The parabolic Mandelbrot set

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Institute of Mathematics of Toulouse

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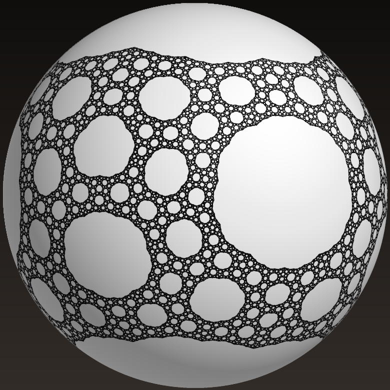

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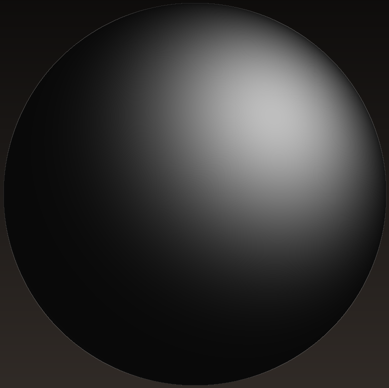
The minimal totally invariant compact set
of cardinality ≥ 3 is called the **Julia set**

Degree 1 rational maps give rise to simple dynamics



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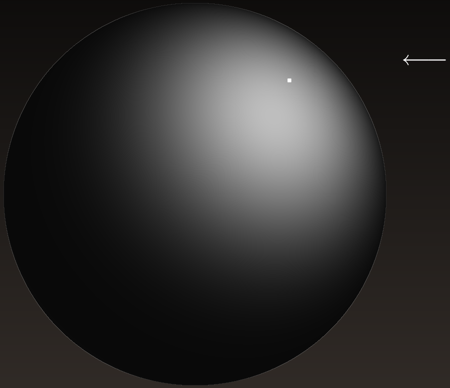

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Milnor's parametrization :

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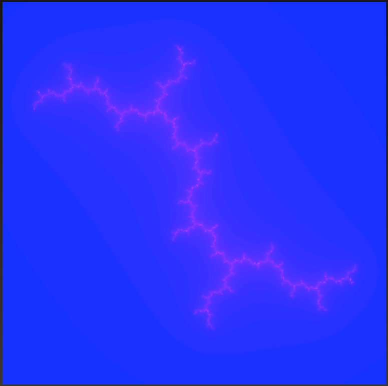

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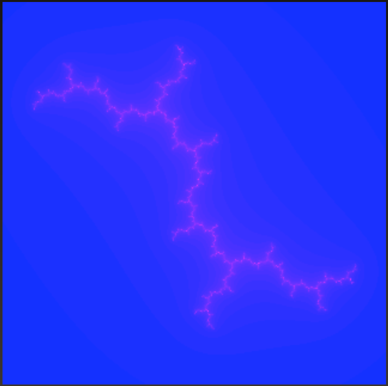

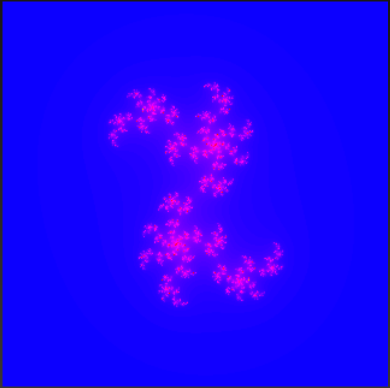

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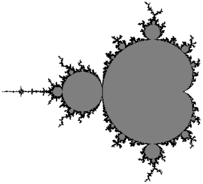
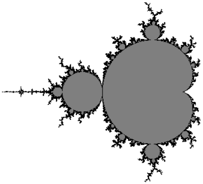
- For $|\lambda| < 1$ any rational map is quasi-conformally conjugate to a quadratic polynomial by the theory of polynomial-like mappings.

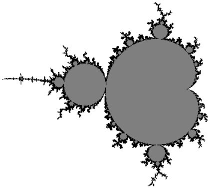
Let $\mathbf{M}_\lambda = \{[f] \in Per_1(\lambda) \mid J(f)$ is connected$\}$ for $\lambda \in \mathbf{D} \cup \{1\}$



$\mathbf{M}_0$: the Mandelbrot set



M_λ when $\lambda \longrightarrow e^{2i\pi/3}$



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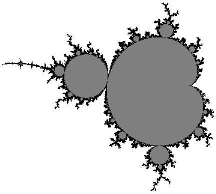

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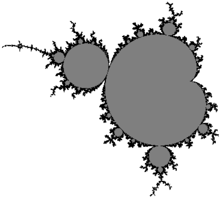

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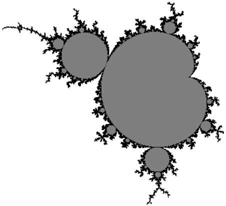

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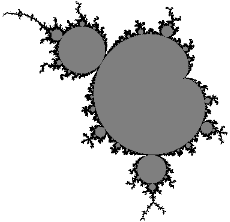

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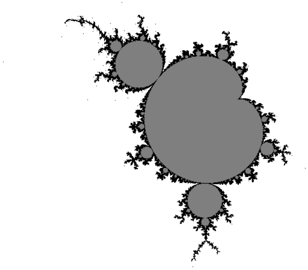

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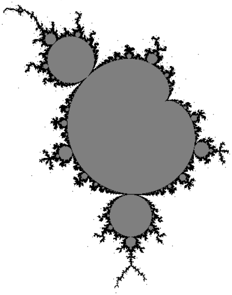
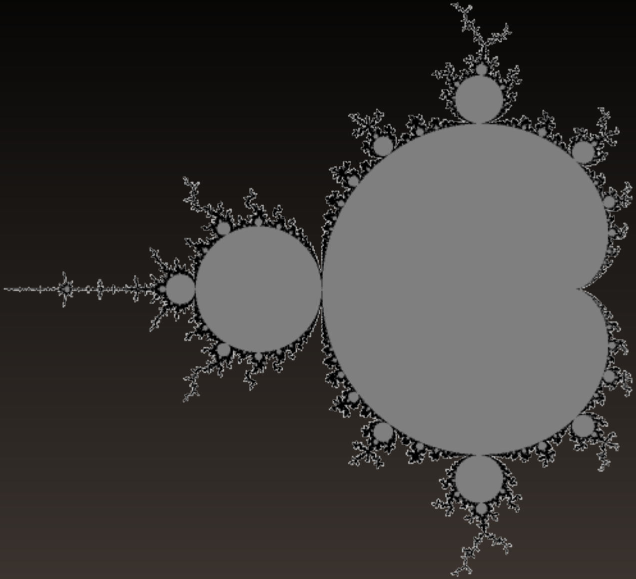


Theorem [Goldberg-Keen, Uhre, Bassanelli-Berteloot]

There exists a map $\Phi : \mathbf{D} \times Per_1(0) \to \mathbf{M}_2$ *such that :*

- $\lambda \mapsto \Phi(\lambda, f)$ *is holomorphic on* $\mathbf{D}$
- $f \mapsto \Phi(\lambda, f)$ *is injective*
- Φ *sends* $Per_1(0)$ *to* $Per_1(\lambda)$ *and* $\mathbf{M}_0$ *to* $\mathbf{M}_\lambda$
- *the maps* $f \in Per_1(0)$ *and* $\Phi(\lambda, f) \in Per_1(\lambda)$ *are conjugate on a neighborhood of their Julia sets.*

Holomorphic motion of the Mandelbrot set



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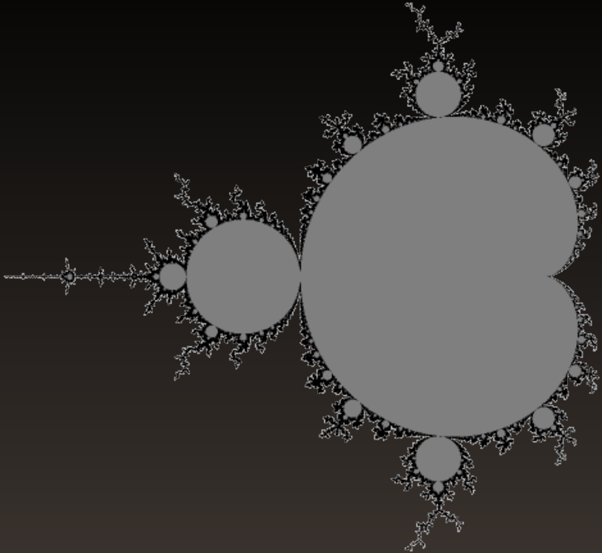

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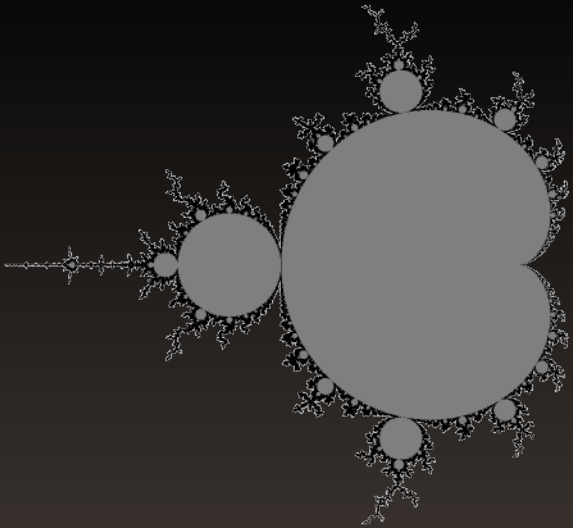

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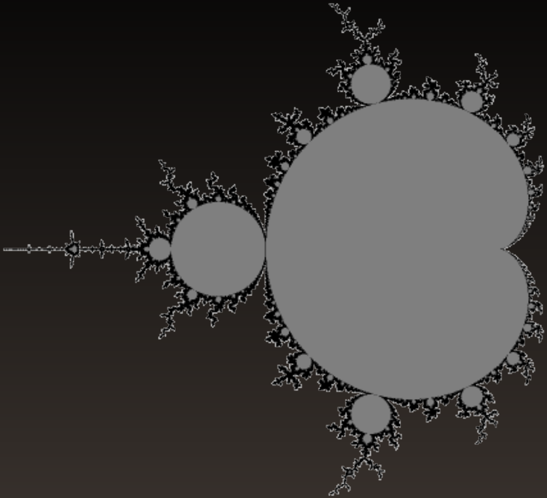

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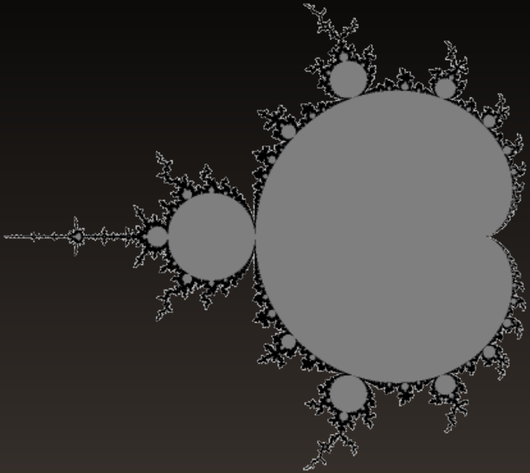

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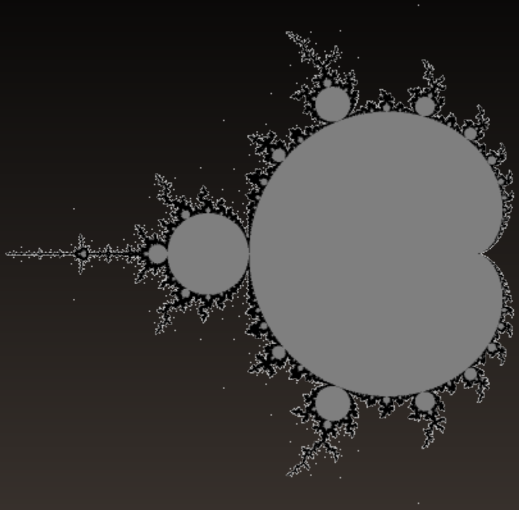

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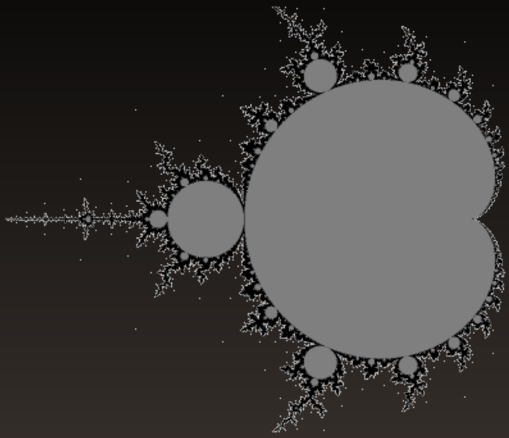
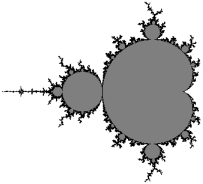
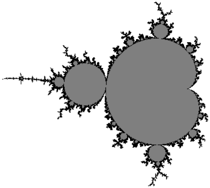
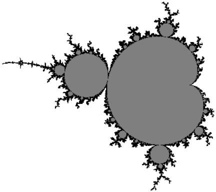
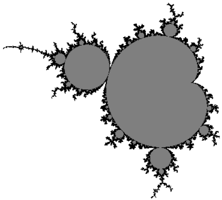


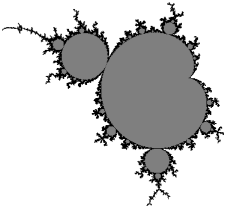
What happens at the boundary of $\mathbf{D}$?

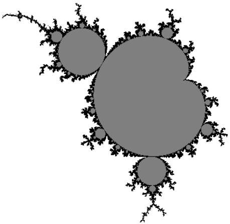


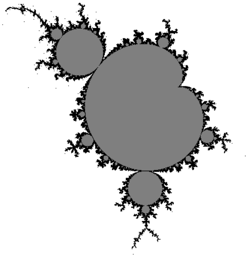


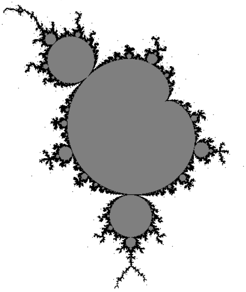


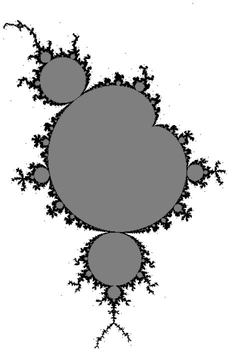


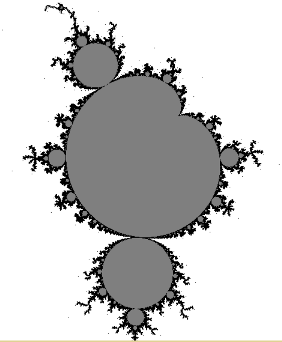


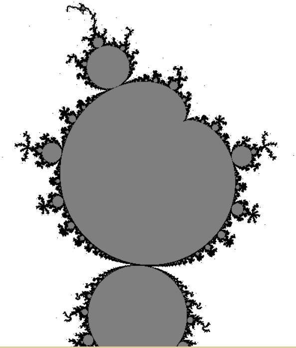


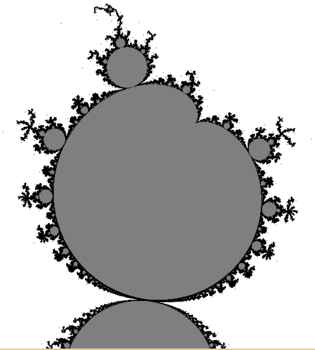


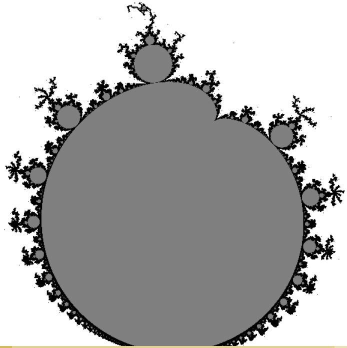


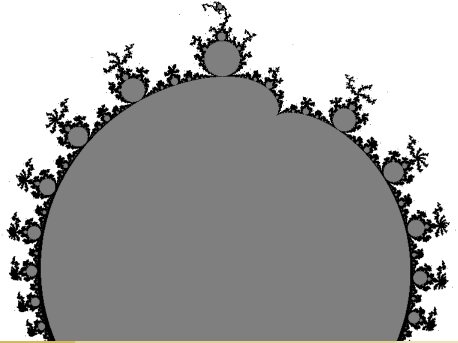


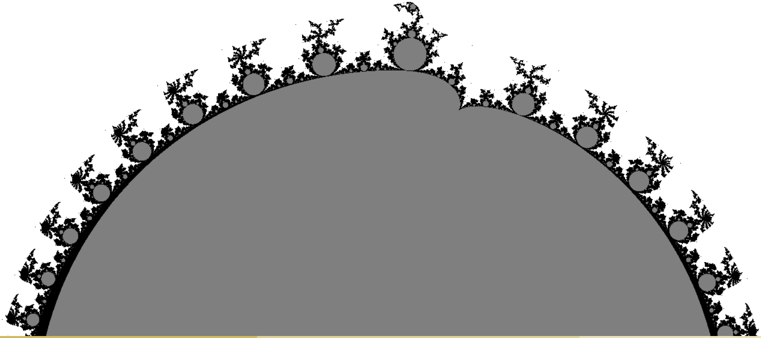


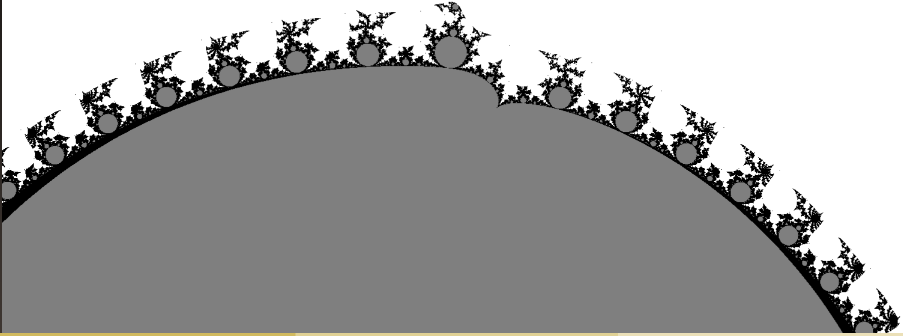


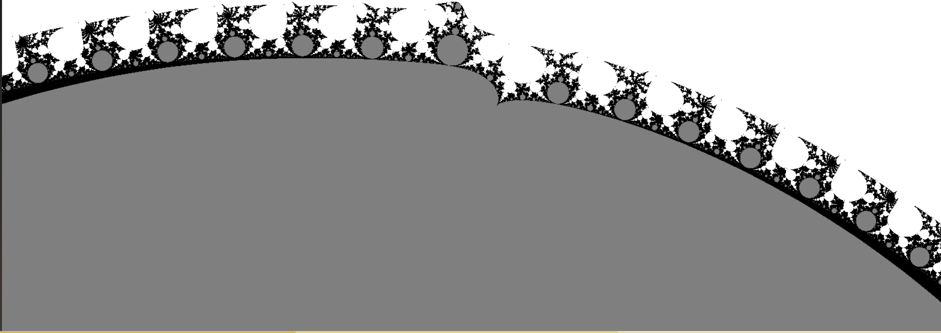


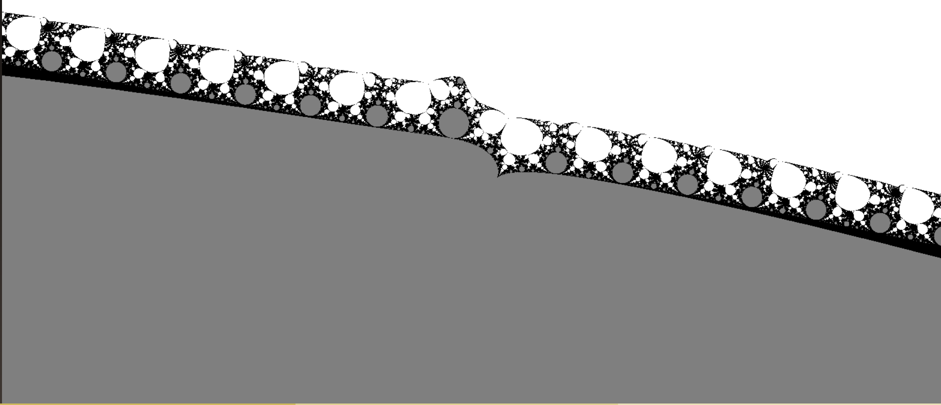


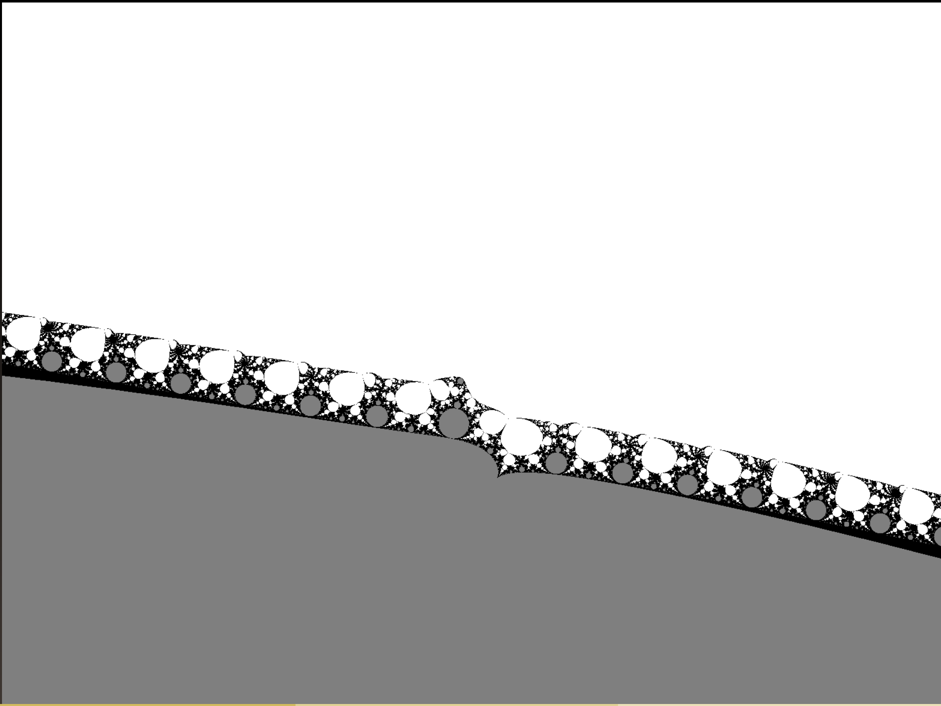












Theorem [Petersen]

If $\lambda \to e^{2i\pi p/q}$ with $p/q \neq 1$ some specific component $L_{-p/q}$ of $\mathbf{M}_0 \setminus \heartsuit$ tends to ∞ in $\mathcal{M}_2$.

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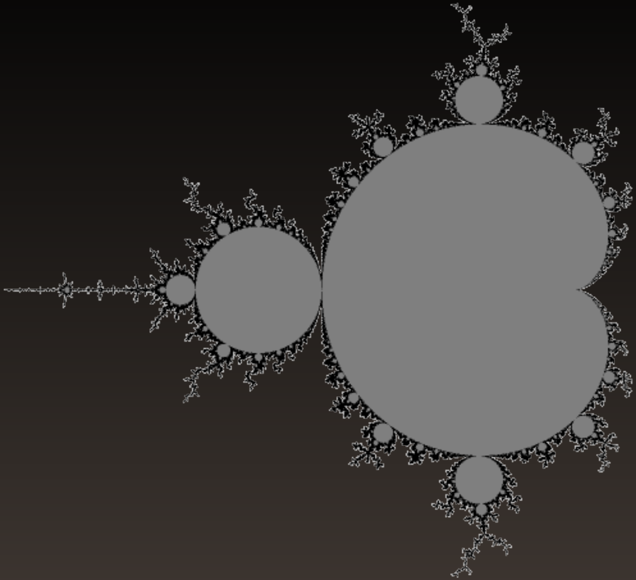
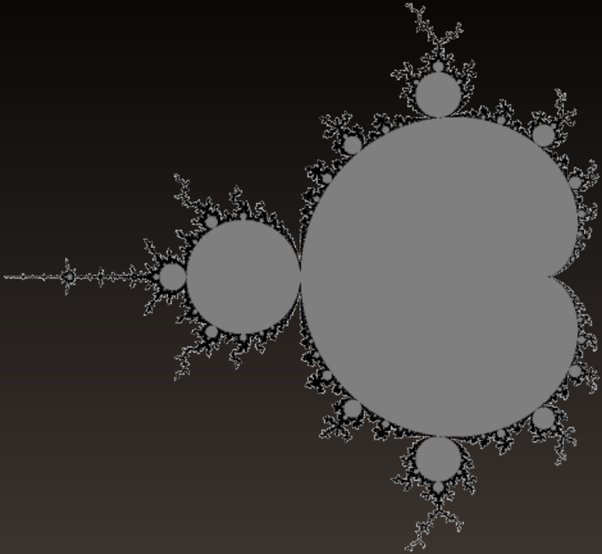
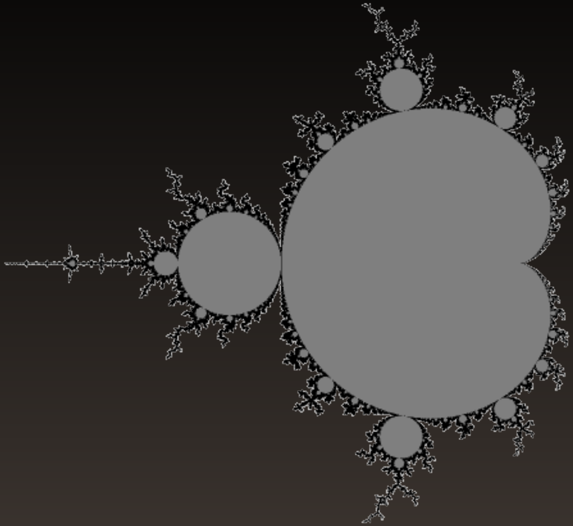
Conjecture [Milnor]

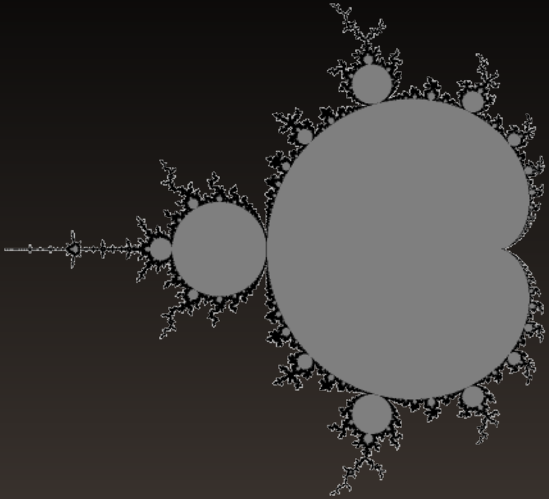
For $\lambda = 1$ the set $\mathbf{M}_1$ is homeomorphic to the Mandelbrot set. Moreover $\mathbf{M}_\lambda$ tends to $\mathbf{M}_1$ when λ tends to 1 for the Hausdorff topology.

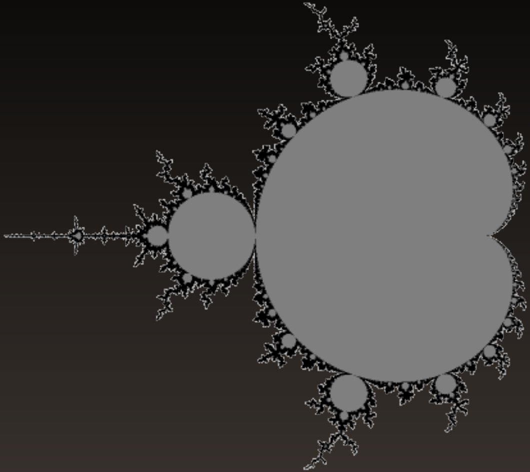
In particular can the possible queer components appear or disappear for $\lambda = 1$?

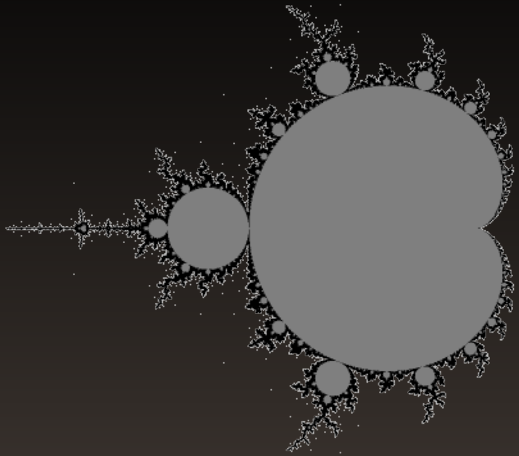


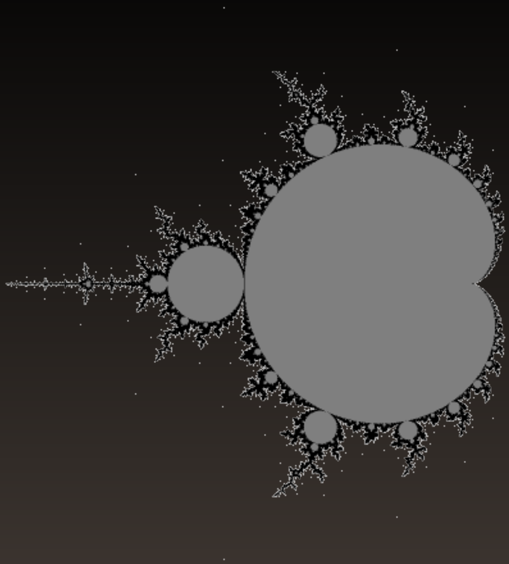


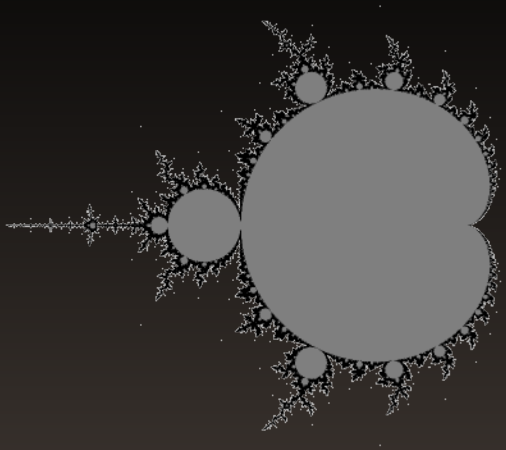


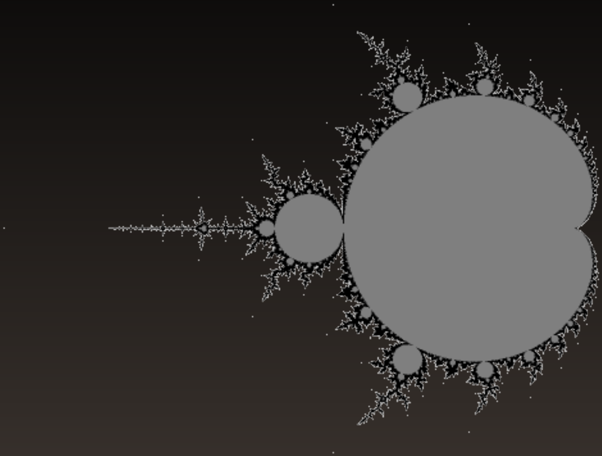


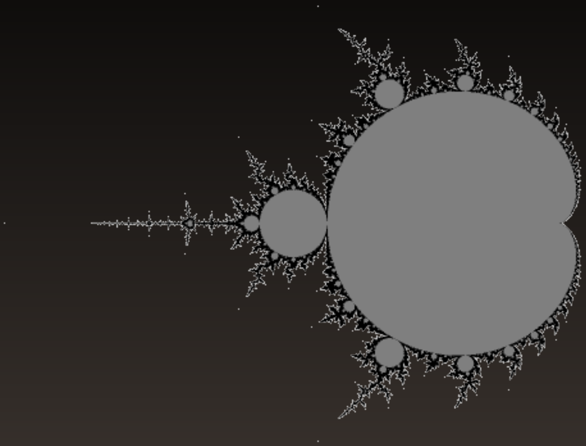


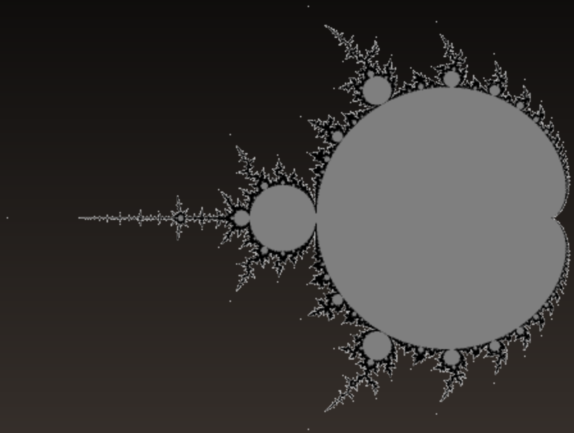


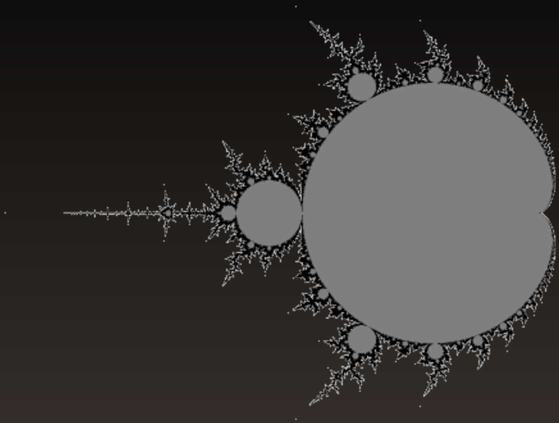


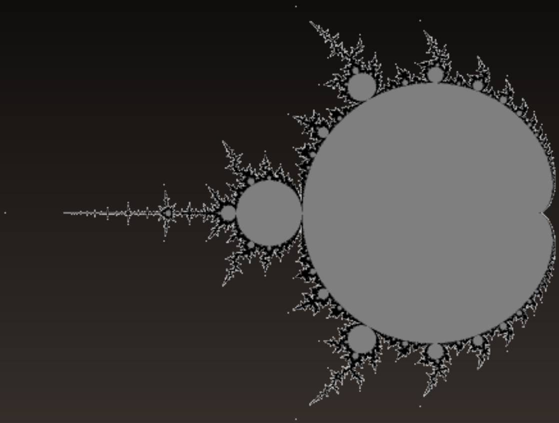


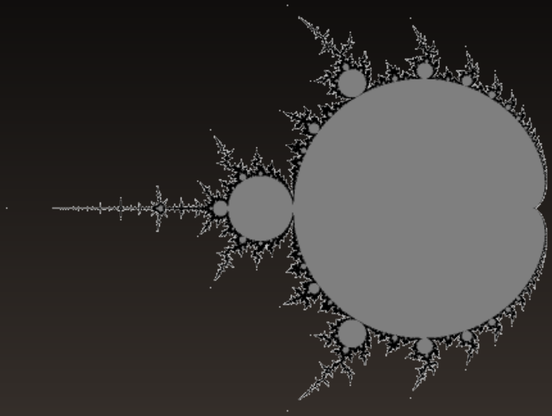


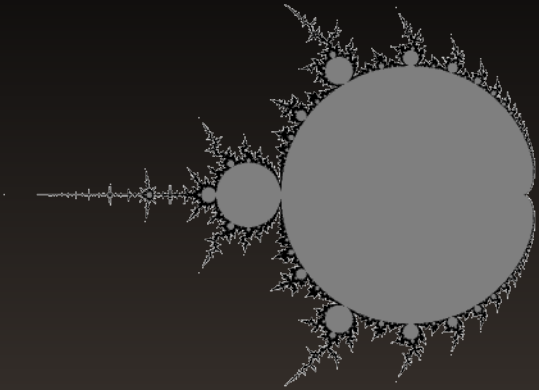


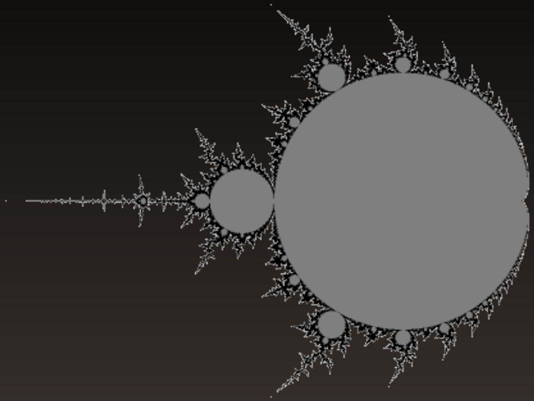


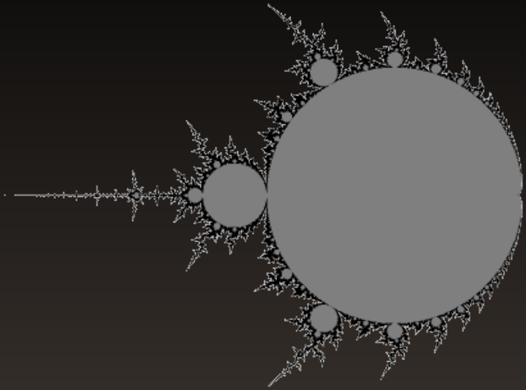


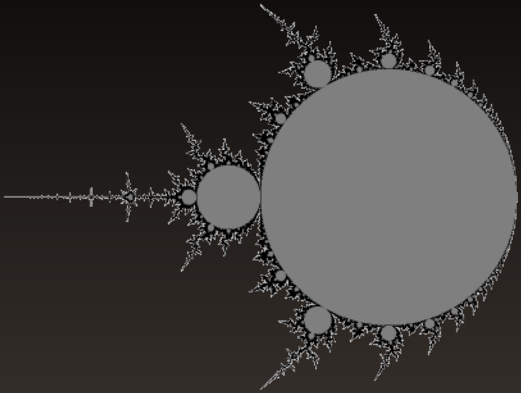






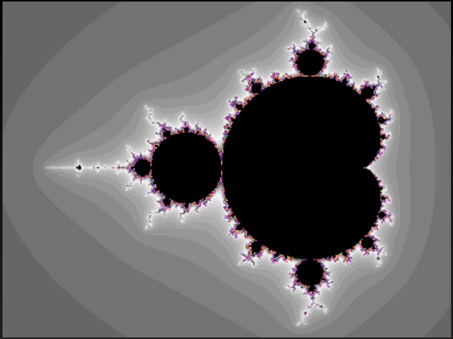


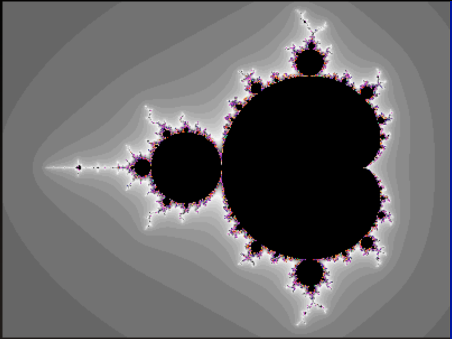


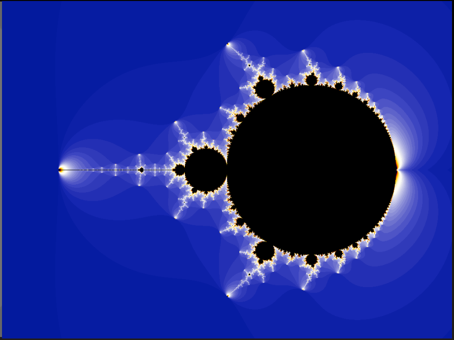


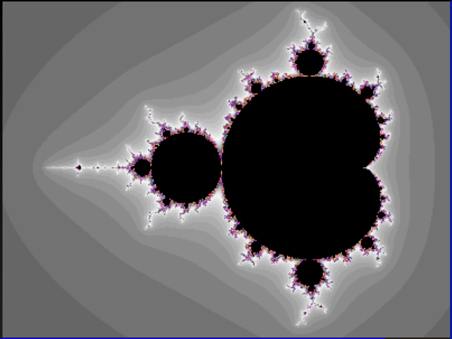
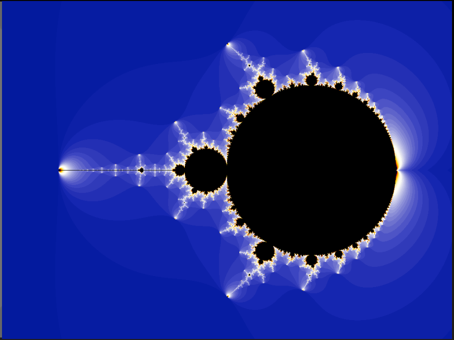
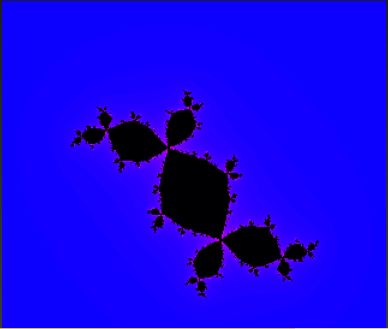
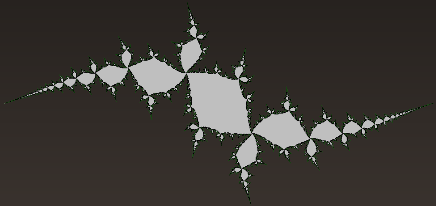
C. Petersen & R.

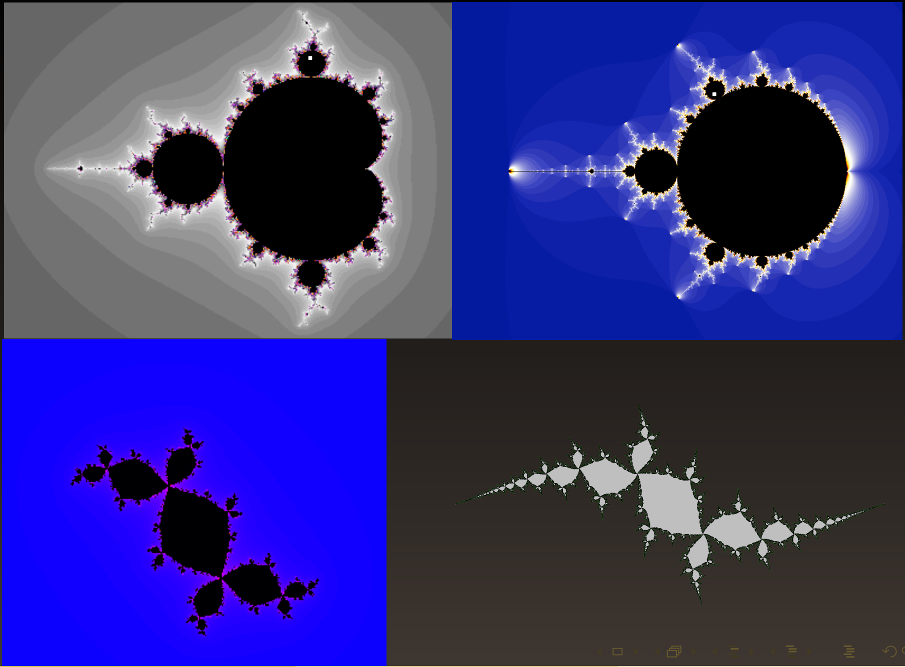
- There exists a homeomorphism between $\mathbf{M}$ and $\mathbf{M}_1$ that induces a (topological) conjugacy between the maps on their Julia sets, except possibly on the main cardioid.
- The maps in $Per_1(1)$ which are finitely renormalizable and without attracting points are rigid. (Topological conjugacy implies conformal conjugacy.)







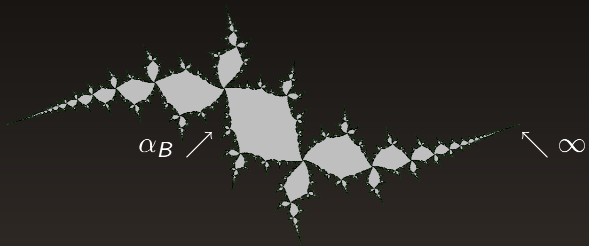







Description of the dynamics in $Per_1(1)$

For $B \in \mathbf{C}$ the map $g_B(z) = z + 1/z + B$ has :

- a double fixed point at ∞ of multiplier 1 ;
- a fixed point at $\alpha_B = -1/B$ of multiplier $1 - B^2$;
- two critical points at ± 1.



$$A = 1 - B^2 \in \mathbf{C} \quad \longrightarrow \quad [g_B] \in Per_1(1) \quad \text{is a biholomorphism.}$$

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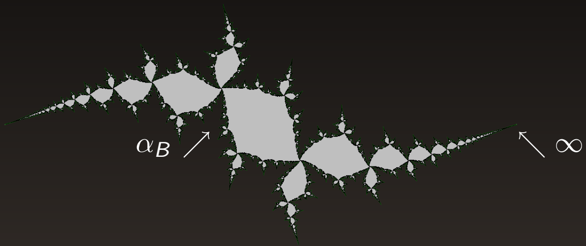

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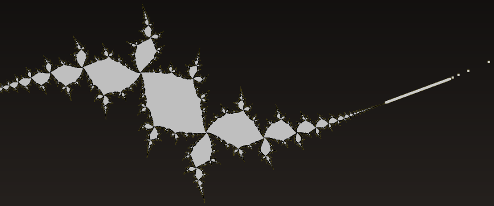
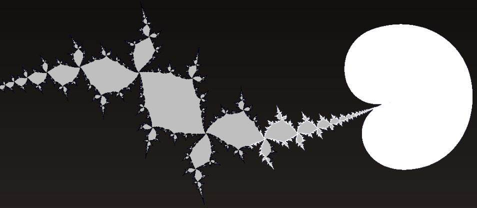
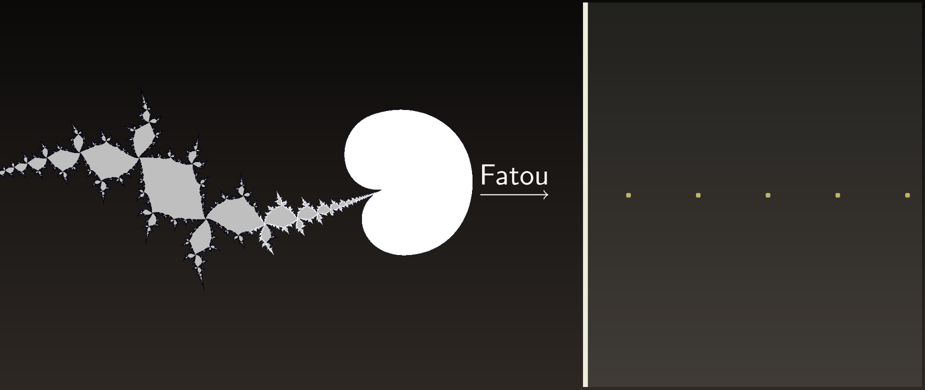
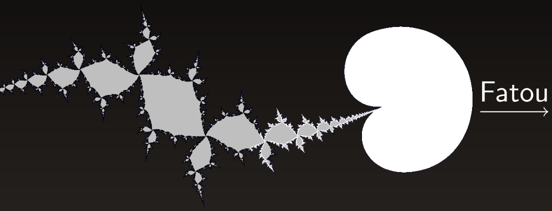
since $g_{-B}(-z) = -g_B(z)$






Fatou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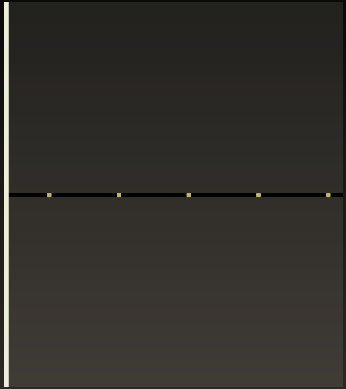

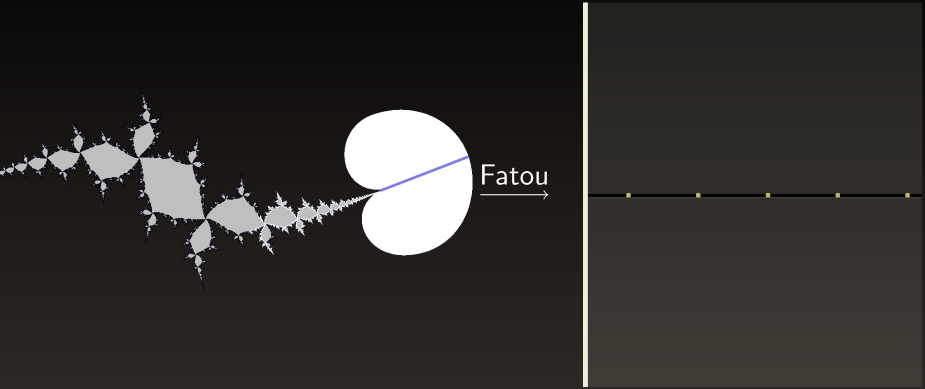
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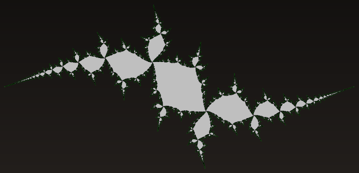
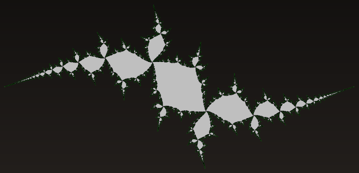
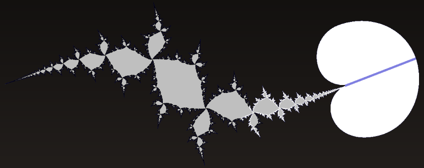


$B_\infty = \{z \mid g^n(z) \to \infty\}$





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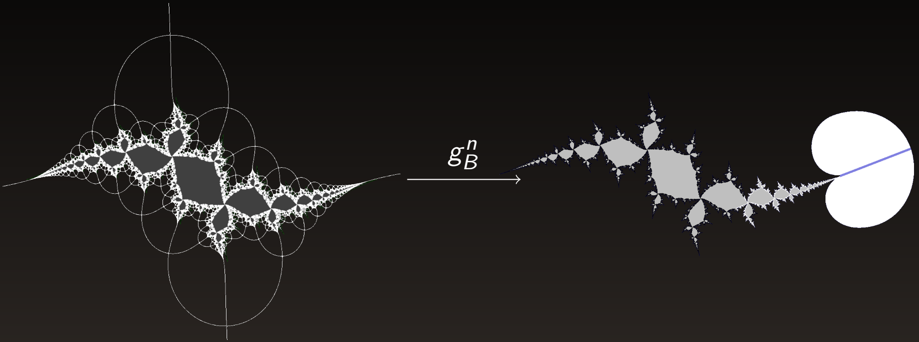
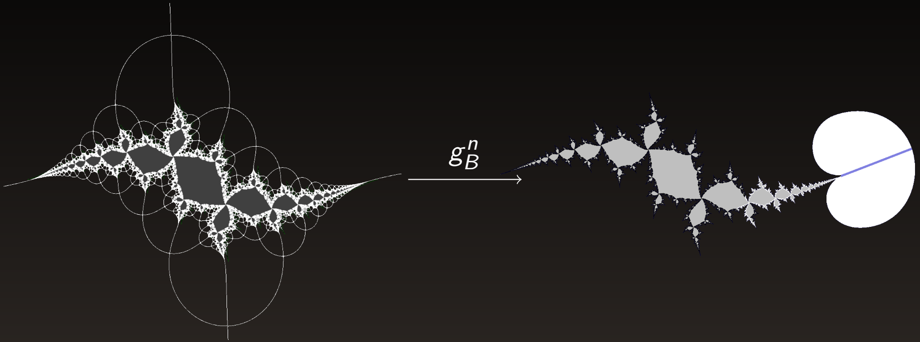
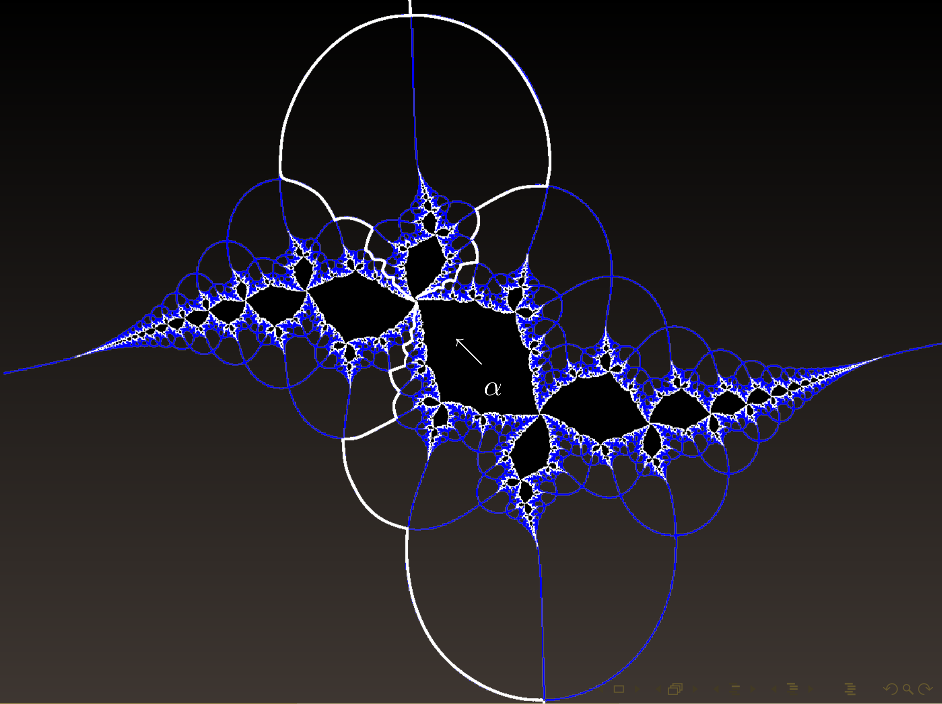


$B_\infty = \{z \mid g^n(z) \to \infty\}$ contains a net by pull-back.

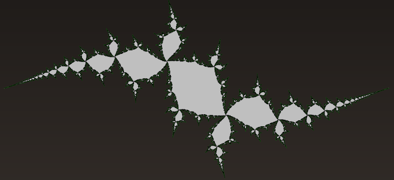


$B_\infty = \{z \mid g^n(z) \to \infty\}$ contains a net by pull-back.
We construct accesses through this net to points of the Julia set.


α

Model

For $[g] \in \mathbf{M}_1$ g is conjugate on B_∞ to $B(z) = \dfrac{z^2 + \frac{1}{3}}{1 + \frac{1}{3}z^2}$ on $\mathbf{D}$ or $\mathbf{C}\setminus\overline{\mathbf{D}}$



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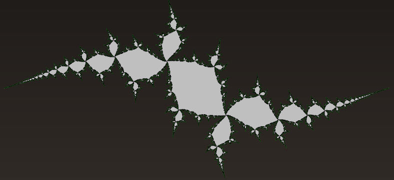

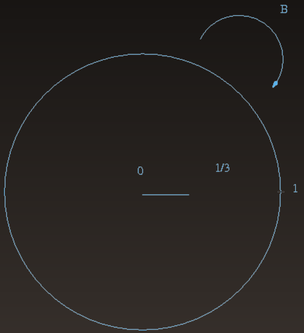

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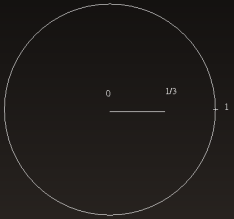
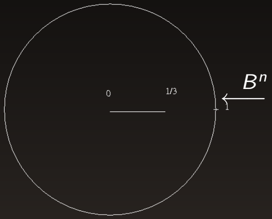


Model



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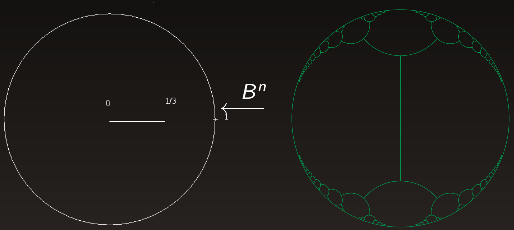

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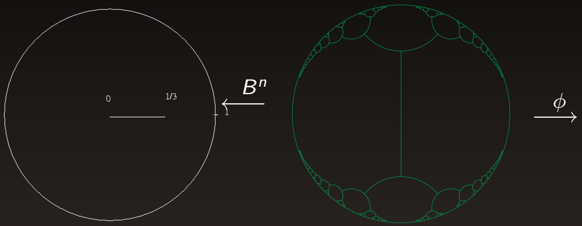

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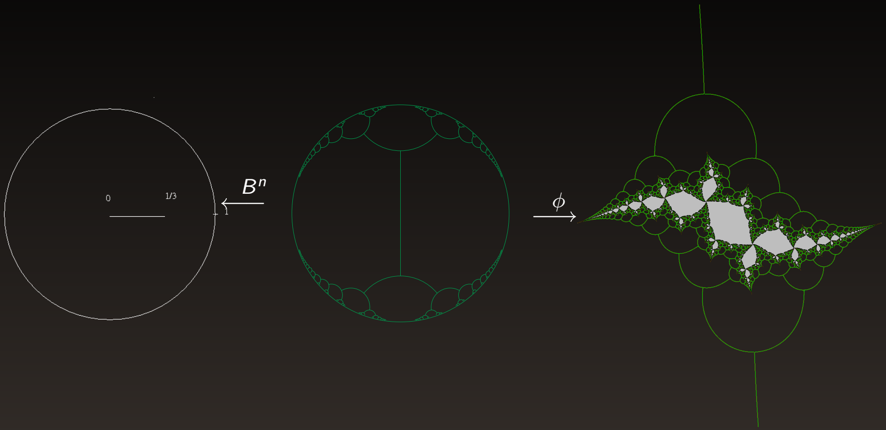
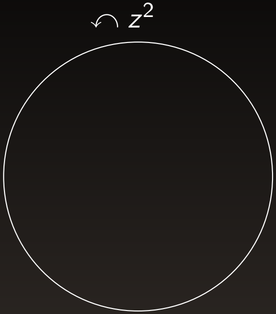


On the circle **S**, the maps B and z^2 are conjugate by some homeomorphism h



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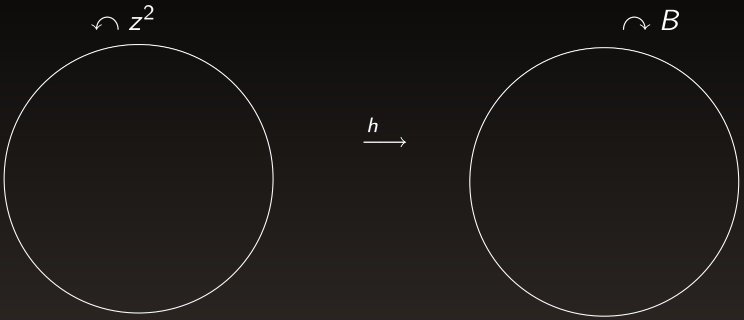
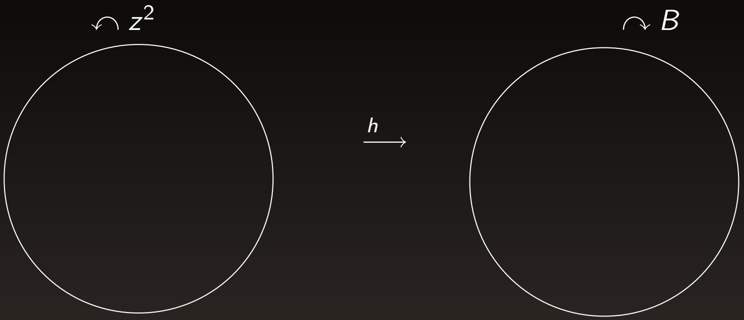
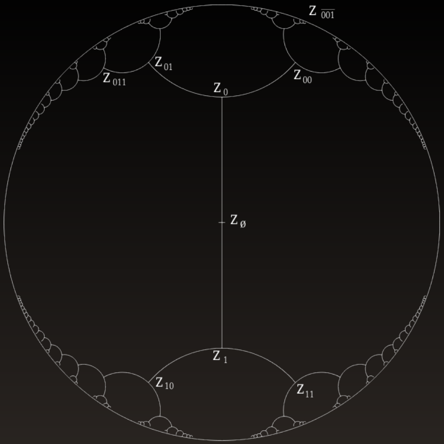


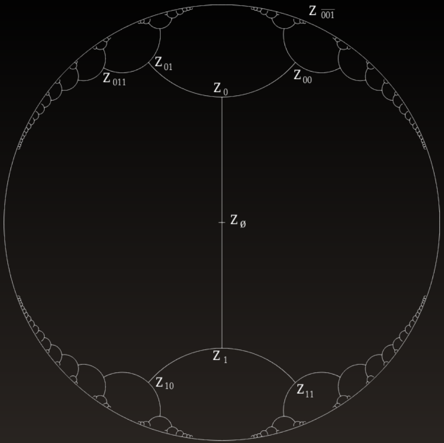
On the circle $\mathbf{S}$, the maps B and z^2 are conjugate by some homeomorphism h

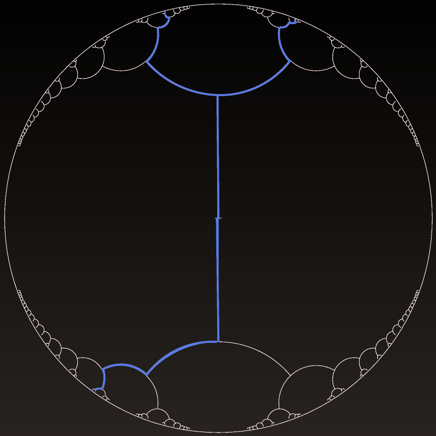


If $\theta = \sum_1^\infty \frac{\varepsilon_k}{2^k}$, the point $Z_{\underline{\varepsilon}} = h(e^{2i\pi\theta})$ has itinerary $\varepsilon_1 \cdots \varepsilon_n \cdots$ with respect to the partition $\mathbf{S} \setminus \{-1, 1\}$



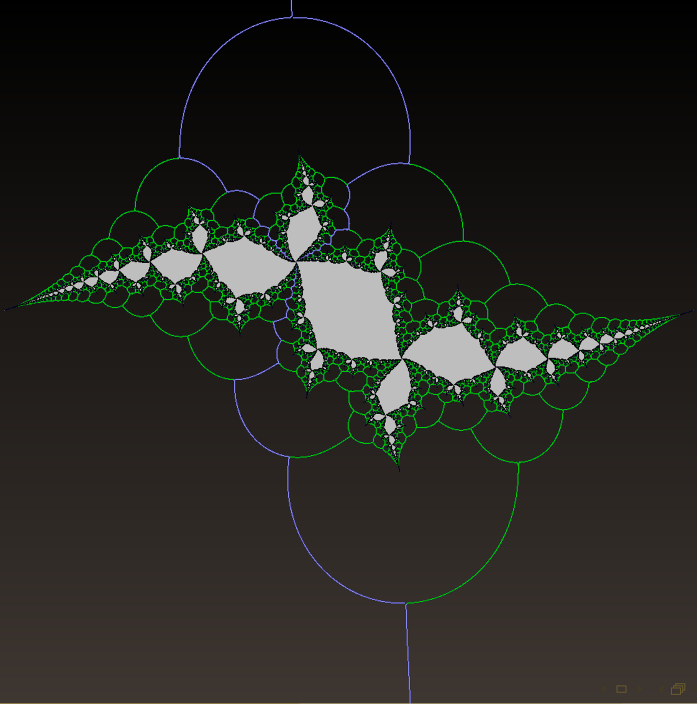
$Z_{\overline{001}}$
Z_{01}
Z_{011}
Z_{00}
Z_0
$Z_\emptyset$
Z_1
Z_{10}
Z_{11}



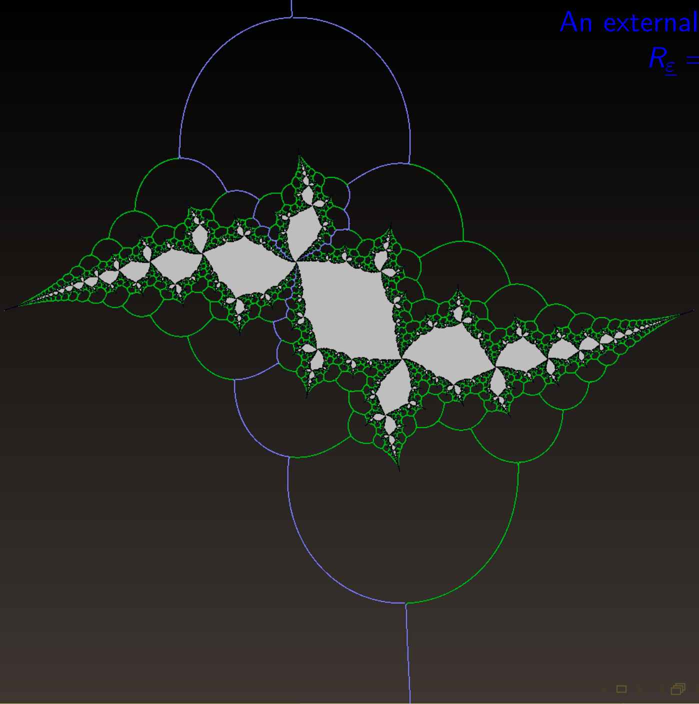


For every itinerary $\underline{\varepsilon} = \varepsilon_1 \cdots \varepsilon_n \cdots$ with $\varepsilon_i \in \{0, 1\}$, a parabolic ray $\gamma_{\underline{\varepsilon}}$ for B is the minimal arc in the tree joining the points $z_{\varepsilon_1 \cdots \varepsilon_n}$ and $z_\emptyset$.

$$B(\gamma_{\underline{\varepsilon}}) = \gamma_{\sigma(\underline{\varepsilon})} \cup [0, \frac{1}{3}] \quad \text{where } \sigma(\varepsilon_1 \varepsilon_2 \cdots) = \varepsilon_2 \cdots$$



An external parabolic ray is
$R_{\underline{\varepsilon}} = \phi(\gamma_{\underline{\varepsilon}})$.



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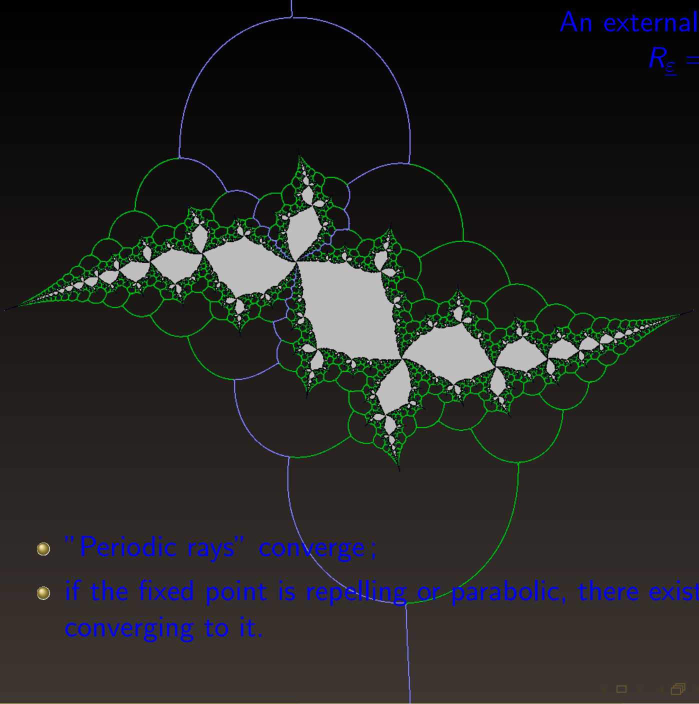
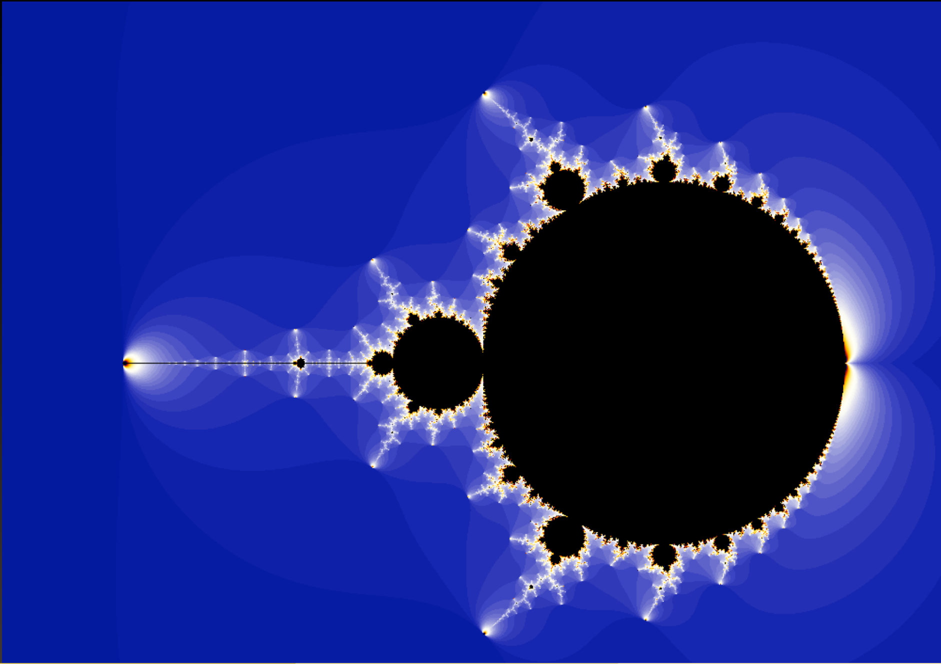
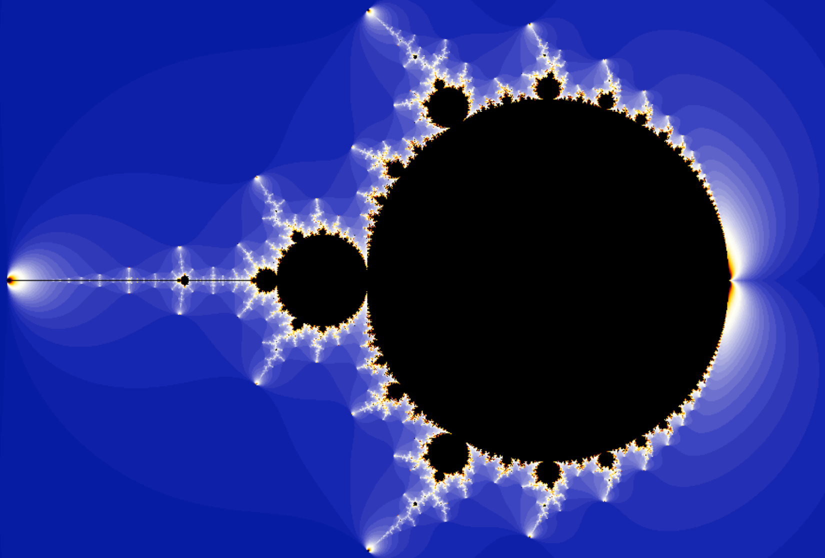


- "Periodic rays" converge;
- if the fixed point is repelling or parabolic, there exists a periodic ray converging to it.

Parameter plane



The parameter $A = 1 - B^2$ is the multiplier of the fixed point $\alpha_B = -1/B$



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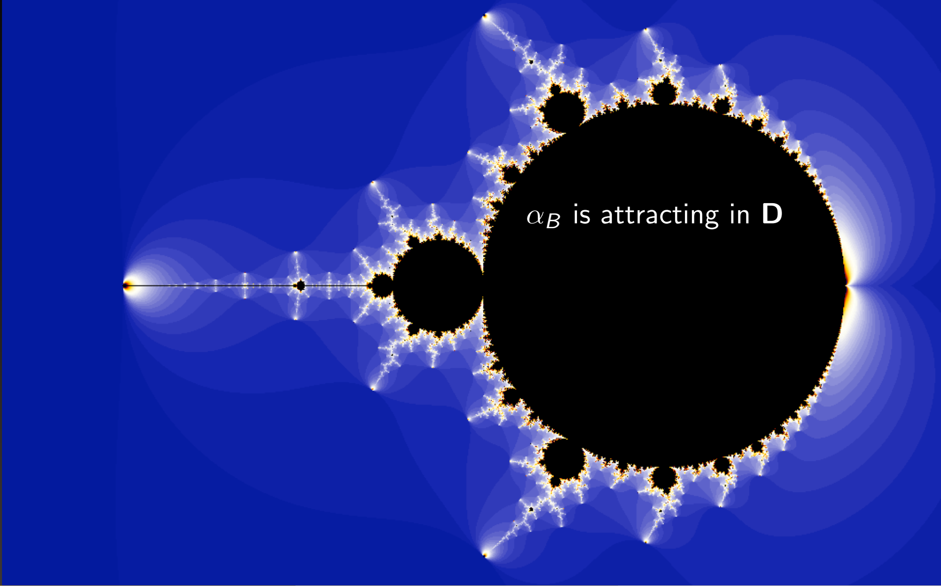

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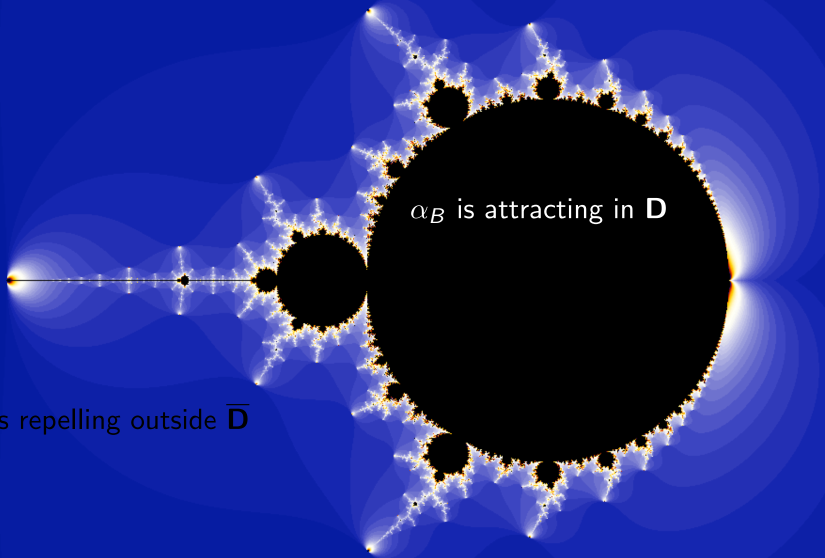


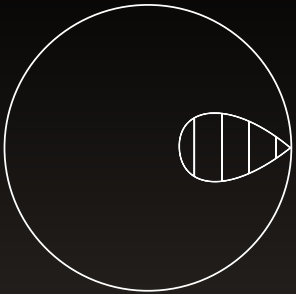
$\mathbf{C} \setminus \mathbf{M}_1$

Milnor

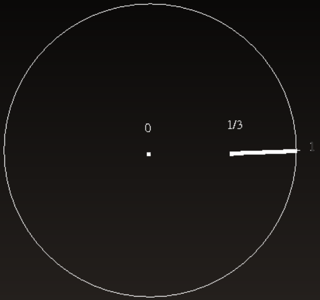
The set $\mathbf{M}_1$ is connected. There is a dynamical holomorphic bijection $\Phi : \mathbf{C} \setminus \mathbf{M}_1 \to \widehat{\mathbf{C}} \setminus (\overline{\mathbf{D}} \cup \{3\})$.

It is given by the position of the "second critical value" in the basin of the model B.

In the basin of B, take out the Fatou petal P bounded by a vertical and passing through the critical value, glue the boundary.
The quotient $(B \setminus P)/\sim$ is conformally equivalent to $\widehat{\mathbf{C}} \setminus \overline{\mathbf{D}}$
$\Pi : B \setminus P \to \widehat{\mathbf{C}} \setminus \overline{\mathbf{D}}$ the projection.



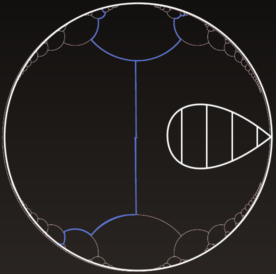
$\overset{\Pi}{\longrightarrow}$

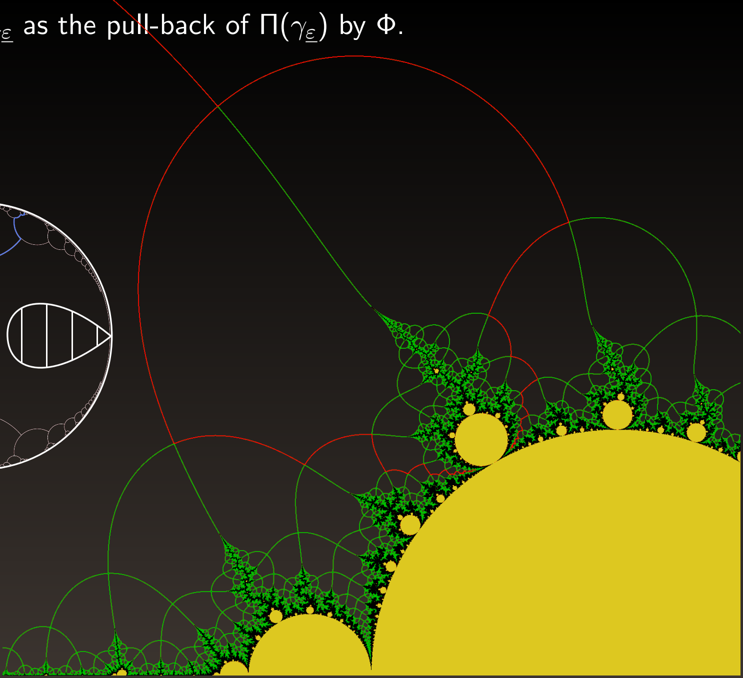


The map ϕ^{-1} is defined until a neighorhood of the second critical value v.

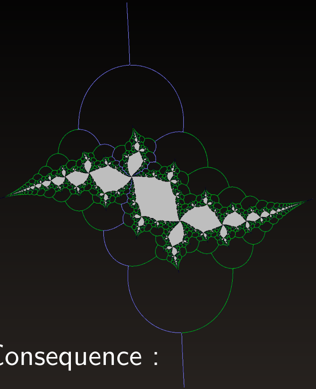
$$\Phi([g]) = \Pi((\phi_g)^{-1}(v_g))$$

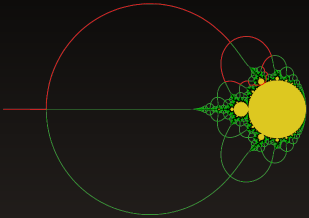
Define rays $\mathcal{R}_{\underline{\varepsilon}}$ as the pull-back of $\Pi(\gamma_{\underline{\varepsilon}})$ by Φ.





In the complement of $\cup_i \mathcal{R}_{\sigma^i(\underline{\varepsilon})}$ the ray $R^B_{\underline{\varepsilon}}$ admits a holomorphic motion.





Consequence :

$$\mathbf{M}_1 = \mathbf{D} \cup \cup_{p/q} L^1_{p/q}$$

- $L^1_{p/q} \cap \mathbf{S}$ is one point $r^1_{p/q}$;
- $L^1_{p/q} \setminus \{r^1_{p/q}\}$ are the connected components of $\mathbf{M}_1 \setminus \overline{\mathbf{D}}$;
- in $L^1_{p/q}$ the fixed point has rotation number p/q.

We use Milnor's argument to prove that there is nothing more "attached" to $\overline{\mathbf{D}}$.

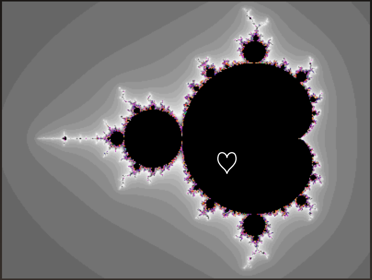
The bijection

$$\Phi_1 : \mathbf{M} \longrightarrow \mathbf{M}_1$$

$$\Phi_1 : [Q_c] \mapsto [g_B]$$

$$\Phi_1 : c \in \heartsuit \mapsto A \in \overline{\mathbf{D}}$$

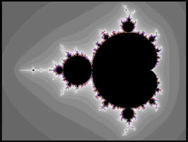
such that $Q_c(z) = z^2 + c$ and g_B (with $A = 1 - B^2$) have a fixed point with the same multiplier.



$\Phi_1 : L_{p/q} \dashrightarrow L^1_{p/q}$ to be define now.

Recall that

$$\mathbf{M} = \heartsuit \cup \cup_{p/q} L_{p/q}$$



- $L_{p/q} \cap \mathbf{S}$ is one point $r_{p/q}$;
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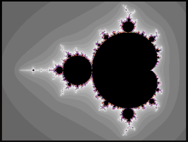

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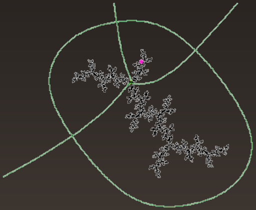

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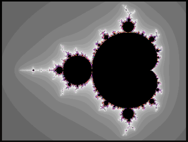

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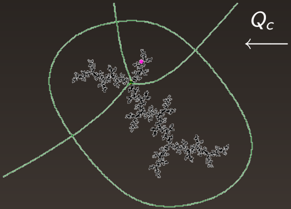

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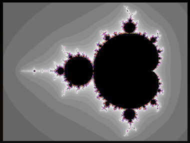

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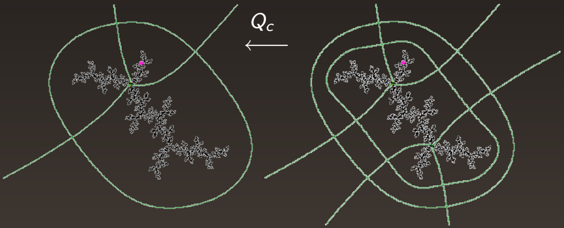

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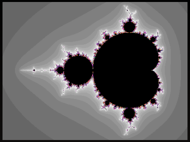

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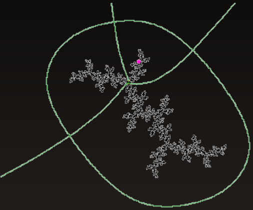
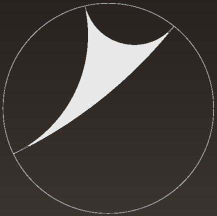
It determines an equivalence relation on $X_n = Q^{-n}(e^{2i\pi\Theta})$ where Θ is the cycle of rotation number p/q and $Q(z) = z^2$.





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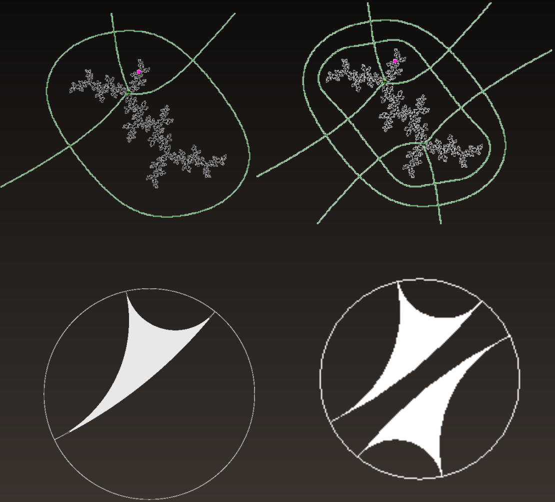

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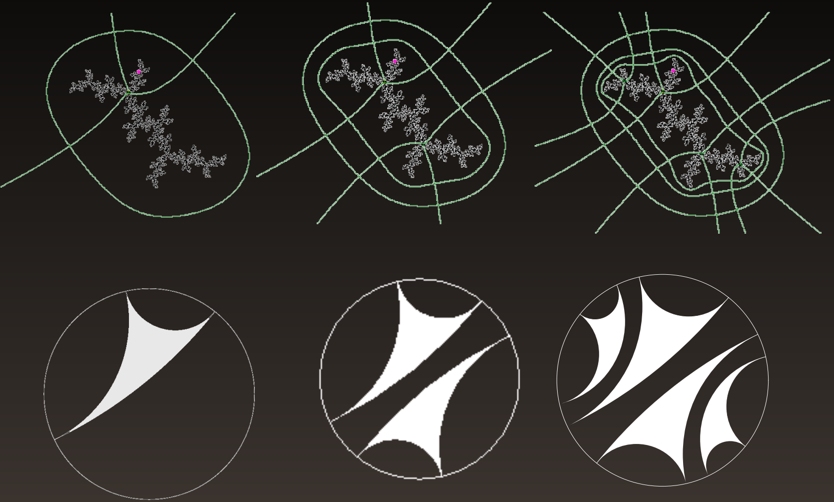
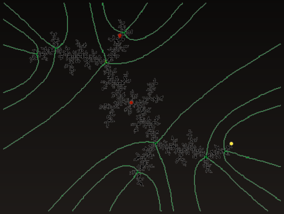


The equivalence relation is the same until some n in $L_{p/q}$ (here $n = 4$).



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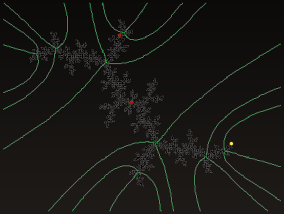

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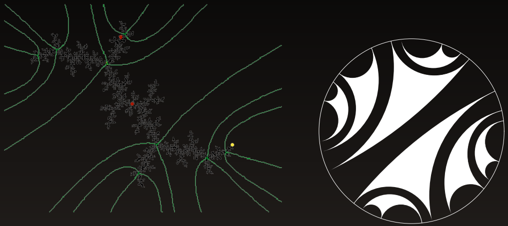

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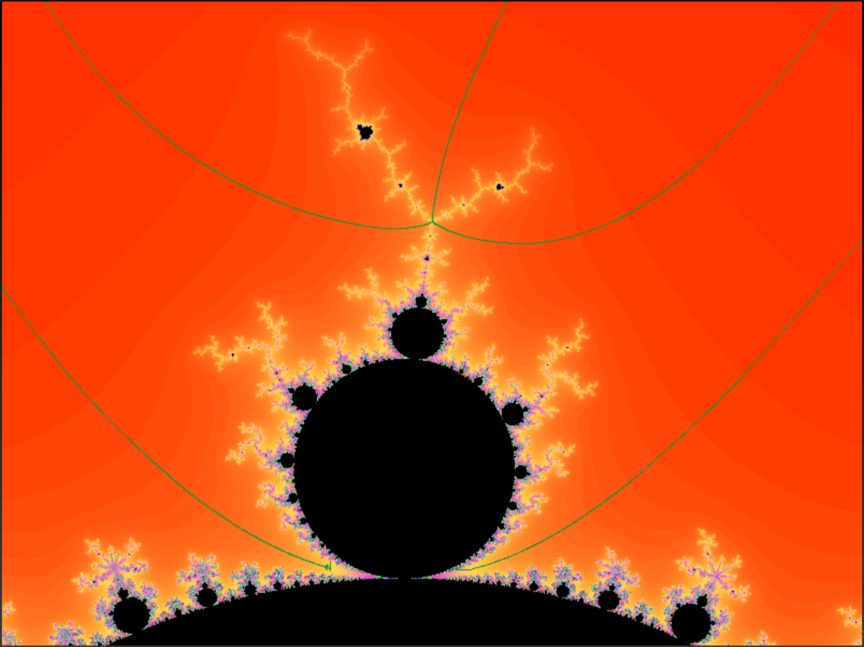

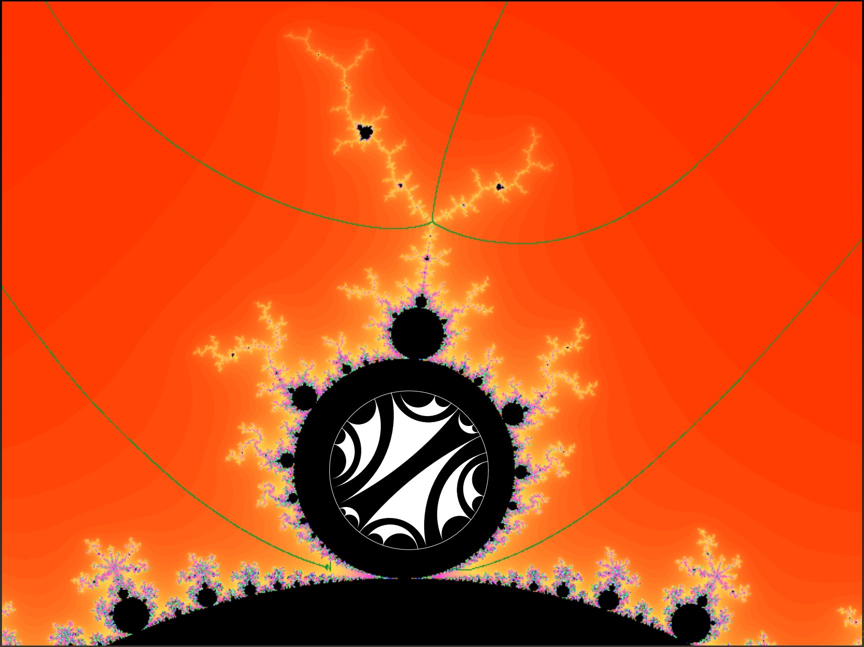

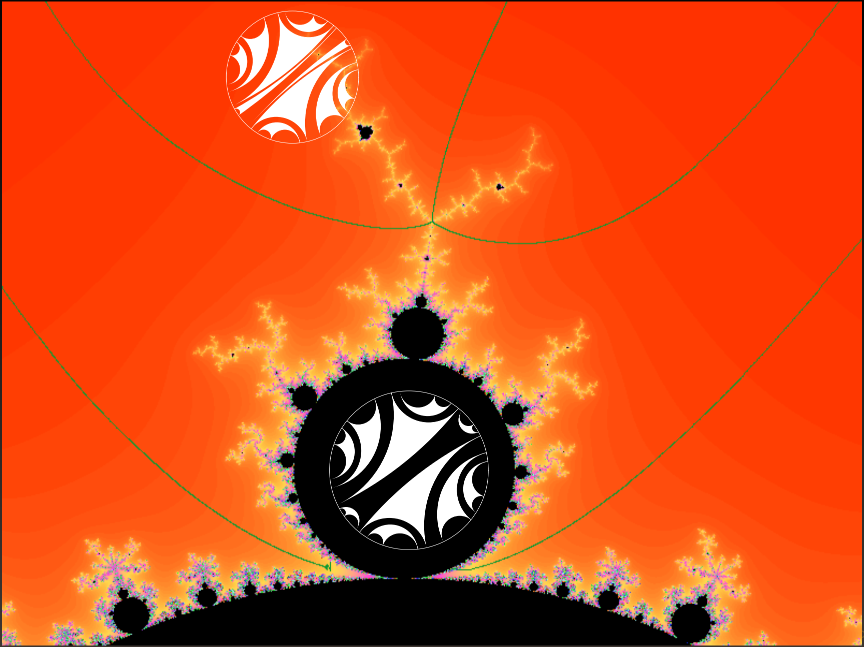

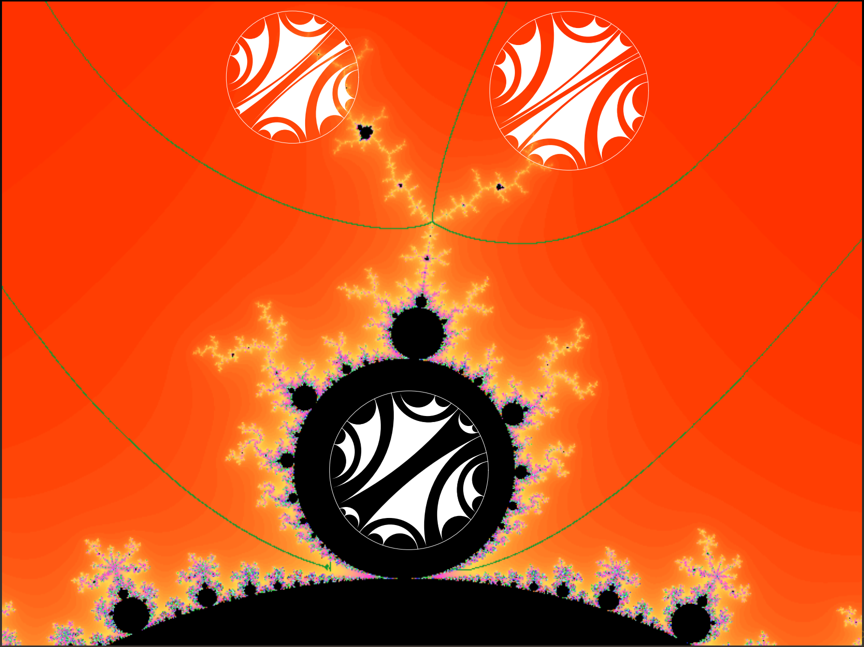

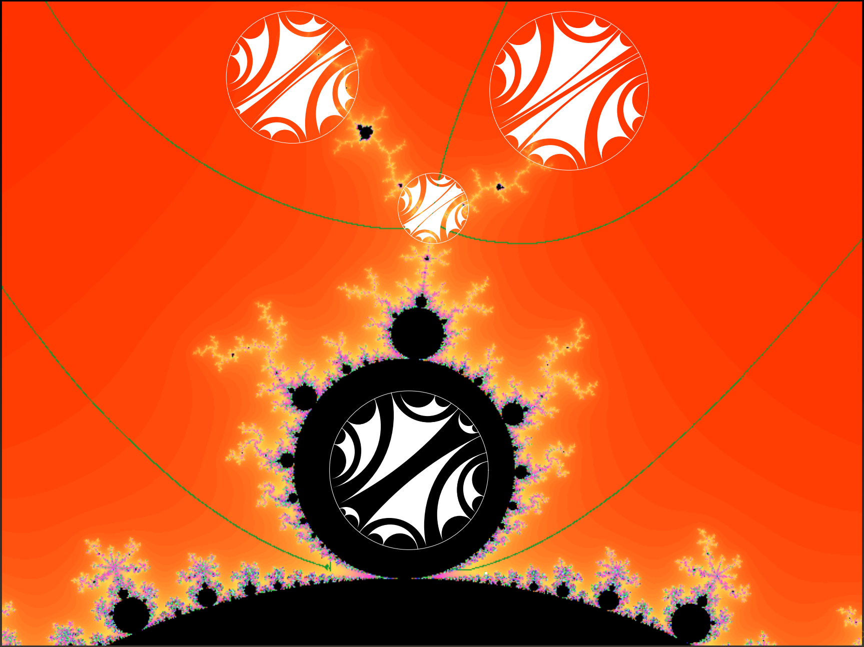

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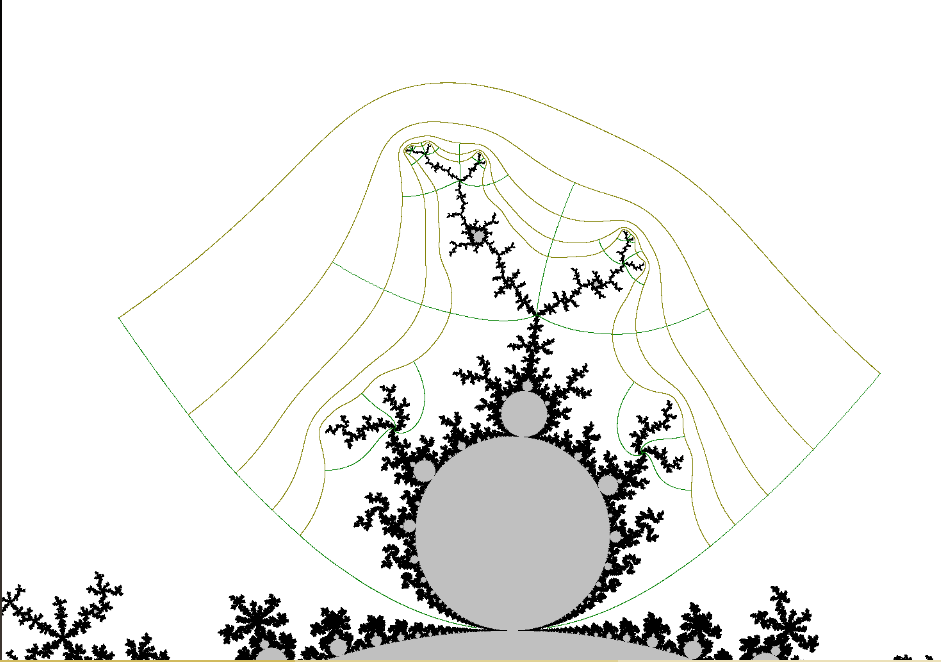
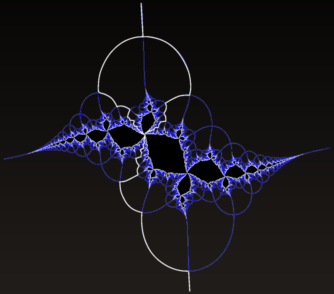


For $[g] \in M_1$ one can define an equivalence relation $\sim_g^0$ on X_0.



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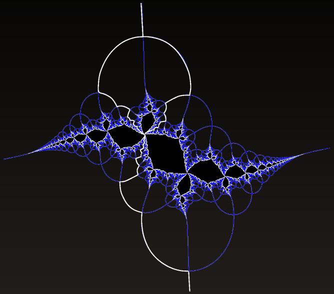

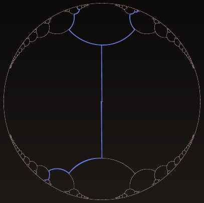

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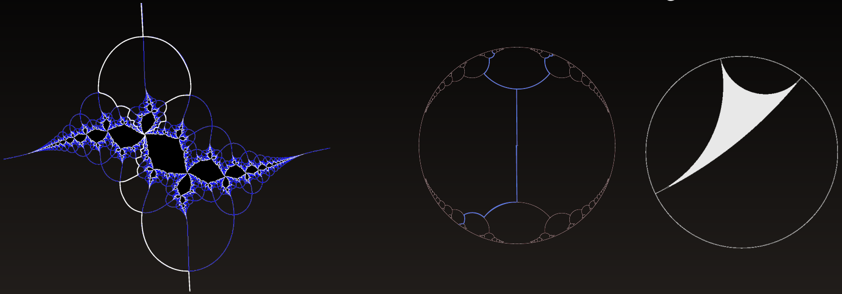

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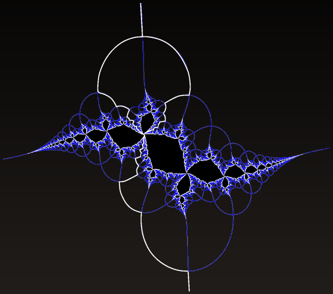
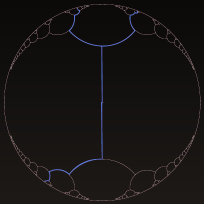


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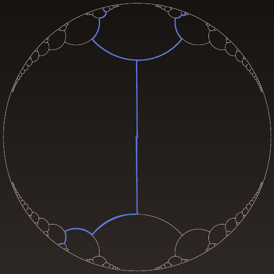
Since there is a conjugacy on $\mathbf{S}$ between B and z^2 we get the same possible equivalence relations by pull back on X_n.
They define pieces in $\mathbf{M}_1$.

Pieces in $Per_1(1)$

Let $\mathcal{G}_0 = \cup_k \gamma_{\sigma^k(\underline{\varepsilon})}$ be the cycle of parabolic rays landing at p/q cycle in $\mathbf{S}$.



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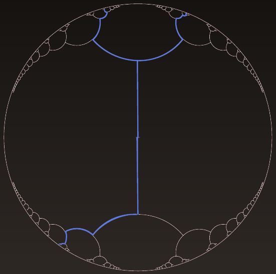
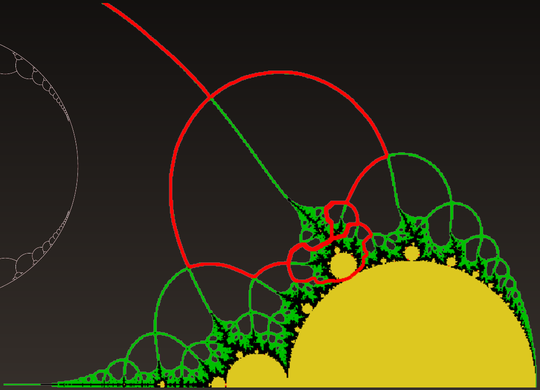
Let $\mathcal{G}_n = B^{-n}(\mathcal{G}_0)$, transport $\mathcal{G}_n$ using the parametrization Φ to $\mathcal{P}\mathcal{G}_n$.





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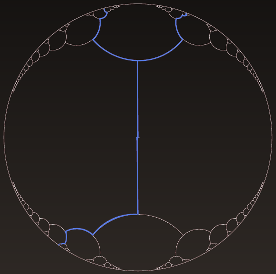
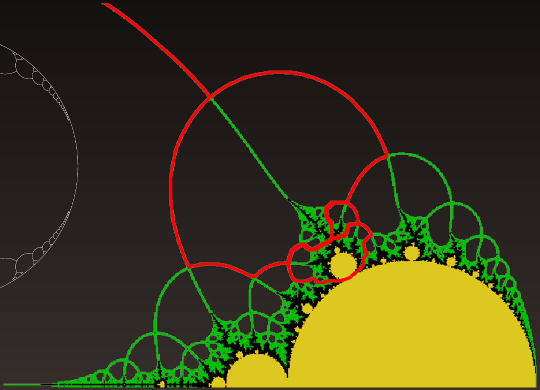
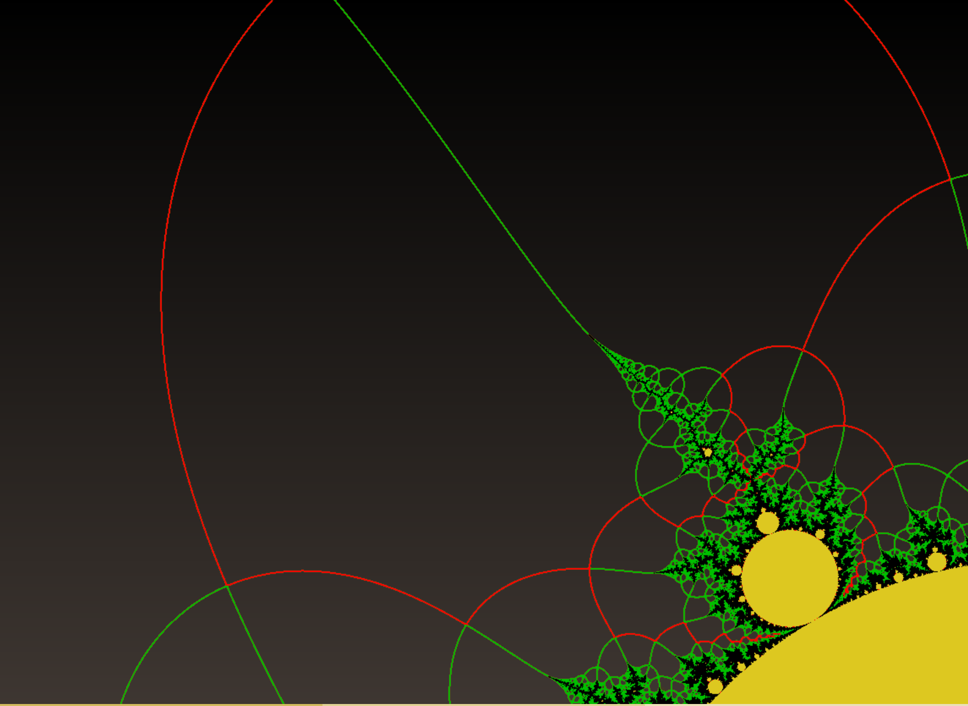
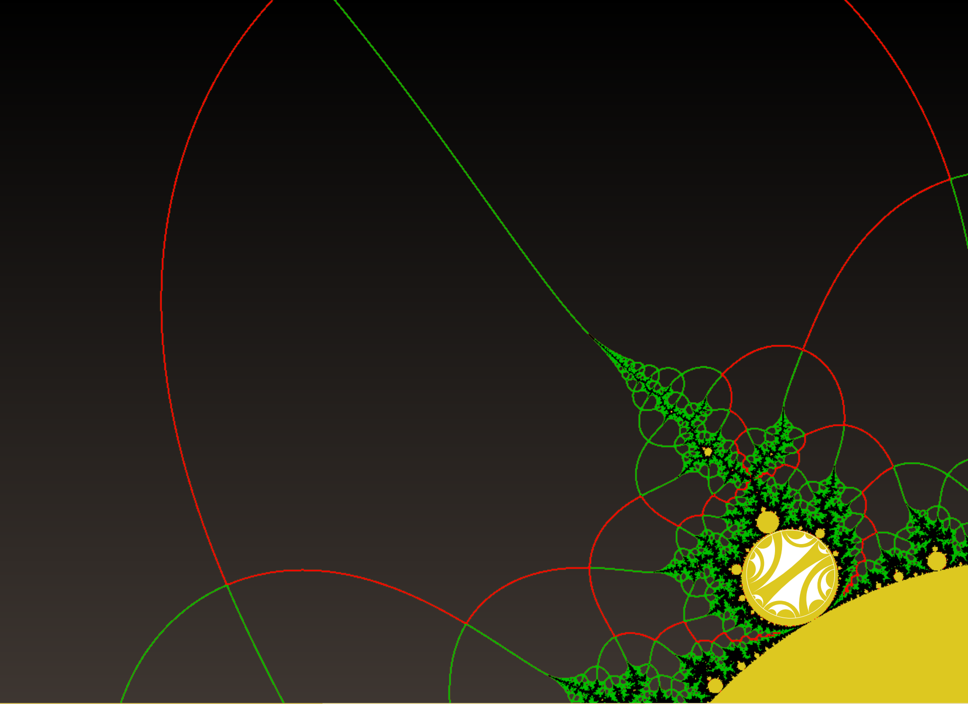
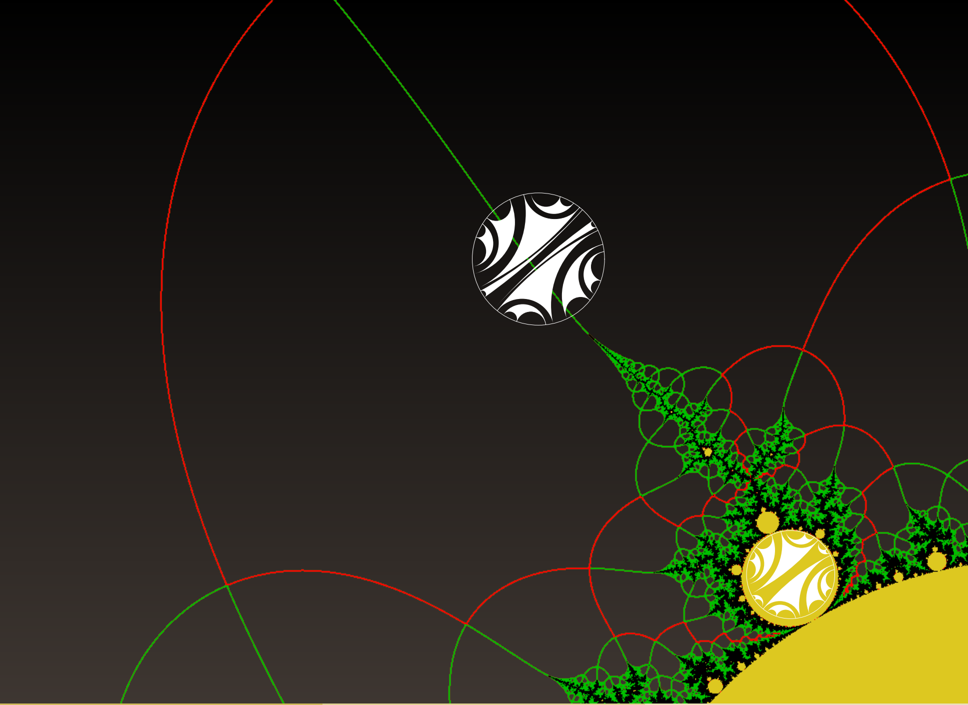


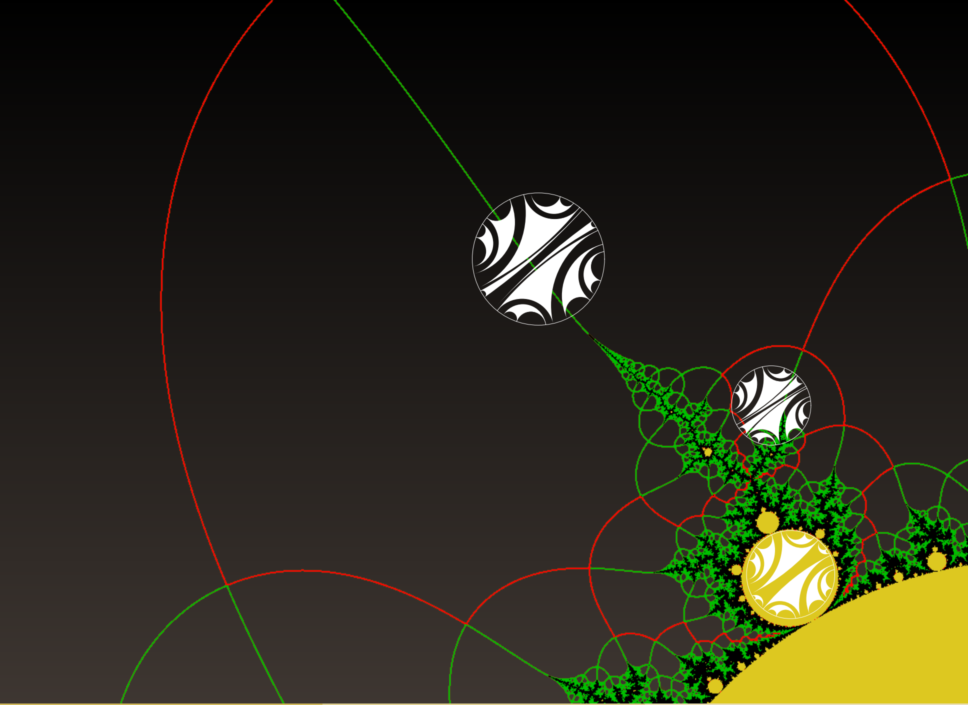


The parameter pieces are the connected components of the complement of $\mathcal{P}\mathcal{G}_n$.









$c \in \mathbf{M} \longrightarrow (\sim_n)_{n \in \mathbf{N}}$ sequence of equivalence relations.
They define in $\mathbf{M}$ and in $\mathbf{M}_1$ nested pieces $(\mathcal{P}(\sim_n))$ and $(\mathcal{P}^1(\sim_n))$.

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- if $(\sim_n)$ is non-renormalizable then $\cap_n \mathcal{P}(\sim_n)$ is one point
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The bijection $c \in \mathbf{M} \longrightarrow (\sim_n)$

- if $(\sim_n)$ is non-renormalizable, $\Phi_1(c) = \Phi_1(\cap_n \mathcal{P}(\sim_n)) = \cap_n \mathcal{P}^1(\sim_n)$
- else $\Phi_1(c) = \chi^1_{\sim_\infty} \circ (\chi_{\sim_\infty})^{-1}(c)$

In the dynamical plane

The sequence of equivalence relations $\sim_g$, $\sim_c$ define pieces in the dynamical plane for g and for Q_c.

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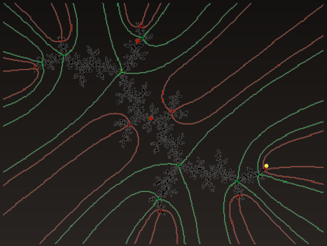

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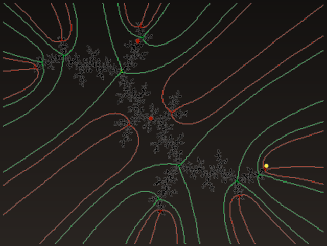
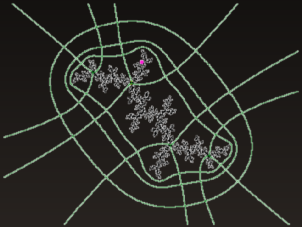


If $\sim_g = \sim_c$ there is a bijection between the set of pieces of level n for g and for Q_c.

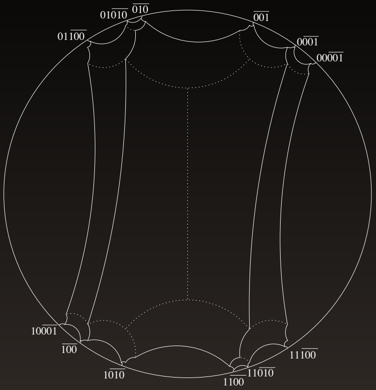
- The bijection sends the critical piece of level n to the critical piece of level n,
- it commutes with the dynamics induced on the pieces.

The dynamical pieces do not shrink to points. One should add equipotentials.



There are no equipotentials for parabolics.

We add some "shortcut" between the parabolic rays.



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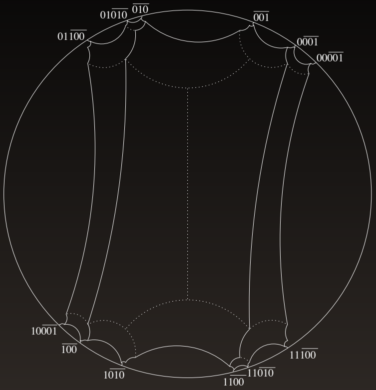

They are preserved by the dynamics (excepted for the nest around the parabolic point at ∞ and its preimage).

The bijection preserves the non degenerate annuli.

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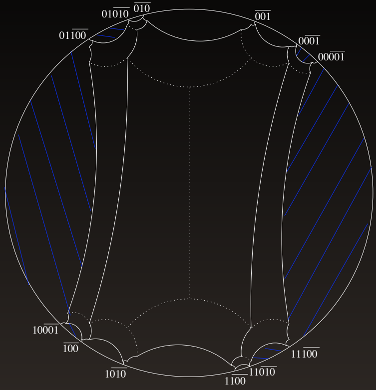

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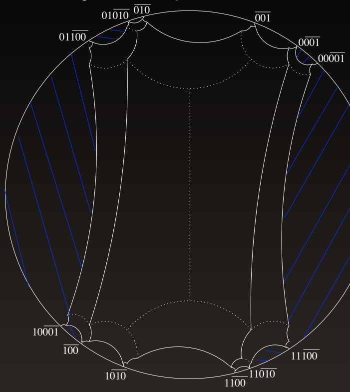

Same combinatorics,

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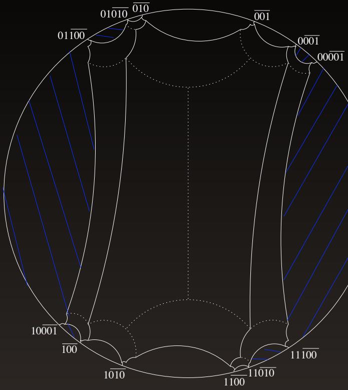

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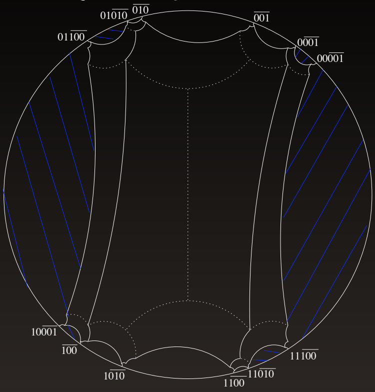

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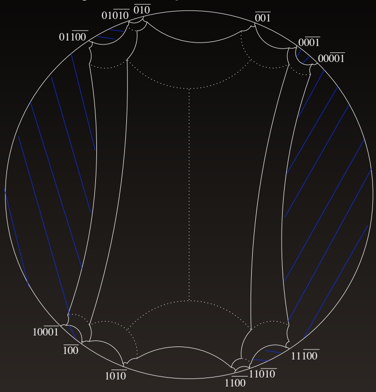

Same combinatorics, same degree, same non-degenerate annuli, the proof of Yoccoz passes to the parabolic case.

Construction of the conjugacy between the two Julia sets:

- In the non renomalizable case, they are both locally connected.
- In the renormalizable case, the conjugacy between the small Julia sets extends to the whole Julia sets by pull back.

Transfer to the parameter plane :

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Transfer to the parameter plane :

We use Shishikura's argument on holomorphic motions to compare the modulus of the annuli in parameter and dynamical planes.

Therefore $\cap_n \mathcal{P}^1(\sim_n)$ is either a point or a copy of the Mandelbrot set.

Continuity

- At non renomalizable maps, parameter pieces of level n define neighborhoods. The continuity follows from $\cap_n \mathcal{P}^1(\sim_n) = \{*\}$.

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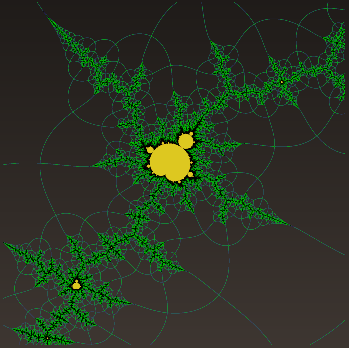
C. PETERSEN & R. If one takes away any small copie of **M** in **M** or in $\mathbf{M}_1$, the diameter of the remaining connected components tends to 0.



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Thank you for your attention.