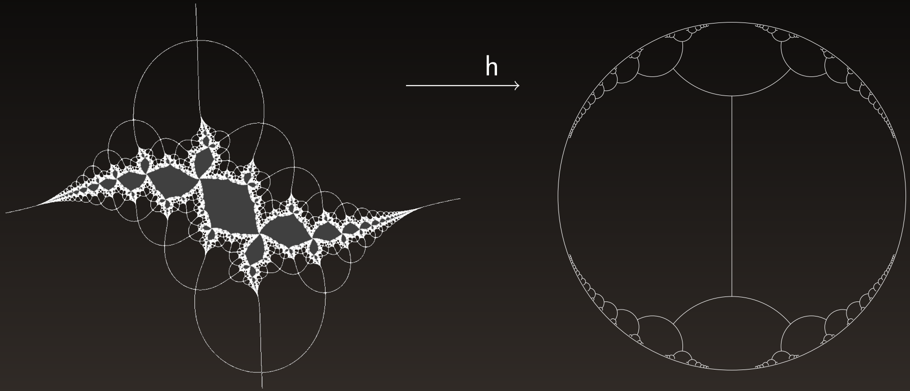
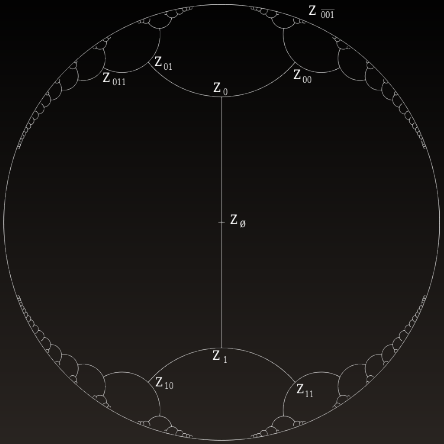
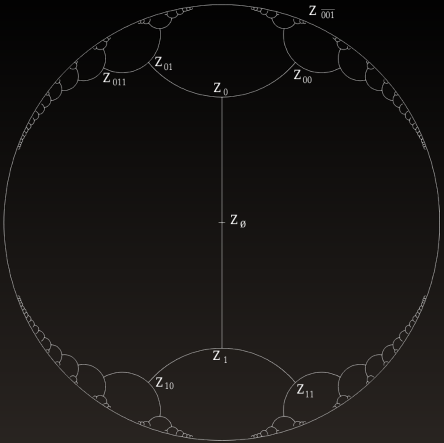
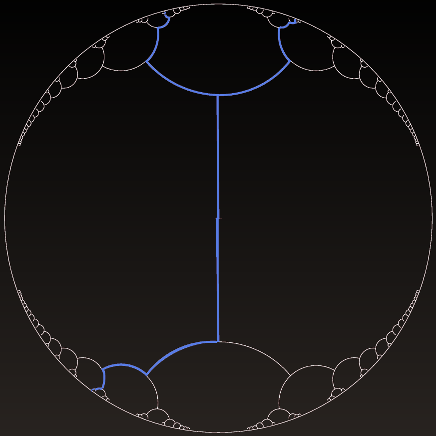


The conjugacy allows to put the tree $\mathcal{T}$ outside of the Julia set.



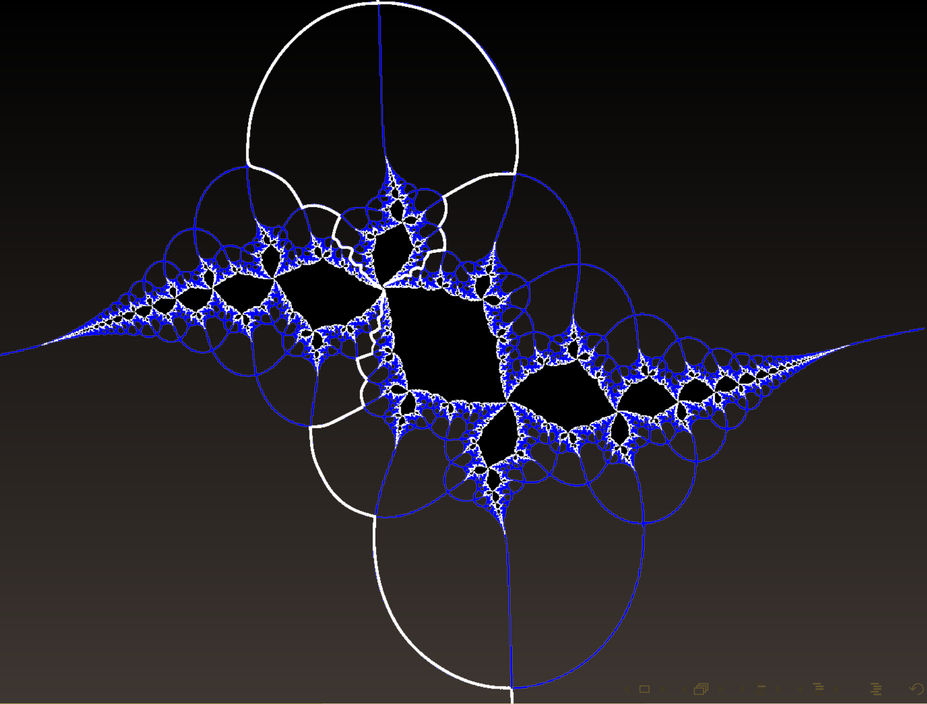
$Z_{\overline{001}}$
Z_{01}
Z_{011}
Z_0
Z_{00}
$Z_\emptyset$
Z_1
Z_{10}
Z_{11}

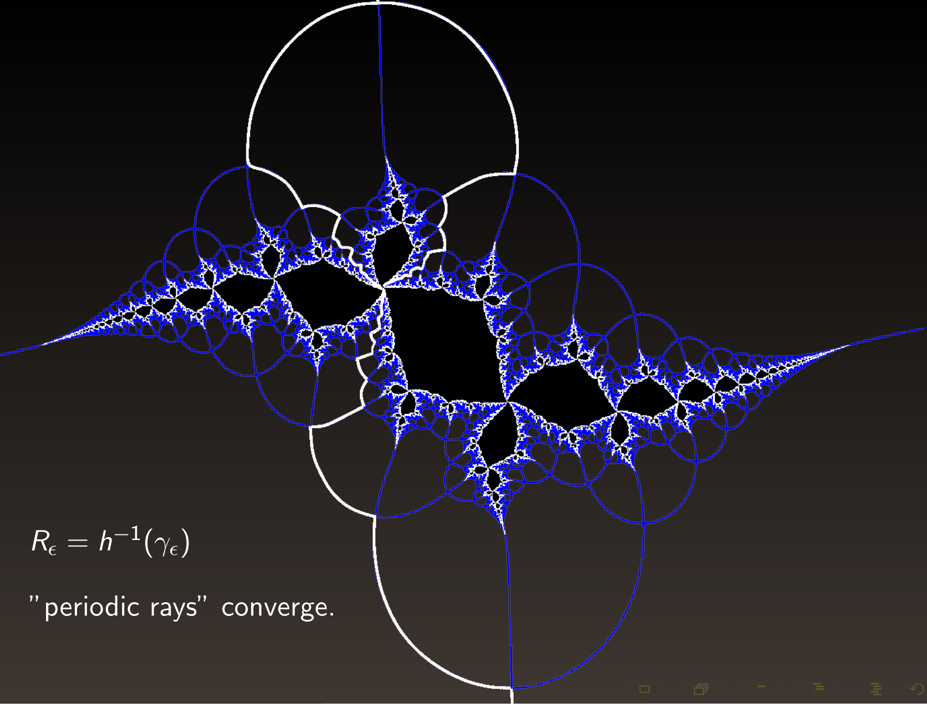




For any itinerary $\underline{\epsilon} = \epsilon_0 \cdots \epsilon_n \cdots$ with $\epsilon_i \in \{0, 1\}$ define the parabolic ray $\gamma_{\underline{\epsilon}}$ to be the minimal arc in the tree joining the points $z_{\epsilon_0 \cdots \epsilon_n}$ and $z_{\emptyset}$.

$$B_2(\gamma_\epsilon) = \gamma_{\sigma(\epsilon)} \cup [0, \frac{1}{3}]$$

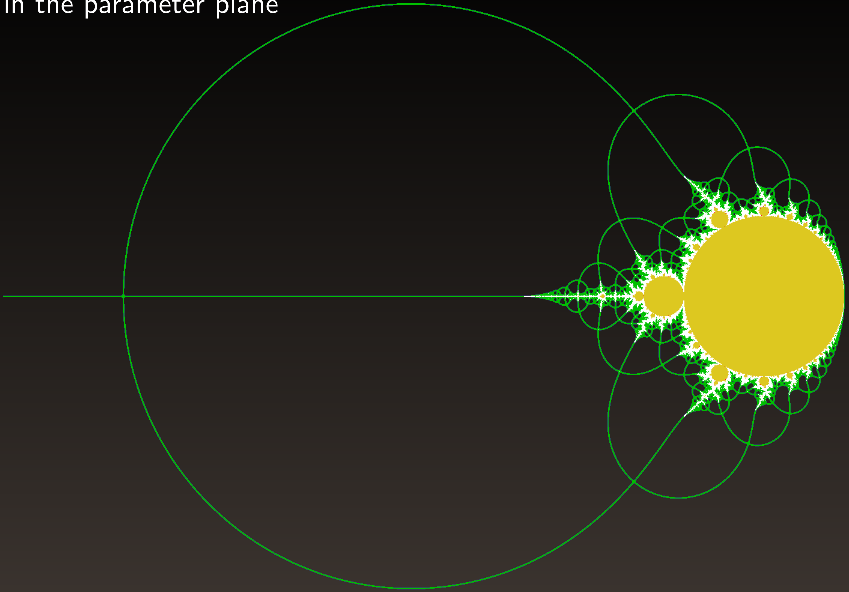




$R_\epsilon = h^{-1}(\gamma_\epsilon)$

"periodic rays" converge.

Milnor parametrization $\Phi : \mathbf{D} \setminus \{1/3\} \to \mathbf{C} \setminus \mathbf{M}_1$ allows to put this tree $\mathcal{T}$ in the parameter plane



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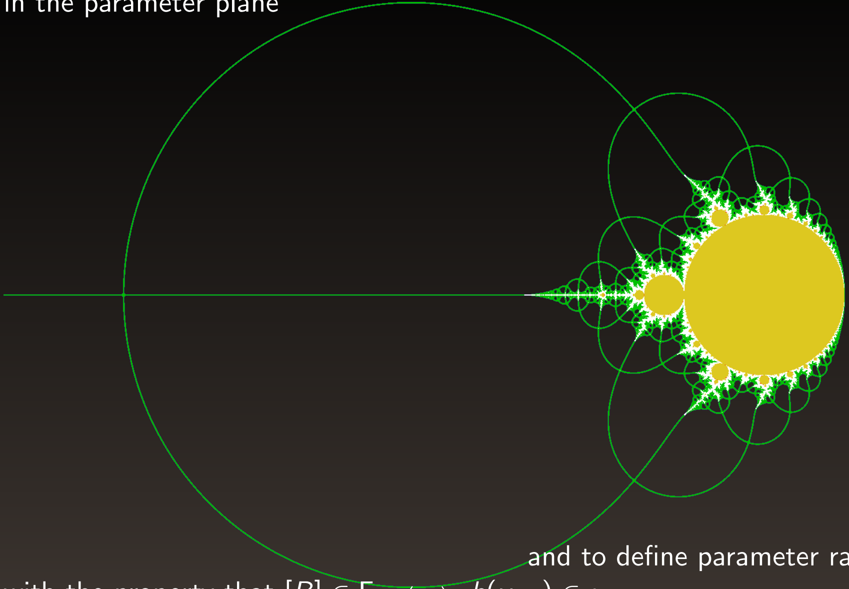
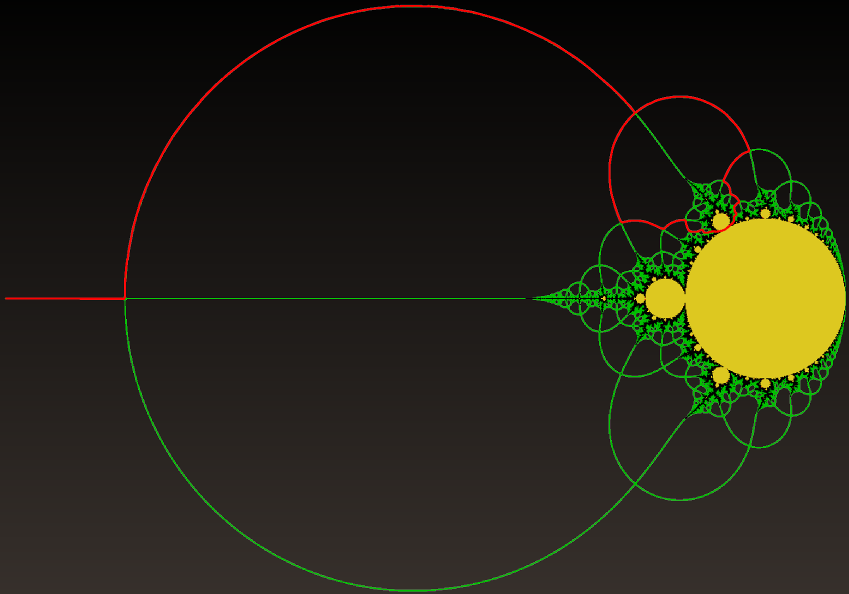
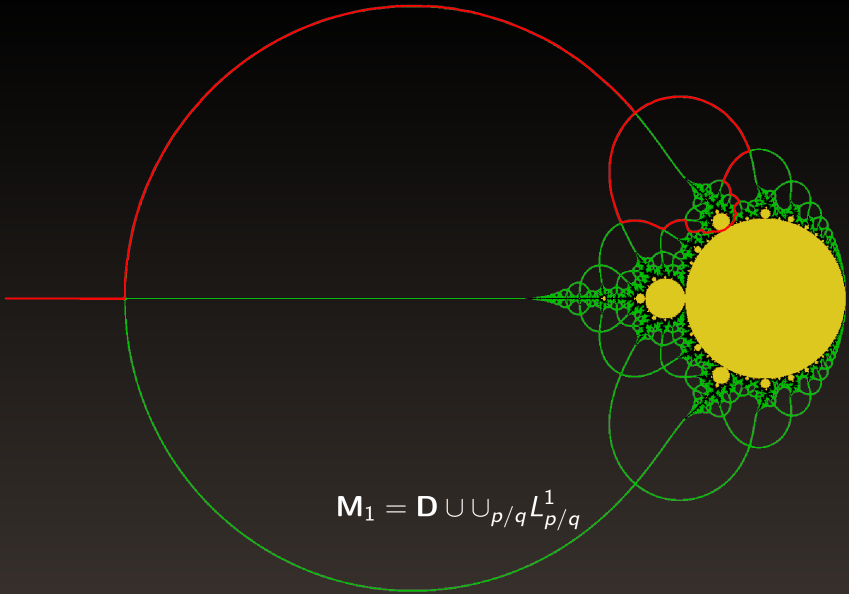


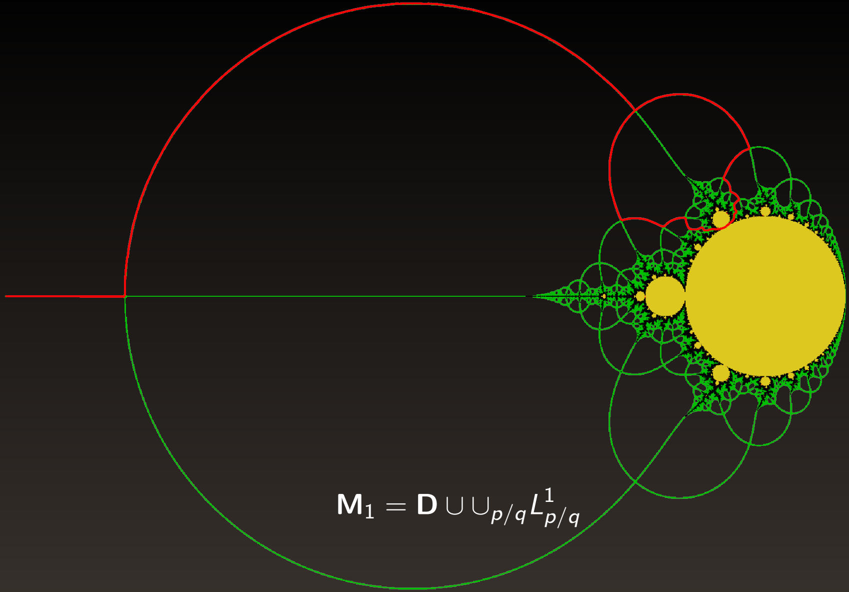
and to define parameter rays $\Gamma_{\underline{\epsilon}}$

with the property that $[B] \in \Gamma_{\underline{\epsilon}} \iff h(v_{[B]}) \in \gamma_{\underline{\epsilon}}$.

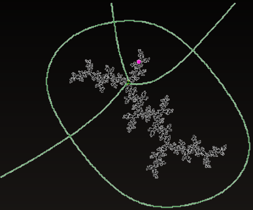


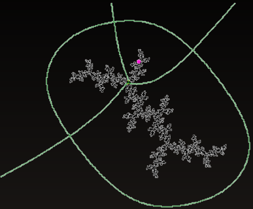


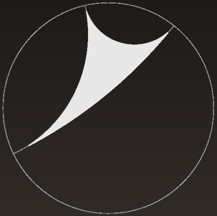
$\mathbf{M}_1 = \mathbf{D} \cup \cup_{p/q} L^1_{p/q}$

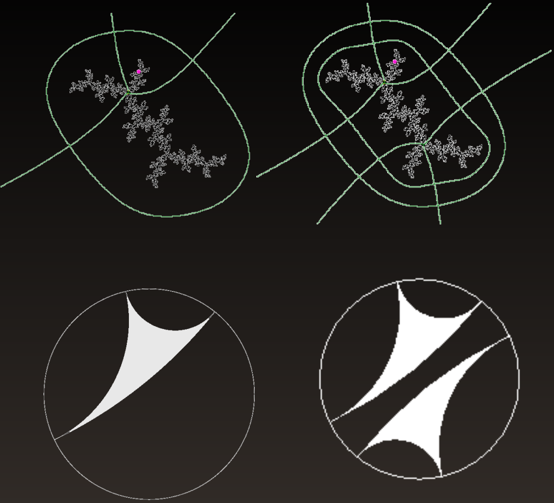


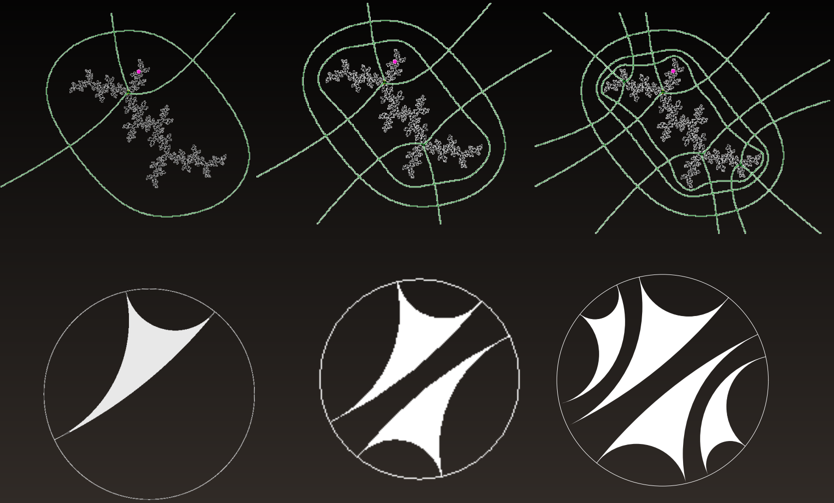
where the fixed point $-\frac{1}{B}$ has rotation number p/q in $L^1_{p/q}$





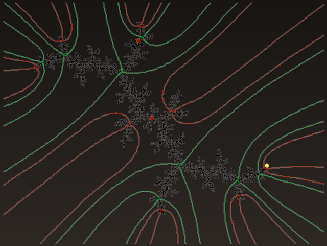


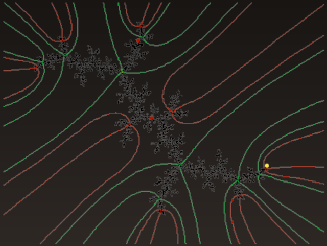


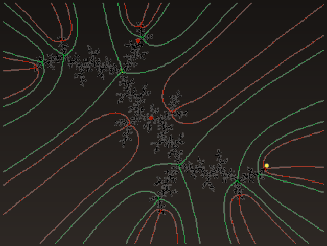


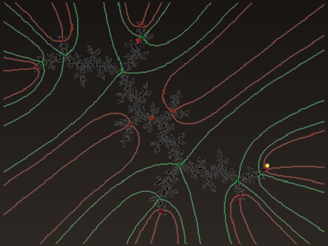
The puzzle of level n determines a lamination on $X_n = Q^{-n}(e^{2i\pi\Theta})$ where Θ is the starting cycle of angles and $Q(z) = z^2$.

It defines puzzle pieces in the parameter plane : different laminations define different puzzle pieces, except the one containing 0 in a class.

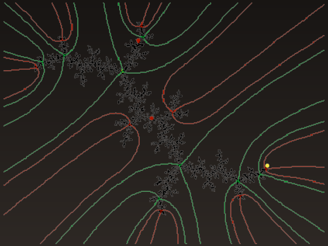




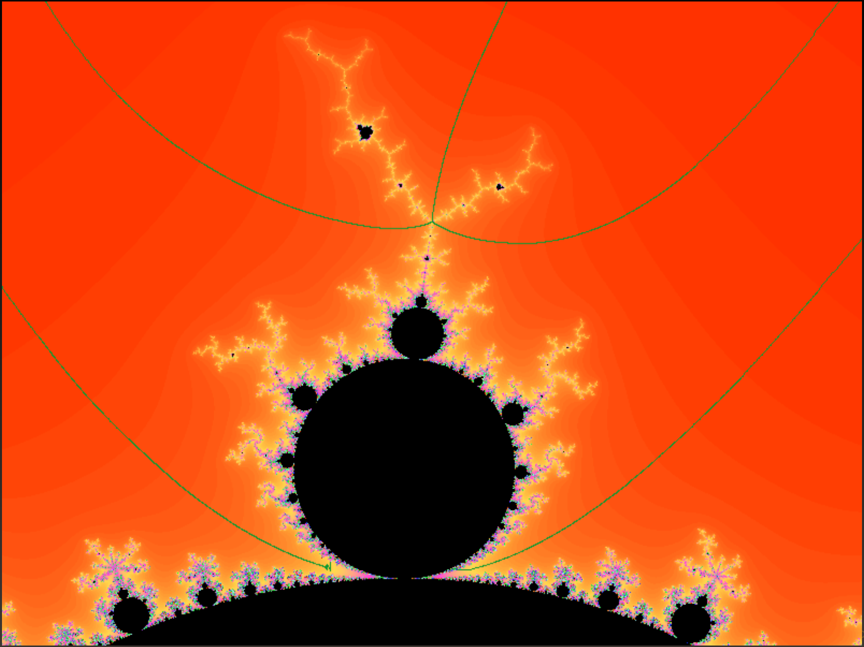


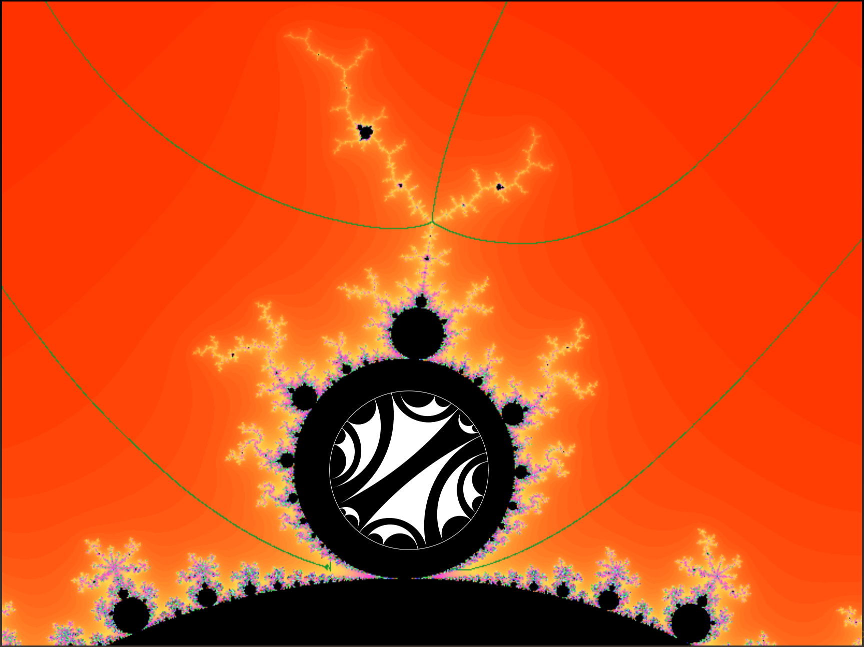


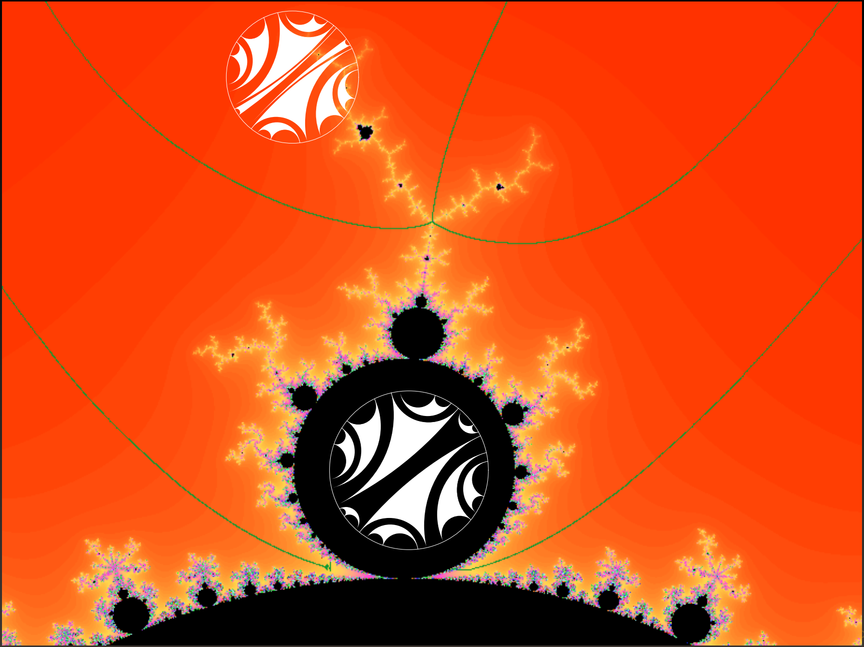


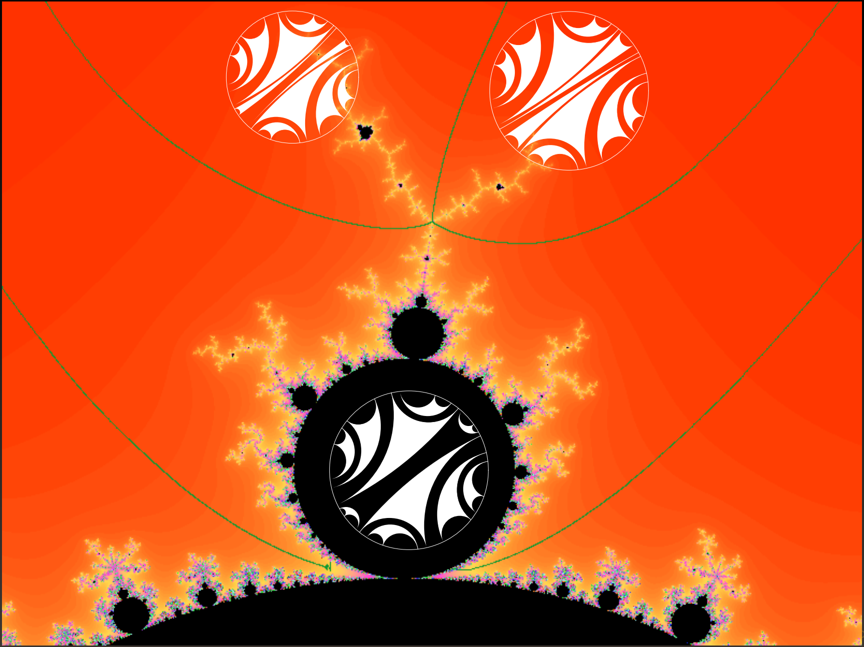


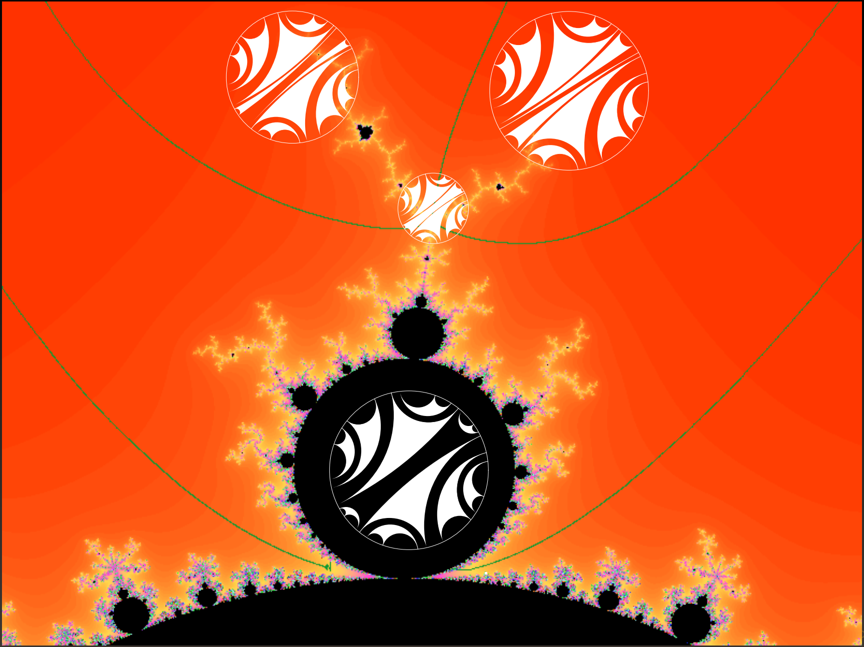






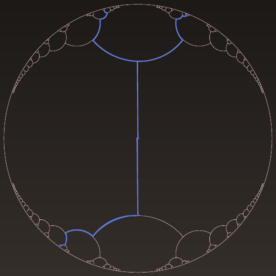






Parapuzzle pieces in $Per_1(1)$

We first define the abstract graph of level n as $\mathcal{G}_n = B_2^{-n}(\mathcal{G}_0)$ where $\mathcal{G}_0 = \cup R_{\sigma^i(\epsilon)}$ with ϵ of rotation number p/q. We transport this graph $\mathcal{G}_n$ using the parametrization Φ to $\mathcal{P}\mathcal{G}_n$.



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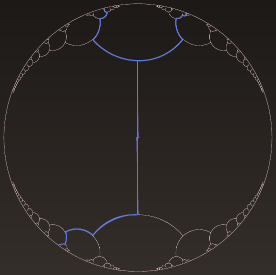
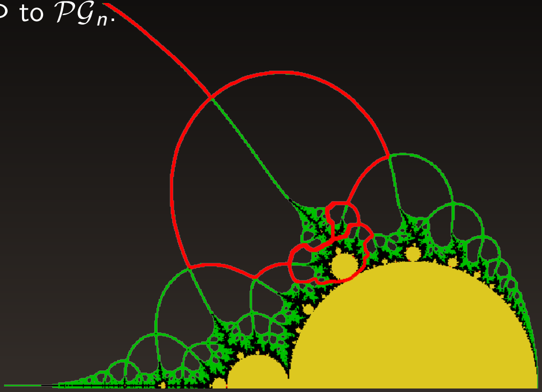
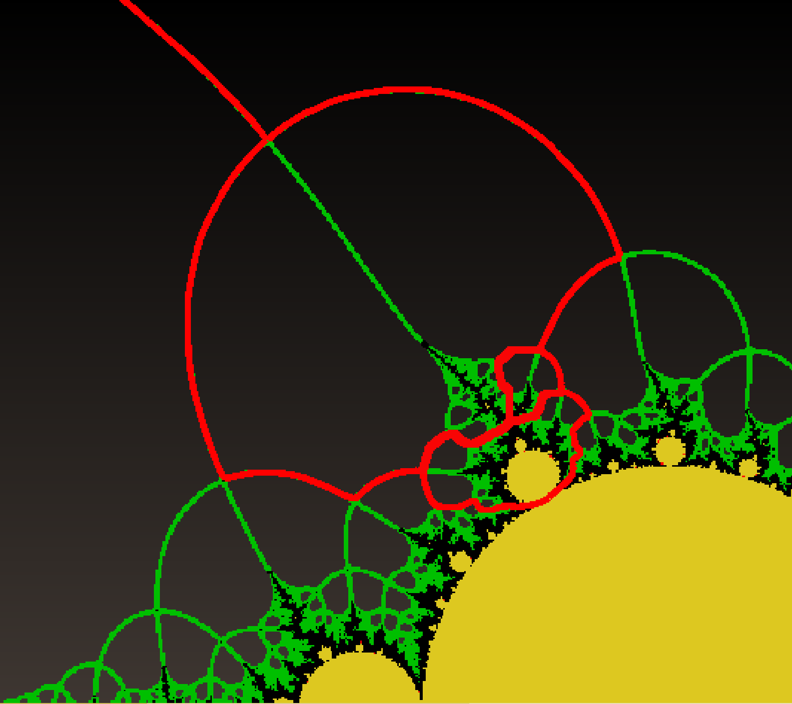
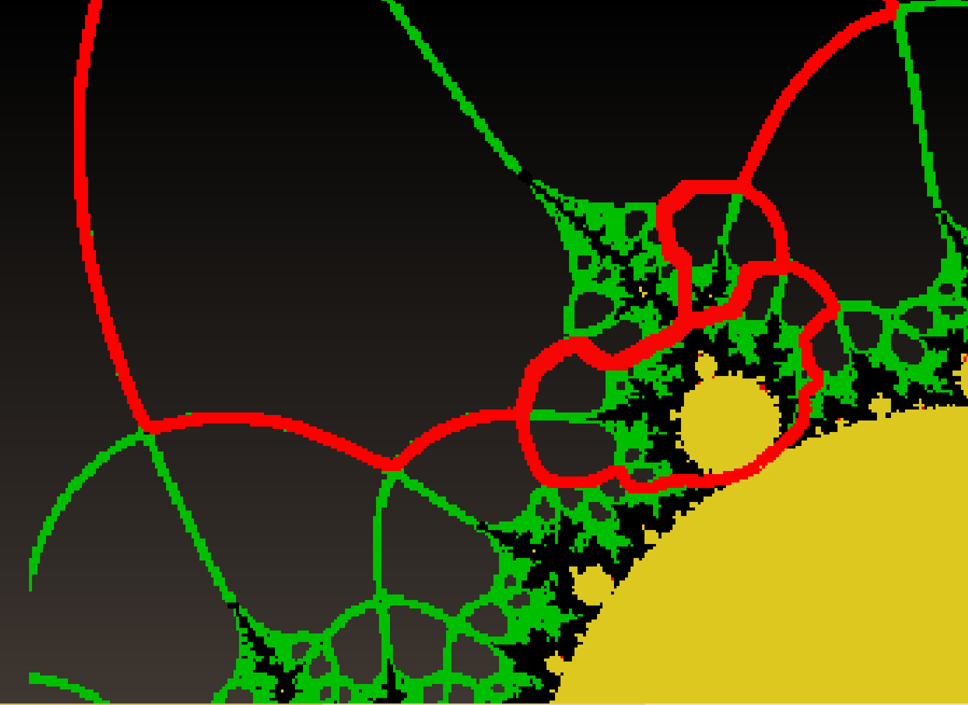
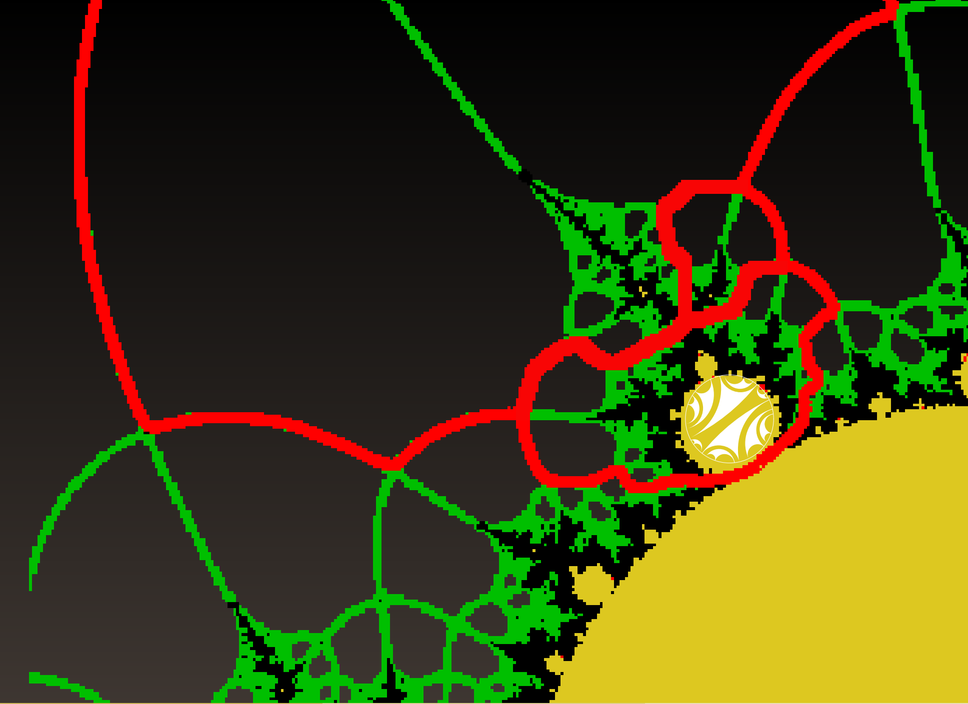


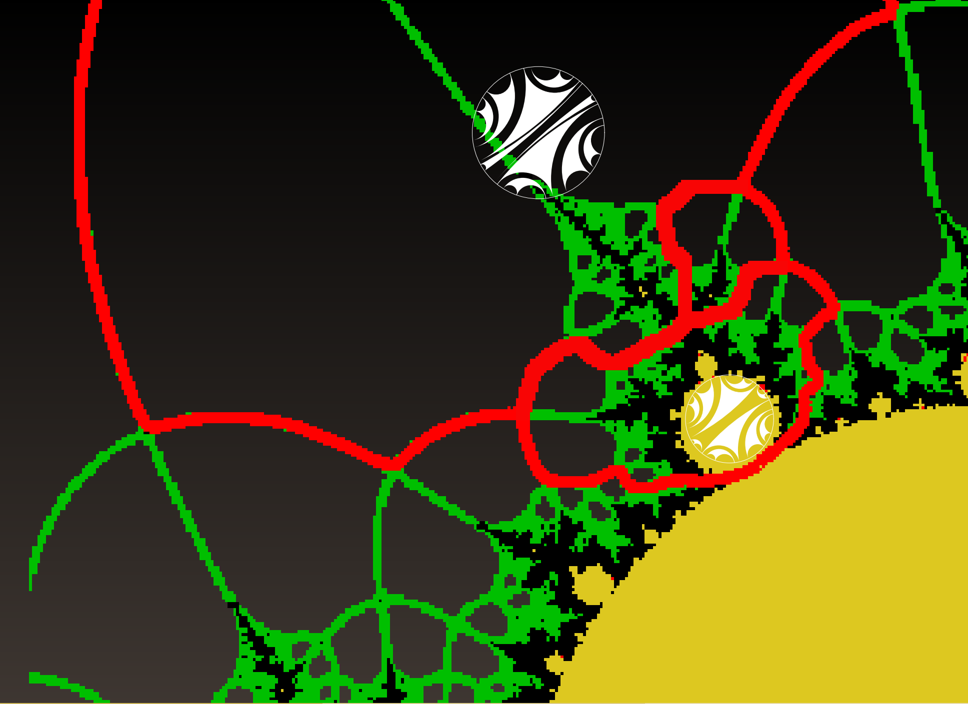


We define the puzzle piece in parameter plane as the connected components of the complement of $\mathcal{PG}_n$.

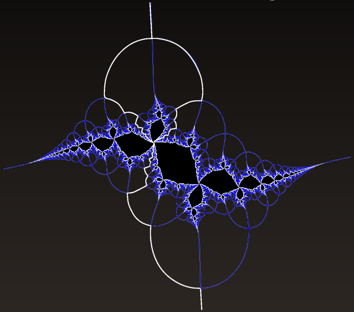








There is a conjugacy on $\mathbb{S}^1$ between B_2 and z^2. For $g \in M_1$ one can define a lamination $\sim_g$ on X_n defined by the rotation number.



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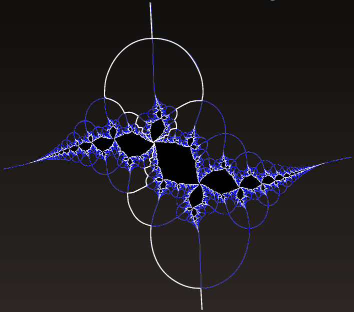

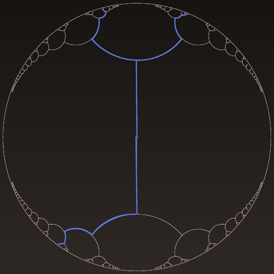

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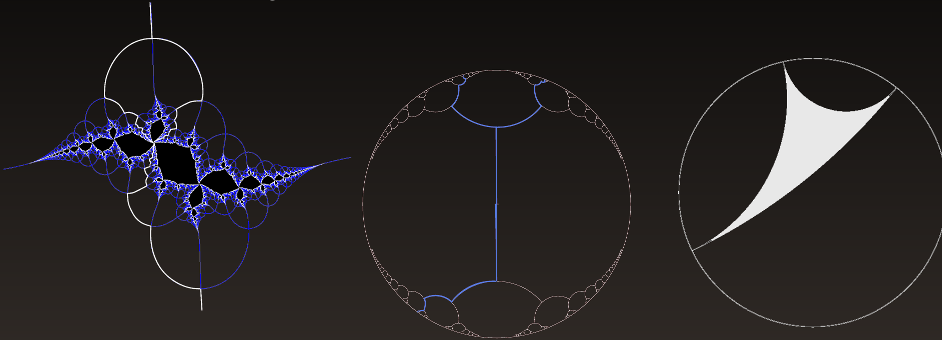

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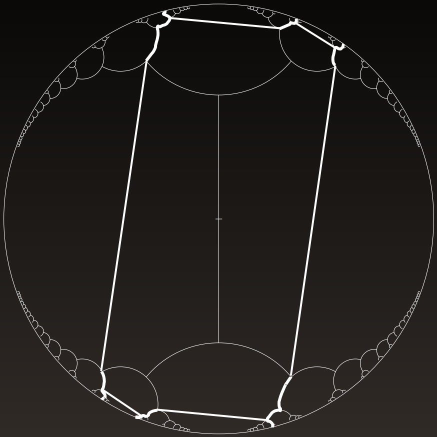
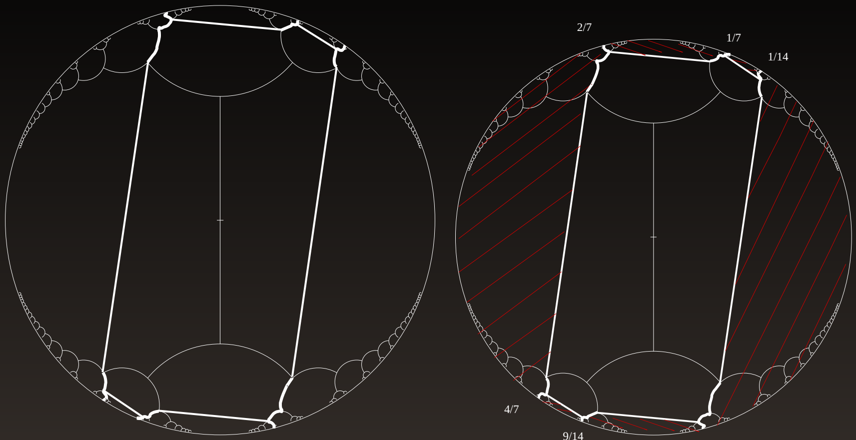
There is no equipotentials in the parabolic case.

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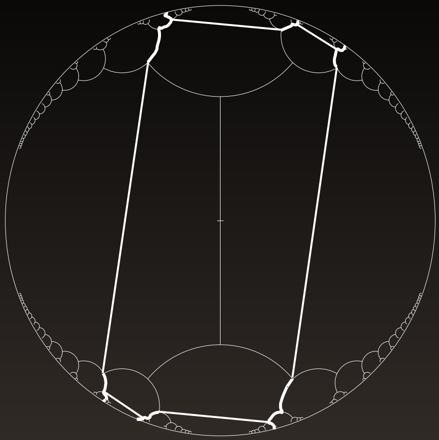

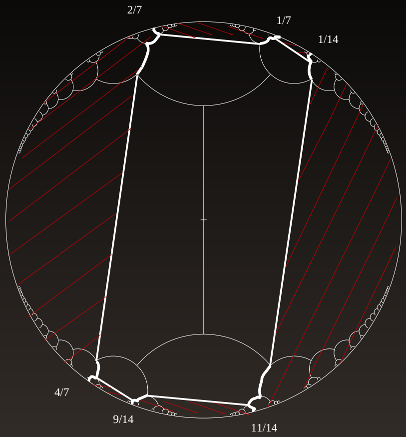
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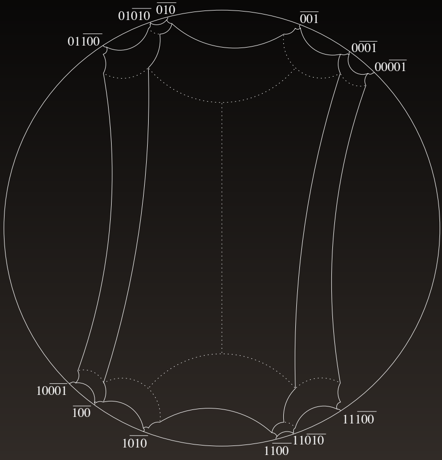
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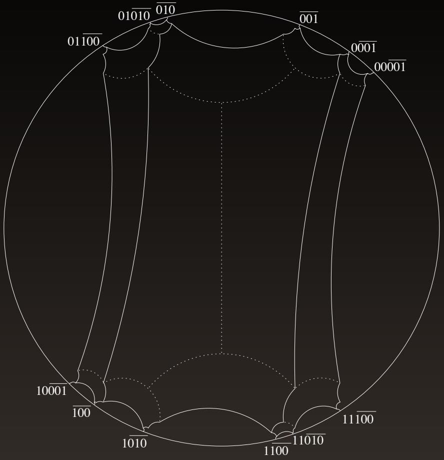

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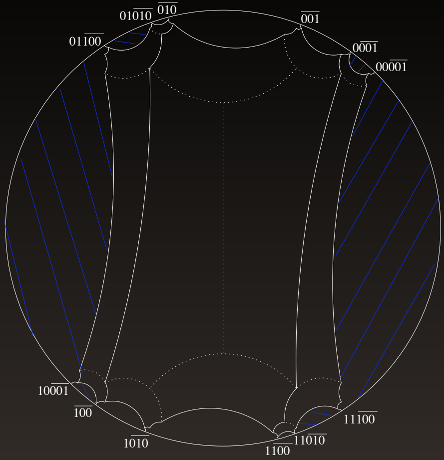

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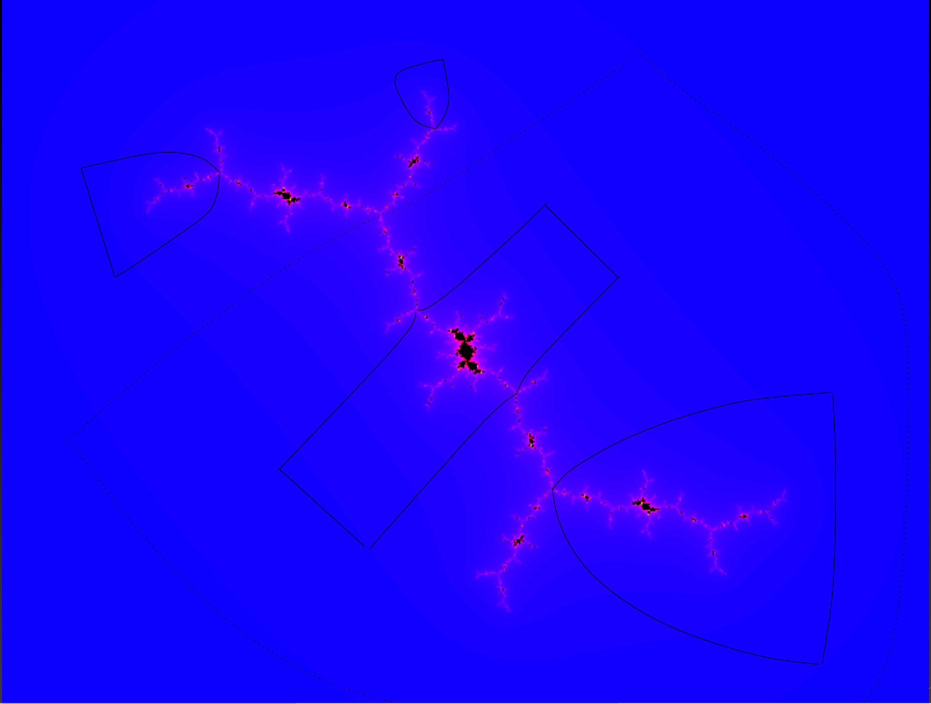
So the proof of Yoccoz translate here to give that the there exist annuli A_{n_i} surrounding the critical value such that $g^{n_i - n_0} A_{n_i} \longrightarrow A_{n_0}$ is a non ramified covering and $\sum \mod A_{n_i} = \infty$ or the map is renormalizable.



Final step : holomorphic motion

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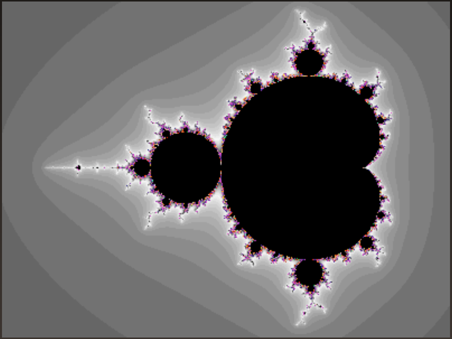
- hence $\sum$ mod $\mathcal{P}_{n_i} \setminus \mathcal{P}_{n_i+1} = \infty$ so $\cap \mathcal{P}_n = \{g\}$.

With some more work ...

C. PETERSEN & R.

There is a homeomorphism between $\mathbf{M}$ and $\mathbf{M}_1$ that has the property to conjugate (topologically) the dynamics on the respective Julia set, except possibly on the cardioid.



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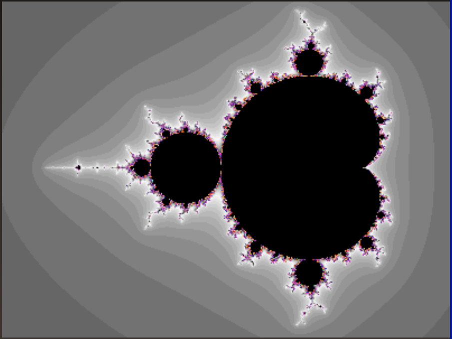

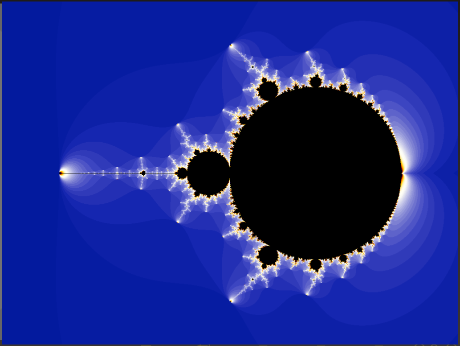

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me with the program of Dan Sørensen