University La Sapienza, Rome 2025-2026
Introduction to micromagnetics

Problem List 4

Problem 1 *Let $\Omega \subset \mathbb{R}^N$ be a connected open set and $u : \Omega \to \mathbb{S}^M$ with $N, M \geqslant 1$.*

a) If $\Delta u = 0$, then prove that u is a constant.

b) If $u \in H^1(\Omega, \mathbb{S}^M)$ is a $\mathbb{S}^M$-valued harmonic map, i.e.,

$$\frac{d}{dt}\bigg|_{t=0} \int_\Omega \left|\nabla\left(\frac{u+t\zeta}{|u+t\zeta|}\right)\right|^2 dx = 0 \quad \text{for every } \zeta \in C_c^\infty(\Omega; \mathbb{R}^{M+1}),$$

then $-\Delta u = u|\nabla u|^2$ in Ω.

c) Assume that $N = 2$ and $M = 1$.

c1) Let $u \in C^2(\Omega, \mathbb{S}^1)$ and $j = u \wedge \nabla u$ be the associated current. Prove that u is a $\mathbb{S}^1$-valued harmonic map if and only if $\operatorname{div} j = 0$ and $\operatorname{curl} j = 0$ in Ω.

c2) If $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ is the canonical harmonic map

$$u(x) = e^{i\varphi} \prod_{k=1}^{n} \left(\frac{x-a_k}{|x-a_k|}\right)^{d_k},$$

for $a_1, \dots, a_n \in \Omega$, $d_1, \dots, d_n \in \mathbb{Z}$ and $\varphi \in W^{1,1}(\Omega, \mathbb{R})$, prove that the current $j = u \wedge \nabla u$ satisfies $\operatorname{div} j = 0$ and $\operatorname{curl} j = 2\pi \sum_{k=1}^n d_k \delta_{a_k}$ in Ω.

Problem 2 *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded open set and $g : \partial\Omega \to \mathbb{R}^2$ be a smooth function.*

a) Prove that for every $\varepsilon > 0$, there exists a minimiser u_ε of the energy E_ε defined by

$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \, dx, \tag{1}$$

over the space

$$H_g^1(\Omega, \mathbb{R}^2) = \{u \in H^1(\Omega, \mathbb{R}^2) \, : \, u = g \text{ on } \partial\Omega\}.$$

Moreover, u_ε satisfies the Euler-Lagrange equation

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \quad \text{in} \quad \Omega. \tag{2}$$

b) Prove that every solution $u_\varepsilon \in H_g^1(\Omega, \mathbb{R}^2)$ to the PDE (2) is smooth in $\bar{\Omega}$.

c) If $|g| = 1$ on $\partial\Omega$, prove that every solution $u_\varepsilon \in H_g^1(\Omega, \mathbb{R}^2)$ to the PDE (2) satisfies $|u_\varepsilon| \leqslant 1$ in Ω. (Hint: start by proving that $\rho = 1 - |u_\varepsilon|^2$ satisfies $-\Delta\rho + \frac{2}{\varepsilon^2}|u_\varepsilon|^2\rho \geqslant 0$ in Ω and conclude by the maximum principle...)

d) Prove that if $\varepsilon > 0$ is large enough, then E_ε is convex over $H^1(\Omega, \mathbb{R}^2)$; as a consequence, there exists a unique solution $u_\varepsilon \in H_g^1(\Omega, \mathbb{R}^2)$ to the PDE (2).

Problem 3 *Let B_1 be the unit disk in $\mathbb{R}^2$, $d \geqslant 1$ be an integer and $g : \mathbb{S}^1 \to \mathbb{S}^1$ be given by $g(e^{i\theta}) = e^{id\theta}$ for every $\theta \in [0, 2\pi]$. Prove that for every $\varepsilon > 0$, there exists a solution $u_\varepsilon \in H^1_g(B_1, \mathbb{R}^2)$ of the form $u_\varepsilon(x) = f_\varepsilon(|x|)e^{id\theta}$ for every $x \in B_1$ to the PDE (2). Moreover, the radial profile f_ε solves the following ODE:*[1]

$$\begin{cases} -f_\varepsilon'' - \frac{1}{r} f_\varepsilon' + \frac{d^2}{r^2} f_\varepsilon = \frac{1}{\varepsilon^2} f_\varepsilon(1 - f_\varepsilon^2) & \text{for every } r \in (0, 1), \\ f_\varepsilon(0) = 0, f_\varepsilon(1) = 1. \end{cases}$$

Problem 4 *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded simply connected domain.*

a) If $k \in \mathbb{N}$ and $u \in C^k(\Omega, \mathbb{S}^1)$ prove that there exists a lifting $\varphi \in C^k(\Omega, \mathbb{R})$ of u, i.e., $u = e^{i\varphi}$ in Ω and φ is unique up to an additive constant $2\pi\mathbb{Z}$.

b) If $p \geqslant 2$ and $u \in W^{1,p}(\Omega, \mathbb{S}^1)$ prove that there exists a lifting $\varphi \in W^{1,p}(\Omega, \mathbb{R})$ of u, i.e., $u = e^{i\varphi}$ in Ω and φ is unique up to an additive constant $2\pi\mathbb{Z}$.

(Hint: Start by proving that $\operatorname{curl}(u \wedge \nabla u) = 0$ and apply Poincaré lemma to obtain $\nabla\varphi = u \wedge \nabla u$...)

c) Using b), prove that for a boundary data $g = e^{i\varphi_0}$ with $\varphi_0 : \partial\Omega \to \mathbb{R}$ smooth, there exists a unique minimizing $\mathbb{S}^1$-valued harmonic map u_* of the problem

$$\min\{\int_\Omega |\nabla u|^2 \, dx \, : \, u \in H^1(\Omega, \mathbb{S}^1), u = g \text{ on } \partial\Omega\}.$$

Moreover, $u_* = e^{i\varphi_*} \in C^\infty(\bar{\Omega}, \mathbb{S}^1)$ where $\varphi_* : \Omega \to \mathbb{R}$ is the unique solution of $\Delta\varphi_* = 0$ in Ω and $\varphi_* = \varphi_0$ on $\partial\Omega$.

Problem 5 *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $u \in C^1(\Omega, \mathbb{R}^2)$ and $j(u) = u \wedge \nabla u$ be the associated current to u.*

a) if $u \neq 0$ in Ω, prove that

$$|\nabla u|^2 = \left|\frac{j(u)}{|u|}\right|^2 + \big|\nabla|u|\big|^2.$$

b) if $|u| = 1$ in Ω and $\rho \in C^1(\Omega, \mathbb{R})$, prove that

$$|\nabla u|^2 = |j(u)|^2, \quad j(\rho u) = \rho^2 j(u).$$

Deduce that

$$|\nabla(\rho u)|^2 = \rho^2 \, |j(u)|^2 + |\nabla\rho|^2 = \rho^2 \, |\nabla u|^2 + |\nabla\rho|^2.$$

c) if $\varphi \in C^1(\Omega, \mathbb{R})$, prove that

$$j(e^{i\varphi} u) = j(u) + |u|^2 \, \nabla\varphi.$$

Problem 6 *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain.*

a) If $u, v \in H^1(\Omega, \mathbb{R}^2)$ and $\zeta \in W^{1,\infty}(\Omega, \mathbb{R})$ with $\zeta = 0$ on $\partial\Omega$, prove that

$$\Big| \int_\Omega (\operatorname{jac}(u) - \operatorname{jac}(v))\zeta \, dx \Big| \leqslant \frac{1}{2} \|u - v\|_{L^2} \big(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2}\big) \|\nabla\zeta\|_{L^\infty}.$$

b) If $u \in H^1(\Omega, \mathbb{S}^1)$, then $\operatorname{jac}(u) = 0$.

c) If $d \in \mathbb{Z}$, $a \in \Omega$, $\varphi \in C^1(\Omega, \mathbb{R})$ and $u(x) = e^{i\varphi}\left(\frac{x-a}{|x-a|}\right)^d$, prove that $\operatorname{jac}(u) = \pi d \delta_a$.

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[1] Such solution f_ε is unique, see e.g. R. Ignat, L. Nguyen, V. Slastikov, A. Zarnescu, *Uniqueness results for an ODE related to a generalized Ginzburg-Landau model for liquid crystals*, SIAM Journal on Mathematical Analysis **46** (2014), 3390-3425.