Exercises on Γ-Convergence

Exercise 1 — Definition and Basic Properties

Let X be a metric space and let (F_n) be a sequence of functionals

$$F_n : X \to \mathbb{R} \cup \{+\infty\}.$$

(1) Show that if $F_n \xrightarrow{\Gamma} F$, then F is lower semicontinuous (l.s.c).

(2) Show that if $F_n \equiv F$ for all n, then $F_n \xrightarrow{\Gamma} \bar{F}$ where $\bar{F}$ is the l.s.c. hull of F, i.e.,

$$\bar{F}(x) = \inf_{(x_n)} \left\{ \liminf_{n\to\infty} F_n(x_n) \, : \, x_n \to x \right\}.$$

(3) Give an example where $F_n \to F$ pointwise but F_n does *not* Γ–converge to F.

Exercise 2 — A Simple Example on $\mathbb{R}$

Let $X = \mathbb{R}$ and define

$$F_n(x) = x^2 + \frac{1}{n}\sin(nx).$$

(1) Show that $F_n \xrightarrow{\Gamma} F$ and identify the limit functional F.

(2) Show that minimizers of F_n converge to the minimizer of F.

(3) What happens if $\frac{1}{n}\sin(nx)$ is replaced by $\sin(nx)$?

Exercise 3 — Stability under Continuous Perturbations

Let X be a metric space. Assume that $F_n \xrightarrow{\Gamma} F$ in X, and let $G : X \to \mathbb{R}$ be continuous.

(1) Show that $F_n + G \xrightarrow{\Gamma} F + G$.

(2) Give an example where G is only lower semicontinuous and the conclusion fails.

Exercise 4 — Quadratic Penalization

Let $X = \mathbb{R}^d$ and $C \subset \mathbb{R}^d$ be a closed, convex, non-empty set. Define

$$F_n(x) = |x|^2 + n\,\mathrm{dist}(x, C)^2.$$

(1) Show that the sequence (F_n) is equi-coercive, i.e., every sequence (x_n) such that $F_n(x_n) \leq M$ for some M admits a convergent subsequence.

(2) Compute the Γ–limit of (F_n).

(3) Explain the connection with the projection onto C.

Exercise 5 — A Simple Elliptic Example

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, and define

$$F_\varepsilon(u) = \int_\Omega \left(|\nabla u|^2 + \varepsilon |u|^2\right)\,dx \quad \text{on } H^1(\Omega).$$

(1) Show that $F_\varepsilon \xrightarrow{\Gamma} F$ as $\varepsilon \to 0$ where the limit functional F is to be determined explicitly.

(2) What happens to minimizers of F_ε under the average constraint $\frac{1}{|\Omega|}\int_\Omega u\,dx = M$ for some $M \in \mathbb{R}$?

Exercise 6 — Indicator Functionals and Constraints

Let (A_n) be a sequence of subsets of a metric space X, and define

$$F_n(x) = \begin{cases} 0 & \text{if } x \in A_n, \\ +\infty & \text{otherwise.} \end{cases}$$

(1) Show that $F_n \xrightarrow{\Gamma} F$, where $F = \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases}$ and the set A is to be determined (in terms of limits of sequences $x_n \in A_n$).

(2) Interpret this result as a convergence of constraints.

Exercise 6 — A Double-Well Energy (Preview)

On $L^2(\Omega)$, consider

$$F_\varepsilon(u) = \int_\Omega \left(\varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right)\,dx,$$

where $W(t) = (t^2 - 1)^2$.

(1) Show that any sequence (u_ε) with uniformly bounded energy is relatively compact in $L^1(\Omega)$.

(2) Describe qualitatively the Γ–limit.

(3) Which geometric object appears in the limit?

(Hint: Modica–Mortola theorem.)