University La Sapienza, Rome 2025-2026
Introduction to micromagnetics

Problem List 3

Problem 1 *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set and $\{e_1, e_2, e_3\}$ be the canonical basis in $\mathbb{R}^3$. For $m \in L^2(\Omega, \mathbb{R}^3)$, we define $P(m)$ as the Helmholtz projection of $m1_\Omega$ to the space of gradient fields $\{\nabla u \,:\, u \in \dot{H}^1(\mathbb{R}^3)\}$. We consider the demagnetized tensor $D = D(\Omega)$ by*

$$D_{ij} = e_i \cdot \frac{1}{|\Omega|} \int_\Omega P(e_j)\, dx, \quad 1 \leqslant i, j \leqslant 3.$$

Show that

1. *D is invariant by translation and dilation of the domain Ω, that is, $D(a+\lambda\Omega) = D(\Omega)$ for $a \in \mathbb{R}^3$ and $\lambda > 0$.*
2. *D preserves the symmetries of Ω, i.e., if $R\Omega = \Omega$ for some $R \in SO(3)$, then $R^t D R = D$.*

Problem 2 (Stoner–Wohlfarth astroid as a sliding segment) *In the Stoner–Wohlfarth model of a single-domain ferromagnetic particle with uniaxial anisotropy, the boundary between regions of stable magnetization in the plane of the applied magnetic field (H_x, H_y) is given by an* astroid.

Consider a rigid segment of fixed length L whose endpoints slide along the coordinate axes: one endpoint remains on the x-axis and the other on the y-axis.

1. *Let the endpoints of the segment be*
$$A = (x, 0), \qquad B = (0, y),$$
with $x > 0$ and $y > 0$. Show that the condition that the length of the segment is constant implies
$$x^{2/3} + y^{2/3} = L^{2/3}.$$
2. *Deduce that the locus of points (x, y) traced by the sliding segment is an* astroid.
3. *Show that the astroid admits the parametrization*
$$x = L\cos^3\theta, \qquad y = L\sin^3\theta,$$
and interpret the parameter θ in terms of the magnetization direction.
4. *Sketch the astroid and indicate the physical meaning of points lying inside and outside this curve.*

Problem 3 *Let X be a metric space and $F : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We define the relaxed functional $\bar{F} : X \to \bar{\mathbb{R}}$ by*

$$\bar{F}(x) = \inf_{(x_n)} \left\{ \liminf_{n\to\infty} F_n(x_n) \,:\, x_n \to x \right\}.$$

Show that

1. *$\bar{F}$ is l.s.c. and $\bar{F} \leqslant F$.*

2. *$\bar{F} = \sup\{G : X \to \bar{\mathbb{R}} : G$ is l.s.c. and $G \leqslant F\}$.*

3. *$\overline{F+K} = \bar{F} + K$ where $K : X \to \mathbb{R}$ is continuous.*

Problem 4 (Legendre–Fenchel transform: basic properties) *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. Its Legendre–Fenchel transform (or convex conjugate) is defined by*

$$f^*(p) := \sup_{x \in \mathbb{R}^n} \big(p \cdot x - f(x)\big), \qquad p \in \mathbb{R}^n.$$

1. *Show that f^* is a convex and lower semicontinuous function, even if f is not.*

2. *(Fenchel–Young inequality) Show that for all $x, p \in \mathbb{R}^n$,*

$$f(x) + f^*(p) \geq p \cdot x,$$

   and characterize the case of equality.

3. *Assume that f is convex, lower semicontinuous and $f \not\equiv \infty$. Define the biconjugate*

$$f^{**}(x) := \sup_{p \in \mathbb{R}^n} \big(p \cdot x - f^*(p)\big).$$

   Show that $f^{**} = f$.

4. *(Order-reversing property) Let $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ satisfy $f \leq g$. Show that $f^* \geq g^*$.*

5. *(Affine changes) Let $a \in \mathbb{R}^n$, $\lambda > 0$, and define*

$$g(x) = f(x - a), \qquad h(x) = \lambda f(x).$$

   Compute g^* and h^* in terms of f^*.

6. *Assume that f is differentiable and strictly convex on $\mathbb{R}^n$. Show that the supremum in the definition of $f^*(p)$ is attained at a unique point $x(p)$, and that*

$$p = \nabla f(x(p)), \qquad x(p) = \nabla f^*(p).$$

7. *Compute the Legendre transform of the following functions:*

   (a) $f(x) = \frac{1}{2}|x|^2$

   (b) *The indicator function of a closed convex set C, ie., $f(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases}$*

   (c) $f(x) = |x|$