University La Sapienza, Rome 2025-2026
Introduction to micromagnetics

# Problem List 1

## Problem 1 Characterization of the stray field.

a. *Prove the Hardy inequality in* $\mathbb{R}^3$, *i.e.,*

$$\int_{\mathbb{R}^3} \frac{\zeta^2}{|x|^2}\, dx \leqslant 4 \int_{\mathbb{R}^3} |\nabla \zeta|^2\, dx, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^3).$$

*(Hint: use the decomposition* $\zeta = \frac{\tilde{\zeta}}{\sqrt{|x|}}$*...)*

b. *We recall the Beppo-Levi space*

$$\mathcal{BL} = \{\zeta : \mathbb{R}^3 \to \mathbb{R} \,:\, \frac{\zeta}{1+|x|} \in L^2(\mathbb{R}^3),\, \nabla\zeta \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}$$

*endowed with the norm* $\|\zeta\|_{\mathcal{BL}} := \|\nabla \zeta\|_{L^2}$. *Prove that* $(\mathcal{BL}, \|\cdot\|_{\mathcal{BL}})$ *is a Hilbert space.*

c. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded open set and* $m \in L^2(\Omega, \mathbb{R}^3)$. *Prove that there exists a unique stray field potential* $U \in H^1(\mathbb{R}^3)$ *satisfying*

$$\int_{\mathbb{R}^3} \nabla U \cdot \nabla \zeta\, dx = \int_\Omega m \cdot \nabla \zeta\, dx, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^3).$$

*The exact expression is given by the convolution*

$$U(x) = -\frac{1}{4\pi|x|} \star \nabla \cdot (m 1_\Omega) \quad in \quad \mathbb{R}^3$$

*of the distribution of compact support* $\nabla\cdot(m1_\Omega)$ *and the tempered distribution* $V(x) = -\frac{1}{4\pi|x|}$ *that is the fundamental solution* $\Delta V = \delta_0$ *in* $\mathbb{R}^3$. *If in addition* $\Omega$ *is Lipschitz and* $m \in H^1(\Omega, \mathbb{R}^3)$, *then*

$$4\pi U(x) = -\int_\Omega \frac{1}{|x-y|} \nabla\cdot m(y)\, dy + \int_{\partial\Omega} \frac{1}{|x-y|} (m\cdot n)(y)\, d\mathcal{H}^2(y), \tag{1}$$

*where* $n$ *is the unit outer normal vector at* $\partial\Omega$ *(see [3, Appendix]).*

## Problem 2 Regularity of $\mathbb{S}^2$-valued harmonic maps in $2D$.

a. *Let* $B$ *be the unit disk in* $\mathbb{R}^2$. *Prove the Wente estimate, i.e., there exists a constant* $C > 0$ *such that for any* $u, v \in H^1(B)$, *if* $\zeta \in H^1_0(B)$ *satisfies*

$$-\Delta\zeta = \nabla u \times \nabla v \quad in\, B$$

*then*

$$\|\zeta\|_{L^\infty} + \|\nabla\zeta\|_{L^2} \leqslant C \|\nabla u\|_{L^2} \|\nabla v\|_{L^2}.$$

*In particular,* $\zeta$ *is continuous in* $B$. *(see [1, Lemma A.1])* *b. Let $m \in H^1(B, \mathbb{S}^2)$ be a weakly harmonic map in the unit disk $B$, i.e., $-\Delta m = |\nabla m|^2 m$ in $B$. By a., deduce that $m$ is continuous in $B$. Finally, deduce that $m$ is smooth in $B$. (see [4, Theorem 7.4.1]).*

**Problem 3 Partial regularity of $\mathbb{S}^2$-valued harmonic maps in $3D$.**

*Let $B$ be the unit ball in $\mathbb{R}^3$ and $m : B \to \mathbb{S}^2$ be a weakly stationary harmonic map in $B$. We denote $E(x_0, r) = \frac{1}{r}\int_{B(x_0,r)} |\nabla m|^2\, dx$ for any ball $B(x_0, r) \subset B$.*

*a. Prove that $E(x_0, \cdot)$ is a nondecreasing function for every $x_0 \in B$.*

*b. Prove that there exist $\varepsilon, t \in (0, 1)$ such that if $E(x, r) \leqslant \varepsilon$ then $E(x, tr) \leqslant \frac{1}{2}E(x, r)$ for every $B(x, r) \subset B$.*

*c. Let $V = \{x \in B : E(x, r) < \varepsilon$ for some $r > 0$ such that $B(x, r) \subset B\}$. Prove that $V$ is open and $\mathcal{H}^1(B \setminus V) = 0$ where $\mathcal{H}^1$ is the $1$-dimensional Hausdorff measure in $\mathbb{R}^3$.*

*d. Conclude that $m$ is smooth in $V$.*

*(see [2]).*

# References


[1] H. Brezis, J.-M. Coron, *Multiple solutions of H-systems and Rellich's conjecture*, Comm. Pure Appl. Math. **37** (1984), 149-187.

[2] L.C. Evans, *Partial regularity for stationary harmonic maps into spheres*, Arch. Rational Mech. Anal. **116** (1991), 101-113.

[3] R. Ignat, M. Kurzke, *An effective model for boundary vortices in thin-film micromagnetics,*, Math. Models Methods Appl. Sci. **33** (2023), 1929-1973

[4] J. Jost, *Riemannian geometry and geometric analysis*, Fifth Edition, Springer