Interaction energy of domain walls in a nonlocal Ginzburg-Landau type model from micromagnetics

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Abstract

We study a variational model from micromagnetics involving a nonlocal Ginzburg-Landau type energy for $\mathbb{S}^1$-valued vector fields. These vector fields form domain walls, called Néel walls, that correspond to one-dimensional transitions between two directions within the unit circle $\mathbb{S}^1$. Due to the nonlocality of the energy, a Néel wall is a two length scale object, comprising a core and two logarithmically decaying tails. Our aim is to determine the energy differences leading to repulsion or attraction between Néel walls. In contrast to the usual Ginzburg-Landau vortices, we obtain a renormalised energy for Néel walls that shows both a tail-tail interaction and a core-tail interaction. This is a novel feature for Ginzburg-Landau type energies that entails attraction between Néel walls of the same sign and repulsion between Néel walls of opposite signs.

Keywords: Néel walls, Ginzburg-Landau, nonlocal, renormalised energy, interaction, micromagnetics

1 Introduction

In this article, we analyse a variational model describing the formation of domain walls in ferromagnetic thin films. These domain walls are called Néel walls and represent one-dimensional transition layers connecting two directions of the magnetisation within the unit circle $\mathbb{S}^1$. Due to dipolar effects, the variational problem is strongly nonlocal and generates Néel walls with an interesting core-and-tail structure. Our aim is to study the repulsive or attractive interaction between the domain walls in terms of their energy. This interaction energy governs the location of the domain walls and is analogous to the renormalised energy in Ginzburg-Landau type problems (see the seminal book [3]). Although our analysis builds to some extent on the theory of Ginzburg-Landau vortices, our model has novel features that have not been studied before. In contrast to the usual Ginzburg-Landau vortices, we obtain a renormalised energy for Néel walls incorporating two types of interaction: a tail-tail interaction and a core-tail interaction. This is due to the nonlocal character of the model and the two distinct length scales of the core and the tails (of logarithmic decay). Moreover, Néel walls of opposite signs repel each other and Néel walls of the same sign attract each other, whereas Ginzburg-Landau vortices show the opposite behaviour. This observation is consistent with the physical prediction (see [12, Section 3.6.(C)]). Furthermore, in typical Ginzburg-Landau systems, most of the energy is contained in the highest order term, whereas in our model, it is the lowest order term that contains most of the energy. From a technical point of view, the lack of a quantised Jacobian is an additional difficulty in the analysis of our model.

1.1 The model

The magnetisation We consider a one-dimensional model for transition layers (incorporating several Néel walls) in the magnetisation of a thin ferromagnetic film. The magnetisation is represented by a continuous map

$$m : (-1,1) \to \mathbb{S}^1.$$

More precisely, we can think of a ferromagnetic thin film of the shape $(-1,1)\times(0,h)\times\mathbb{R}$ (with very small thickness $h > 0$ in the x_2-direction) and a magnetisation vector field $M : (-1,1)\times(0,h)\times\mathbb{R} \to \mathbb{S}^2$ of the form $M(x_1,x_2,x_3) = (m(x_1),0)$. Here, the non-dependence of M on x_2 is a natural assumption for a thin film, whereas the non-dependence on x_3 represents a simplification of the problem. (It implies that the walls appear in planes parallel to the x_2x_3-plane and we assume that the magnetisation depends only on the normal direction x_1.) The strip over $x_1 \in (-1,1)$ does not necessarily represent the whole ferromagnetic sample, but merely a region that contains the Néel walls in question. The assumption that the third component M_3 vanishes is consistent with the fact that Néel walls correspond to an in-plane magnetisation. Another characteristic feature of Néel walls is that the magnetisations on either side (represented by $m(-1)$ and $m(1)$ in our model) differ by a vector parallel to the wall plane (in this case the x_2-direction). Thus there exists a number $\alpha \in (0,\pi)$ such that

$$m_1(-1) = m_1(1) = \cos\alpha. \tag{1}$$

Moreover, we will sometimes assume that

$$m(-1) = m(1) = (\cos\alpha, \sin\alpha), \tag{2}$$

so that a winding number can be defined.

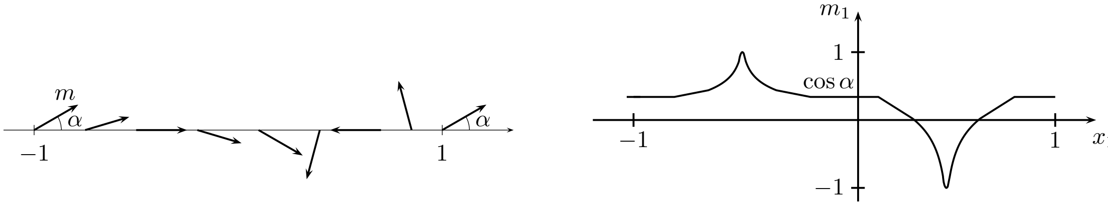

Figure 1: A magnetisation $m = (m_1, m_2)$ of winding number -1 consisting of a positive Néel wall of angle 2α and a negative Néel wall of angle $2(\pi-\alpha)$ (right).

More precisely, since m is continuous, there exists a continuous function $\varphi : (-1,1) \to \mathbb{R}$, called a lifting of m, such that

$$m = (\cos\varphi, \sin\varphi) \quad \text{in} \quad (-1,1)$$

and $\varphi(-1) = \alpha$. If (2) holds, then the winding number (or topological degree) of m is defined as

$$\deg(m) = \frac{\varphi(1) - \varphi(-1)}{2\pi} \in \mathbb{Z}.$$

The angle $\alpha \in (0,\pi)$ will stay fixed throughout this paper. (The case $\alpha \in \{0,\pi\}$ is geometrically different and is not studied here.) However, our arguments do not require that $m_2(-1) = m_2(1) = \sin\alpha$ in principle and we will present our results in a wider generality, i.e., with $m_2(\pm1) \in \{\pm\sin\theta\}$.

The energy The energy for our model comprises two terms, called the exchange energy and the magnetostatic energy (or stray-field energy), respectively. The exchange energy is modelled by the following expression involving the L^2-norm of the derivative m':

$$\frac{\epsilon}{2}\int_{-1}^{1}|m'|^2\,dx_1 = \frac{\epsilon}{2}\int_{-1}^{1}(\varphi')^2\,dx_1 = \frac{\epsilon}{2}\int_{-1}^{1}\frac{(m_1')^2}{1-m_1^2}\,dx_1.$$

Here $\epsilon > 0$ is a ratio between a material constant called the exchange length and the length scale of the thin film. (This is a model obtained after rescaling, i.e., the length scale of the ferromagnetic sample has been set to unit size.) The number ϵ is assumed to be small, and we will eventually study the limit $\epsilon \searrow 0$.

We write $x = (x_1, x_2)$ for a generic point in the upper half-plane $\mathbb{R}^2_+ = \mathbb{R} \times (0, \infty)$. In order to compute the magnetostatic energy, we need to solve the boundary value problem[1]

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^2_+, \tag{3}$$

$$\frac{\partial u}{\partial x_2} = -m_1' \quad \text{on } (-1,1)\times\{0\}, \tag{4}$$

$$\frac{\partial u}{\partial x_2} = 0 \qquad \text{on } (-\infty,-1)\times\{0\} \text{ and on } (1,\infty)\times\{0\}. \tag{5}$$

Equivalently, if we extend m_1 by the constant $\cos\alpha$ on $\mathbb{R}\setminus(-1,1)$, then

$$\int_{\mathbb{R}^2_+}\nabla u\cdot\nabla\zeta\,dx = \int_{-\infty}^{\infty} m_1'\zeta(\,\cdot\,,0)\,dx_1 \quad \text{for every } \zeta \in C_0^\infty(\mathbb{R}^2). \tag{6}$$

Let $\dot{W}^{1,2}(\mathbb{R}^2)$ be the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|\zeta\|_{\dot{W}^{1,2}(\mathbb{R}^2)} = \|\nabla\zeta\|_{L^2(\mathbb{R}^2)}.$$

(We sometimes abuse notation and treat elements of $\dot{W}^{1,2}(\mathbb{R}^2)$ as functions, even though the completion process identifies any two functions that differ by a constant.) For an open set $\Omega \subset \mathbb{R}^2$, we write $\dot{W}^{1,2}(\Omega)$ for the set of all restrictions of functions in $\dot{W}^{1,2}(\mathbb{R}^2)$ to Ω and

$$\|\zeta\|_{\dot{W}^{1,2}(\Omega)} = \|\nabla\zeta\|_{L^2(\Omega)}.$$

By the Lax-Milgram theorem, solutions of (6) are unique in $\dot{W}^{1,2}(\mathbb{R}^2_+)$ (i.e., up to a constant). Thus the quantity

$$\frac{1}{2}\int_{\mathbb{R}^2_+}|\nabla u|^2\,dx$$

depends only on m_1. This is the term representing the magnetostatic energy. It is worth remarking that the solutions u of (3)–(5) in $\dot{W}^{1,2}(\mathbb{R}^2_+)$ have a limit for $|x| \to \infty$. Indeed, if we extend u to $\mathbb{R}^2$ by even reflection, then we obtain a harmonic function near ∞ with finite Dirichlet energy, and it is well-known that the limit exists at ∞. Then we normalise this constant and define $U(m)$ (sometimes also denoted $U(m_1)$) to be the unique solution of (6) in $\dot{W}^{1,2}(\mathbb{R}^2_+)$ with

$$U(m) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Moreover, in view of (6), using the extension of m_1 by the constant $\cos\alpha$ on $\mathbb{R}\setminus(-1,1)$, we may express the magnetostatic energy in terms of the homogeneous $\|\cdot\|_{\dot{H}^{1/2}}$-seminorm of m_1 (see e.g. [8, 13]):

$$\frac{1}{2}\int_{\mathbb{R}^2_+}|\nabla U(m)|^2\,dx = \frac{1}{2}\int_{\mathbb{R}}\left|\left|\frac{d}{dx_1}\right|^{1/2} m_1\right|^2 dx_1. \tag{7}$$

[1] Here, ∇u represents the stray-field associated to M, which is also invariant in the x_3-direction.

To summarise, we study the energy functional

$$E_\epsilon(m) = \frac{\epsilon}{2} \int_{-1}^{1} |m'|^2 \, dx_1 + \frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla U(m)|^2 \, dx$$

for $m \in W^{1,2}\big((-1,1), \mathbb{S}^1\big)$ satisfying (1). We are interested in the behaviour of m and of its energy $E_\epsilon(m)$ as $\epsilon \searrow 0$, especially under conditions that force the nucleation of several Néel walls.

Néel walls If we trace m from -1 to 1, we may well find that m winds around the circle $\mathbb{S}^1$ one or several times. If that happens, then there necessarily exist two points $a_+, a_- \in (-1,1)$ such that $m_1(a_+) = 1$ and $m_1(a_-) = -1$. But even if the topology of m is trivial (i.e., if $\deg(m) = 0$), a transition from $(\cos\alpha, \sin\alpha)$ to $(\cos\alpha, -\sin\alpha)$ may occur, giving rise to a point in between where m_1 reaches one of the values ± 1. We think of any such transition as a Néel wall and we use these points in order to track them. Obviously, it is possible for m_1 to attain ± 1 when no proper transition occurs, but from the energetics point of view, this makes no difference and we call this a Néel wall anyway. We speak of a positive or negative Néel wall depending on the sign of m_1 (see Figure 1). We will see that a Néel wall has a two-length scale structure comprising a core of size $\delta = \epsilon \log \frac{1}{\epsilon}$ around the transition point and two tails of size $O(1)$, where m_1 decays logarithmically to $\cos\alpha$ (see Theorem 22 below). The total change of the phase during the transition is called the rotation angle of the Néel wall (which may be 0 by the above convention).[2] For more physical background, we refer to [12, 10].

We will assume in the following that there are certain points $a_1, \ldots, a_N \in (-1,1)$ such that

$$-1 < a_1 < \cdots < a_N < 1 \tag{8}$$

and certain numbers $d_1, \ldots, d_N \in \{-1, 1\}$ such that

$$m_1(a_n) = d_n \quad \text{for} \quad n = 1, \ldots, N. \tag{9}$$

These points $(a_n)_{1 \le n \le N}$ represent the positions of the Néel walls that we study, while $(d_n)_{1 \le n \le N}$ indicate whether a Néel wall is positive or negative. We keep the number N of walls fixed throughout the paper. Let

$$A_N = \left\{ a = (a_1, \ldots, a_N) \in (-1,1)^N \text{ with (8)} \right\}.$$

For $a \in A_N$ and $d \in \{\pm 1\}^N$, we consider the set

$$M(a,d) = \left\{ m \in W^{1,2}((-1,1); \mathbb{S}^1) \text{ with (1) and (9)} \right\}.$$

Our aim is to answer the following question.

Question. *For a given $a \in A_N$ and $d \in \{\pm 1\}^N$, what is the behaviour of*

$$\inf_{M(a,d)} E_\epsilon \quad \text{as} \quad \epsilon \searrow 0?$$

That is, if we prescribe Néel walls at the positions $a_1, \ldots, a_N$ with signs $d_1, \ldots, d_N$, what energy does it take to achieve such a configuration? We first note that a minimal configuration m always exists and that its first component m_1 is unique. (Obviously, $|m_2|$ is also unique, but the sign of the m_2 component can change between a_n and a_{n+1} for two different minimisers.)

[2]When studying the interaction between a pair of walls, the physics literature (see [12]) distinguishes between winding walls, which refers to a pair with the same rotation sense, and unwinding walls, which refers to a pair with opposite rotation sense. Except for degenerate cases, a pair of Néel walls with opposite signs according to our terminology corresponds to winding walls and a pair with the same sign corresponds to unwinding walls.

Proposition 1. *There exists a minimiser of* $\inf_{M(a,d)} E_\epsilon$ *for any* $\epsilon > 0$. *Moreover, any minimiser* m *is smooth on* $(-1,1) \setminus \{a_1, \ldots, a_N\}$ *and has a unique* m_1*-component.*

Proof. The direct method in the calculus of variations yields a minimiser m of E_ϵ in $M(a,d)$. The regularity of m is standard (see, e.g., [14]). The uniqueness of m_1 follows from the strict convexity of (7) and of the function $(v,w) \mapsto \frac{v^2}{1-w^2}$ for $(v,w) \in \mathbb{R} \times (-1,1)$. □

We look for an expansion of $\inf_{M(a,d)} E_\epsilon$ similar to [9], where it is shown that

$$\inf_{M(a,d)} E_\epsilon = \sum_{n=1}^{N} \frac{\pi(d_n - \cos\alpha)^2}{2\log\frac{1}{\epsilon}} + O\left(\frac{\log\log\frac{1}{\epsilon}}{\left(\log\frac{1}{\epsilon}\right)^2}\right) \tag{10}$$

for $\epsilon > 0$ small. Since this is not good enough to understand the interaction between domain walls, we need to determine the second term in such an expansion completely and identify the third term as well. This problem is analogous to finding the "renormalised energy" in Ginzburg-Landau type problems, but in the context of Néel walls, it has remained open until now.

We give the answer to this question in Theorem 2. The key is to identify the contributions to the renormalised energy coming from the interaction between two tails and between a core and a tail of two different walls. It turns out that the above expansion is easier to understand when we replace ϵ by $\delta = \epsilon \log\frac{1}{\epsilon}$ (recall that this is the typical length scale of the core of a Néel wall). The first two terms of the expansion (10) are then united in a single leading order term in the expansion in $1/|\log\delta|$ (see (11) below). The next-to-leading order term corresponds to the renormalised energy.

1.2 Motivation

There are several reasons for asking the above question. First, we may want to study the positions of Néel walls in equilibrium. Once we have determined the renormalised energy, we can find the likely positions by minimising it. Second, we may want to study the dynamics of Néel walls (see, e.g., [4, 6]). The dynamics of the magnetisation is described by the Landau-Lifshitz-Gilbert equation, which is derived from the micromagnetic energy through a variational principle. For reasons that are explained below, understanding the asymptotic behaviour of the energy is expected to be a key step towards deriving an effective motion law for the walls in the limit $\epsilon \searrow 0$. A third reason for studying these long range interactions is that we want to understand some phenomena in thin ferromagnetic films where they matter, such as cross-tie walls. A cross-tie wall is a typical domain wall that consists in an ensemble of Néel walls and micromagnetic vortices (similar to Ginzburg-Landau vortices), see [9, 1, 29, 30]. It has an internal length scale, the size of which is not predicted by any existing theory, and our analysis on the interaction energy of Néel walls could represent an significant step forward here.

A related question concerns the analysis of general transition layers m carrying a winding number when the location of the Néel walls is no longer prescribed. More precisely, suppose that the lifting $\varphi : (-1,1) \to \mathbb{R}$ of $m = (\cos\varphi, \sin\varphi)$ satisfies the boundary conditions $\varphi(-1) = \alpha$ and $\varphi(1) = 2\ell\pi + \alpha$, so that we have winding number ℓ, i.e. $\ell = \deg(m)$. Hence the magnetisation performs ℓ full rotations, so that (2) is satisfied. Then by continuity, we necessarily have a certain number of transitions between $(\cos\alpha, \sin\alpha)$ and $(\cos\alpha, -\sin\alpha)$.

Open problem. *For a prescribed winding number and given suitable control of* $E_\epsilon(m)$*, what can we say about the profile of* m *and of the stray field potential* $U(m)$*?*

As mentioned before, a prescribed degree ℓ will automatically give rise to certain Néel walls. But it is not obvious, for example, that these Néel walls stay separate from one another (uniformly as $\epsilon \to 0$) and that one can rule out other transitions. In fact, it is an open question whether the lifting of m is monotone even for minimisers (which would exclude unexpected transitions). However, assuming good control of the energy, we expect to have exactly 2ℓ transitions (corresponding to the expected Néel walls) and no extraordinary behaviour of the magnetisation in between. For the stray field energy, it is expected that the energy density concentrates at the walls. Such information would be useful in the study of compactness properties in the appropriate function spaces, for example with a view to Γ-convergence.

1.3 Main results

For any $\epsilon \in (0, \frac{1}{2}]$, let

$$\delta = \epsilon \log \frac{1}{\epsilon}$$

and define the metric ϱ on $(-1,1)$ by[3]

$$\varrho(b,c) = \frac{|b-c|}{1-bc} \in [0,1) \quad \text{for} \quad b,c \in (-1,1).$$

We have the following result, answering the question on page 4.

Theorem 2. *There exists a function $e : \{\pm 1\} \to \mathbb{R}$ such that for any $a \in A_N$ and $d \in \{\pm 1\}^N$, the following holds true. Let $\gamma_n = d_n - \cos\alpha$ for $n = 1, \ldots, N$ and let*

$$\mathbb{W}(a,d) = \sum_{n=1}^{N} e(d_n) - \frac{\pi}{2}\sum_{n=1}^{N} \gamma_n^2 \log(2 - 2a_n^2) - \frac{\pi}{2}\sum_{n=1}^{N}\sum_{k\neq n} \gamma_k\gamma_n \log\left(\frac{1+\sqrt{1-\varrho(a_k,a_n)^2}}{\varrho(a_k,a_n)}\right).$$

Then

$$\mathbb{W}(a,d) = \lim_{\epsilon\searrow 0}\left(\left(\log\frac{1}{\delta}\right)^2 \inf_{M(a,d)} E_\epsilon - \frac{\pi}{2}\log\frac{1}{\delta}\sum_{n=1}^{N}\gamma_n^2\right).$$

In analogy to the theory of Ginzburg-Landau vortices, we call $\mathbb{W}(a,d)$ the renormalised energy for the N walls placed at $a = (a_1, \ldots a_N)$ with signs $d = (d_1, \ldots, d_N)$. As the theorem shows, $\mathbb{W}(a,d)$ represents the next-to-leading order term in the expansion of $\inf_{M(a,d)} E_\epsilon$ in $1/|\log\delta|$. If we express these asymptotics in terms of ϵ, our result improves (10) by determining the precise second and third coefficients:

$$\inf_{M(a,d)} E_\epsilon = \frac{1}{\left(\log\frac{1}{\epsilon}\right)^2}\left(\frac{1}{2}\sum_{n=1}^{N}\pi(d_n - \cos\alpha)^2\left(\log\frac{1}{\epsilon} + \log\log\frac{1}{\epsilon}\right) + \mathbb{W}(a,d)\right) + o\left(\frac{1}{\left(\log\frac{1}{\epsilon}\right)^2}\right). \quad (11)$$

We now briefly discuss how the above expression comes about. Suppose that for a given $a \in A_N$, we study minimisers m of E_ϵ in $M(a,d)$. When ϵ is small, we expect to have a typical Néel wall profile near each of the points $a_1, \ldots, a_N$ with the prescribed signs $d_1, \ldots, d_N$, and the full transition layer m is essentially a superposition of all of these. As discussed previously, we can think of a Néel wall as consisting of two parts: a small core around a_n and two logarithmically decaying tails. In our situation, the walls are confined in the relatively short interval $(-1,1)$ and each tail will interact with the other walls and with the boundary as well. We can then account for the full energy $\inf_{M(a,d)} E_\epsilon$ (at leading and next-to-leading order) as follows.

Core energy. The core of each wall requires a certain amount of energy, namely

$$\frac{e(1)}{\left(\log\frac{1}{\delta}\right)^2} \quad \text{and} \quad \frac{e(-1)}{\left(\log\frac{1}{\delta}\right)^2}$$

[3] We will not use the fact that ϱ is a metric, but if we want to verify it, we can use that $\varrho(\Phi_d(b), \Phi_d(c)) = \varrho(b,c)$ for the Möbius transforms Φ_d defined in (26) below for every $d \in (-1,1)$. For the triangle inequality, it then suffices to show that $\varrho(c,0) \le \varrho(b,0) + \varrho(b,c)$ for $b,c \in (-1,1)$, which is not difficult.

for a positive and a negative wall, respectively. The constants $e(\pm 1)$ are given in Definition 26 below as limits of a rescaled energy of the core profile as $\epsilon \to 0$. Then the sum accounts for the term

$$\frac{\sum_{n=1}^{N} e(d_n)}{\left(\log \frac{1}{\delta}\right)^2}.$$

This is the only term where we have a contribution from the exchange energy and it appears only at next-to-leading order in the full energy. All the remaining terms below come from the magnetostatic energy alone.

Tail energy. The two tails of the wall at a_n give rise to the energy

$$\frac{\pi \gamma_n^2}{2 \log \frac{1}{\delta}},$$

leading to a total of

$$\frac{\pi \sum_{n=1}^{N} \gamma_n^2}{2 \log \frac{1}{\delta}}.$$

This is the leading order term of the full energy.

Tail-boundary interaction. Moving a wall relative to the boundary points ± 1 will deform the tail profile, resulting in a change of the energy. This phenomenon gives rise to the energy

$$\frac{\pi \gamma_n^2 \log(2 - 2a_n^2)}{2 \left(\log \frac{1}{\delta}\right)^2}$$

for the wall at a_n. Summing up these contributions, we obtain

$$\frac{\pi}{2 \left(\log \frac{1}{\delta}\right)^2} \sum_{n=1}^{N} \gamma_n^2 \log(2 - 2a_n^2).$$

(The sign here is not a mistake; it is the opposite of the sign of the corresponding expression in Theorem 2.) This means that the tails are attracted by the boundary, in the sense that the energy decreases if a_n approaches ± 1.

Tail-tail interaction. There is an energy contribution coming from reinforcement or cancellation between the stray fields generated by different walls. For the walls at a_k and a_n with $k \neq n$, this amounts to

$$\frac{\pi \gamma_k \gamma_n}{2 \left(\log \frac{1}{\delta}\right)^2} \log \left(\frac{1 + \sqrt{1 - \varrho(a_k, a_n)^2}}{\varrho(a_k, a_n)} \right).$$

The total contribution is

$$\frac{\pi}{2 \left(\log \frac{1}{\delta}\right)^2} \sum_{n=1}^{N} \sum_{k \neq n} \gamma_k \gamma_n \log \left(\frac{1 + \sqrt{1 - \varrho(a_k, a_n)^2}}{\varrho(a_k, a_n)} \right).$$

(Again we have the opposite sign relative to the above theorem.) A conclusion is that the tails of two walls attract each other if they have opposite signs and repel each other if they have the same sign.[4]

[4]This is because the function $t \mapsto \frac{1+\sqrt{1-t^2}}{t}$ is decreasing on $(0, 1)$.

Tail-core interaction. Since the profile of a Néel wall decays only logarithmically, it will change the turning angle of the neighbouring walls slightly. This has an effect on the energy as well (at the next-to-leading order). Indeed, the tail of the wall at a_k and the core of the wall at a_n with $k \neq n$ lead to a contribution of

$$-\frac{\pi\gamma_k\gamma_n}{\left(\log\frac{1}{\delta}\right)^2}\log\left(\frac{1+\sqrt{1-\varrho(a_k,a_n)^2}}{\varrho(a_k,a_n)}\right).$$

We also have an interaction between the two tails of a wall and its own core: if $k = n$, then we obtain the energy

$$-\frac{\pi\gamma_n^2\log(2-2a_n^2)}{\left(\log\frac{1}{\delta}\right)^2}.$$

This gives a total of

$$-\frac{\pi}{\left(\log\frac{1}{\delta}\right)^2}\sum_{n=1}^{N}\left(\gamma_n^2\log(2-2a_n^2)+\sum_{k\neq n}\gamma_k\gamma_n\log\left(\frac{1+\sqrt{1-\varrho(a_k,a_n)^2}}{\varrho(a_k,a_n)}\right)\right).$$

This is twice the size of the terms from the tail-boundary interaction and tail-tail interaction, but with the opposite signs, resulting in a net repulsion between walls of opposite signs and a net attraction between walls of the same sign. Furthermore, we have a net repulsion of the walls by the boundary.

Notwithstanding the term 'energy' used in this description, strictly speaking, these are energy differences and therefore some of them may be negative. All except one of these contributions occur similarly in the theory of Ginzburg-Landau vortices. The core-tail interaction, on the other hand, is new and more delicate to handle.

1.4 Physical relevance

Our result represents a rigorous proof of the physical prediction on the interaction energy between Néel walls. Indeed, Hubert and Schäfer ([12, Section 3.6. (C)]) predict the following behaviour in the case of a pair of Néel walls: *"The extended tails of Néel walls lead to strong interactions between them [...] The interactions become important as soon as the tail regions overlap. The sign of the interaction depends on the wall rotation sense. Néel walls of opposite rotation sense (so-called unwinding walls) attract each other because they generate opposite charges in their overlapping tails. If they are not pinned, they can annihilate. Néel walls of equal rotation sense (winding walls) repel each other."* (We recall that unwinding walls correspond—according to our definition in Section 1.1—to a pair of Néel walls with the same sign, while winding walls correspond to a pair of walls with the opposite signs as in Figure 1.)

1.5 Comparison with a linear model

If we replace the exchange energy by the simpler expression

$$\frac{\epsilon}{2}\int_{-1}^{1}(m_1')^2\,dx_1,$$

then the energy functional, now given by

$$\tilde{E}_\epsilon(m_1)=\frac{\epsilon}{2}\int_{-1}^{1}(m_1')^2\,dx_1+\frac{1}{2}\int_{\mathbb{R}^2_+}|\nabla U(m_1)|^2\,dx,\quad m_1:(-1,1)\to\mathbb{R},$$

becomes a quadratic form and the Euler-Lagrange equation for its critical points becomes linear. This functional has been used as a tool for studying the energy of Néel walls [9, 14]. Since the exchange energy in E_ϵ does not enter the expansion (10) at the leading order, we may expect good approximations from the linear model involving the energy functional $\tilde{E}_\epsilon$. The exchange energy does have an effect on the next-to-leading order term however, even though it is not through a direct contribution but rather by changing the core width of a domain wall. For the linear model, the core width of a domain wall is of order ϵ. Accordingly, for the functional $\tilde{E}_\epsilon$, the expansion that corresponds to (10) is of the form

$$\inf_{m \in M(a,d)} \tilde{E}_\epsilon(m_1) = \sum_{n=1}^{N} \frac{\pi(d_n - \cos\alpha)^2}{2\log\frac{1}{\epsilon}} + \frac{\tilde{\mathbb{W}}(a,d)}{\left(\log\frac{1}{\epsilon}\right)^2} + o\left(\frac{1}{\left(\log\frac{1}{\epsilon}\right)^2}\right)$$

as $\epsilon \searrow 0$. Here, $\tilde{\mathbb{W}}$ is nearly the same as the function $\mathbb{W}$ from Theorem 2, except that it may differ by a number depending only on N and d. That is, there exists a function $\tilde{e} : \{\pm 1\} \to \mathbb{R}$ such that

$$\tilde{\mathbb{W}}(a,d) \;=\; \sum_{n=1}^{N} \tilde{e}(d_n) \;-\; \frac{\pi}{2}\sum_{n=1}^{N} \gamma_n^2 \log(2 \;-\; 2a_n^2) \;-\; \frac{\pi}{2}\sum_{n=1}^{N}\sum_{k\neq n} \gamma_k\gamma_n \log\left(\frac{1+\sqrt{1-\varrho(a_k,a_n)^2}}{\varrho(a_k,a_n)}\right).$$

As for the full model, we may regard $\tilde{e}(\pm 1)$ as the core energy of a transition of sign ± 1. Our analysis does not give an explicit expression, but for variational principles where the number and signs of the Néel walls does not change, this part of the limiting energy is irrelevant.

The formula can be proved with the same arguments as in the proof of Theorem 2 below, although the linearity allows a few shortcuts. Therefore, we do not give a separate proof but leave it to the reader to make the necessary changes.

As a consequence, the linear model does not describe the interaction between Néel walls accurately, but the discrepancy is easily corrected by adjusting the core width (i.e., replacing ϵ by δ). Although we study only the energy of interacting Néel walls in this paper, the analogy to the theory of Ginzburg-Landau vortices (see Sect. 1.6) suggests that the same may be true for the dynamics of Néel walls. The simplified model may therefore be useful as a test case for future analysis, or, with the necessary care and the appropriate corrections, even be used for quantitative predictions.

1.6 Comparison with Ginzburg-Landau vortices

The interaction between topological singularities has been intensively studied in the last two decades in the context of Ginzburg-Landau problems. The work was pioneered by Bethuel, Brezis, and Hélein [2, 3], and an overview of later developments can be found in a book by Sandier and Serfaty [31]. These problems are designed to describe phenomena in superconductors and Bose-Einstein condensates, and a simple model that captures some of the main features is based on the functionals

$$G_\epsilon(f) = \int_\Omega \left(\frac{1}{2}|\nabla f|^2 + \frac{1}{4\epsilon^2}(1-|f|^2)^2\right) dx \tag{12}$$

for a domain $\Omega \subset \mathbb{R}^2$ and a function $f : \Omega \to \mathbb{R}^2$. We identify $\mathbb{R}^2$ with $\mathbb{C}$. Then in the limit $\epsilon \searrow 0$, the analysis in the aforementioned papers leads to a limiting function $f : \Omega \to \mathbb{C}$ of the form

$$f(z) = e^{i\theta(z)} \prod_{n=1}^{N} \left(\frac{z-a_n}{|z-a_n|}\right)^{d_n}$$

for certain points $a_1, \ldots, a_N \in \Omega$, integers $d_1, \ldots, d_N \in \mathbb{Z}\backslash\{0\}$, and a function $\theta : \Omega \to \mathbb{R}$. The renormalised energy

$$\liminf_{r \searrow 0} \left(\frac{1}{2}\int_{\Omega \backslash \bigcup_{n=1}^{N} B_r(a_n)} |\nabla f|^2\, dx - \pi \log\frac{1}{r} \sum_{n=1}^{N} d_n^2\right)$$

appears in a result similar to Theorem 2 (together with an additional term describing the core energy). Here $B_r(a)$ stands for the open ball of radius r and centre a. We have topological singularities at $a_1, \ldots, a_n$ with vortex structures and with topological degrees $d_1, \ldots, d_n$. These data are encoded in the distributional Jacobian

$$J(f) = \frac{1}{2}\operatorname{curl}(f^\perp \cdot \nabla f),$$

where $f^\perp = (-f_2, f_1)$.

The renormalised energy gives information about the vortex positions in equilibrium, but it is also important for their dynamics. Typically, if f evolves by a variational equation derived from G_ϵ, then on an appropriate time scale, the limiting motion law for the vortices (as $\epsilon \searrow 0$) is described by an analogous equation derived from the renormalised energy. This is true for gradient flows [21, 20, 17], Schrödinger type equations [5, 23], as well as nonlinear wave equations [22, 16].

It has been observed before that certain phenomena from micromagnetics give rise to similar models [11, 19, 18, 26, 27]. The connection to our model is less obvious, but can be seen once we show that under assumptions such as in Theorem 2, we obtain a limiting function from the rescaled stray field potential $(\log \frac{1}{\delta})U(m)$ of the form

$$u^*_{a,d}(x) = u_*(x) + \sum_{n=1}^N \gamma_n \left(\arctan\left(\frac{x_2}{x_1 - a_n}\right) - \frac{\pi(x_1 - a_n)}{2|x_1 - a_n|}\right)$$

for some $a \in A_N$ and $d \in \{\pm 1\}^N$ and a harmonic function $u_* : \mathbb{R}^2_+ \to \mathbb{R}$ that is smooth near $(-1, 1) \times \{0\}$ (see Sect. 2 for details). Examining $u^*_{a,d}$ near the point $(a_n, 0) \in \mathbb{R}^2$, we see that it behaves like the phase of a vortex in the upper half-plane, up to the constant γ_n. The expression

$$\liminf_{r \searrow 0} \left(\frac{1}{2}\int_{\mathbb{R}^2_+ \setminus \bigcup_{n=1}^N B_r(a_n,0)} |\nabla u^*_{a,d}|^2 \, dx - \frac{\pi}{2}\log\frac{1}{r}\sum_{n=1}^N \gamma_n^2\right)$$

also plays a role, although for our problem, it only accounts for a part of renormalised energy in Theorem 2 (even after adding the core energy). In fact, a Néel wall at a_n behaves like a vortex of "degree" γ_n in many respects, which is why the toolbox from the theory of Ginzburg-Landau vortices is very useful for the analysis.

There are, however, significant differences to Ginzburg-Landau vortices as well. A Néel wall is a two-length scale object, comprising a core and two tails, each with its own characteristic length. In contrast, in the standard Ginzburg-Landau problem, a vortex has a single length scale characterising its core and the renormalised energy between the vortices comes essentially from the interaction of its out-of-core structure. For Néel walls, we have a renormalised energy consisting of two parts. The interaction between the tails of two walls is similar to the interaction between Ginzburg-Landau vortices and gives rise to the above expression. But in addition, we have an interaction between the core of one wall and the tail of another, which is a novel feature for Ginzburg-Landau type systems. This interaction is responsible for the fact that we have attraction for walls of the same sign and repulsion in the case of opposite signs, whereas for Ginzburg-Landau vortices, we have attraction for degrees of opposite signs and repulsion for degrees of the same sign. Finally, our "degree" γ_n is not quantised in the same way as the degree of Ginzburg-Landau vortices. It does take only two values $(\pm 1 - \cos\alpha)$, but these depend on the choice of the angle α and are not topological invariants. As a consequence, the Jacobian becomes a much less powerful tool. To overcome these difficulties, we use duality arguments and "logarithmically failing" interpolation inequalities (see [7, 15]), Γ-convergence methods, and refined elliptic estimates.

In our model, the magnetostatic energy, being of order $O(1/|\log\delta|)$, dominates the higher order exchange energy (of order $O(1/(\log\delta)^2)$ for small values of ϵ. This is in contrast to most Ginzburg-Landau systems, where the highest order term is dominant. This is the case for the functionals

G_ϵ in (12), but also for similar models coming from micromagnetics. For example, the model for boundary vortices studied by Kurzke [18] contains a term coming from the exchange energy and one coming from the magnetostatic energy as well, but in the analysis, the roles of the two are reversed relative to the model for Néel walls.

1.7 Notation

We now introduce some notation that we will use frequently throughout this paper. As mentioned previously, we define $\delta = \epsilon \log \frac{1}{\epsilon}$. This is the scale of the Néel walls' typical core width.

If $x_0 \in \mathbb{R}^2$ and $r > 0$, then we write $B_r(x_0)$ for the open ball in $\mathbb{R}^2$ of radius r centred at x_0. Furthermore, we write $B_r^+(x_0) = B_r(x_0) \cap \mathbb{R}^2_+$. For a set $S \subset \mathbb{R}^2$, we also use the notation $\partial^+ S = \partial S \cap \mathbb{R}^2_+$.

For a vector $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we write $\xi^\perp = (-\xi_2, \xi_1)$. If $f : \Omega \to \mathbb{R}$ is a function on a domain $\Omega \subset \mathbb{R}^2$ with gradient $\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2})$, then $\nabla^\perp f = (-\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1})$.

For $a \in A_N$, we define

$$\rho(a) = \frac{1}{2} \min\{2a_1 + 2, a_2 - a_1, \ldots, a_N - a_{N-1}, 2 - 2a_N\}.$$

Thus this is a quantity that controls the distance between two points of a and the distance to the boundary. For $r > 0$, also define

$$B_r^*(a) = \bigcup_{n=1}^N B_r^+(a_n, 0)$$

and

$$\Omega_r(a) = \mathbb{R}^2_+ \backslash B_r^*(a).$$

Given a function $f : \mathbb{R}^2_+ \to \mathbb{R}$ with a well-defined trace on $(-1,1) \times \{0\}$, we often use the shorthand notation f' for $\frac{\partial f}{\partial x_1}(\,\cdot\,, 0)$ and

$$\int_{-1}^1 f \, dx_1 = \int_{-1}^1 f(x_1, 0) \, dx_1.$$

In addition to the space $\dot{W}^{1,2}(\mathbb{R}^2_+)$ introduced in Sect. 1.1, we also define $\dot{W}^{1,2}_*(\mathbb{R}^2_+; a)$ for $a \in A_N$, which is the space of all $u \in \bigcap_{r>0} \dot{W}^{1,2}(\Omega_r(a))$ such that

$$\sup_{r>0} \frac{\|\nabla u\|^2_{L^2(\Omega_r(a))}}{\log\left(\frac{1}{r} + 2\right)} < \infty.$$

If $b \in (-1,1)$ is a single point, then $\dot{W}^{1,2}_*(\mathbb{R}^2_+; b)$ is defined similarly.

2 The renormalised energy

When we study minimisers of E_ϵ in $M(a,d)$ and let ϵ tend to 0, then we expect the suitably rescaled stray field potential to converge to a harmonic function with specific boundary data depending on a and d. We compute this function here, in order to obtain the limiting magnetostatic energy. This corresponds to the sum of the tail self-energies and the contributions of the core-tail, tail-tail, and tail-boundary interactions. As a side product, we will also obtain information about the expected logarithmic profile of the Néel walls.

2.1 The limiting rescaled stray-field potential

Fix $a \in A_N$ and $d \in \{\pm 1\}^N$. We recall that

$$\gamma_n = d_n - \cos\alpha \quad \text{for } n = 1, \ldots, N$$

and we denote

$$\sigma_n = \frac{\pi}{2}\left(\sum_{k=n+1}^{N} \gamma_k - \sum_{k=1}^{n} \gamma_k\right), \quad n = 0, \ldots, N.$$

We look for a solution of the following boundary value problem:

$$\Delta u^*_{a,d} = 0 \quad \text{in } \mathbb{R}^2_+, \tag{13}$$

$$u^*_{a,d} = \sigma_0 \quad \text{on } (-1, a_1) \times \{0\}, \tag{14}$$

$$u^*_{a,d} = \sigma_n \quad \text{on } (a_n, a_{n+1}) \text{ for } n = 1, \ldots, N-1, \tag{15}$$

$$u^*_{a,d} = \sigma_N \quad \text{on } (a_N, 1) \times \{0\}, \tag{16}$$

$$\frac{\partial u^*_{a,d}}{\partial x_2} = 0 \quad \text{on } (-\infty, -1) \times \{0\} \text{ and on } (1, \infty) \times \{0\}. \tag{17}$$

The boundary data are chosen so that we have a jump of size $-\pi\gamma_n$ at a_n for $n = 1, \ldots, N$. We also require that $u^*_{a,d} \in \dot{W}^{1,2}_*(\mathbb{R}^2_+; a)$ so that the boundary conditions make sense. However, the Dirichlet energy cannot be finite on all of $\mathbb{R}^2_+$, since any solution $u^*_{a,d}$ will behave similarly to the function[5]

$$\sum_{n=1}^{N} \gamma_n \left(\arctan\left(\frac{x_2}{x_1 - a_n}\right) - \frac{\pi(x_1 - a_n)}{2|x_1 - a_n|}\right)$$

near the boundary $(-1, 1) \times \{0\}$ (see Proposition 5 below). But we can compensate by considering the expression

$$E^*_{a,d}(u^*_{a,d}) = \frac{1}{2} \liminf_{r \searrow 0} \left(\int_{\Omega_r(a)} |\nabla u^*_{a,d}|^2 \, dx - \pi \log \frac{1}{r} \sum_{n=1}^{N} \gamma_n^2\right), \tag{18}$$

and this will be part of the limiting energy. We will see that the unique solution $u^*_{a,d}$ of (13)–(17) is the minimizer of $E^*_{a,d}$ and corresponds to the limit as $\epsilon \searrow 0$ of the rescaled stray field potential $(\log \frac{1}{\delta})U(m_\epsilon)$ associated to the magnetisation m_ϵ with walls at $a_1, \ldots, a_N$ with prescribed signs $d_1, \ldots, d_N$ (see Proposition 5 and Theorem 22).

In order to solve (13)–(17), we first study the simpler problem

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2_+, \tag{19}$$

$$u = \frac{\pi}{2} \quad \text{on } (-1, 0) \times \{0\}, \tag{20}$$

$$u = -\frac{\pi}{2} \quad \text{on } (0, 1) \times \{0\}, \tag{21}$$

$$\frac{\partial u}{\partial x_2} = 0 \quad \text{on } (-\infty, -1) \times \{0\} \text{ and } (1, \infty) \times \{0\}. \tag{22}$$

We can obtain a solution $u^*_{a,d}$ of (13)–(17) by a linear combination of solutions to problems of the type (19)–(22) (see Proposition 5).

[5] This function satisfies (13)–(16), but not (17). It does not belong to $\dot{W}^{1,2}_*(\mathbb{R}^2_+; a)$ due to its behaviour at ∞.

We first construct an explicit solution, using the fact that harmonic functions remain harmonic upon precomposition with a conformal map. We identify $\mathbb{R}^2$ with $\mathbb{C}$. Consider the following domain in the complex plane $\mathbb{C}$:

$$S = \{w_1 + iw_2 \in \mathbb{C} \colon w_1 > 0 \text{ and } 0 < w_2 < \pi\}.$$

Also consider the conformal map $F : S \to \mathbb{R}^2_+$ with

$$F(w) = -\frac{1}{\cosh w}, \quad w \in S.$$

Extend F continuously to the boundary $\partial S \setminus \{\frac{\pi i}{2}\}$. Assuming that u solves (19)–(22), set

$$\hat{u} = u \circ F.$$

Then $\hat{u}$ solves the boundary value problem

$$\begin{aligned} \Delta \hat{u} &= 0 && \text{in } S, \\ \hat{u} &= \frac{\pi}{2} && \text{on } (0,\infty), \\ \hat{u} &= -\frac{\pi}{2} && \text{on } \pi i + (0,\infty), \\ \frac{\partial \hat{u}}{\partial x_1} &= 0 && \text{on } i(0,\pi). \end{aligned}$$

This problem has an obvious solution,

$$\hat{u}(w) = \frac{\pi}{2} - \operatorname{Im} w,$$

with $\operatorname{Im} w = w_2$ for $w = w_1 + iw_2$. Thus we obtain a solution of (19)–(22) by

$$u(z) = \hat{u}(F^{-1}(z)) = \frac{\pi}{2} - \operatorname{Im} F^{-1}(z), \quad z \in \mathbb{R}^2_+,$$

which can be written as

$$u = \operatorname{Im} f \quad \text{for} \quad f(z) = \frac{\pi i}{2} - F^{-1}(z), \quad z \in \mathbb{R}^2_+, \tag{23}$$

where f is an holomorphic function. Since $\lim_{|z|\to\infty} F^{-1}(z) = \frac{\pi i}{2}$, we conclude that $\lim_{|x|\to\infty} u(x) = 0$.

Proposition 3. *The function u from (23) is the unique solution of (19)–(22) in $\dot{W}^{1,2}_*(\mathbb{R}^2_+;0)$. It satisfies $|u| \leq \frac{\pi}{2}$ in $\mathbb{R}^2_+$ and $\lim_{|x|\to\infty} u(x) = 0$. Moreover, it is odd in x_1, that is, $u(x_1,x_2) = -u(-x_1,x_2)$ for every $x \in \mathbb{R}^2_+$. Furthermore, there exists a constant C such that*

$$\left| \nabla u(x) - \frac{x^\perp}{|x|^2} \right| \leq C|x| \tag{24}$$

for all $x \in B^+_{1/2}(0)$.

Proof. It is clear from the construction that $|u| \leq \frac{\pi}{2}$ and $\lim_{|x|\to\infty} u(x) = 0$. Since the Dirichlet energy is conformally invariant, we have

$$\int_{R^2_+\setminus B_r(0)} |\nabla u|^2 \, dz = \int_{F^{-1}(R^2_+\setminus B_r(0))} |\nabla \hat{u}|^2 \, dw.$$

Note that

$$
\begin{aligned}
|\cosh w|^2 &= \frac{1}{4}\left(e^{w}+e^{-w}\right)\left(e^{\bar{w}}+e^{-\bar{w}}\right) \\
&= \frac{1}{4}\left(e^{2\operatorname{Re} w}+e^{-2\operatorname{Re} w}+e^{2i\operatorname{Im} w}+e^{-2i\operatorname{Im} w}\right) \\
&= \frac{1}{2}\cosh(2\operatorname{Re} w)+\frac{1}{2}\cos(2\operatorname{Im} w) \quad \text{ for } w\in\mathbb{C}.
\end{aligned} \tag{25}
$$

Thus

$$
F^{-1}(R^2_+\backslash B_r(0)) \subset \left\{ w\in S\colon\ \operatorname{Re} w < \frac{1}{2}\operatorname{arcosh}\left(\frac{2}{r^2}+1\right)\right\}.
$$

It follows that

$$
\int_{\mathbb{R}^2_+\backslash B_r(0)} |\nabla u|^2\, dz \le \frac{\pi}{2}\operatorname{arcosh}\left(\frac{2}{r^2}+1\right).
$$

In particular, we have $u\in \dot{W}^{1,2}_*(\mathbb{R}^2_+;0)$.

If $\tilde{u}\in \dot{W}^{1,2}_*(\mathbb{R}^2_+;0)$ is another solution of (19)–(22), then the difference $v=u-\tilde{u}$ belongs to $\dot{W}^{1,2}_*(\mathbb{R}^2_+;0)$ and thus to $W^{1,p}(B_1^+(0))\cap \dot{W}^{1,2}(\mathbb{R}^2_+\backslash B_{1/2}(0))$ for any $p\in[1,2)$. Furthermore, it is harmonic in $\mathbb{R}^2_+$ and satisfies $v=0$ on $(-1,1)\times\{0\}$ and $\frac{\partial v}{\partial x_2}=0$ on $(-\infty,-1)\times\{0\}$ and on $(1,\infty)\times\{0\}$. Standard regularity theory then implies that $v\in \dot{W}^{1,2}(\mathbb{R}^2_+)$, and with an integration by parts, we obtain

$$
\int_{\mathbb{R}^2_+} |\nabla v|^2\, dx = 0.
$$

Hence u is the unique solution of (19)–(22) in $\dot{W}^{1,2}_*(\mathbb{R}^2_+;0)$.

The odd symmetry of u is a consequence of the uniqueness, as $x\mapsto -u(-x_1,x_2)$ is another solution of the boundary value problem.

Finally, we consider the function

$$
\tilde{v}(x) = u(x) - \arctan\left(\frac{x_2}{x_1}\right) + \frac{\pi x_1}{2|x_1|},
$$

which is harmonic in $\mathbb{R}^2_+$ as well with $\tilde{v}=0$ on $(-1,1)\times\{0\}$. Invoking standard regularity theory again, we conclude that $\tilde{v}\in C^\infty(\overline{B^+_{1/2}(0)})$. By the odd symmetry, we have $\nabla\tilde{v}(0)=0$. Hence we obtain inequality (24). □

Definition 4. *For every $b\in(-1,1)$, we introduce the Möbius transform*

$$
\Phi_b(z) = \frac{z+b}{1+bz}, \quad z\in\mathbb{C}, \tag{26}
$$

and its inverse $\Phi_b^{-1}=\Phi_{-b}$. We also define $u_b:\mathbb{R}^2_+\to\mathbb{R}$ by

$$
u_b = u\circ\Phi_{-b} \quad \text{in} \quad \mathbb{R}^2_+, \tag{27}
$$

which, by Proposition 3, is the unique solution of the boundary value problem

$$
\begin{aligned}
\Delta u_b &= 0 && \text{in } \mathbb{R}^2_+,\\
u_b &= \frac{\pi}{2} && \text{on } (-1,b)\times\{0\},\\
u_b &= -\frac{\pi}{2} && \text{on } (b,1)\times\{0\},\\
\frac{\partial u_b}{\partial x_2} &= 0 && \text{on } (-\infty,-1)\times\{0\} \text{ and on } (1,\infty)\times\{0\},
\end{aligned}
$$

in the space $\dot{W}^{1,2}_*(\mathbb{R}^2_+;b)$.

Now we can also construct a solution of (13)–(17) by superposition.

Proposition 5. *The function* $u^*_{a,d}$*, defined by*

$$u^*_{a,d} = \sum_{n=1}^{N} \gamma_n u_{a_n}, \tag{28}$$

is the unique minimiser of $E^*_{a,d}$ *among all* $u \in \dot{W}^{1,2}_*(\mathbb{R}^2_+; a)$ *satisfying* (14)–(16)*. Moreover,* $u^*_{a,d}$ *is the unique solution of* (13)–(17) *in* $\dot{W}^{1,2}_*(\mathbb{R}^2_+; a)$*.*

Proof. By Proposition 3, we know that $u^*_{a,d} \in \dot{W}^{1,2}_*(\mathbb{R}^2_+; a)$, and it satisfies (14)–(16). Let u be another function with these properties. Since $u^*_{a,d}$ is harmonic in $\mathbb{R}^2_+$, integration by parts leads to

$$\begin{aligned}\int_{\Omega_r(a)} \left|\nabla u - \nabla u^*_{a,d}\right|^2 dx &= \int_{\Omega_r(a)} |\nabla u|^2 dx - \int_{\Omega_r(a)} |\nabla u^*_{a,d}|^2 dx - 2\int_{\Omega_r(a)} \left(\nabla u - \nabla u^*_{a,d}\right) \cdot \nabla u^*_{a,d}\, dx \\ &= \int_{\Omega_r(a)} |\nabla u|^2 dx - \int_{\Omega_r(a)} |\nabla u^*_{a,d}|^2 dx - 2\int_{\partial\Omega_r(a)} \left(u - u^*_{a,d}\right) \nu \cdot \nabla u^*_{a,d}\, d\sigma.\end{aligned}$$

Using (24), we find that there exists a constant C such that for any $n = 1, \ldots, N$ and for $x \in B^+_{\rho(a)}(a_n, 0)$,

$$\left|\frac{\partial u^*_{a,d}}{\partial x_1}(x) + \frac{\gamma_n x_2}{(x_1 - a_n)^2 + x_2^2}\right| + \left|\frac{\partial u^*_{a,d}}{\partial x_2}(x) - \frac{\gamma_n (x_1 - a_n)}{(x_1 - a_n)^2 + x_2^2}\right| \leq C.$$

Hence the boundary integral will tend to 0 when we let $r \searrow 0$. Therefore,

$$E^*_{a,d}(u^*_{a,d}) = E^*_{a,d}(u) - \frac{1}{2}\lim_{r \searrow 0} \int_{\Omega_r(a)} \left|\nabla u - \nabla u^*_{a,d}\right|^2 dx.$$

The limit on the right hand side exists, because the quantity is monotone in r.

This implies that $u^*_{a,d}$ is the unique minimizer of $E^*_{a,d}$. The uniqueness of solutions of (13)–(17) follows with the same arguments as for Proposition 3. □

Remark 6. *Since we have an explicit representation of* $u^*_{a,d}$*, an easy computation shows that the liminf in the definition of* $E^*_{a,d}(u^*_{a,d})$ *is in fact a limit. By the preceding computation, the same holds for* $E^*_{a,d}(u)$ *for any* $u \in \dot{W}^{1,2}_*(\mathbb{R}^2_+; a)$ *satisfying* (14)–(16)*.*

2.2 The energy of the limiting rescaled stray field

Next we want to compute the energy $E^*_{a,d}(u^*_{a,d})$ defined in (18). Since this depends only on a and d, we use the abbreviation

$$W_1(a, d) = E^*_{a,d}(u^*_{a,d}).$$

This quantity corresponds to the tail-boundary and tail-tail interaction energy.

Proposition 7. *If* ϱ *is the metric on* $(-1, 1)$ *defined in Section 1.3, then*

$$W_1(a, d) = \frac{\pi}{2}\sum_{n=1}^{N} \gamma_n^2 \log(2 - 2a_n^2) + \frac{\pi}{2}\sum_{k \neq n} \gamma_k \gamma_n \log\left(\frac{1 + \sqrt{1 - \varrho(a_k, a_n)^2}}{\varrho(a_k, a_n)}\right). \tag{29}$$

For the proof, we need the following two lemmas. First we compute the rescaled tail-tail interaction energy for two Néel walls located at two points $b \neq c$.

Lemma 8. *For all $b, c \in (-1, 1)$ with $b \neq c$, if u_b and u_c are defined by (27), then*

$$\int_{\mathbb{R}^2_+} \nabla u_b \cdot \nabla u_c \, dx = \pi \log\left(\frac{1 + \sqrt{1 - \varrho(b,c)^2}}{\varrho(b,c)}\right). \tag{30}$$

Then we compute the rescaled tail self-energy together with the tail-boundary interaction of a Néel wall located at a point $b \in (-1, 1)$.

Lemma 9. *There exists a constant $C > 0$ such that for every $b \in (-1, 1)$ and $r \in (0, 1 - |b|)$,*

$$\left| \int_{\mathbb{R}^2_+ \setminus B_r(b)} |\nabla u_b|^2 \, dx - \pi \log\left(\frac{2 - 2b^2}{r}\right) \right| \leq C \left(\frac{|b| r}{1 - b^2} + \frac{r^2}{(1 - b^2 - |b| r)^2} \right). \tag{31}$$

In particular,

$$\lim_{r \searrow 0} \left(\int_{\mathbb{R}^2_+ \setminus B_r(b)} |\nabla u_b|^2 \, dx - \pi \log \frac{1}{r} \right) = \pi \log(2 - 2b^2).$$

Proof of Lemma 8. Using the definition of the Möbius transform Φ_b, it is easy to check that

$$\Phi_b \circ \Phi_c = \Phi_{\frac{b+c}{1+bc}}.$$

Set $q = \frac{b-c}{1-bc}$. Then $|q| = \varrho(b, c)$ and

$$\begin{aligned}
\int_{\mathbb{R}^2_+} \nabla u_b \cdot \nabla u_c \, dx &= \int_{\mathbb{R}^2_+} \nabla u \cdot \nabla (u_c \circ \Phi_b) \, dx \\
&= \int_{\mathbb{R}^2_+} \nabla u \cdot \nabla u_q \, dx \\
&= -\operatorname{PV} \int_{-1}^{1} u_q \frac{\partial u}{\partial x_2} \, dx_1 \\
&= -\frac{\pi}{2} \operatorname{PV} \int_{-1}^{q} \frac{\partial u}{\partial x_2} \, dx_1 + \frac{\pi}{2} \operatorname{PV} \int_{q}^{1} \frac{\partial u}{\partial x_2} \, dx_1.
\end{aligned}$$

In order to determine $\frac{\partial u}{\partial x_2}$, we recall that the function f defined in (23) is holomorphic in $\mathbb{R}^2_+$. Hence we have

$$f'(z) = \frac{\partial u}{\partial x_2} + i \frac{\partial u}{\partial x_1}, \quad z \in \mathbb{R}^2_+.$$

In particular,

$$\frac{\partial u}{\partial x_2} = \operatorname{Re} f'(z), \quad z \in \mathbb{R}^2_+.$$

Differentiating both sides of the equation

$$\frac{\pi i}{2} - w = f(F(w)),$$

we calculate

$$f'(F(w)) = -\frac{1}{F'(w)} = -\frac{\cosh^2 w}{\sinh w}.$$

Let $t \in (0, 1)$ and $s = \operatorname{arcosh} \frac{1}{t}$. Set $w = s + i\pi$, so that $F(w) = t$. Then $\cosh w = -\frac{1}{t}$ and $\sinh w = -\sinh s = -\sqrt{t^{-2} - 1}$. Hence

$$f'(t) = \frac{1}{t\sqrt{1 - t^2}} = \frac{d}{dt}\left(\log|t| - \log\left(1 + \sqrt{1 - t^2}\right)\right).$$

That is,

$$\frac{\partial u}{\partial x_2}(x_1,0) = \frac{d}{dx_1}\left(\log|x_1| - \log\left(1+\sqrt{1-x_1^2}\right)\right), \quad x_1 \in (0,1). \tag{32}$$

By the odd symmetry of u in x_1 (see Proposition 3), we have the same equality on $(-1,0)\times\{0\}$. Thus

$$\mathrm{PV}\int_{-1}^{q} \frac{\partial u}{\partial x_2}\,dx_1 = \log|q| - \log\left(1+\sqrt{1-q^2}\right)$$

and

$$\mathrm{PV}\int_{q}^{1} \frac{\partial u}{\partial x_2}\,dx_1 = -\log|q| + \log\left(1+\sqrt{1-q^2}\right),$$

where the principal value can be ignored for exactly one of these integrals because there is no singularity. It follows that (30) holds. ☐

Proof of Lemma 9. Let $r\in(0,1)$ and consider the integral

$$I_r = \int_{\mathbb{R}^2_+\setminus B_r(0)} |\nabla u|^2\,dx.$$

In order to estimate I_r, we first study the image of $\mathbb{R}^2_+\setminus B_r(0)$ under F^{-1} and then perform the change of variables $x = F(w) = -1/\cosh(w)$. Recall identity (25), which implies that

$$\left\{w\in S\colon\ \operatorname{Re} w < \frac{1}{2}\operatorname{arcosh}\left(\frac{2}{r^2}-1\right)\right\} \subset F^{-1}(\mathbb{R}^2_+\setminus B_r(0)) \subset \left\{w\in S\colon\ \operatorname{Re} w < \frac{1}{2}\operatorname{arcosh}\left(\frac{2}{r^2}+1\right)\right\},$$

where S is the domain defined on page 13. Recalling the function $\hat{u}(w) = u(x)$ for $x = F(w)$, we see that

$$I_r = \int_{F^{-1}(\mathbb{R}^2_+\setminus B_r(0))} |\nabla \hat{u}|^2\,dw \in \left[\frac{\pi}{2}\operatorname{arcosh}\left(\frac{2}{r^2}-1\right), \frac{\pi}{2}\operatorname{arcosh}\left(\frac{2}{r^2}+1\right)\right].$$

Thus

$$\frac{4}{r^2} - 2 \le e^{2I_r/\pi} + e^{-2I_r/\pi} \le \frac{4}{r^2}+2.$$

As $0 < e^{-2I_r/\pi} < 1$, this means that

$$\frac{4}{r^2} - 3 \le e^{2I_r/\pi} \le \frac{4}{r^2}+2$$

and

$$\frac{\pi}{2}\log\left(\frac{4}{r^2}-3\right) \le I_r \le \frac{\pi}{2}\log\left(\frac{4}{r^2}+2\right).$$

In particular, there exists a universal constant $C_1>0$ such that

$$\left|\int_{\mathbb{R}^2_+\setminus B_r(0)} |\nabla u|^2\,dx - \pi\log\frac{2}{r}\right| \le C_1 r^2, \quad r\in(0,1). \tag{33}$$

For $b\in(-1,1)$ and $r\in(0,1-|b|)$, we have

$$\int_{\mathbb{R}^2_+\setminus B_r(b)} |\nabla u_b|^2\,dx = \int_{\mathbb{R}^2_+\setminus \Phi_{-b}(B_r^+(b))} |\nabla u|^2\,dx.$$

Thus we now examine the set $\Phi_{-b}(B_r^+(b))$. Since Φ_{-b} is a Möbius transform, it maps the semicircle $\partial^+ B_r(b)$ to a semicircle, which contains the points

$$\Phi_{-b}(b-r) = -\frac{r}{1-b^2+br} < 0 \quad \text{and} \quad \Phi_{-b}(b+r) = \frac{r}{1-b^2-br} > 0.$$

Since $\Phi_{-b}(b) = 0$, we obtain

$$B^+_{r/(1-b^2+|b|r)}(0) \subset \Phi_{-b}(B_r^+(b)) \subset B^+_{r/(1-b^2-|b|r)}(0).$$

It then follows from (33) that

$$\begin{aligned}\pi \log \frac{2}{r} + \pi \log(1-b^2-|b|r) - \frac{C_1 r^2}{(1-b^2-|b|r)^2} \\ \leq \int_{\mathbb{R}^2_+ \setminus B_r(b)} |\nabla u_b|^2 \, dx \\ \leq \pi \log \frac{2}{r} + \pi \log(1-b^2+|b|r) + \frac{C_1 r^2}{(1-b^2-|b|r)^2}.\end{aligned}$$

Hence (31) holds. □

Proof of Proposition 7. The formula follows directly from the definition of W_1, the definition (18) of $E^*_{a,d}$, Proposition 5, and the last two lemmas. □

2.3 The rescaled tail profile

Since $u^*_{a,d}$ is the expected limit of the rescaled stray field potential, assuming that condition (4) will be preserved in the limit, we can derive an expected profile $\mu^*_{a,d}$ for the rescaled first component of the tails of the Néel walls.

Using the unique solution u of the problem (19)–(22), we first define the logarithmically rescaled tail profiles

$$\mu(x_1) := -\operatorname{PV}\int_{-1}^{x_1} \frac{\partial u}{\partial x_2}(t,0)\,dt \overset{(32)}{=} \log\left(1+\sqrt{1-x_1^2}\right) - \log|x_1|, \quad x_1 \in (-1,1)\setminus\{0\}, \tag{34}$$

and

$$\mu_b(x_1) := \mu \circ \Phi_{-b}(x_1) = \mu\left(\frac{x_1-b}{1-bx_1}\right), \quad x_1 \in (-1,1)\setminus\{b\}. \tag{35}$$

Then for $u_b = u \circ \Phi_{-b}$, we find $\frac{\partial u_b}{\partial x_2} = \frac{\partial u}{\partial x_2}\Phi'_{-b} = -\mu_b'$ on $(-1,b)\times\{0\}$ and $(b,1)\times\{0\}$, using the conformality of Φ_{-b} and the fact that Φ'_{-b} is real on $(-1,b)\times\{0\}$ and on $(b,1)\times\{0\}$. If we define

$$\mu^*_{a,d} = \sum_{n=1}^{N} \gamma_n \mu_{a_n}, \quad x_1 \in (-1,1)\setminus\{a_1,\ldots,a_N\},$$

then we also have

$$\frac{\partial u^*_{a,d}}{\partial x_2} = -(\mu^*_{a,d})' \text{ on } (-1,1)\times\{0\} \text{ except at the singularities } a_k,\ k=1,\ldots,N.$$

Note that

$$\mu_{a_n}(a_k) = \mu_{a_k}(a_n) = \log\left(\frac{1+\sqrt{1-\varrho(a_n,a_k)^2}}{\varrho(a_n,a_k)}\right), \quad n \neq k.$$

We need to examine the behaviour of $\mu^*_{a,d}$ near the points a_n, as this will be important for determining the energy of the tail-core interaction.

Proposition 10. *Set*

$$\lambda_n = \gamma_n \log(2 - 2a_n^2) + \sum_{k \neq n} \gamma_k \mu_{a_k}(a_n)$$

for $n = 1, \ldots, N$*. Then there exists a constant* $C = C(a) > 0$ *such that*

$$\left| \mu_{a,d}^*(a_n \pm r) - \lambda_n - \gamma_n \log \frac{1}{r} \right| \leq Cr \tag{36}$$

for any $r \in (0, \rho(a)]$ *and any* $n = 1, \ldots, N$*.*

Proof. First we study μ_b again for a fixed b. Obviously, for $c \in (-1, 1)$ with $b \neq c$, this function is smooth at c, and therefore there exists a constant $C_1 > 0$, depending only on b and c, such that

$$|\mu_b(c \pm r) - \mu_b(c)| \leq C_1 r$$

for any sufficiently small $r > 0$. We also have, by (34) and (35), the formula

$$\mu_b(b \pm r) = \log\left(1 + \sqrt{1 - \frac{r^2}{(1 - b^2 \mp br)^2}}\right) + \log(1 - b^2 \mp br) + \log \frac{1}{r}.$$

Thus there exists a constant $C_2 = C_2(b)$ such that

$$\left| \mu_b(b \pm r) - \log(2 - 2b^2) - \log \frac{1}{r} \right| \leq C_2 r$$

for $r \in (0, 1 - |b|)$. If we sum up for $b = a_k$, $k = 1, \ldots, N$, then the conclusion follows. $\square$

We will prove in Section 6.2 that the function

$$\cos\alpha + \frac{\mu_{a,d}^*}{\log \frac{1}{\delta}}$$

gives an approximation for the profile of the first component $m_{1,\epsilon}$ for a minimiser m_ϵ of E_ϵ over $M(a, d)$. In a standard Néel wall with rotation angle 2α (or $2\pi - 2\alpha$), the function $m_{1,\epsilon}$ would be required to make a transition from $\cos\alpha$ to 1 (or -1) and back. If we superimpose several Néel walls, then this is no longer true. Instead, in an interval $(a_n - r, a_n + r)$, up to a small error, $m_{1,\epsilon}$ is now required, by (36), to make a transition from

$$\cos\alpha + \frac{\lambda_n + \gamma_n \log \frac{1}{r}}{\log \frac{1}{\delta}}$$

to ± 1 and back. We discount the term involving $\log \frac{1}{r}$, as it will be cancelled by a similar term elsewhere. The contribution of the remaining term to the energy (to leading order, rescaled by $(\log \delta)^2$) is then $-\pi \gamma_n \lambda_n$ near the point a_n (see Remark 19 below). Thus in total, we have a correction term of the form

$$W_2(a, d) = -\pi \sum_{n=1}^N \gamma_n \lambda_n = -\pi \sum_{n=1}^N \left(\gamma_n^2 \log(2 - 2a_n^2) + \sum_{k \neq n} \gamma_n \gamma_k \mu_{a_k}(a_n) \right).$$

Using the above explicit expression for $\mu_{a_k}(a_n)$, we find

$$W_2(a, d) = -\pi \sum_{n=1}^N \gamma_n^2 \log(2 - 2a_n^2) - \pi \sum_{n=1}^N \sum_{k \neq n} \gamma_k \gamma_n \log\left(\frac{1 + \sqrt{1 - \varrho(a_k, a_n)^2}}{\varrho(a_k, a_n)} \right).$$

This corresponds to the sum of the tail-core interactions described in Section 1.3. Note that

$$W_2(a,d) = -2W_1(a,d).$$

We finally define $W(a,d) := W_1(a,d) + W_2(a,d)$. That is,

$$W(a,d) = -\frac{\pi}{2}\sum_{n=1}^{N}\gamma_n^2\log(2-2a_n^2) - \frac{\pi}{2}\sum_{n=1}^{N}\sum_{k\neq n}\gamma_k\gamma_n\log\left(\frac{1+\sqrt{1-\varrho(a_k,a_n)^2}}{\varrho(a_k,a_n)}\right).$$

3 The Euler-Lagrange equation

Since we study the number $\inf_{M(a,d)} E_\epsilon$ for a given $a \in A_N$ and $d \in \{\pm 1\}^N$, it is useful to consider minimisers of E_ϵ in $M(a,d)$ (see Proposition 1). The problem gives rise to an Euler-Lagrange equation, which is most easily expressed in terms of the continuous function $\varphi : (-1,1) \to \mathbb{R}$ with $\varphi(-1) = \alpha$ and $m = (\cos\varphi, \sin\varphi) \in W^{1,2}((-1,1),\mathbb{S}^1)$. Let $u = U(m)$ be the function defined on page 3. Then the equation is

$$\epsilon\varphi''(x_1) = \frac{\partial u}{\partial x_1}(x_1,0)\sin\varphi(x_1) \quad \text{for } x_1 \in (-1,1)\backslash\{a_1,\ldots,a_N\}. \tag{37}$$

For the derivation of (37), let $\zeta \in C_0^\infty(-1,1)$ with $\zeta(a_k) = 0$ for $k = 1,\ldots,N$ be an infinitesimal variation of m_1. Then

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\left(\frac{1}{2}\int_{\mathbb{R}^2_+}|\nabla U(m_1+t\zeta)|^2\,dx\right) &= \int_{\mathbb{R}^2_+}\nabla U(m_1)\cdot\nabla U(\zeta)\,dx \\ &\overset{(6)}{=} \int_{-1}^{1}\zeta' U(m_1)\,dx_1 = -\int_{-1}^{1}\zeta\frac{\partial u}{\partial x_1}\,dx_1.\end{aligned}$$

Equation (37) is now obtained with the usual computations.

We use the abbreviation u' for the derivative of the trace of u with respect to $(-1,1)\times\{0\}$. Then we have the following shorthand form of (37):

$$\epsilon\varphi'' = u'\sin\varphi \quad \text{on } (-1,1)\backslash\{a_1,\ldots,a_N\}.$$

We now analyse this equation. More precisely, we prove an interior $W^{2,2}$-estimate for solutions of (37) and we prove a Pohozaev identity. As a consequence, we eventually find that the exchange energy in the core of a Néel wall is of order $O(1/(\log\delta)^2)$.

3.1 An interior $W^{2,2}$-estimate

We have the following interior $W^{2,2}$-estimate for a solution of the Euler-Lagrange equation and for the corresponding stray field potential. This estimate will be used in Theorem 22 below in order to find the specific profile of the Néel wall.

Lemma 11. *Let $0 \le r < r' < R' < R$. There exists a constant $C > 0$ (depending only on $r' - r$ and $R - R'$) such that the following holds true. Let $\epsilon > 0$ and set $I = (-R,-r)\cup(r,R)$. Suppose that the functions $u \in W^{1,2}(B_R^+(0)\setminus B_r(0))$ and $\varphi \in W^{1,2}(I)$ solve the system*

$$\begin{aligned}\Delta u &= 0 && \text{in } B_R^+(0)\setminus B_r(0),\\ \frac{\partial u}{\partial x_2} &= \varphi'\sin\varphi && \text{on } I\times\{0\},\\ \epsilon\varphi'' &= u'\sin\varphi && \text{in } I.\end{aligned}$$

Further suppose that $\sin\varphi \neq 0$ *in* I*. Then*

$$\int_{B^+_{R'}(0)\setminus B_{r'}(0)} |\nabla^2 u|^2\,dx + \epsilon \int_{(-R',-r')\cup(r',R')} \Big((\varphi'')^2 + (\varphi')^4 (1+\cot^2\varphi) \Big)\,dx_1 \\ \leq C\bigg(\epsilon \int_I (\varphi')^2\,dx_1 + \int_{B^+_R(0)\setminus B_r(0)} |\nabla u|^2\,dx \bigg).$$

Proof. We first note that φ is smooth on I (see [14]) and therefore u is smooth in $B^+_R(0)\setminus B_r(0)$ up to the boundary $I\times\{0\}$. Consider the function $v = \frac{\partial u}{\partial x_1}$ on $B^+_R(0)\setminus B_r(0)$. We have

$$v = \frac{\epsilon\varphi''}{\sin\varphi} \quad \text{and} \quad \frac{\partial v}{\partial x_2} = -\frac{\partial^2}{\partial x_1^2}\big(\cos\varphi\big) \;\text{ on } I\times\{0\}.$$

Let $\eta \in C^\infty_0(B^+_R(0)\setminus B_r(0))$. Since v is harmonic in $B^+_R(0)\setminus B_r(0)$, integration by parts yields

$$\begin{aligned}
\int_{B^+_R(0)} \eta^2|\nabla v|^2\,dx &= \int_I \eta^2 v \frac{d^2}{dx_1^2}(\cos\varphi)\,dx_1 - 2\int_{B^+_R(0)} \eta v \nabla\eta\cdot\nabla v\,dx \\
&= -\epsilon\int_I \eta^2 \frac{\varphi''}{\sin\varphi}\left(\varphi''\sin\varphi + (\varphi')^2\cos\varphi\right)\,dx_1 - 2\int_{B^+_R(0)} \eta v\nabla\eta\cdot\nabla v\,dx \\
&= -\epsilon\int_I \eta^2(\varphi'')^2\,dx_1 - \epsilon\int_I \eta^2\frac{\big((\varphi')^3\big)'}{3}\cot\varphi\,dx_1 - 2\int_{B^+_R(0)} \eta v\nabla\eta\cdot\nabla v\,dx \\
&= -\epsilon\int_I \eta^2(\varphi'')^2\,dx_1 - \frac{\epsilon}{3}\int_I \eta^2(\varphi')^4(1+\cot^2\varphi)\,dx_1 + \frac{2\epsilon}{3}\int_I \eta\eta'(\varphi')^3\cot\varphi\,dx_1 \\
&\quad - 2\int_{B^+_R(0)} \eta v\nabla\eta\cdot\nabla v\,dx.
\end{aligned}$$

By Young's inequality,

$$\int_I \eta\eta'(\varphi')^3\cot\varphi\,dx_1 \leq \frac{1}{4}\int_I \eta^2(\varphi')^4\cot^2\varphi\,dx_1 + \int_I (\eta')^2(\varphi')^2\,dx_1$$

and

$$-\int_{B^+_R(0)} \eta v\nabla\eta\cdot\nabla v\,dx \leq \frac{1}{4}\int_{B^+_R(0)} \eta^2|\nabla v|^2\,dx + \int_{B^+_R(0)} |\nabla\eta|^2 v^2\,dx.$$

Therefore,

$$\int_{B^+_R(0)} \eta^2|\nabla v|^2\,dx + \epsilon\int_I \eta^2\left((\varphi'')^2 + (\varphi')^4(1+\cot^2\varphi)\right)\,dx_1 \\ \leq 12\int_{B^+_R(0)} v^2|\nabla\eta|^2\,dx + 4\epsilon\int_I (\eta')^2(\varphi')^2\,dx_1. \tag{38}$$

As u is harmonic, we observe that that

$$\frac{\partial^2 u}{\partial x_2^2} = -\frac{\partial v}{\partial x_1},$$

so that

$$\frac{1}{2}|\nabla^2 u|^2 = \left|\frac{\partial v}{\partial x_1}\right|^2 + \left|\frac{\partial v}{\partial x_2}\right|^2 \quad \text{in } B^+_R(0)\setminus B_r(0).$$

Choosing suitable cut-off functions η, we can now easily derive the desired inequality. $\square$

3.2 A Pohozaev identity

As in the theory of Ginzburg-Landau vortices, a variant of the identity due to Pohozaev [28] is useful for our problem. For a function $u : B_1^+(0) \to \mathbb{R}$, we use the notation

$$\partial_\rho u = \frac{x}{|x|} \cdot \nabla u.$$

Lemma 12. *Let $\epsilon > 0$. Suppose that the functions $u \in W^{1,2}(B_1^+(0))$ and $\varphi \in W^{1,2}(-1,1)$ solve the system*

$$\Delta u = 0 \qquad \text{in } B_1^+(0), \tag{39}$$

$$\frac{\partial u}{\partial x_2} = \varphi' \sin\varphi \quad \text{on } (-1,1) \times \{0\}, \tag{40}$$

$$\epsilon\varphi'' = u' \sin\varphi \quad \text{in } (-1,0) \text{ and } (0,1). \tag{41}$$

Then for any $r \in (0,1)$,

$$\epsilon \int_{-r}^{r} (\varphi')^2 \, dx_1 = r\epsilon(\varphi'(r))^2 + r\epsilon(\varphi'(-r))^2 + r \int_{\partial^+ B_r(0)} \left(|\nabla u|^2 - 2\,(\partial_\rho u)^2 \right) d\sigma.$$

Proof. As u is harmonic, we calculate

$$\operatorname{div}\left(\frac{1}{2}|\nabla u|^2 x - (x \cdot \nabla u)\nabla u \right) = 0 \quad \text{in } B_1^+(0).$$

For any $r, s \in (0,1)$ with $s < r$, it follows that

$$\begin{aligned} 0 = &\int_{\partial^+ B_r(0)} \left(\frac{r}{2}|\nabla u|^2 - r\,(\partial_\rho u)^2 \right) d\sigma - \int_{\partial^+ B_s(0)} \left(\frac{s}{2}|\nabla u|^2 - s\,(\partial_\rho u)^2 \right) d\sigma \\ &+ \int_{-r}^{-s} x_1 \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} \, dx_1 + \int_{s}^{r} x_1 \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} \, dx_1. \end{aligned}$$

Using (40) and (41), we compute

$$\begin{aligned} \int_{-r}^{-s} x_1 \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} \, dx_1 &= \epsilon \int_{-r}^{-s} x_1 \varphi''\varphi' \, dx_1 = \frac{\epsilon}{2} \int_{-r}^{-s} x_1 \frac{d}{dx_1}(\varphi')^2 \, dx_1 \\ &= \frac{\epsilon r}{2}(\varphi'(-r))^2 - \frac{\epsilon s}{2}(\varphi'(-s))^2 - \frac{\epsilon}{2}\int_{-r}^{-s} (\varphi')^2 \, dx_1. \end{aligned}$$

Similarly,

$$\int_{s}^{r} x_1 \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} \, dx_1 = \frac{\epsilon r}{2}(\varphi'(r))^2 - \frac{\epsilon s}{2}(\varphi'(s))^2 - \frac{\epsilon}{2}\int_{s}^{r} (\varphi')^2 \, dx_1.$$

Hence

$$\begin{aligned} \frac{\epsilon}{2}\int_{-r}^{-s} (\varphi')^2 \, dx_1 + \frac{\epsilon}{2}\int_{s}^{r} (\varphi')^2 \, dx_1 = &\frac{r}{2}\left(\epsilon(\varphi'(r))^2 + \epsilon(\varphi'(-r))^2 + \int_{\partial^+ B_r(0)} \left(|\nabla u|^2 - 2\,(\partial_\rho u)^2 \right) d\sigma \right) \\ &- \frac{s}{2}\left(\epsilon(\varphi'(s))^2 + \epsilon(\varphi'(-s))^2 + \int_{\partial^+ B_s(0)} \left(|\nabla u|^2 - 2\,(\partial_\rho u)^2 \right) d\sigma \right). \end{aligned}$$

Since $\varphi' \in L^2(-1,1)$ and $|\nabla u| \in L^2(B_1^+(0))$, there exists a sequence $s_k \searrow 0$ such that

$$s_k \left(\epsilon(\varphi'(s_k))^2 + \epsilon(\varphi'(-s_k))^2 + \int_{\partial^+ B_{s_k}(0)} |\nabla u|^2 \, d\sigma \right) \to 0.$$

Therefore, we obtain the desired identity. □

As a consequence, we obtain the following estimate, which implies that the exchange energy in the core of a Néel wall is of order $O(1/(\log \delta)^2)$.

Lemma 13. *Suppose that $\epsilon \in (0, \frac{1}{2}]$ and the functions $u \in W^{1,2}(B_1^+(0))$ and $\varphi \in W^{1,2}(-1,1)$ solve the system* (39)–(41)*. Let*

$$F = \frac{\epsilon}{2} \int_{-1}^{1} (\varphi')^2 \, dx_1 + \frac{1}{2} \int_{B_1^+(0)} |\nabla u|^2 \, dx.$$

Then

$$\frac{\epsilon}{2} \int_{-\delta}^{\delta} (\varphi')^2 \, dx_1 \leq \frac{F}{\log \frac{1}{\delta}}. \tag{42}$$

If $\varphi(0) \in \pi\mathbb{Z}$, then

$$\int_{-\delta}^{\delta} \sin^2 \varphi \, dx_1 \leq 8\delta F. \tag{43}$$

Proof. It follows from Lemma 12 that

$$\frac{\epsilon}{2} \int_{\delta}^{1} \frac{1}{r} \int_{-r}^{r} (\varphi')^2 \, dx_1 \, dr \leq F.$$

Thus there exists an $r \in (\delta, 1]$ such that

$$\frac{\epsilon}{2} \int_{-r}^{r} (\varphi')^2 \, dx_1 \leq \frac{F}{\log \frac{1}{\delta}}.$$

Inequality (42) then follows immediately.

For the second inequality, we note that for $|x_1| \leq \delta$,

$$|\sin \varphi(x_1)| = \left| \int_0^{x_1} \varphi'(t) \cos \varphi(t) \, dt \right| \leq \left(|x_1| \int_0^{x_1} (\varphi')^2 \, dt \right)^{1/2} \leq \sqrt{\frac{2\delta F}{\epsilon \log \frac{1}{\delta}}} \leq 2\sqrt{F}.$$

(Here we use that fact that $\log \frac{1}{\epsilon} \leq 2 \log \frac{1}{\delta}$ for $0 < \epsilon \leq 1$.) Thus (43) follows immediately as well. □

Later we will prove estimates similar to (42) and (43) even without making use of the Euler-Lagrange equation, but assuming suitable control of the energy instead (see Theorem 28 and Remark 30).

4 The core

In this section, we study what happens near a single Néel wall, rescaled to unit size. Several technical difficulties arise in the analysis of the local behaviour of the magnetization, due to the nonlocal nature of the magnetostatic energy. For this reason, we first introduce a modified energy functional where the stray field is considered on a half ball (instead of $\mathbb{R}^2_+$) with a Neumann boundary condition. We prove energy estimates (upper and lower bounds) for this modified functional together with statements about its behaviour under small perturbations of the boundary data. The analysis of the minimisers of this functional is essential because it yields good approximation results for the Néel wall profile (see Theorem 22) and for the core energy (see Theorem 25).

4.1 A modified functional

For $\gamma \in (0, 2)$, we define the convex set

$$M_\gamma = \{\mu \in W^{1,2}(-1, 1) \,:\, \mu(0) = 1,\ \mu(\pm 1) \leq 1 - \gamma\}.$$

We think of μ as the first component of the magnetisation near a Néel wall with transition angle $2\arccos(1-\gamma)$. For $\mu \in M_\gamma$, consider the convex set

$$W^{1,2}_\mu(B_1^+(0)) = \{w \in W^{1,2}(B_1^+(0))\colon w(x_1, 0) = \mu(x_1) \text{ for } x_1 \in (-1, 1) \text{ and } w \leq 1-\gamma \text{ on } \partial^+ B_1(0)\}.$$

Clearly $W^{1,2}_\mu(B_1^+(0)) \neq \emptyset$ and there exists a unique function where the infimum

$$\inf_{w \in W^{1,2}_\mu(B_1^+(0))} \int_{B_1^+(0)} |\nabla w|^2 \, dx$$

is attained, owing to the strict convexity of the Dirichlet functional.

Now we define the following modified energy functional for $\mu \in M_\gamma$:

$$E^\gamma_\epsilon(\mu) = \frac{\epsilon}{2} \int_{-1}^1 \frac{(\mu')^2}{1-\mu^2} \, dx_1 + \frac{1}{2} \inf_{w \in W^{1,2}_\mu(B_1^+(0))} \int_{B_1^+(0)} |\nabla w|^2 \, dx.$$

Proposition 14. *The functional E^γ_ϵ admits a unique minimizer $\mu \in M_\gamma$, which satisfies $\mu(\pm 1) = 1-\gamma$ and $1 \geq \mu \geq 1-\gamma$ in $(-1, 1)$. Moreover, if $v \in W^{1,2}_\mu(B_1^+(0))$ is the unique function with*

$$\int_{B_1^+(0)} |\nabla v|^2 \, dx = \inf_{w \in W^{1,2}_\mu(B_1^+(0))} \int_{B_1^+(0)} |\nabla w|^2 \, dx,$$

then $1-\gamma \leq v \leq 1$ in $B_1^+(0)$ and v is the unique solution in $W^{1,2}(B_1^+(0))$ of the boundary value problem

$$\Delta v = 0 \quad \text{in } B_1^+(0), \tag{44}$$

$$v(x_1, 0) = \mu(x_1) \quad \text{for } x_1 \in (-1, 1), \tag{45}$$

$$v = 1-\gamma \quad \text{on } \partial^+ B_1(0). \tag{46}$$

Proof. The direct method of the calculus of variations leads to existence of a minimizer $\mu \in M_\gamma$ of E^γ_ϵ. Moreover, if $\mu \in M_\gamma$ is a minimiser of E^γ_ϵ, then using a simple cut-off argument at $1-\gamma$ and 1, we see that $\mu(\pm 1) = 1-\gamma$ and $1 \geq \mu \geq 1-\gamma$ in $(-1, 1)$. This implies that the corresponding minimizer v of the Dirichlet energy satisfies $1-\gamma \leq v \leq 1$ in $B_1^+(0)$ and solves (44)–(46). Obviously, this boundary value problem has a unique solution in $W^{1,2}(B_1^+(0))$. Since the function $(x_1, x_2) \mapsto \frac{x_1^2}{1-x_2^2}$, for $(x_1, x_2) \in \mathbb{R} \times (-1, 1)$, is strictly convex, we deduce that E^γ_ϵ admits in fact a *unique* minimizer $\mu \in M_\gamma$. □

Let $\mu \in M_\gamma$ be the minimizer of E^γ_ϵ and v be the solution of (44)–(46). Since $\operatorname{curl} \nabla^\perp v = 0$, there exists a function $u \in W^{1,2}(B_1^+(0))$ with $\nabla^\perp v = -\nabla u$. This is then a solution of

$$\Delta u = 0 \quad \text{in } B_1^+(0), \tag{47}$$

$$\frac{\partial u}{\partial x_2} = -\mu' \quad \text{on } (-1, 1) \times \{0\}, \tag{48}$$

$$x \cdot \nabla u = 0 \quad \text{on } \partial B_1^+(0). \tag{49}$$

Note that u is determined uniquely up to a constant by these conditions. On the other hand, given a function $u \in W^{1,2}(B_1^+(0))$ satisfying (47)–(49), we can reconstruct a corresponding function

$v \in W^{1,2}_\mu(B_1^+(0))$ with $\nabla^\perp v = -\nabla u$. It follows that the minimiser μ of E_ϵ^γ automatically minimises the quantity

$$\frac{\epsilon}{2}\int_{-1}^{1} \frac{(\mu')^2}{1-\mu^2}\,dx_1 + \frac{1}{2}\int_{B_1^+(0)} |\nabla u|^2\,dx,$$

where u is determined (up to a constant) by (47)–(49). Since μ plays the role of the first component of the magnetisation near a Néel wall, then u roughly corresponds to a stray-field potential associated to μ in $B_1^+(0)$.

The Euler-Lagrange equation for the minimiser of E_ϵ^γ is therefore

$$\epsilon\varphi'' = u'\sin\varphi \quad \text{in } (-1,0) \text{ and } (0,1) \tag{50}$$

for any continuous function $\varphi : (-1,1) \to \mathbb{R}$ with $\mu = \cos\varphi$, similarly to Section 3.

4.2 Energy estimates

We first prove the following preliminary estimate with a direct construction. This result is similar to [7] (see also [13]).

Proposition 15. *Let $\beta \in (0,1)$. Then there exists a constant $C_0 > 0$ (depending on β) such that for every $\gamma \in (\beta, 2-\beta)$ and every $\epsilon \in (0, \frac{1}{2}]$,*

$$\inf_{M_\gamma} E_\epsilon^\gamma \le \frac{\pi\gamma^2}{2\log\frac{1}{\delta}} + \frac{C_0}{\left(\log\frac{1}{\delta}\right)^2}.$$

Proof. Consider the function $\mu : (-1,1) \to [1-\gamma, 1]$ given by

$$\mu(x_1) = 1 - \gamma\frac{\log(x_1^2+\delta^2) - \log\delta^2}{\log(1+\delta^2) - \log\delta^2} \in [1-\gamma, 1] \quad \text{for } x_1 \in (-1,1).$$

Then $1 + \mu \ge 2 - \gamma$ and thus

$$1 - (\mu(x_1))^2 \ge \gamma(2-\gamma)\frac{\log(x_1^2+\delta^2) - \log\delta^2}{\log(1+\delta^2) - \log\delta^2}.$$

Therefore,

$$\begin{aligned}\int_{-1}^{1} \frac{(\mu')^2}{1-\mu^2}\,dx_1 &\le \frac{2\gamma^2}{\gamma(2-\gamma)\log\sqrt{\frac{1}{\delta^2}+1}}\int_{-1}^{1}\frac{x_1^2}{(x_1^2+\delta^2)^2\log\left(\frac{x_1^2}{\delta^2}+1\right)}\,dx_1 \\ &\le \frac{2\gamma^2}{\delta\gamma(2-\gamma)\log\sqrt{\frac{1}{\delta^2}+1}}\int_{-\infty}^{\infty}\frac{t^2}{(t^2+1)^2\log(t^2+1)}\,dt.\end{aligned}$$

Define $w : B_1^+(0) \to \mathbb{R}$ by

$$w(r\cos\theta, r\sin\theta) = \mu(r)$$

for $0 < r \le 1$ and $0 < \theta < \pi$. Then

$$\begin{aligned}\int_{B_1^+(0)} |\nabla w|^2\,dx &= \frac{\pi\gamma^2}{\left(\log\sqrt{\frac{1}{\delta^2}+1}\right)^2}\int_0^1 \frac{r^3}{(r^2+\delta^2)^2}\,dr \\ &= \frac{\pi\gamma^2}{2\left(\log\sqrt{\frac{1}{\delta^2}+1}\right)^2}\int_{\delta^2}^{1+\delta^2}\frac{t-\delta^2}{t^2}\,dt \le \frac{\pi\gamma^2}{\log\sqrt{\frac{1}{\delta^2}+1}}.\end{aligned}$$

Hence

$$E_\epsilon^\gamma(\mu) \le \frac{\pi\gamma^2}{2\log\sqrt{\frac{1}{\delta^2}+1}} + \frac{\gamma^2}{\gamma(2-\gamma)\log\frac{1}{\epsilon}\log\sqrt{\frac{1}{\delta^2}+1}}\int_{-\infty}^{\infty}\frac{t^2}{(t^2+1)^2\log(t^2+1)}\,dt,$$

which implies the statement of the proposition. □

We can match the leading order term in this inequality with an estimate from below. Moreover, we obtain more information about the behaviour of the minimiser μ of E_ϵ^γ: in particular, we have a uniform $\dot{W}^{1,2}$-estimate for the difference between the rescaled stray field potential $u\log\frac{1}{\delta}$ (associated to μ via (47)–(49)) and the map

$$(x_1,x_2)\mapsto\gamma\left(\arctan\left(\frac{x_2}{x_1}\right)-\frac{\pi x_1}{2|x_1|}\right),\quad (x_1,x_2)\in B_1^+(0).$$

First, however, we need some information on the regularity of solutions of the Euler-Lagrange equation, especially at the boundary.

Lemma 16. *Let $\gamma\in(0,2)$. Suppose that $\varphi\in W^{1,2}(-1,1)$ with $\cos\varphi(-1)=\cos\varphi(1)=1-\gamma$. Further suppose that $u\in W^{1,2}(B_1^+(0))$ is a function such that (47)–(50) are satisfied for $\mu=\cos\varphi$. Then ∇u is continuous in $B_1^+(0)$ and has a continuous extension to $\partial B_1^+(0)\backslash\{0\}$.*

Proof. Let $\tilde\mu=\cos\varphi-1+\gamma$ and consider the extension of u to $\mathbb{R}_+^2$ and of $\tilde\mu$ to $\mathbb{R}$ given by

$$u(x)=u\left(\frac{x}{|x|^2}\right)\quad\text{for } x\in\mathbb{R}_+^2 \text{ with } |x|>1$$

and

$$\tilde\mu(x_1)=-\tilde\mu\left(\frac{1}{x_1}\right)\quad\text{for } x_1\in(-\infty,-1)\cup(1,\infty).$$

Then we have $\Delta u=0$ in $\mathbb{R}_+^2$ and $\frac{\partial u}{\partial x_2}=-\tilde\mu'$ on $\mathbb{R}\times\{0\}$. Since $\tilde\mu'\in L^2_{\mathrm{loc}}(\mathbb{R})$, we conclude that $u'(\,\cdot\,,0)\in L^p_{\mathrm{loc}}(\mathbb{R})$ for any $p\in[1,2)$ with standard regularity theory. Thus by (50), we have $\varphi''\in L^p(-1,1)$, and it follows that $\tilde\mu'$ is Hölder continuous on $[-1,-r]\cup[r,1]$ for every $r\in(0,1)$. The extension is then locally Hölder continuous in $\mathbb{R}\backslash\{0\}$, and using standard regularity theory once more, we conclude that ∇u is continuous away from 0. □

Theorem 17. *For any $\beta\in(0,1)$, there exists a constant $C>0$ (depending on β) such that the following holds true for every $\epsilon\in(0,\frac{1}{2}]$. Suppose that $\gamma\in(\beta,2-\beta)$ and $\mu\in M_\gamma$ is the minimiser of E_ϵ^γ. Then*

$$\left|E_\epsilon^\gamma(\mu)-\frac{\pi\gamma^2}{2\log\frac{1}{\delta}}\right|\le\frac{C}{\left(\log\frac{1}{\delta}\right)^2}. \tag{51}$$

Let $u\in W^{1,2}(B_1^+(0))$ be the solution of (47)–(49). Then

$$\epsilon\int_{-1}^{1}\frac{(\mu')^2}{1-\mu^2}\,dx_1+\int_{B_1^+(0)\backslash B_\delta(0)}\left|\nabla u(x)-\frac{\gamma x^\perp}{|x|^2\log\frac{1}{\delta}}\right|^2dx+\int_{B_\delta(0)}|\nabla u|^2\,dx\le\frac{C}{\left(\log\frac{1}{\delta}\right)^2} \tag{52}$$

and

$$\int_{\partial^+B_1(0)}|\nabla u|^2\,d\sigma\le\frac{C}{\left(\log\frac{1}{\delta}\right)^2}. \tag{53}$$

Proof. We already know, by Proposition 15, that there exists a constant $C_0 > 0$, depending only on β, such that

$$E_\epsilon^\gamma(\mu) \le \frac{\pi\gamma^2}{2\log\frac{1}{\delta}} + \frac{C_0}{\left(\log\frac{1}{\delta}\right)^2}. \tag{54}$$

We want to prove an estimate from below of the same order. To this end, we choose some $\varphi \in W^{1,2}(-1,1)$ with $\mu = \cos\varphi$. Consider the function $\xi : B_1^+(0) \to \mathbb{R}$, defined in polar coordinates by

$$\xi(r\cos\theta, r\sin\theta) = \begin{cases} \dfrac{\gamma(\theta - \pi/2)}{\log\frac{1}{\delta}} & \text{if } r \ge \delta, \\ \dfrac{\gamma r(\theta - \pi/2)}{\delta\log\frac{1}{\delta}} & \text{if } 0 < r < \delta. \end{cases}$$

According to Proposition 14, we have

$$\begin{aligned} \frac{\gamma^2\pi}{\log\frac{1}{\delta}} &= \frac{\gamma\pi}{2\log\frac{1}{\delta}}\left(\int_{-1}^0 \mu'\,dx_1 - \int_0^1 \mu'\,dx_1\right) \\ &= \int_{-1}^1 \xi(x_1,0)\mu'(x_1)\,dx_1 - \int_{-\delta}^{\delta}\left(\frac{\gamma\pi x_1}{2|x_1|\log\frac{1}{\delta}} + \xi(x_1,0)\right)\mu'(x_1)\,dx_1. \end{aligned}$$

Using Lemma 13 and (54), we find a constant $C_1 = C_1(\beta)$ such that

$$\left|\int_{-\delta}^{\delta}\left(\frac{\gamma\pi x_1}{2|x_1|\log\frac{1}{\delta}} + \xi(x_1,0)\right)\mu'(x_1)\,dx_1\right| \le \frac{\gamma\pi}{2\log\frac{1}{\delta}}\int_{-\delta}^{\delta}|\varphi'||\sin\varphi|\,dx_1 \le \frac{C_1}{\left(\log\frac{1}{\delta}\right)^2}.$$

Moreover, by (47)–(49), we have

$$\int_{-1}^1 \xi(x_1,0)\mu'(x_1)\,dx_1 = -\int_{(-1,1)\times\{0\}} \xi\frac{\partial u}{\partial x_2}\,dx_1 = \int_{B_1^+(0)} \nabla\xi\cdot\nabla u\,dx.$$

Combining these estimates, we obtain

$$\int_{B_1^+(0)} \nabla\xi\cdot\nabla u\,dx \ge \frac{\gamma^2\pi}{\log\frac{1}{\delta}} - \frac{C_1}{\left(\log\frac{1}{\delta}\right)^2}. \tag{55}$$

We compute

$$|\nabla\xi|^2 = \frac{\gamma^2}{\delta^2(\log\delta)^2}\left(1 + (\theta - \frac{\pi}{2})^2\right) \quad \text{if } r < \delta$$

and

$$|\nabla\xi|^2 = \frac{\gamma^2}{r^2\left(\log\frac{1}{\delta}\right)^2} \quad \text{if } r > \delta.$$

This implies

$$\int_{B_1^+(0)} |\nabla\xi|^2\,dx = \frac{\gamma^2\pi}{\log\frac{1}{\delta}} + \frac{\gamma^2\pi}{\left(\log\frac{1}{\delta}\right)^2}\left(\frac{1}{2} + \frac{\pi^2}{24}\right) \le \frac{\gamma^2\pi}{\log\frac{1}{\delta}} + \frac{\gamma^2\pi}{\left(\log\frac{1}{\delta}\right)^2}.$$

Hence, by (54) and (55),

$$\int_{B_1^+(0)} |\nabla u - \nabla\xi|^2\,dx \le 2E_\epsilon^\gamma(\mu) - \frac{\gamma^2\pi}{\log\frac{1}{\delta}} + \frac{2C_1 + \pi\gamma^2}{\left(\log\frac{1}{\delta}\right)^2} \le \frac{C_2}{\left(\log\frac{1}{\delta}\right)^2},$$

where $C_2 = 2C_0 + 2C_1 + \pi\gamma^2$. It follows in particular that

$$\int_{B_\delta^+(0)} |\nabla u|^2 \, dx \leq 2 \int_{B_\delta^+(0)} |\nabla u - \nabla \xi|^2 \, dx + 2 \int_{B_\delta^+(0)} |\nabla \xi|^2 \, dx \leq \frac{2C_2 + 2\pi\gamma^2}{\left(\log \frac{1}{\delta}\right)^2}$$

and

$$\int_{B_1^+(0) \setminus B_\delta(0)} \left| \nabla u(x) - \frac{\gamma x^\perp}{|x|^2 \log \frac{1}{\delta}} \right|^2 dx \leq \frac{C_2}{\left(\log \frac{1}{\delta}\right)^2},$$

since $\nabla \xi = \frac{\gamma x^\perp}{|x|^2 \log \frac{1}{\delta}}$ if $r > \delta$. Furthermore, we have

$$\begin{aligned} \int_{B_1^+(0)} |\nabla u|^2 \, dx &= \int_{B_1^+(0)} |\nabla u - \nabla \xi|^2 \, dx - \int_{B_1^+(0)} |\nabla \xi|^2 \, dx + 2 \int_{B_1^+(0)} \nabla \xi \cdot \nabla u \, dx \\ &\geq \frac{\gamma^2 \pi}{\log \frac{1}{\delta}} - \frac{\gamma^2 \pi + 2C_1}{\left(\log \frac{1}{\delta}\right)^2}. \end{aligned}$$

If we combine this with (54), then we obtain

$$\epsilon \int_{-1}^1 \frac{(\mu')^2}{1 - \mu^2} \, dx_1 \leq 2E_\epsilon^\gamma(\mu) - \int_{B_1^+(0)} |\nabla u|^2 \, dx \leq \frac{2C_0 + 2C_1 + \gamma^2 \pi}{\left(\log \frac{1}{\delta}\right)^2}.$$

Thus we have proved inequalities (51) and (52).

Finally, we apply Lemma 12 and let $r \nearrow 1$. Then by Lemma 16, we obtain (53). □

4.3 Behaviour under small perturbations

Next we want to understand how the number $\inf_{M_\gamma} E_\epsilon^\gamma$ changes when we vary γ, and how the energy changes when we perturb the boundary condition (49). In particular, we prove the following statements.

Proposition 18. *Let $\beta \in (0, 1)$. There exists a constant $C(\beta) > 0$ such that for all $\gamma_1, \gamma_2 \in (\beta, 2-\beta)$ and every $\epsilon \in (0, \frac{1}{2}]$,*

$$\inf_{M_{\gamma_2}} E_\epsilon^{\gamma_2} - \frac{\pi\gamma_2^2}{2\log\frac{1}{\delta}} \leq \inf_{M_{\gamma_1}} E_\epsilon^{\gamma_1} - \frac{\pi\gamma_1^2}{2\log\frac{1}{\delta}} + \frac{C|\gamma_2 - \gamma_1|}{\left(\log\frac{1}{\delta}\right)^2}.$$

In other words, the function

$$g : (0, 1) \to \mathbb{R}, \ \gamma \mapsto \inf_{M_\gamma} E_\epsilon^\gamma - \frac{\pi\gamma^2}{2\log\frac{1}{\delta}}, \tag{56}$$

is locally Lipschitz continuous with Lipschitz constant of order $O(1/(\log \delta)^2)$.

Proof. First note that it suffices to prove the inequality when $|\gamma_2 - \gamma_1|$ is small, for otherwise it follows from Theorem 17.

Let $\mu_1 \in M_{\gamma_1}$ be the minimiser of $E_\epsilon^{\gamma_1}$ and let $v_1 \in W^{1,2}(B_1^+(0))$ be the corresponding solution of (44)–(46) in Proposition 14. Define

$$\mu_2 = \frac{\gamma_2}{\gamma_1}(\mu_1 - 1) + 1$$

and

$$v_2 = \frac{\gamma_2}{\gamma_1}(v_1 - 1) + 1.$$

Then $\mu_2 \in M_{\gamma_2}$ and v_2 is the unique solution of (44)–(46) associated to μ_2, so that, by Proposition 14,

$$E_\epsilon^{\gamma_2}(\mu_2) = \frac{\epsilon}{2} \int_{-1}^{1} \frac{(\mu_2')^2}{1-\mu_2^2}\, dx_1 + \frac{1}{2} \int_{B_1^+(0)} |\nabla v_2|^2\, dx.$$

We have

$$1 + \mu_2 = \frac{\gamma_2}{\gamma_1}(1 + \mu_1) + \frac{2(\gamma_1 - \gamma_2)}{\gamma_1}$$

and

$$1 - \mu_2 = \frac{\gamma_2}{\gamma_1}(1 - \mu_1).$$

Hence

$$\frac{(\mu_2')^2}{1-\mu_2^2} = \frac{\frac{(\mu_1')^2}{1-\mu_1^2}}{1 + \frac{2(\gamma_1-\gamma_2)}{\gamma_2(1+\mu_1)}}.$$

Under the assumptions of the proposition, we have $\beta < \gamma_1, \gamma_2 < 2-\beta$ and $\beta < 1+\mu_1 \leq 2$ throughout $(-1, 1)$. Therefore, we have a constant $C_1 = C_1(\beta)$ such that

$$\frac{1}{1 + \frac{2(\gamma_1-\gamma_2)}{\gamma_2(1+\mu_1)}} \leq \frac{\gamma_2^2}{\gamma_1^2} + C_1|\gamma_2 - \gamma_1|,$$

provided that $|\gamma_2 - \gamma_1|$ is small enough.

We now have

$$E_\epsilon^{\gamma_2}(\mu_2) \leq \left(\frac{\gamma_2^2}{\gamma_1^2} + C_1|\gamma_2 - \gamma_1|\right) \frac{\epsilon}{2} \int_{-1}^{1} \frac{(\mu_1')^2}{1-\mu_1^2}\, dx_1 + \frac{\gamma_2^2}{2\gamma_1^2} \int_{B_+^1(0)} |\nabla v_1|^2\, dx.$$

Combining this with (51) and (52), we obtain the inequality

$$g(\gamma_2) \leq g(\gamma_1) + \frac{\gamma_2^2 - \gamma_1^2}{\gamma_1^2} g(\gamma_1) + \frac{C_2|\gamma_1 - \gamma_2|}{\left(\log \frac{1}{\delta}\right)^2}$$

for a constant $C_2 = C_2(\beta)$ by Theorem 17, where g is the function defined in (56). Finally, we know that

$$g(\gamma_1) \leq \frac{C_3}{\left(\log \frac{1}{\delta}\right)^2}$$

for another constant C_3 depending only on β (by Theorem 17 again). Hence we obtain the desired inequality. □

Remark 19. *Recall the discussion in Section 2.3. Consider a Néel wall at the point a_n such that m_1 makes a transition from $1-\gamma_n$ to 1 and back. If we modify the wall, changing γ_n to $\gamma_n - \zeta_n$ with $\zeta_n = \frac{\lambda_n}{\log \frac{1}{\delta}}$, then the change of the energy (to leading order, rescaled by $(\log \delta)^2$) is*

$$\left(\log \frac{1}{\delta}\right)^2 \left(\inf_{M_{\gamma_n - \zeta_n}} E_\epsilon^{\gamma_n - \zeta_n} - \inf_{M_{\gamma_n}} E_\epsilon^{\gamma_n}\right) = -\pi \gamma_n \lambda_n + o(1) \quad \text{as} \quad \epsilon \searrow 0.$$

This is the phenomenon that leads to the core-tail interaction term in the renormalised energy.

Lemma 20. *Let $C_0 > 0$, $\beta \in (0,1)$ and $q > 2$. Then there exists a constant $C > 0$ (depending only on C_0, β and q) such that for any $\gamma \in (\beta, 2-\beta)$, any $\epsilon \in (0, \frac{1}{2}]$, and any $\eta \in (0, C_0)$, the following holds true. Suppose that $\mu \in M_\gamma$ and $u \in W^{1,2}(B_1^+(0))$ with $\Delta u = 0$ in $B_1^+(0)$ and $\frac{\partial u}{\partial x_2} = -\mu'$ on $(-1,1) \times \{0\}$. Suppose further that*

$$\|x \cdot \nabla u\|_{L^q(\partial^+ B_1(0))} \leq \frac{\eta}{\log \frac{1}{\delta}}$$

and

$$\left\| \nabla u - \frac{\gamma x^\perp}{|x|^2 \log \frac{1}{\delta}} \right\|_{L^2(B_1^+(0) \setminus B_\delta(0))} + \|\nabla u\|_{L^2(B_\delta^+(0))} \leq \frac{C_0}{\log \frac{1}{\delta}}. \tag{57}$$

Then

$$\frac{\epsilon}{2} \int_{-1}^{1} \frac{(\mu')^2}{1-\mu^2} \, dx_1 + \frac{1}{2} \int_{B_1^+(0)} |\nabla u|^2 \, dx \geq \inf_{M_\gamma} E_\epsilon^\gamma - \frac{C\eta}{\left(\log \frac{1}{\delta}\right)^2}.$$

Proof. Consider the function $\tilde{w} \in W^{1,2}(B_1^+(0))$ with $\nabla \tilde{w} = \nabla^\perp u$ and $\tilde{w}(x_1, 0) = \mu(x_1)$ for $x_1 \in (-1,1)$. Let $S_\theta = \{(\cos t, \sin t) \colon t \in (0, \theta)\}$ for $0 < \theta < \frac{\pi}{2}$. Note that

$$\tilde{w}(\cos\theta, \sin\theta) - \mu(1) = \int_{S_\theta} x \cdot \nabla u \, d\sigma \leq \frac{\theta^{1-1/q}\eta}{\log \frac{1}{\delta}}, \quad \theta \in (0, \frac{\pi}{2}).$$

Similarly, we find that

$$\tilde{w}(-\cos\theta, \sin\theta) - \mu(-1) \leq \frac{\theta^{1-1/q}\eta}{\log \frac{1}{\delta}}, \quad \theta \in (0, \frac{\pi}{2}).$$

Thus if we define

$$g(x) = \frac{2\eta x_2^{1-1/q}}{\log \frac{1}{\delta}}$$

and

$$w = \tilde{w} - g,$$

then we have that $w \leq \max\{\mu(\pm 1)\} \leq 1 - \gamma$ on $\partial B_1^+(0)$ (because $2 \sin\theta \geq \theta$ for $\theta \in (0, \frac{\pi}{2})$). Moreover, $w \in W_\mu^{1,2}(B_1^+(0))$. Indeed, we have

$$\int_{B_1^+(0)} |\nabla w|^2 \, dx = \int_{B_1^+(0)} |\nabla u|^2 \, dx + 2 \int_{B_1^+(0)} \nabla u \cdot \nabla^\perp g \, dx + \int_{B_1^+(0)} |\nabla g|^2 \, dx.$$

For any $p \in [1, q)$, we have a constant $C_1 = C_1(p)$ such that

$$\|\nabla g\|_{L^p(B_1^+(0))} \leq \frac{C_1 \eta}{\log \frac{1}{\delta}}.$$

For $p > 2$, inequality (57) gives another constant $C_2 = C_2(p, C_0)$ such that

$$\|\nabla u\|_{L^{p/(p-1)}(B_1^+(0))} \leq \frac{C_2}{\log \frac{1}{\delta}}.$$

We conclude, using Hölder's inequality, that

$$\int_{B_1^+(0)} \nabla u \cdot \nabla^\perp g \, dx \leq \frac{C_1 C_2 \eta}{\left(\log \frac{1}{\delta}\right)^2},$$

using some fixed $p \in (2, q)$. Therefore,

$$\int_{B_1^+(0)} |\nabla w|^2 \, dx \le \int_{B_1^+(0)} |\nabla u|^2 \, dx + \frac{C_3 \eta}{\left(\log \frac{1}{\delta}\right)^2}$$

for a number $C_3 > 0$ that depends only on q and C_0. Now the statement of the lemma follows. ☐

Corollary 21. *Let $\beta \in (0, \frac{2}{3})$, $C_0 > 0$, and $q > 2$. Then there exists a constant $C > 0$ such that for any $\gamma \in (2\beta, 2 - 2\beta)$, any $\epsilon \in (0, \frac{1}{2}]$, any $\eta \in (0, C_0)$, and any $\zeta \in (-C_0, C_0)$, the following holds true. Suppose that $\mu \in W^{1,2}(-1,1)$ with $\mu(\pm 1) \le 1 - \gamma + \frac{\zeta}{\log \frac{1}{\delta}}$ and $\mu(0) = 1$. Let $u \in W^{1,2}(B_1^+(0))$ be a function with $\Delta u = 0$ in $B_1^+(0)$ and $\frac{\partial u}{\partial x_2} = -\mu'$ on $(-1,1) \times \{0\}$. Suppose that*

$$\|x \cdot \nabla u\|_{L^q(\partial^+ B_1(0))} \le \frac{\eta}{\log \frac{1}{\delta}}$$

and

$$\left\| \nabla u - \frac{\gamma x^\perp}{|x|^2 \log \frac{1}{\delta}} \right\|_{L^2(B_1^+(0) \setminus B_\delta(0))} + \|\nabla u\|_{L^2(B_\delta^+(0))} \le \frac{C_0}{\log \frac{1}{\delta}}.$$

Then

$$\frac{\epsilon}{2} \int_{-1}^1 \frac{(\mu')^2}{1 - \mu^2} \, dx_1 + \frac{1}{2} \int_{B_1^+(0)} |\nabla u|^2 \, dx \ge \inf_{M_\gamma} E_\epsilon^\gamma - \frac{\pi \gamma \zeta + C\eta}{\left(\log \frac{1}{\delta}\right)^2} - \frac{C}{\left(\log \frac{1}{\delta}\right)^3}.$$

Proof. Set $\tilde{\gamma} = \zeta / \log \frac{1}{\delta}$. We have $\mu \in M_{\gamma - \tilde{\gamma}}$, thus for ϵ small enough, we can first apply Lemma 20 for $\gamma - \tilde{\gamma} \in (\beta, 2 - \beta)$ instead of γ. We obtain

$$\frac{\epsilon}{2} \int_{-1}^1 \frac{(\mu')^2}{1 - \mu^2} \, dx_1 + \frac{1}{2} \int_{B_1^+(0)} |\nabla u|^2 \, dx \ge \inf_{M_{\gamma - \tilde{\gamma}}} E_\epsilon^{\gamma - \tilde{\gamma}} - \frac{C_1 \eta}{\left(\log \frac{1}{\delta}\right)^2}$$

for a constant $C_1 = C_1(\beta, C_0, q)$. From Proposition 18, it follows that

$$\inf_{M_{\gamma - \tilde{\gamma}}} E_\epsilon^{\gamma - \tilde{\gamma}} - \inf_{M_\gamma} E_\epsilon^\gamma \ge -\frac{\pi \gamma \zeta}{\left(\log \frac{1}{\delta}\right)^2} - \frac{C_2}{\left(\log \frac{1}{\delta}\right)^3}$$

for another constant $C_2 = C_2(\beta, C_0)$. Now it suffices to combine these estimates. ☐

4.4 The profile near the core

Minimisers of E_ϵ^γ give a good approximation of the behaviour of Néel walls near the core. When we let $\epsilon \searrow 0$, then, after renormalisation, we have convergence of those minimisers to a specific profile. More precisely, we have the following.

Theorem 22. *For $\epsilon \in (0, \frac{1}{2}]$ and $\gamma \in (0, 2)$, let $\mu_\epsilon \in M_\gamma$ be the minimiser of E_ϵ^γ. Let u_ϵ be the corresponding solution of (47)–(49) with*

$$\int_{B_1^+(0) \setminus B_{1/2}(0)} u_\epsilon \, dx = 0. \tag{58}$$

Suppose that $0 < r < R < 1$. Then, as $\epsilon \searrow 0$,

$$u_\epsilon(x) \log \frac{1}{\delta} \rightharpoonup \gamma \left(\arctan\left(\frac{x_2}{x_1}\right) - \frac{\pi x_1}{2|x_1|} \right) \quad \text{weakly in } W^{2,2}(B_R^+(0) \setminus B_r(0))$$

and

$$(\mu_\epsilon(x_1) - 1 + \gamma) \log \frac{1}{\delta} \to \gamma \log \frac{1}{|x_1|} \quad \text{strongly in } W^{1,2}((-R, -r) \cup (r, R)).$$

Remark 23. *For a related problem (concerning Néel walls in the presence of an anisotropy but without the confinement to an interval), Melcher [24, 25] proved similar results on the profile of Néel walls (with methods different from ours).*

The proof of Theorem 22 relies on Lemma 11. Therefore, we first need to show that the assumption $\sin\varphi_\epsilon \neq 0$ in that lemma is satisfied for a function φ_ϵ with $\mu_\epsilon = \cos\varphi_\epsilon$, at least away from $x_1 = 0$, for sufficiently small values of ϵ. Because we will use this result in a slightly different context as well, we formulate it more generally.

Lemma 24. *There exists a constant $C > 0$ such that the following holds true. Suppose that $\Omega \subset \mathbb{R}^2_+$ is a Lipschitz domain with $(-1,1)\times\{0\} \subset \partial\Omega$ and let ν be the outer normal vector on $\partial\Omega$. Let $\epsilon \in (0, \frac{1}{2}]$. Suppose that $\mu \in W^{1,2}(-1,1)$ and let $u \in W^{1,2}(\Omega)$ be a solution of*

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial x_2} &= -\mu' && \text{on } (-1,1)\times\{0\}, \\ \nu\cdot\nabla u &= 0 && \text{on } \partial\Omega\backslash((-1,1)\times\{0\}). \end{aligned}$$

Let $x_1 \in (\delta-1, 1-\delta)$ and set

$$\Sigma = \left(B_\delta^+(x_1,0)\cap\Omega\right) \cup \left\{y \in \Omega\cap B_2^+(0)\colon y_2 \geq |y_1 - x_1| \text{ or } |y| \in (\tfrac{3}{2}, 2)\right\}.$$

Then

$$|\mu(x_1) - \mu(-1)| \leq C\left(\log\frac{1}{\delta}\int_\Sigma |\nabla u|^2\,dx\right)^{1/2} + C\left(\log\frac{1}{\delta}\int_{x_1-\delta}^{x_1+\delta} \epsilon(\mu')^2\,dt\right)^{1/2}.$$

Proof. Define

$$\psi(t) = \begin{cases} 1 & \text{if } t \leq x_1 - \delta, \\ \frac{x_1 - t}{2\delta} + \frac{1}{2} & \text{if } x_1 - \delta < t < x_1 + \delta, \\ 0 & \text{if } t \geq x_1 + \delta. \end{cases}$$

Furthermore, let

$$\eta(\theta) = \begin{cases} 0 & \text{if } \theta \leq \frac{\pi}{4}, \\ \frac{2\theta}{\pi} - \frac{1}{2} & \text{if } \frac{\pi}{4} < \theta < \frac{3\pi}{4}, \\ 1 & \text{if } \theta \geq \frac{3\pi}{4}, \end{cases}$$

and define $\tilde{\chi} : \mathbb{R}^2_+ \to \mathbb{R}$ by

$$\tilde{\chi}(x_1 + r\cos\theta, r\sin\theta) = (1 - \eta(\theta))\psi(x_1 + r) + \eta(\theta)\psi(x_1 - r).$$

Finally, let

$$\chi(x) = \begin{cases} \tilde{\chi}(x) & \text{if } |x| \leq \frac{3}{2}, \\ 2(2 - |x|)\tilde{\chi}(x) & \text{if } \frac{3}{2} < |x| < 2, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Then $\Omega \cap \operatorname{supp}\nabla\chi \subset \overline{\Sigma}$ and

$$\int_{\mathbb{R}^2_+} |\nabla\chi|^2\,dx \leq C_1 \log\frac{1}{\delta}$$

for some universal constant C_1.

We have

$$\mu(x_1) - \mu(-1) = \int_{-1}^{x_1} \mu'(t)\,dt.$$

Moreover,

$$\left|\int_{-1}^{x_1} \mu'(t)\,dt - \int_{-1}^{1} \chi(t,0)\mu'(t)\,dt\right| \leq \int_{x_1-\delta}^{x_1+\delta} |\mu'|\,dt \\ \leq \left(2\delta \int_{x_1-\delta}^{x_1+\delta} (\mu')^2\,dx\right)^{1/2}.$$

Using the boundary value problem for u, we find that

$$\left|\int_{-1}^{1} \chi(t,0)\mu'(t)\,dt\right| = \left|\int_\Omega \nabla\chi \cdot \nabla u\,dx\right| \leq \left(C_1 \log\frac{1}{\delta}\int_\Sigma |\nabla u|^2\,dx\right)^{1/2}.$$

Combining these estimates, we obtain the desired inequality. □

Proof of Theorem 22. We choose $\varphi_\epsilon : (-1,1) \to [0,\pi)$ such that $\varphi_\epsilon(0) = 0$ and $\mu_\epsilon = \cos\varphi_\epsilon$. Using Theorem 17 and Lemma 24 (applied for $\Omega = B_1^+(0)$), we see that for any $r, R \in (0,1)$ and any sufficiently small $\beta > 0$, we have $|\mu_\epsilon - 1 + \gamma| < \beta$ and $|\sin\varphi_\epsilon| \geq \beta$ in $(-R,-r) \cup (r,R)$ if $\epsilon > 0$ is small enough. Therefore, we can apply Lemma 11 in $(-R,-r)$ and in (r,R).

By Theorem 17, the Poincaré inequality, and (58), the functions $u_\epsilon \log\frac{1}{\delta}$ are uniformly bounded in the space $W^{1,2}(B_1^+(0)\backslash B_r(0))$ for all $r > 0$. Hence, by Lemma 11, they are uniformly bounded in $W^{2,2}(B_R^+(0)\backslash B_r(0))$ whenever $0 < r < R < 1$. Hence there exists a sequence $\epsilon_k \to 0$ as $k \to \infty$ such that we have weak convergence

$$u_{\epsilon_k} \log\frac{1}{\delta_k} \rightharpoonup w$$

in $W^{2,2}(B_R^+(0)\backslash B_r(0))$ for all $r, R \in (0,1)$, where $\delta_k = \epsilon_k \log\frac{1}{\epsilon_k}$. The limit

$$w \in \bigcap_{0<r<R<1} W^{2,2}(B_R^+(0)\backslash B_r(0))$$

is harmonic in $B_1^+(0)$ and satisfies the boundary condition $x \cdot \nabla w = 0$ on $\partial^+ B_1(0)$. According to Lemma 11, we also have

$$\limsup_{k\to\infty}\left(\left(\log\frac{1}{\delta_k}\right)^2 \epsilon_k \int_r^R (\varphi_{\epsilon_k}'')^2\,dx_1\right) < \infty.$$

It follows from (50) that

$$\limsup_{k\to\infty}\left(\frac{\left(\log\frac{1}{\delta_k}\right)^2}{\epsilon_k}\int_r^R (u_{\epsilon_k}')^2\,dx_1\right) < \infty.$$

Thus w is constant on $(0,1)\times\{0\}$, and we can prove the same on $(-1,0)\times\{0\}$.

Define

$$\tilde{w}(x) = w(x) - \gamma\left(\arctan\left(\frac{x_2}{x_1}\right) - \frac{\pi x_1}{2|x_1|}\right), \quad x \in B_1^+(0).$$

Since

$$\int_{B_1^+(0)} \left|\nabla w - \frac{\gamma x^\perp}{|x|^2}\right|^2 dx < \infty$$

by Theorem 17, it follows that $\tilde{w} \in W^{1,2}(B_1^+(0))$. We conclude that $\tilde{w}(\,\cdot\,,0) \in H^{1/2}(-1,1)$. Moreover, the trace $\tilde{w}(\,\cdot\,,0)$ is constant on $(-1,0)$ and on $(0,1)$. But $H^{1/2}(-1,1)$ does not allow any jumps; hence $\tilde{w}(\cdot, 0)$ is in fact constant on $(-1,1)$. We also have $\Delta \tilde{w} = 0$ in $B_1^+(0)$ and $x \cdot \nabla \tilde{w} = 0$ on $\partial^+ B_1(0)$. Thus it follows that $\tilde{w}$ is constant in $B_1^+(0)$. Because of (58), we have $\tilde{w} = 0$. That is,

$$w(x) = \gamma \left(\arctan\left(\frac{x_2}{x_1}\right) - \frac{\pi x_1}{2|x_1|} \right) \quad \text{in} \quad B_1^+(0).$$

As the limit is thus independent of the sequence ϵ_k, this implies the first claim.

We have

$$\log \frac{1}{\delta} \frac{\partial u_\epsilon}{\partial x_2}(x_1, 0) \to \frac{\gamma}{x_1}$$

strongly in $L^2(-R,-r)$ and in $L^2(r,R)$. But since $\mu_\epsilon' = -\frac{\partial u_\epsilon}{\partial x_2}$, it follows that $(\mu_\epsilon - 1 + \gamma) \log \frac{1}{\delta}$ converges strongly in $W^{1,2}(-R,-r)$ to $\lambda_- - \gamma \log |x_1|$ and in $W^{1,2}(r,R)$ and to $\lambda_+ - \gamma \log |x_1|$ for two constants $\lambda_-, \lambda_+ \in [-\infty, \infty]$. It remains to determine these constants.

Choose a function $\chi \in C_0^\infty(B_1(-1,0))$ with $(-1,0) \notin \operatorname{supp} \nabla\chi$. Then

$$\lim_{\epsilon \searrow 0} \left(\log \frac{1}{\delta} \int_{B_1^+(0)} \nabla\chi \cdot \nabla u_\epsilon \, dx \right) = \gamma \int_{B_1^+(0)} \nabla \chi \cdot \frac{x^\perp}{|x|^2} \, dx = -\gamma \int_{-1}^1 \frac{\chi(x_1)}{x_1} \, dx_1.$$

On the other hand,

$$\int_{B_1^+(0)} \nabla\chi \cdot \nabla u_\epsilon \, dx = \int_{-1}^1 \mu_\epsilon' \chi \, dx_1 = -\int_{-1}^1 (\mu_\epsilon - 1 + \gamma)\chi' \, dx_1$$

by an integration by parts. Hence

$$\begin{aligned} \lim_{\epsilon \searrow 0} \left(\log \frac{1}{\delta} \int_{B_1^+(0)} \nabla\chi \cdot \nabla u_\epsilon \, dx \right) &= \int_{-1}^1 (\gamma \log|x_1| - \lambda_-)\chi' \, dx_1 \\ &= \chi(-1,0)\lambda_- - \gamma \int_{-1}^1 \frac{\chi(x_1)}{x_1} \, dx_1, \end{aligned}$$

and we obtain $\lambda_- = 0$. Similarly we show that $\lambda_+ = 0$. □

4.5 The core energy

We can now determine the values of the function e in Theorem 2, albeit not explicitly. They arise as the limits in the following result for $\gamma_\pm = 1 \mp \cos\alpha$.

Theorem 25. *For any $\gamma \in (0,2)$, the limit*

$$e_\gamma = \lim_{\epsilon \searrow 0} \left(\left(\log \frac{1}{\delta} \right)^2 \inf_{M_\gamma} E_\epsilon^\gamma - \frac{\pi \gamma^2}{2} \log \frac{1}{\delta} \right)$$

exists.

Definition 26. *The function $e : \{\pm 1\} \to \mathbb{R}$ is defined by*

$$e(-1) = e_{1+\cos\alpha} \quad \textit{and} \quad e(1) = e_{1-\cos\alpha}.$$

Proof of Theorem 25. Define

$$f(\epsilon) = \left(\log \frac{1}{\delta} \right)^2 \inf_{M_\gamma} E_\epsilon^\gamma - \frac{\pi\gamma^2}{2} \log \frac{1}{\delta}.$$

Fix $\epsilon > 0$ small enough and choose a number $R \in (1, \frac{1}{\delta})$. Let $\mu \in M_\gamma$ be the minimiser of E_ϵ^γ and let $v \in W^{1,2}(B_1^+(0))$ be the solution of (44)–(46). Define

$$\tilde{\mu}(x_1) = \begin{cases} \dfrac{\gamma \log \frac{1}{|x_1|}}{\log \frac{1}{\delta}} + 1 - \gamma & \text{if } \frac{1}{R} \leq |x_1| < 1, \\ \left(1 - \dfrac{\log R}{\log \frac{1}{\delta}}\right) \mu(Rx_1) + \dfrac{\log R}{\log \frac{1}{\delta}} & \text{if } |x_1| \leq \frac{1}{R}, \end{cases}$$

and

$$\tilde{v}(x) = \begin{cases} \dfrac{\gamma \log \frac{1}{|x|}}{\log \frac{1}{\delta}} + 1 - \gamma & \text{if } \frac{1}{R} \leq |x| < 1, \\ \left(1 - \dfrac{\log R}{\log \frac{1}{\delta}}\right) v(Rx) + \dfrac{\log R}{\log \frac{1}{\delta}} & \text{if } |x| \leq \frac{1}{R}. \end{cases}$$

Then we have $\tilde{\mu} \in M_\gamma$ and $\tilde{v} \in W_{\tilde{\mu}}^{1,2}(B_1^+(0))$. Moreover, by Proposition 14,

$$1 + \tilde{\mu}(x_1) \geq 1 + \mu(Rx_1) \geq 2 - \gamma > 0$$

and

$$1 - \tilde{\mu}(x_1) = \left(1 - \frac{\log R}{\log \frac{1}{\delta}}\right) (1 - \mu(Rx_1)) > 0$$

for $|x_1| \leq \frac{1}{R}$. Therefore, we compute

$$\frac{\epsilon}{R} \int_{-1/R}^{1/R} \frac{(\tilde{\mu}')^2}{1 - \tilde{\mu}^2} \, dx_1 \leq \left(1 - \frac{\log R}{\log \frac{1}{\delta}}\right) \epsilon \int_{-1}^{1} \frac{(\mu')^2}{1 - \mu^2} \, dx_1.$$

Furthermore,

$$\int_{B_{1/R}^+(0)} |\nabla \tilde{v}|^2 \, dx = \left(1 - \frac{\log R}{\log \frac{1}{\delta}}\right)^2 \int_{B_1^+(0))} |\nabla v|^2 \, dx.$$

We also observe that for $x_1 \in [-1, -\frac{1}{R}) \cup (\frac{1}{R}, 1]$,

$$1 - \tilde{\mu}^2 = \gamma \left(1 - \frac{\log \frac{1}{|x_1|}}{\log \frac{1}{\delta}}\right) \left(2 - \gamma + \frac{\gamma \log \frac{1}{|x_1|}}{\log \frac{1}{\delta}}\right) \geq \gamma(2 - \gamma) \left(1 - \frac{\log \frac{1}{|x_1|}}{\log \frac{1}{\delta}}\right).$$

Hence

$$\begin{aligned} \int_{(-1,-1/R) \cup (1/R,1)} \frac{(\tilde{\mu}')^2}{1 - \tilde{\mu}^2} \, dx_1 &\leq \frac{2\gamma}{(2 - \gamma) \log \frac{1}{\delta}} \int_{1/R}^{1} \frac{dx_1}{x_1^2 \left(\log \frac{1}{\delta} - \log \frac{1}{x_1}\right)} \\ &= \frac{2\gamma}{(2 - \gamma) \log \frac{1}{\delta}} \int_{1}^{R} \frac{ds}{\log \frac{1}{\delta} - \log s}. \end{aligned}$$

Define

$$g(R) = \frac{\gamma}{(2 - \gamma) \log \frac{1}{\delta}} \int_{1}^{R} \frac{ds}{\log \frac{1}{\delta} - \log s}.$$

Finally, we compute

$$\int_{B_1^+(0) \setminus B_{1/R}(0)} |\nabla \tilde{v}|^2 \, dx = \frac{\pi \gamma^2 \log R}{\left(\log \frac{1}{\delta}\right)^2}.$$

Let

$$\tilde{\epsilon} = \tilde{\epsilon}(R) = \frac{\epsilon}{R}\left(1 - \frac{\log R}{\log \frac{1}{\delta}}\right) < \epsilon.$$

Then we have

$$\begin{aligned}
E_{\tilde{\epsilon}}^{\gamma}(\tilde{\mu}) &\leq \frac{\tilde{\epsilon}}{2}\int_{-1}^{1} \frac{(\tilde{\mu}')^2}{1-\tilde{\mu}^2}\,dx_1 + \frac{1}{2}\int_{B_1^+(0)} |\nabla \tilde{v}|^2\,dx \\
&\leq \left(1 - \frac{\log R}{\log \frac{1}{\delta}}\right)^2 E_{\epsilon}^{\gamma}(\mu) + \frac{\epsilon g(R)}{R}\left(1 - \frac{\log R}{\log \frac{1}{\delta}}\right) + \frac{\pi\gamma^2 \log R}{2\left(\log \frac{1}{\delta}\right)^2}.
\end{aligned}$$

It follows that

$$f(\tilde{\epsilon}) \leq \left(\log \frac{1}{\tilde{\delta}}\right)^2 \left(\left(1 - \frac{\log R}{\log \frac{1}{\delta}}\right)^2 E_{\epsilon}^{\gamma}(\mu) + \frac{\epsilon g(R)}{R}\left(1 - \frac{\log R}{\log \frac{1}{\delta}}\right) + \frac{\pi\gamma^2 \log R}{2\left(\log \frac{1}{\delta}\right)^2}\right) - \frac{\pi\gamma^2}{2}\log \frac{1}{\tilde{\delta}},$$

where $\tilde{\delta} = \tilde{\epsilon}\log\frac{1}{\tilde{\epsilon}}$. Since we have equality for $R = 1$, we can use this inequality to estimate the left-hand superdifferential

$$f_-'(\epsilon) = \liminf_{s \nearrow \epsilon} \frac{f(s) - f(\epsilon)}{s - \epsilon}.$$

Indeed, note first that

$$\left.\frac{d\tilde{\epsilon}}{dR}\right|_{R=1} = -\epsilon\left(1 + \frac{1}{\log\frac{1}{\delta}}\right)$$

and

$$\left.\frac{d\tilde{\delta}}{dR}\right|_{R=1} = \epsilon\left(1 - \log\frac{1}{\epsilon}\right)\left(1 + \frac{1}{\log\frac{1}{\delta}}\right).$$

The above inequality therefore implies that for all $\epsilon \in (0, e^{-2}]$,

$$\begin{aligned}
-\epsilon\left(1 + \frac{1}{\log\frac{1}{\delta}}\right) f_-'(\epsilon) &\leq 2\left(\frac{\log\frac{1}{\delta}}{\log\frac{1}{\epsilon}}\left(\log\frac{1}{\epsilon} - 1\right)\left(1 + \frac{1}{\log\frac{1}{\delta}}\right) - \log\frac{1}{\delta}\right) E_{\epsilon}^{\gamma}(\mu) \\
&\quad + \frac{\gamma\epsilon}{2-\gamma} + \frac{\pi\gamma^2}{2} + \frac{\pi\gamma^2}{2\log\frac{1}{\epsilon}}\left(1 - \log\frac{1}{\epsilon}\right)\left(1 + \frac{1}{\log\frac{1}{\delta}}\right) \\
&= \left(\log\log\frac{1}{\epsilon} - 1\right)\left(\frac{2E_{\epsilon}^{\gamma}(\mu)}{\log\frac{1}{\epsilon}} - \frac{\pi\gamma^2}{2\log\frac{1}{\epsilon}\log\frac{1}{\delta}}\right) + \frac{\epsilon\gamma}{2-\gamma} \\
&\leq \frac{C\log\log\frac{1}{\epsilon}}{\left(\log\frac{1}{\epsilon}\right)^2}
\end{aligned}$$

for a constant $C = C(\gamma)$ by Proposition 15. Hence

$$f_-'(\epsilon) \geq -\frac{C\log\log\frac{1}{\epsilon}}{\epsilon\left(\log\frac{1}{\epsilon}\right)^2}.$$

Note that

$$\int_0^{e^{-2}} \frac{\log\log\frac{1}{\epsilon}}{\epsilon\left(\log\frac{1}{\epsilon}\right)^2}\,d\epsilon < \infty.$$

Thus if we denote

$$e_\gamma = \liminf_{\epsilon \searrow 0} f(\epsilon),$$

then for any $\eta > 0$ we can find a number $\epsilon_0 > 0$ such that

$$f(\epsilon_0) \leq e_\gamma + \frac{\eta}{2}$$

and at the same time,

$$\int_\epsilon^{\epsilon_0} f'_-(s)\,ds \geq -\frac{\eta}{2}$$

for any $\epsilon \in (0, \epsilon_0]$. It then follows that

$$f(\epsilon) \leq f(\epsilon_0) - \int_\epsilon^{\epsilon_0} f'_-(s)\,ds \leq e_\gamma + \eta.$$

Hence we have in fact

$$e_\gamma = \lim_{\epsilon \searrow 0} f(\epsilon),$$

as required. □

5 Several walls

We now consider a given $a \in A_N$ and $d \in \{\pm 1\}^N$ and we study magnetisations $m \in M(a,d)$. In particular, we want to estimate $\inf_{M(a,d)} E_\epsilon$ and derive some inequalities for the magnetisation and the stray field in terms of the energy excess

$$E_\epsilon(m) - \inf_{M(a,d)} E_\epsilon.$$

5.1 Upper bound for the minimal energy

The purpose of this section is to prove the following upper bound for the energy by a direct construction. It is a generalization of Proposition 15 to configurations with several walls $a \in A_N$.

Proposition 27. *For every $R \in (0,1]$ there exists a constant $C_0 > 0$ such that for all $\epsilon \in (0, \frac{1}{2}]$, $a \in A_N$ with $\rho(a) \geq R$, and $d \in \{\pm 1\}^N$, the inequality*

$$\inf_{M(a,d)} E_\epsilon \leq \frac{\pi}{2 \log \frac{1}{\delta}} \sum_{n=1}^N (d_n - \cos\alpha)^2 + \frac{C_0}{\left(\log \frac{1}{\delta}\right)^2}$$

holds true.

Proof. We will in fact prove a more explicit estimate, showing that there exists an $m \in M(a,d)$ with

$$E_\epsilon(m) \leq \frac{\sum_{n=1}^N (d_n - \cos\alpha)^2}{2 \log \sqrt{\frac{R^2}{\delta^2} + 1}} \left(\pi + \frac{2}{\sin^2\alpha \log \frac{1}{\epsilon}} \int_{-\infty}^\infty \frac{t^2}{(t^2+1)^2 \log(t^2+1)}\,dt \right). \qquad (59)$$

This will clearly imply the statement of the proposition. The proof is similar to the proof of Proposition 15. Let $\gamma_n = d_n - \cos\alpha$. Define

$$f(x_1) = \frac{\log(x_1^2 + \delta^2) - \log(R^2 + \delta^2)}{\log \delta^2 - \log(R^2 + \delta^2)}$$

and $m_1 : (-1,1) \to [-1,1]$, given by

$$m_1(x_1) = \begin{cases} \cos\alpha + \gamma_n f(x_1 - a_n) & \text{if } x_1 \in (a_n - R, a_n + R) \text{ for } n = 1, \ldots, N, \\ \cos\alpha & \text{else.} \end{cases}$$

Then there exists a function $m_2 : (-1,1) \to [-1,1]$ such that $m = (m_1, m_2) \in M(a,d)$. Suppose that $x_1 \in (a_n - R, a_n + R)$. If $d_n = -1$, then $1 - m_1(x_1) \geq 1 - \cos\alpha$ and

$$1 + m_1(x_1) = (1 + \cos\alpha)\,(1 - f(x_1 - a_n))\,.$$

If $d_n = 1$, then $1 + m_1(x_1) \geq 1 + \cos\alpha$ and

$$1 - m_1(x_1) = (1 - \cos\alpha)\,(1 - f(x_1 - a_n))\,.$$

In both cases,

$$1 - (m_1(x_1))^2 \geq \sin^2\alpha\,(1 - f(x_1 - a_n))\,.$$

Thus as in the proof of Proposition 15, we can estimate

$$\begin{aligned}\int_{a_n - R}^{a_n + R} |m'|^2\,dx_1 &\leq \frac{2\gamma_n^2}{\sin^2\alpha \log\sqrt{\frac{R^2}{\delta^2} + 1}} \int_{-R}^{R} \frac{x_1^2}{(x_1^2 + \delta^2)^2 \log\left(\frac{x_1^2}{\delta^2} + 1\right)}\,dx_1 \\ &\leq \frac{2\gamma_n^2}{\delta \sin^2\alpha \log\sqrt{\frac{R^2}{\delta^2} + 1}} \int_{-\infty}^{\infty} \frac{t^2}{(t^2+1)^2 \log(t^2+1)}\,dt.\end{aligned}$$

Summing over n, we find

$$\int_{-1}^{1} |m'|^2\,dt \leq \frac{2\sum_{n=1}^N \gamma_n^2}{\delta \sin^2\alpha \log\sqrt{\frac{R^2}{\delta^2} + 1}} \int_{-\infty}^{\infty} \frac{t^2}{(t^2+1)^2 \log(t^2+1)}\,dt.$$

Now consider the function $u = U(m)$ as defined on page 3. Since $\operatorname{curl} \nabla^\perp u = 0$, there exists a function $v \in \dot{W}^{1,2}(\mathbb{R}^2_+)$ such that $\nabla v = \nabla^\perp u$ in $\mathbb{R}^2_+$. Since this means that $v' = m_1'$ on $(-1,1) \times \{0\}$, we can choose v such that $v = m_1 - \cos\alpha$ on $(-1,1) \times \{0\}$. Then we also have $v = 0$ on $(-\infty,-1) \times \{0\}$ and on $(1,\infty) \times \{0\}$, and of course $\Delta v = 0$ in $\mathbb{R}^2_+$. Furthermore, the function has finite Dirichlet energy, and it follows that it is the unique minimiser of the Dirichlet energy under these boundary conditions.

Define $w : \mathbb{R}^2_+ \to \mathbb{R}$ by

$$w(a_n + r\cos\theta, r\sin\theta) = m_1(a_n + r) - \cos\alpha \quad \text{for } 0 < r \leq R \text{ and } 0 \leq \theta \leq \pi,\ n = 1, \ldots, N,$$

while $w = 0$ in $\Omega_R(a)$. Then we compute, similarly to the proof of Proposition 15, that

$$\begin{aligned}\int_{B_R^+(a_n,0)} |\nabla w|^2\,dx &= \frac{\pi\gamma_n^2}{\left(\log\sqrt{\frac{R^2}{\delta^2}+1}\right)^2} \int_0^R \frac{r^3}{(r^2+\delta^2)^2}\,dr \\ &= \frac{\pi\gamma_n^2}{2\left(\log\sqrt{\frac{R^2}{\delta^2}+1}\right)^2} \int_{\delta^2}^{R^2+\delta^2} \frac{r - \delta^2}{r^2}\,dr \leq \frac{\pi\gamma_n^2}{\log\sqrt{\frac{R^2}{\delta^2}+1}}.\end{aligned}$$

Hence

$$\int_{\mathbb{R}^2_+} |\nabla w|^2\,dx \leq \frac{\pi\sum_{n=1}^N \gamma_n^2}{\log\sqrt{\frac{R^2}{\delta^2}+1}}.$$

In particular

$$\int_{\mathbb{R}^2_+} |\nabla u|^2\,dx = \int_{\mathbb{R}^2_+} |\nabla v|^2\,dx \leq \int_{\mathbb{R}^2_+} |\nabla w|^2\,dx \leq \frac{\pi\sum_{n=1}^N \gamma_n^2}{\log\sqrt{\frac{R^2}{\delta^2}+1}}.$$

If we combine these inequalities, then we obtain (59). □

5.2 Stray field estimates

The following is the main result of this section and one of the key ingredients for the proof of Theorem 2. We recall the function $u^*_{a,d}$ from Section 2, solving $\Delta u^*_{a,d} = 0$ in $\mathbb{R}^2_+$ and $\frac{\partial u^*_{a,d}}{\partial x_2} = 0$ on $(-\infty, -1) \times \{0\}$ and $(1, \infty) \times \{0\}$, and with a piecewise constant trace on $(-1, 1) \times \{0\}$ given by the values

$$\sigma_n = \frac{\pi}{2} \left(\sum_{k=n+1}^{N} (d_k - \cos\alpha) - \sum_{k=1}^{n} (d_k - \cos\alpha) \right).$$

It has the property that for $n = 1, \ldots, N$, the function

$$x \mapsto u^*_{a,d}(x) - (d_n - \cos\alpha) \left(\arctan\left(\frac{x_2}{x_1 - a_n} \right) - \frac{\pi(x_1 - a_n)}{2|x_1 - a_n|} \right)$$

is harmonic in $B^+_{\rho(a)}(a_n, 0)$ and constant on $(a_n - \rho(a), a_n + \rho(a)) \times \{0\}$. Standard elliptic estimates then imply that this function is smooth near $(a_n, 0)$. In view of the energy estimates (30) and (31), we can make more quantitative statements as well: if $\rho(a) \geq R > 0$, then

$$\left| \frac{\partial u^*_{a,d}}{\partial x_1}(x) + \frac{(d_n - \cos\alpha)x_2}{(x_1 - a_n)^2 + x_2^2} \right| + \left| \frac{\partial u^*_{a,d}}{\partial x_2}(x) - \frac{(d_n - \cos\alpha)(x_1 - a_n)}{(x_1 - a_n)^2 + x_2^2} \right| \leq C \tag{60}$$

for $x \in B^+_{\rho(a)}(a_n, 0)$, where $C = C(\alpha, N, R)$.

Theorem 28. *For any $R \in (0, \frac{1}{2}]$ and $C_0 > 0$, there exists a constant $C_1 > 0$ such the following holds true. Let $a \in A_N$ with $\rho(a) \geq R$ and $d \in \{\pm 1\}^N$. Set*

$$\Gamma = \sum_{n=1}^{N} (d_n - \cos\alpha)^2.$$

Suppose that $\epsilon \in (0, \frac{1}{2}]$ with $\delta \leq R$ and $m \in M(a, d)$ with

$$E_\epsilon(m) \leq \frac{\pi\Gamma}{2\log\frac{1}{\delta}} + \frac{C_0}{\left(\log\frac{1}{\delta}\right)^2}. \tag{61}$$

Let $u = U(m)$ be the function defined on page 3. Then

$$\int_{\Omega_\delta(a)} \left| \nabla u - \frac{\nabla u^*_{a,d}}{\log\frac{1}{\delta}} \right|^2 dx \leq \frac{C_1}{\left(\log\frac{1}{\delta}\right)^2} \tag{62}$$

and

$$\int_{\Omega_\delta(a)} |\nabla u|^2 \, dx \geq \frac{\pi\Gamma}{\log\frac{1}{\delta}} - \frac{C_1}{\left(\log\frac{1}{\delta}\right)^2}. \tag{63}$$

This statement is somewhat similar to Theorem 17. The main difference, apart from the fact that we consider several Néel walls here, is that we only assume a suitable bound for the energy, whereas in Theorem 17, we study minimisers. Before we prove the theorem, we establish the following auxiliary result.

Lemma 29. *Let $s > 0$ and $\mu \in W^{1,2}(-s, s)$. If $\mu(0) = 1$ and $|\mu| \leq 1$, then*

$$\int_{-s}^{s} |\mu'| \, dx_1 \leq 2s \int_{-s}^{s} \frac{(\mu')^2}{1 - \mu^2} \, dx_1.$$

Proof. By the Cauchy-Schwarz inequality, we have

$$\int_{-s}^{s} |\mu'|\, dx_1 \le \left(\int_{-s}^{s} \frac{(\mu')^2}{1-\mu^2}\, dx_1\right)^{1/2} \left(\int_{-s}^{s} (1-\mu^2)\, dx_1\right)^{1/2}. \tag{64}$$

Since $\mu(0) = 1$, any $x_1 \in (-s, s)$ will satisfy

$$\begin{aligned} 1 - (\mu(x_1))^2 &= -2 \int_0^{x_1} \mu(t)\mu'(t)\, dt \\ &\le 2 \left| \int_0^{x_1} \frac{(\mu')^2}{1-\mu^2}\, dt \right|^{1/2} \left| \int_0^{x_1} \mu^2(1-\mu^2)\, dt \right|^{1/2}. \end{aligned}$$

Integrating over $(-s, s)$, recalling that $|\mu| \le 1$, and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \int_{-s}^{s} (1-\mu^2)\, dx_1 &\le 2 \left(\int_{-s}^{s} \left| \int_0^{x_1} \frac{(\mu')^2}{1-\mu^2}\, dt \right| dx_1\right)^{1/2} \left(\int_{-s}^{s} \left| \int_0^{x_1} (1-\mu^2)\, dt \right| dx_1\right)^{1/2} \\ &\le 2s \left(\int_{-s}^{s} \frac{(\mu')^2}{1-\mu^2}\, dx_1\right)^{1/2} \left(\int_{-s}^{s} (1-\mu^2)\, dx_1\right)^{1/2}, \end{aligned}$$

which leads to

$$\int_{-s}^{s} (1-\mu^2)\, dx_1 \le 4s^2 \int_{-s}^{s} \frac{(\mu')^2}{1-\mu^2}\, dx_1.$$

Combining this with (64), we obtain the desired inequality. ∎

Proof of Theorem 28. It is clear that it suffices to prove the inequalities for small values of ϵ. We modify the functions $u^*_{a,d}$ as follows: for a fixed $s \in (0, R]$, let $\xi_s \in \dot{W}^{1,2}(\mathbb{R}^2_+)$ be such that

$$\xi_s(x) = \frac{u^*_{a,d}(x)}{\log \frac{1}{\delta}}$$

for $x \in \Omega_s(a)$ and

$$\xi_s(a_n + r\cos\theta, r\sin\theta) = \frac{r u^*_{a,d}(a_n + s\cos\theta, s\sin\theta)}{s \log \frac{1}{\delta}} + \left(1 - \frac{r}{s}\right) \frac{\sigma_{n-1} + \sigma_n}{2\log \frac{1}{\delta}}$$

for $0 < r < s$, $0 < \theta < \pi$, and $n = 1, \ldots, N$. Then we have

$$\int_{\mathbb{R}^2_+} |\nabla \xi_s|^2\, dx \le \frac{\pi\Gamma \log \frac{1}{s} + C_1}{\left(\log \frac{1}{\delta}\right)^2} \tag{65}$$

for a constant $C_1 = C_1(\alpha, N, R) > 0$. This follows from (30), (31), and (60).

We observe that

$$\begin{aligned} \frac{\pi\Gamma}{\log \frac{1}{\delta}} &= \int_{-1}^{1} \frac{u^*_{a,d}(x_1, 0)}{\log \frac{1}{\delta}} m_1'(x_1)\, dx_1 \\ &= \int_{-1}^{1} \xi_s(x_1, 0) m_1'(x_1)\, dx_1 - \int_{-1}^{1} \left(\xi_s(x_1, 0) - \frac{u^*_{a,d}(x_1, 0)}{\log \frac{1}{\delta}}\right) m_1'(x_1)\, dx_1 \\ &= \int_{\mathbb{R}^2_+} \nabla \xi_s \cdot \nabla u\, dx - \int_{-1}^{1} \left(\xi_s(x_1, 0) - \frac{u^*_{a,d}(x_1, 0)}{\log \frac{1}{\delta}}\right) m_1'(x_1)\, dx_1. \end{aligned}$$

We have a constant $C_2 = C_2(\alpha, N, R) > 0$ such that

$$\left| \int_{-1}^{1} \left(\xi_s(x_1, 0) - \frac{u^*_{a,d}(x_1, 0)}{\log \frac{1}{\delta}} \right) m_1'(x_1)\, dx_1 \right| \leq \frac{C_2}{\log \frac{1}{\delta}} \sum_{n=1}^{N} \int_{a_n - s}^{a_n + s} |m_1'|\, dx_1.$$

Thus by Lemma 29,

$$\left| \int_{-1}^{1} \left(\xi_s(x_1, 0) - \frac{u^*_{a,d}(x_1, 0)}{\log \frac{1}{\delta}} \right) m_1'(x_1)\, dx_1 \right| \leq \frac{2C_2 s}{\epsilon \log \frac{1}{\delta}} \left(2E_\epsilon(m) - \|\nabla u\|^2_{L^2(\mathbb{R}^2_+)} \right).$$

We conclude that

$$\frac{\pi\Gamma}{\log \frac{1}{\delta}} \leq \frac{2C_2 s}{\epsilon \log \frac{1}{\delta}} \left(2E_\epsilon(m) - \|\nabla u\|^2_{L^2(\mathbb{R}^2_+)} \right) + \int_{\mathbb{R}^2_+} \nabla \xi_s \cdot \nabla u\, dx. \tag{66}$$

Using the Cauchy-Schwarz inequality and (65), we obtain

$$\frac{\pi\Gamma}{\log \frac{1}{\delta}} \leq \frac{2C_2 s}{\epsilon \log \frac{1}{\delta}} \left(2E_\epsilon(m) - \|\nabla u\|^2_{L^2(\mathbb{R}^2_+)} \right) + \frac{\sqrt{\pi\Gamma \log \frac{1}{s} + C_1}}{\log \frac{1}{\delta}} \|\nabla u\|_{L^2(\mathbb{R}^2_+)}. \tag{67}$$

We want to use this inequality to prove (63) first. For this purpose, we choose $s \in (0, R]$ such that

$$\|\nabla u\|^2_{L^2(\mathbb{R}^2_+)} = \frac{\pi\Gamma}{\log \frac{1}{s}} - \frac{2C_1}{\left(\log \frac{1}{s}\right)^2}. \tag{68}$$

This is possible whenever ϵ is sufficiently small because of (61). Then (67) and (61) imply

$$\begin{aligned}
\frac{\pi\Gamma}{\log \frac{1}{\delta}} &\leq \frac{2C_2 s}{\epsilon \log \frac{1}{\delta}} \left(\frac{\pi\Gamma}{\log \frac{1}{\delta}} - \frac{\pi\Gamma}{\log \frac{1}{s}} + \frac{2C_0}{\left(\log \frac{1}{\delta}\right)^2} + \frac{2C_1}{\left(\log \frac{1}{s}\right)^2} \right) \\
&\quad + \frac{\sqrt{\pi^2\Gamma^2 \left(\log \frac{1}{s}\right)^2 - C_1 \pi\Gamma \log \frac{1}{s} - 2C_1^2}}{\log \frac{1}{\delta} \log \frac{1}{s}} \\
&\leq \frac{2C_2 s}{\epsilon \log \frac{1}{\delta}} \left(\frac{\pi\Gamma}{\log \frac{1}{\delta}} - \frac{\pi\Gamma}{\log \frac{1}{s}} + \frac{2C_0}{\left(\log \frac{1}{\delta}\right)^2} + \frac{2C_1}{\left(\log \frac{1}{s}\right)^2} \right) \\
&\quad + \frac{\pi\Gamma}{\log \frac{1}{\delta}} \left(1 - \frac{C_1}{2\pi\Gamma \log \frac{1}{s}} - \frac{C_1^2}{\pi^2\Gamma^2 \left(\log \frac{1}{s}\right)^2} \right),
\end{aligned}$$

because $\sqrt{1 - \beta} \leq 1 - \frac{\beta}{2}$ for $\beta \in (0, 1)$. That is,

$$\frac{C_1}{2\pi\Gamma} + \frac{C_1^2}{\pi^2\Gamma^2 \log \frac{1}{s}} \leq \frac{2C_2 s}{\epsilon \log \frac{1}{\delta}} \left(\log \frac{1}{s} - \log \frac{1}{\delta} + \frac{2C_0 \log \frac{1}{s}}{\pi\Gamma \log \frac{1}{\delta}} + \frac{2C_1 \log \frac{1}{\delta}}{\pi\Gamma \log \frac{1}{s}} \right).$$

In particular, there exist certain constants C_3, C_4, C_5, C_6, all of them positive and depending only on α, N, C_0, and R, such that

$$C_3 + \frac{C_4}{\log \frac{1}{s}} \leq \frac{s}{\delta} \left(\log \frac{\delta}{s} + \frac{C_5 \log \frac{1}{s}}{\log \frac{1}{\delta}} + \frac{C_6 \log \frac{1}{\delta}}{\log \frac{1}{s}} \right). \tag{69}$$

We want to use this inequality to show that there exists a constant $C_7 = C_7(\alpha, R, N, C_0) > 0$ such that $C_7 s \geq \delta$. Then (63) follows immediately from (68), because the right hand side is increasing in s when $\log \frac{1}{s} \geq 4C_1/(\pi\Gamma)$, which is the case for ϵ small due to (61).

To this end, let $c = \min\{1, \frac{C_3}{4C_6}\}$. We distinguish three cases.

Case 1. If $s \geq c\delta$, then the claim is obvious.

Case 2. If $s < c\delta$ and

$$s \log \frac{1}{s} \geq \frac{C_3}{2C_5} \delta \log \frac{1}{\delta},$$

then it follows that $s \geq \delta^2$, provided that ϵ is small enough. (Otherwise we would have an immediate contradiction to the assumptions for this case.) Hence $\log \frac{1}{s} \leq 2 \log \frac{1}{\delta}$ and

$$s \geq \frac{C_3 \delta}{4C_5}.$$

Case 3. If $s < c\delta$ and

$$s \log \frac{1}{s} < \frac{C_3}{2C_5} \delta \log \frac{1}{\delta},$$

then we obtain

$$\frac{C_3}{2} + \frac{C_4}{\log \frac{1}{s}} \leq \frac{s}{\delta} \left(\log \frac{\delta}{s} + \frac{C_6 \log \frac{1}{\delta}}{\log \frac{1}{s}} \right)$$

from (69). We also have $\log \frac{1}{s} > \log \frac{1}{\delta}$ (since $c \leq 1$). Hence

$$\frac{s \log \frac{1}{\delta}}{\delta \log \frac{1}{s}} \leq c \leq \frac{C_3}{4C_6}.$$

Hence

$$\frac{C_3}{4} \leq \frac{s}{\delta} \log \frac{\delta}{s},$$

which implies the claim.

This concludes the proof of (63).

Now we go back to inequality (66) and use it for $s = \delta$ in order to prove (62). Since we now have (63), the inequality implies that

$$\int_{\mathbb{R}^2_+} \nabla \xi_\delta \cdot \nabla u \, dx \geq \frac{\pi \Gamma}{\log \frac{1}{\delta}} - \frac{C_8}{\left(\log \frac{1}{\delta}\right)^2} \tag{70}$$

for a constant $C_8 = C_8(\alpha, N, C_0, R)$. Hence

$$\begin{aligned} \int_{\mathbb{R}^2_+} |\nabla u - \nabla \xi_\delta|^2 \, dx &\leq 2E_\epsilon(m) - 2 \int_{\mathbb{R}^2_+} \nabla \xi_\delta \cdot \nabla u \, dx + \int_{\mathbb{R}^2_+} |\nabla \xi_\delta|^2 \, dx \\ &\leq \frac{2C_0 + 2C_8 + C_1}{\left(\log \frac{1}{\delta}\right)^2} \end{aligned}$$

by (61), (65) and (70). Since ξ_δ coincides with $u^*_{a,d} / \log \frac{1}{\delta}$ in $\Omega_\delta(a)$, this finally implies (62). □

Remark 30. *Once we have* (61) *and* (63) *in Theorem 28, we can also derive the inequalities*

$$\epsilon \int_{-1}^{1} (\varphi')^2 \, dx_1 \leq \frac{C}{\left(\log \frac{1}{\delta}\right)^2} \quad \text{and} \quad \int_{a_n - \delta}^{a_n + \delta} \sin^2 \varphi \, dx_1 \leq \frac{C\delta}{\log \frac{1}{\delta}}$$

for a lifting φ of m and for $n = 1, \ldots, N$, where $C = C(\alpha, N, R, C_0)$. The first inequality is an immediate consequence of (61) *and* (63)*, and the second follows with the same arguments as in the proof of Lemma 13. These estimates are similar to* (42) *and* (43)*, but now we know that they hold true for non-minimizing configurations (under the energy control* (61)*) and the Pohozaev identity previously used for the proof of* (42) *and* (43) *is no longer needed.*

Remark 31. *Inequalities* (61) *and* (63) *further imply that*

$$\int_{B_\delta^*(a)} |\nabla u|^2 \, dx \le \frac{2C_0 + C_1}{\left(\log \frac{1}{\delta}\right)^2}.$$

If we combine this estimate with (62)*, then we also obtain*

$$\int_{B_r^*(a)} |\nabla u|^2 \, dx \le \frac{1}{\left(\log \frac{1}{\delta}\right)^2} \left(2C_0 + 3C_1 + 2 \int_{B_r^*(a) \setminus B_\delta^*(a)} |\nabla u_{a,d}^*|^2 \, dx \right)$$

for any $r > 0$. Furthermore, since $u_{a,d}^*$ is known quite explicitly, the last integral is typically not too difficult to estimate.

6 Proof of the main result

We now prove Theorem 2. To this end, fix $a \in A_N$ and $d \in \{\pm 1\}^N$. Set $\gamma_n = d_n - \cos\alpha$ for $n = 1, \ldots, N$ and

$$\Gamma = \sum_{n=1}^N \gamma_n^2.$$

Furthermore, set $\gamma_\pm = 1 \mp \cos\alpha$ and recall Definition 26, which introduces the function $e : \{\pm 1\} \to \mathbb{R}$ with

$$e(\pm 1) = \lim_{\epsilon \searrow 0} \left(\left(\log \frac{1}{\delta} \right)^2 \inf_{M_{\gamma_\pm}} E_\epsilon^{\gamma_\pm} - \frac{\pi \gamma_\pm^2}{2} \log \frac{1}{\delta} \right).$$

We divide the identity from Theorem 2 into two inequalities, which are proved in Section 6.2 and Section 6.3, respectively, after some preparation. Throughout the proof, we indiscriminately write C for various positive constants that depend only on α, N, a, d, and occasionally on the exponents of L^p-spaces appearing in the context (always denoted by p or q).

6.1 Preparation

Define

$$w_0(x) = \arctan\left(\frac{x_2}{x_1}\right) - \frac{\pi x_1}{2|x_1|}$$

for $x \in \mathbb{R}^2_+$. Recall the functions

$$u_{a,d}^* = \sum_{n=1}^N \gamma_n u_{a_n} \quad \text{and} \quad \mu_{a,d}^* = \sum_{n=1}^N \gamma_n \mu_{a_n}$$

from Section 2. Consider a number $r \in (0, \rho(a)]$. For $n = 1, \ldots, N$, let

$$\lambda_n = \gamma_n \log(2 - 2a_n^2) + \sum_{k \neq n} \gamma_k \mu_{a_k}(a_n)$$

again and recall estimate (36), which implies that

$$\left| \mu_{a,d}^*(x_1) - \lambda_n - \gamma_n \log \frac{1}{|x_1 - a_n|} \right| \le Cr \tag{71}$$

for $x_1 \in [a_n - r, a_n + r]$. Also define

$$\omega_n = \sum_{k \neq n} \gamma_k u_{a_k}(a_n).$$

Then

$$|u^*_{a,d}(x) - \omega_n - \gamma_n w_0(x_1 - a_n, x_2)| \leq Cr \quad \text{in } B_r(a_n, 0) \tag{72}$$

and

$$|\nabla u^*_{a,d}(x) - \gamma_n \nabla w_0(x_1 - a_n, x_2)| \leq C \quad \text{in } B_r(a_n, 0) \tag{73}$$

for $1 \leq n \leq N$, because $u^*_{a,d}(x) - \omega_n - \gamma_n w_0(x_1 - a_n, x_2)$ is a smooth function that vanishes at $(a_n, 0)$.

We now study how δ changes when we replace ϵ by ϵ/r for a number $r \in (\epsilon, 1]$, since we will have to rescale the magnetisation about the centres of the Néel walls. We have

$$\log\left(\frac{\epsilon}{r}\log\frac{r}{\epsilon}\right) = \log\delta - \log r + \log\left(1 - \frac{\log r}{\log\epsilon}\right).$$

Since $\log(1-\xi) \leq -\xi$ for $\xi \in (0,1)$, we obtain

$$\log\left(\frac{\epsilon}{r}\log\frac{r}{\epsilon}\right) \leq \log\delta - \log r - \frac{\log r}{\log\epsilon}. \tag{74}$$

Similarly, if ϵ is sufficiently small (for a fixed r), then

$$\log\left(\frac{\epsilon}{r}\log\frac{r}{\epsilon}\right) \geq \log\delta - \log r - \frac{2\log r}{\log\epsilon}. \tag{75}$$

6.2 A lower bound for the interaction energy

The purpose of this section is to prove the inequality

$$\liminf_{\epsilon \searrow 0}\left(\left(\log\frac{1}{\delta}\right)^2 \inf_{M(a,d)} E_\epsilon - \frac{\pi\Gamma}{2}\log\frac{1}{\delta}\right) \geq \sum_{n=1}^N e(d_n) + W(a,d) \tag{76}$$

for the function W defined at the end of Section 2, which amounts to half of the statement of Theorem 2.

First step: use minimisers Clearly it is sufficient to consider functions $m_\epsilon \in W^{1,2}((-1,1);\mathbb{S}^1)$ that minimise E_ϵ in $M(a,d)$. Then

$$\limsup_{\epsilon \searrow 0}\left(\left(\log\frac{1}{\delta}\right)^2 E_\epsilon(m_\epsilon) - \frac{\pi\Gamma}{2}\log\frac{1}{\delta}\right) < \infty \tag{77}$$

by Proposition 27.

Second step: prove convergence away from the walls This part of the proof is similar to the proof of Theorem 22. Let $\varphi_\epsilon \in W^{1,2}(-1,1)$ such that $m_\epsilon = (\cos\varphi_\epsilon, \sin\varphi_\epsilon)$. Define $u_\epsilon = U(m_\epsilon)$ and $v_\epsilon = u_\epsilon \log\frac{1}{\delta}$. Then by Theorem 28, we have a sequence $\epsilon_k \searrow 0$ such that $v_{\epsilon_k} \rightharpoonup v$ weakly in $\dot{W}^{1,2}(\Omega_r(a))$ for every $r > 0$ for some function

$$v \in u^*_{a,d} + \dot{W}^{1,2}(\mathbb{R}^2_+).$$

In fact, for any fixed $r > 0$, owing to Lemma 24 (with $\Omega = \mathbb{R}^2_+$), Theorem 28, and Remark 30, there exists a number $\beta > 0$ (depending on r) such that $|\sin \varphi_\epsilon| \geq \beta$ at distance at least r away from $-1, a_1, \ldots, a_N, 1$ for ϵ small enough. For $r, R > 0$, define $\Sigma_{r,R}(a) = (\Omega_r(a) \cap B^+_R(0)) \backslash (B_r(-1,0) \cup B_r(1,0))$. Then we can use Lemma 11 and standard elliptic estimates to obtain uniform estimates in $W^{2,2}(\Sigma_{r,R}(a))$ for any $r, R > 0$ and any sufficiently small ϵ. Therefore, we even have $v_{\epsilon_k} \rightharpoonup v$ weakly in $W^{2,2}(\Sigma_{r,R}(a))$ for all $r, R > 0$. Furthermore, we have $v_{\epsilon_k} \to v$ uniformly in $\mathbb{R}^2_+ \backslash B_2(0)$ by standard estimates for the Laplace equation. It follows that $\lim_{|x|\to\infty} v(x) = 0$.

Obviously $\Delta v = 0$ in $\mathbb{R}^2_+$ and $\frac{\partial v}{\partial x_2} = 0$ on $(-1, -\infty) \times \{0\}$ and on $(1, \infty) \times \{0\}$. By Lemma 11, we also have

$$\limsup_{\epsilon \searrow 0} \left(\left(\log \frac{1}{\delta} \right)^2 \epsilon \int_{a_n + r}^{a_{n+1} - r} (\varphi''_\epsilon)^2 \, dx_1 \right) < \infty$$

for $n = 1, \ldots, N-1$ and any $r > 0$, and we have similar inequalities in $(r-1, a_1 - r)$ and in $(a_N + r, 1 - r)$. Since

$$u'_\epsilon = \epsilon \varphi''_\epsilon / \sin \varphi_\epsilon$$

by (37), we conclude that $v(\,\cdot\,, 0)$ is locally constant in $(-1,1) \backslash \{a_1, \ldots, a_N\}$. But there is only one function in the space $u^*_{a,d} + \dot{W}^{1,2}(\mathbb{R}^2_+)$ with these properties (which can be seen with the arguments from the proof of Theorem 22), and thus we have

$$v = u^*_{a,d} \quad \text{in } \mathbb{R}^2_+.$$

Since $\frac{\partial v_\epsilon}{\partial x_2}(\,\cdot\,, 0) = -m'_{1\epsilon} \log \frac{1}{\delta}$ on $(-1,1) \backslash \{a_1, \ldots, a_N\}$, it also follows that there exists a sequence $\epsilon_k \searrow 0$ such that

$$(m_{1\epsilon_k} - \cos \alpha) \log \frac{1}{\delta_k} \to \nu$$

locally uniformly in $(-1,1) \backslash \{a_1, \ldots, a_N\}$ for a function $\nu : (-1,1) \to [-\infty, \infty]$ such that $\mu^*_{a,d} - \nu$ is locally constant in $(-1,1) \backslash \{a_1, \ldots, a_N\}$, where $\delta_k = \epsilon_k \log \frac{1}{\epsilon_k}$. With the same arguments as in the proof of Theorem 22, we show that $\nu = \mu^*_{a,d}$ and

$$(m_{1\epsilon} - \cos \alpha) \log \frac{1}{\delta} \to \mu^*_{a,d} \quad \text{locally uniformly in } (-1,1) \backslash \{a_1, \ldots, a_N\}.$$

Now for any $r \in (0, \rho(a)]$, we have

$$\int_{\Omega_r(a)} |\nabla u^*_{a,d}|^2 \, dx \leq \liminf_{\epsilon \searrow 0} \int_{\Omega_r(a)} |\nabla v_\epsilon|^2 \, dx = \liminf_{\epsilon \searrow 0} \left(\left(\log \frac{1}{\delta} \right)^2 \int_{\Omega_r(a)} |\nabla u_\epsilon|^2 \, dx \right). \tag{78}$$

Furthermore, by (71), we have

$$\left| m_{1\epsilon}(a_n \pm r) - \cos \alpha - \frac{\gamma_n \log \frac{1}{r} + \lambda_n}{\log \frac{1}{\delta}} \right| \leq \frac{Cr}{\log \frac{1}{\delta}} + \frac{o(1)}{\log \frac{1}{\delta}}. \tag{79}$$

Here and subsequently, we use the notation $o(1)$ for any quantity that converges to 0 as $\epsilon \searrow 0$, with a rate of convergence possibly depending on r.

Third step: rescale the cores Fix $n \in \{1, \ldots, N\}$ and $r \in (0, \rho(a)]$, and define the functions $\tilde{m}_\epsilon : (-1,1) \to \mathbb{S}^1$ and $\tilde{u}_\epsilon : \mathbb{R}^2_+ \to \mathbb{R}$ and the number $\tilde{\epsilon}$ by

$$\begin{aligned} \tilde{m}_\epsilon(x_1) &= m_\epsilon(rx_1 + a_n), \\ \tilde{u}_\epsilon(x) &= u_\epsilon(rx_1 + a_n, rx_2), \\ \tilde{\epsilon} &= \frac{\epsilon}{r}. \end{aligned}$$

Then we have

$$\tilde{\epsilon} \int_{-1}^{1} |\tilde{m}_\epsilon'|^2 \, dx_1 = \epsilon \int_{a_n - r}^{a_n + r} |m_\epsilon'|^2 \, dx_1 \tag{80}$$

and

$$\int_{B_1^+(0)} |\nabla \tilde{u}_\epsilon|^2 \, dx = \int_{B_r^+(a_n, 0)} |\nabla u_\epsilon|^2 \, dx. \tag{81}$$

Moreover, if we denote $\tilde{u}^*(x) = u_{a,d}^*(rx_1 + a_n, rx_2)$ and $\tilde{v}_\epsilon = \tilde{u}_\epsilon \log \frac{1}{\delta}$, then by the observations in the second step, we have $\tilde{v}_\epsilon - \tilde{u}^* \rightharpoonup 0$ weakly in $W^{2,2}(B_1^+(0) \backslash B_{1/2}(0))$. In particular, if we fix a number $q > 2$, then we have strong convergence of the boundary data in $W^{1,q}(\partial^+ B_1^+(0))$. Because of (73), we have

$$\|x \cdot \nabla \tilde{u}^*(x)\|_{L^\infty(\partial^+ B_1(0))} \leq Cr.$$

Hence

$$\|x \cdot \nabla \tilde{u}_\epsilon\|_{L^q(\partial^+ B_1(0))} \leq \frac{Cr + o(1)}{\log \frac{1}{\delta}}.$$

From Theorem 28, Remark 31, and inequality (73), we also obtain the inequality

$$\left\| \nabla \tilde{u}_\epsilon - \frac{\gamma_n x^\perp}{|x|^2 \log \frac{1}{\delta}} \right\|_{L^2(B_1^+(0) \backslash B_{\tilde{\delta}}(0))} + \|\nabla \tilde{u}_\epsilon\|_{L^2(B_{\tilde{\delta}}^+(0))} \leq \frac{C}{\log \frac{1}{\delta}},$$

where $\tilde{\delta} = \tilde{\epsilon} \log \frac{1}{\tilde{\epsilon}}$. We then also obtain

$$\|x \cdot \nabla \tilde{u}_\epsilon\|_{L^q(\partial^+ B_1(0))} \leq \frac{Cr + o(1)}{\log \frac{1}{\tilde{\delta}}}$$

and

$$\left\| \nabla \tilde{u}_\epsilon - \frac{\gamma_n x^\perp}{|x|^2 \log \frac{1}{\tilde{\delta}}} \right\|_{L^2(B_1^+(0) \backslash B_{\tilde{\delta}}(0))} + \|\nabla \tilde{u}_\epsilon\|_{L^2(B_{\tilde{\delta}}^+(0))} \leq \frac{C}{\log \frac{1}{\tilde{\delta}}}.$$

Because of this and (79), we may apply Corollary 21 to $d_n \tilde{m}_{1\epsilon}$ with

$$\gamma = d_n \gamma_n, \quad \eta = Cr + o(1), \quad \text{and} \quad \zeta = d_n \left(\lambda_n + \gamma_n \log \frac{1}{r} \right) + Cr + o(1).$$

In view of Definition 26, we conclude that

$$\left(\log \frac{1}{\tilde{\delta}} \right)^2 \left(\tilde{\epsilon} \int_{-1}^{1} |\tilde{m}_\epsilon'|^2 \, dx_1 + \int_{B_1^+(0)} |\nabla \tilde{u}_\epsilon|^2 \, dx \right) \geq \pi \gamma_n^2 \log \frac{1}{\tilde{\delta}} + 2e(d_n) - 2\pi \gamma_n^2 \log \frac{1}{r} - 2\pi \gamma_n \lambda_n - Cr - o(1). \tag{82}$$

We recall (74) and (75). Now (80), (81), and (82) imply that

$$\left(\log \frac{1}{\delta} - \log \frac{1}{r} \right)^2 \left(\epsilon \int_{a_n - r}^{a_n + r} |m_\epsilon'|^2 \, dx_1 + \int_{B_r^+(a_n, 0)} |\nabla u_\epsilon|^2 \, dx \right) - \pi \gamma_n^2 \left(\log \frac{1}{\delta} - \log \frac{1}{r} \right) \geq 2e(d_n) - 2\pi \gamma_n^2 \log \frac{1}{r} - 2\pi \gamma_n \lambda_n - Cr - o(1).$$

Since this means in particular that

$$\log \frac{1}{\delta} \left(\epsilon \int_{a_n - r}^{a_n + r} |m_\epsilon'|^2 \, dt + \int_{B_r^+(a_n,0)} |\nabla u_\epsilon|^2 \, dx \right) \geq \pi \gamma_n^2 + o(1),$$

and since we have (77), the inequality also yields

$$\left(\log \frac{1}{\delta} \right)^2 \left(\epsilon \int_{a_n - r}^{a_n + r} |m_\epsilon'|^2 \, dt + \int_{B_r^+(a_n,0)} |\nabla u_\epsilon|^2 \, dx \right) - \pi \gamma_n^2 \left(\log \frac{1}{\delta} - \log \frac{1}{r} \right)$$
$$\geq 2e(d_n) - 2\pi \gamma_n \lambda_n - Cr - o(1).$$

Fourth step: combine the estimates If we sum over n and use (78), we obtain

$$\left(\log \frac{1}{\delta} \right)^2 E_\epsilon(m_\epsilon) - \frac{\pi \Gamma}{2} \left(\log \frac{1}{\delta} - \log \frac{1}{r} \right) \geq \sum_{n=1}^N (e(d_n) - \pi \gamma_n \lambda_n) + \frac{1}{2} \int_{\Omega_r(a)} |\nabla u_{a,d}^*|^2 \, dx - Cr - o(1).$$

Thus

$$\liminf_{\epsilon \searrow 0} \left(\left(\log \frac{1}{\delta} \right)^2 E_\epsilon(m_\epsilon) - \frac{\pi \Gamma}{2} \log \frac{1}{\delta} \right) \geq \sum_{n=1}^N (e(d_n) - \pi \gamma_n \lambda_n) + \frac{1}{2} \int_{\Omega_r(a)} |\nabla u_{a,d}^*|^2 \, dx - \frac{\pi \Gamma}{2} \log \frac{1}{r} - Cr.$$

Finally we let $r \searrow 0$. Recall that

$$\frac{1}{2} \lim_{r \searrow 0} \left(\int_{\Omega_r(a)} |\nabla u_{a,d}^*|^2 \, dx - \pi \Gamma \log \frac{1}{r} \right) = E_{a,d}^*(u_{a,d}^*) = W_1(a,d)$$

and

$$-\pi \sum_{n=1}^N \gamma_n \lambda_n = W_2(a,d)$$

for the functions W_1 and W_2 defined in Section 2.3. Hence we conclude that

$$\liminf_{\epsilon \searrow 0} \left(\left(\log \frac{1}{\delta} \right)^2 E_\epsilon(m_\epsilon) - \frac{\pi \Gamma}{2} \log \frac{1}{\delta} \right) \geq \sum_{n=1}^N e(d_n) + W(a,d).$$

That is, inequality (76) is indeed satisfied.

6.3 An upper bound for the interaction energy

We now want to prove the inequality

$$\limsup_{\epsilon \searrow 0} \left(\left(\log \frac{1}{\delta} \right)^2 \inf_{M(a,d)} E_\epsilon - \frac{\pi \Gamma}{2} \log \frac{1}{\delta} \right) \leq \sum_{n=1}^N e(d_n) + W(a,d), \tag{83}$$

which complements (76).

First step: glue energy minimising cores into the tail profile Define

$$\kappa_n^r = \frac{\lambda_n + \gamma_n \log\frac{1}{r}}{\gamma_n \log\frac{1}{\delta}}.$$

Fix $r \in (0, \rho(a)]$ small enough so that $\kappa_n^r \in (0,1)$ for sufficiently small values of ϵ. Choose minimisers $\hat{\mu}_\epsilon^n \in M_{|\gamma_n|}$ of the functionals $E_\epsilon^{|\gamma_n|}$ and let $\hat{u}_\epsilon^n$ be the solutions of the corresponding boundary value problem (47)–(49) with

$$\int_{B_1^+(0)\backslash B_{1/2}(0)} \hat{u}_\epsilon^n \, dx = 0.$$

Define $\mu_\epsilon^n = d_n \hat{\mu}_\epsilon^n$ and $u_\epsilon^n = d_n \hat{u}_\epsilon^n$.

Let $\eta \in C^\infty(\mathbb{R})$ with $\eta \equiv 1$ in $(-\infty, \frac{1}{2}]$ and $\eta \equiv 0$ in $[\frac{3}{4}, \infty)$. Set

$$\varphi_n(x_1) = \eta\left(\frac{|x_1 - a_n|}{r}\right) \quad \text{and} \quad \psi_n(x) = \eta\left(\frac{\sqrt{(x_1-a_n)^2 + x_2^2}}{r}\right).$$

Note that $\frac{\partial \psi_n}{\partial x_2} = 0$ on $\mathbb{R} \times \{0\}$. Now define

$$\begin{aligned} m_{1\epsilon}(x_1) = \cos\alpha + \frac{\mu_{a,d}^*(x_1)}{\log\frac{1}{\delta}} \\ + \sum_{n=1}^N \varphi_n(x_1)\left((1-\kappa_n^r)\mu_{\epsilon/r}^n\left(\frac{x_1 - a_n}{r}\right) + \kappa_n^r d_n - \cos\alpha - \frac{\mu_{a,d}^*(x_1)}{\log\frac{1}{\delta}}\right) \end{aligned}$$

and

$$\tilde{u}_\epsilon(x) = \frac{u_{a,d}^*(x)}{\log\frac{1}{\delta}} + \sum_{n=1}^N \psi_n(x)\left((1-\kappa_n^r)u_{\epsilon/r}^n\left(\frac{(x_1-a_n, x_2)}{r}\right) - \frac{u_{a,d}^*(x) - \omega_n}{\log\frac{1}{\delta}}\right).$$

Then $m_{1\epsilon}(a_n) = d_n$ for $n = 1, \ldots, N$. If ϵ is sufficiently small, we have $-1 \le m_{1\epsilon} \le 1$. Hence there exists a function $m_{2\epsilon}$ such that $m_\epsilon = (m_{1\epsilon}, m_{2\epsilon}) \in M(a,d)$. Let $u_\epsilon = U(m_\epsilon)$.

Second step: estimate the magnetostatic energy in terms of $\tilde{u}_\epsilon$ Since $u_{a,d}^*$ is harmonic on $\mathbb{R}_+^2$, we compute

$$\begin{aligned} \Delta\tilde{u}_\epsilon(x) = \sum_{n=1}^N \Delta\psi_n(x)\left((1-\kappa_n^r)u_{\epsilon/r}^n\left(\frac{(x_1-a_n,x_2)}{r}\right) - \frac{u_{a,d}^*(x)-\omega_n}{\log\frac{1}{\delta}}\right) \\ + 2\sum_{n=1}^N \nabla\psi_n(x)\left(r^{-1}(1-\kappa_n^r)\nabla u_{\epsilon/r}^n\left(\frac{(x_1-a_n,x_2)}{r}\right) - \frac{\nabla u_{a,d}^*(x)}{\log\frac{1}{\delta}}\right). \end{aligned}$$

Let $\Sigma_n^r = B_{3r/4}^+(a_n, 0)\backslash B_{r/2}(a_n, 0)$. Using Theorem 22 and the inequalities (72) and (73), we infer

$$\left\| u_{\epsilon/r}^n\left(\frac{(x_1-a_n,x_2)}{r}\right) - \frac{u_{a,d}^*(x) - \omega_n}{\log\frac{1}{\delta}}\right\|_{L^\infty(\Sigma_n^r)} \le \frac{Cr + o(1)}{\log\frac{1}{\delta}} \tag{84}$$

and

$$\left\| \nabla u_{\epsilon/r}^n\left(\frac{(x_1-a_n,x_2)}{r}\right) - \frac{r\nabla u_{a,d}^*(x)}{\log\frac{1}{\delta}}\right\|_{L^p(\Sigma_n^r)} \le \frac{Cr^{1+2/p} + o(1)}{\log\frac{1}{\delta}} \tag{85}$$

for any $p < \infty$ and all $n = 1, \ldots, N$. (Here $o(1)$ again stands for any quantity that converges to 0 as $\epsilon \searrow 0$, with a rate of convergence that may possibly depend on r.) Therefore, we have

$$\|\Delta \tilde{u}_\epsilon\|_{L^p(\mathbb{R}^2_+)} \leq \frac{Cr^{2/p-1} + o(1)}{\log \frac{1}{\delta}}.$$

We also compute

$$\begin{aligned}\frac{\partial \tilde{u}_\epsilon}{\partial x_2}(x_1, 0) &= -\frac{\frac{d}{dx_1}\mu^*_{a,d}(x_1)}{\log \frac{1}{\delta}} - \sum_{n=1}^N \varphi_n(x_1) \left(\frac{1 - \kappa_n^r}{r} \frac{d\mu^n_{\epsilon/r}}{dx_1} \left(\frac{x_1 - a_n}{r} \right) - \frac{\frac{d}{dx_1}\mu^*_{a,d}(x_1)}{\log \frac{1}{\delta}} \right) \\ &= -m'_{1\epsilon}(x_1) + \sum_{n=1}^N \varphi'_n(x_1) \left((1 - \kappa_n^r)\mu^n_{\epsilon/r} \left(\frac{x_1 - a_n}{r} \right) + d_n \kappa_n^r - \cos\alpha - \frac{\mu^*_{a,d}(x_1)}{\log \frac{1}{\delta}} \right).\end{aligned}$$

Using Theorem 22, we then see that

$$\left| \mu^n_\epsilon(x_1) - \cos\alpha + \frac{\gamma_n \log |x_1|}{\log \frac{1}{\delta}} \right| \leq \frac{o(1)}{\log \frac{1}{\delta}}$$

for $x_1 \in [-\frac{3}{4}, -\frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}]$. This, together with (71), implies that

$$\left| (1 - \kappa_n^r)\mu^n_{\epsilon/r} \left(\frac{x_1 - a_n}{r} \right) + d_n \kappa_n^r - \cos\alpha - \frac{\mu^*_{a,d}(x_1)}{\log \frac{1}{\delta}} \right| \leq \frac{Cr + o(1)}{\log \frac{1}{\delta}} \tag{86}$$

for any $x_1 \in [a_n - \frac{3r}{4}, a_n - \frac{r}{2}] \cup [a_n + \frac{r}{2}, a_n + \frac{3r}{4}]$.

Recall that u_ϵ is the solution of

$$\begin{aligned} \Delta u_\epsilon &= 0 && \text{in } \mathbb{R}^2_+, \\ \frac{\partial u_\epsilon}{\partial x_2} &= -m'_{1\epsilon} && \text{on } (-1, 1) \times \{0\}, \\ \frac{\partial u_\epsilon}{\partial x_2} &= 0 && \text{on } (-\infty, 1) \times \{0\} \text{ and } (1, \infty) \times \{0\}. \end{aligned}$$

Thus we have

$$\|\Delta(u_\epsilon - \tilde{u}_\epsilon)\|_{L^p(\mathbb{R}^2_+)} \leq \frac{Cr^{2/p-1} + o(1)}{\log \frac{1}{\delta}} \tag{87}$$

for an arbitrary (but fixed) $p \in (1, 2)$ and

$$\left\| \frac{\partial}{\partial x_2}(u_\epsilon - \tilde{u}_\epsilon) \right\|_{L^\infty(\mathbb{R})} \leq \frac{C + o(1)}{\log \frac{1}{\delta}}. \tag{88}$$

Also note that the support of $\Delta(u_\epsilon - \tilde{u}_\epsilon)$ is contained in $B^*_r(a)$ and the support of $\frac{\partial}{\partial x_2}(u_\epsilon - \tilde{u}_\epsilon)$ is contained in $\bigcup_{n=1}^N (a_n - r, a_n + r)$. Thus if $\mathcal{M}(\mathbb{R}^2)$ denotes the space of Radon measures on $\mathbb{R}^2$, then after extending to $\mathbb{R}^2$ by an even reflection on $\mathbb{R} \times \{0\}$, we have

$$\|\Delta(u_\epsilon - \tilde{u}_\epsilon)\|_{\mathcal{M}(\mathbb{R}^2)} \leq \frac{Cr + o(1)}{\log \frac{1}{\delta}},$$

whence

$$\|\nabla(u_\epsilon - \tilde{u}_\epsilon)\|_{L^q(B^+_2(0))} \leq \frac{Cr + o(1)}{\log \frac{1}{\delta}}$$

for any fixed $q \in [1,2)$. Theorem 17 and (84) imply that

$$\|\nabla \tilde{u}_\epsilon\|_{L^q(B_2^+(0))} \le \frac{C + o(1)}{\log \frac{1}{\delta}}.$$

It then follows that

$$\|\nabla u_\epsilon\|_{L^q(B_2^+(0))} \le \frac{C + o(1)}{\log \frac{1}{\delta}} \tag{89}$$

as well. We will use this inequality for $q = \frac{2p}{3p-2}$ in conjunction with (87).

Let

$$\bar{u}_\epsilon = \fint_{B_1^+(0)} u_\epsilon \, dx.$$

Then it follows that

$$\begin{aligned}\int_{\mathbb{R}^2_+} |\nabla u_\epsilon|^2 \, dx &= -\int_{(-1,1)\times\{0\}} (u_\epsilon - \bar{u}_\epsilon) \frac{\partial u_\epsilon}{\partial x_2} \, dx_1 \\ &= \int_{\mathbb{R}^2_+} \nabla \tilde{u}_\epsilon \cdot \nabla u_\epsilon \, dx - \int_{(-1,1)\times\{0\}} (u_\epsilon - \bar{u}_\epsilon) \frac{\partial}{\partial x_2} (u_\epsilon - \tilde{u}_\epsilon) \, dx_1 + \int_{\mathbb{R}^2_+} (u_\epsilon - \bar{u}_\epsilon) \Delta \tilde{u}_\epsilon \, dx.\end{aligned}$$

Now with the help of (88), (89), and the continuous embedding $W^{1,q}(B_2^+(0)) \to L^{q/(2-q)}(-1,1)$, we derive the estimate

$$-\int_{(-1,1)\times\{0\}} (u_\epsilon - \bar{u}_\epsilon) \frac{\partial}{\partial x_2} (u_\epsilon - \tilde{u}_\epsilon) \, dx_1 \le \frac{C(r^{2-2/q} + o(1))}{\left(\log \frac{1}{\delta}\right)^2}.$$

(Note that for $q = \frac{2p}{3p-2}$, we have $2 - \frac{2}{q} = \frac{2}{p} - 1$.) Furthermore, by (87), (89), and the Sobolev inequality,

$$\int_{\mathbb{R}^2_+} (u_\epsilon - \bar{u}_\epsilon) \Delta \tilde{u}_\epsilon \, dx \le \frac{C(r^{2/p-1} + o(1))}{\left(\log \frac{1}{\delta}\right)^2}.$$

If we choose $p = \frac{4}{3}$, then we obtain

$$\int_{\mathbb{R}^2_+} |\nabla u_\epsilon|^2 \, dx \le \int_{\mathbb{R}^2_+} |\nabla \tilde{u}_\epsilon|^2 \, dx + \frac{C\sqrt{r} + o(1)}{\left(\log \frac{1}{\delta}\right)^2}.$$

Hence

$$E_\epsilon(m_\epsilon) \le \frac{\epsilon}{2} \int_{-1}^1 |m_\epsilon'|^2 \, dt + \frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla \tilde{u}_\epsilon|^2 \, dx + \frac{C\sqrt{r} + o(1)}{\left(\log \frac{1}{\delta}\right)^2}. \tag{90}$$

Third step: estimate $\|\nabla \tilde{u}_\epsilon\|_{L^2(\mathbb{R}^2_+)}$ We clearly have

$$\int_{\Omega_r(a)} |\nabla \tilde{u}_\epsilon|^2 \, dx = \frac{1}{\left(\log \frac{1}{\delta}\right)^2} \int_{\Omega_r(a)} |\nabla u_{a,d}^*|^2 \, dx. \tag{91}$$

In $B_r^+(a_n, 0)$ for $n = 1, \ldots, N$, we have

$$\begin{aligned}\nabla \tilde{u}_\epsilon(x) &= r^{-1}(1 - \kappa_n^r) \nabla u_{\epsilon/r}^n \left(\frac{(x_1 - a_n, x_2)}{r} \right) \\ &\quad + (\psi_n(x) - 1) \left(r^{-1}(1 - \kappa_n^r) \nabla u_{\epsilon/r}^n \left(\frac{(x_1 - a_n, x_2)}{r} \right) - \frac{\nabla u_{a,d}^*(x)}{\log \frac{1}{\delta}} \right) \\ &\quad + \nabla \psi_n(x) \left((1 - \kappa_n^r) u_{\epsilon/r}^n \left(\frac{(x_1 - a_n, x_2)}{r} \right) - \frac{u_{a,d}^*(x) - \omega_n}{\log \frac{1}{\delta}} \right).\end{aligned}$$

Thus using (84), (85) and Theorem 22, and observing that

$$\|\nabla u^n_{\epsilon/r}\|^2_{L^2(B_1^+(0)\setminus B_{1/2}(0))} \leq \frac{C}{\left(\log\frac{1}{\delta}\right)^2}$$

by Theorem 17, we obtain

$$\|\nabla \tilde{u}_\epsilon\|^2_{L^2(B_r^+(a_n,0))} \leq (1-\kappa_n^r)^2 \|\nabla u^n_{\epsilon/r}\|^2_{L^2(B_1^+(0))} + \frac{Cr+o(1)}{\left(\log\frac{1}{\delta}\right)^2}. \tag{92}$$

Combining (91) and (92), we now find that

$$\int_{\mathbb{R}^2_+} |\nabla \tilde{u}_\epsilon|^2\,dx \leq \frac{1}{\left(\log\frac{1}{\delta}\right)^2}\int_{\Omega_r(a)} |\nabla u^*_{a,d}|^2\,dx + \sum_{n=1}^N (1-2\kappa_n^r)\int_{B_1^+(0)} |\nabla u^n_{\epsilon/r}|^2\,dx + \frac{Cr+o(1)}{\left(\log\frac{1}{\delta}\right)^2}.$$

Since

$$\int_{B_1^+(0)} |\nabla u^n_{\epsilon/r}|^2\,dx \geq \frac{\pi\gamma_n^2}{\log\frac{1}{\delta}} - \frac{C}{\left(\log\frac{1}{\delta}\right)^2}$$

by Theorem 17, it follows that

$$\begin{aligned}\int_{\mathbb{R}^2_+} |\nabla \tilde{u}_\epsilon|^2\,dx \leq \frac{1}{\left(\log\frac{1}{\delta}\right)^2}\int_{\Omega_r(a)} |\nabla u^*_{a,d}|^2\,dx + \sum_{n=1}^N \int_{B_1^+(0)} |\nabla u^n_{\epsilon/r}|^2\,dx \\ - \sum_{n=1}^N \frac{2\pi\gamma_n(\lambda_n + \gamma_n\log\frac{1}{r})}{\left(\log\frac{1}{\delta}\right)^2} + \frac{Cr+o(1)}{\left(\log\frac{1}{\delta}\right)^2}.\end{aligned} \tag{93}$$

Fourth step: estimate the exchange energy Note that we have $|m_{1\epsilon} - \cos\alpha| \leq \frac{C}{\log\frac{1}{\delta}}$ in $(-1, a_1 - \frac{r}{2}]$, in $[a_n + \frac{r}{2}, a_{n+1} - \frac{r}{2}]$ for $n = 1, \ldots, N$, and in $[a_N - \frac{r}{2}, 1)$ by Theorem 22. Moreover, by (86), we have

$$\begin{aligned} d_n - m_{1\epsilon}(x_1) &= (1-\kappa_n^r)\left(d_n - \mu^n_{\epsilon/r}\left(\frac{x_1 - a_n}{r}\right)\right) \\ &\quad + (1-\varphi_n(x_1))\left((1-\kappa_n^r)\mu^n_{\epsilon/r}\left(\frac{x_1-a_n}{r}\right) - \cos\alpha - \frac{\mu^*_{a,d}(x_1)}{\log\frac{1}{\delta}} + d_n\kappa_n^r\right) \\ &= \left(d_n - \mu^n_{\epsilon/r}\left(\frac{x_1-a_n}{r}\right)\right)\left(1 - \frac{O(1)}{\log\frac{1}{\delta}}\right) \end{aligned}$$

and

$$\begin{aligned} d_n + m_{1\epsilon}(x_1) &= (1-\kappa_n^r)\left(d_n + \mu^n_{\epsilon/r}\left(\frac{x_1 - a_n}{r}\right)\right) + 2d_n\kappa_n^r \\ &\quad + (1-\varphi_n(x_1))\left((\kappa_n^r - 1)\mu^n_{\epsilon/r}\left(\frac{x_1-a_n}{r}\right) - d_n\kappa_n^r + \cos\alpha + \frac{\mu^*_{a,d}(x_1)}{\log\frac{1}{\delta}}\right) \\ &= \left(d_n + \mu^n_{\epsilon/r}\left(\frac{x_1-a_n}{r}\right)\right)\left(1 - \frac{O(1)}{\log\frac{1}{\delta}}\right) \end{aligned}$$

in $(a_n - r, a_n + r)$. We also have

$$
\begin{aligned}
m'_{1\epsilon}(x_1) = {} & \frac{1-\kappa_n^r}{r} \frac{d\mu^n_{\epsilon/r}}{dx_1} \left(\frac{x_1 - a_n}{r} \right) \\
& + \varphi_n'(x_1) \left((1-\kappa_n^r) \mu^n_{\epsilon/r} \left(\frac{x_1 - a_n}{r} \right) + d_n \kappa_n^r - \cos\alpha - \frac{\mu^*_{a,d}(x_1)}{\log \frac{1}{\delta}} \right) \\
& - (1 - \varphi_n(x_1)) \left(\frac{1-\kappa_n^r}{r} \frac{d\mu^n_{\epsilon/r}}{dx_1} \left(\frac{x_1 - a_n}{r} \right) - \frac{\frac{d}{dx_1}\mu^*_{a,d}(x_1)}{\log \frac{1}{\delta}} \right)
\end{aligned}
$$

near a_n. We have

$$
\frac{d\mu^*_{a,d}}{dx_1}(x_1) = -\frac{\partial u^*_{a,d}}{\partial x_2}(x_1, 0).
$$

Hence defining $T_n^r = (a_n - r, a_n - \frac{r}{2}) \cup (a_n + \frac{r}{2}, a_n + r)$, we have

$$
\left\| \frac{1-\kappa_n^r}{r} \frac{d\mu^n_{\epsilon/r}}{dx_1} \left(\frac{x_1 - a_n}{r} \right) - \frac{\frac{d}{dx_1}\mu^*_{a,d}(x_1)}{\log \frac{1}{\delta}} \right\|_{L^2(T_n^r)} \leq \frac{C\sqrt{r} + o(1)}{\log \frac{1}{\delta}}
$$

by Theorem 22 and (73). Recalling (86), we then compute

$$
\frac{\epsilon}{2} \int_{-1}^1 |m_\epsilon'|^2 \, dt = \sum_{n=1}^N \frac{\epsilon}{2r} \int_{-1}^1 \frac{(\frac{d}{dx_1}\mu^n_{\epsilon/r})^2}{1 - (\mu^n_{\epsilon/r})^2} \, dt + \frac{o(1)}{\left(\log \frac{1}{\delta}\right)^2}. \tag{94}
$$

Combining (90), (93), and (94), we now find

$$
E_\epsilon(m_\epsilon) \leq \frac{1}{2\left(\log \frac{1}{\delta}\right)^2} \int_{\Omega_r(a)} |\nabla u^*_{a,d}|^2 \, dx + \sum_{n=1}^N \inf_{M_{|\gamma_n|}} E^{|\gamma_n|}_{\epsilon/r} - \sum_{n=1}^N \frac{\pi\gamma_n(\lambda_n + \gamma_n \log \frac{1}{r})}{\left(\log \frac{1}{\delta}\right)^2} + \frac{C\sqrt{r} + o(1)}{\left(\log \frac{1}{\delta}\right)^2}.
$$

Fifth step: estimate the core energy Recalling Definition 26, we see that

$$
\inf_{M_{|\gamma_n|}} E^{|\gamma_n|}_{\epsilon/r} \leq \frac{\pi\gamma_n^2}{2\log \frac{1}{\tilde{\delta}}} + \frac{e(d_n) + o(1)}{\left(\log \frac{1}{\tilde{\delta}}\right)^2},
$$

where

$$
\tilde{\delta} = \frac{\epsilon}{r} \log \frac{r}{\epsilon}.
$$

Using the estimates (74) and (75), we obtain

$$
\inf_{M_{|\gamma_n|}} E^{|\gamma_n|}_{\epsilon/r} \leq \frac{\pi\gamma_n^2}{2\log \frac{1}{\delta}} + \frac{\pi\gamma_n^2 \log \frac{1}{r}}{2\left(\log \frac{1}{\delta}\right)^2} + \frac{e(d_n) + o(1)}{\left(\log \frac{1}{\delta}\right)^2}.
$$

Sixth step: combine the estimates It follows that

$$
\begin{aligned}
\left(\log \frac{1}{\delta}\right)^2 E_\epsilon(m_\epsilon) \leq {} & \frac{1}{2} \int_{\Omega_r(a)} |\nabla u^*_{a,d}|^2 \, dx + \frac{\pi\Gamma}{2} \log \frac{1}{\delta} - \frac{\pi\Gamma}{2} \log \frac{1}{r} + \sum_{n=1}^N e(d_n) + W_2(a,d) \\
& + C\sqrt{r} + o(1).
\end{aligned}
$$

That is,

$$\limsup_{\epsilon\searrow 0}\left(\left(\log\frac{1}{\delta}\right)^2 E_\epsilon(m_\epsilon)-\frac{\pi\Gamma}{2}\log\frac{1}{\delta}\right)\le\frac{1}{2}\int_{\Omega_r(a)}|\nabla u^*_{a,d}|^2\,dx-\frac{\pi\Gamma}{2}\log\frac{1}{r}+\sum_{n=1}^N e(d_n)+W_2(a,d)+C\sqrt{r}.$$

Letting $r\searrow 0$, we finally obtain

$$\limsup_{\epsilon\searrow 0}\left(\left(\log\frac{1}{\delta}\right)^2 E_\epsilon(m_\epsilon)-\frac{d\pi\Gamma}{2}\log\frac{1}{\delta}\right)\le\sum_{n=1}^N e(d_n)+W(a,d),$$

which amounts to inequality (83). This completes the proof of Theorem 2.

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