Lecture: Jacobian and degree theory

(1)

Ginzburg-Landau model: $\Omega \subset \mathbb{R}^2$, $\varepsilon > 0$

$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \, dx, \quad u : \Omega \to \mathbb{R}^2$$

Minimisers: $g : \partial\Omega \to \mathbb{S}^1$ smooth

$$u_\varepsilon \in \arg\min \{ E_\varepsilon(u) : u = g \ \partial\Omega \}$$

Case 1: $\deg(g) = \frac{1}{2\pi} \int_{\partial\Omega} g \wedge \partial_\tau g \, d\mathcal{H}^1 = 0 \Rightarrow g = e^{i\varphi_0}$, $\varphi_0 : \partial\Omega \to \mathbb{R}$ smooth

as $\varepsilon \to 0$, $u_\varepsilon \to u_*$ $H^1 \cap C^\infty_{loc} \cap C^{1,\alpha}(\Omega)$

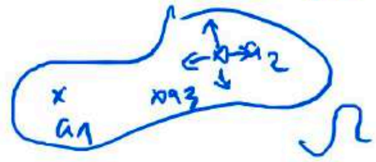
where $u_* = e^{i\varphi_*}$, $\begin{cases} \Delta \varphi_* = 0 & \Omega \\ \varphi_* = \varphi_0 & \partial\Omega \end{cases}$

$$E_\varepsilon(u_\varepsilon) = \frac{1}{2}\int_\Omega |\nabla u_*|^2 + o(1) \qquad \text{NO VORTICES}$$

Case 2: $d = \deg(g) > 0$

for a subseq $\varepsilon \to 0$, $u_\varepsilon \to u_*$ $W^{1,1} \cap C^\infty_{loc}(\Omega \setminus \{a_k\}) \cap C^{1,\alpha}_{loc}(\Omega \setminus \{a_k\})$

where $u_* = e^{i\varphi_*} \frac{x-a_1}{|x-a_1|} \cdot \ldots \cdot \frac{x-a_d}{|x-a_d|}$



with $a_1, \ldots, a_d$ vortices of degree 1.

Jacobian: $\to$ detect vortices ②

$$\underline{\underline{\mathrm{jac}(u_\varepsilon)}} = \pi \sum_{k=1}^{d} \delta_{a_k}, \quad a_k = \text{vortex of degree } 1.$$

Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be C^1; then

$$\mathrm{jac}(u) = \det(\nabla u) = \partial_1 u \wedge \partial_2 u \quad \text{continuous as } \Omega \to \mathbb{R}.$$

$\to$ extension $\underline{H^1(\Omega, \mathbb{R}^2)}$

if $u \in H^1(\Omega, \mathbb{R}^2)$, $\mathrm{jac}(u) = \underbrace{\partial_1 u}_{\in L^2} \wedge \underbrace{\partial_2 u}_{\in L^2}$ is $L^1(\Omega, \mathbb{R})$.

<u>Q</u>: How about $u_\varepsilon \notin H^1$?

<u>Idea</u>: if $u \in C^2(\Omega, \mathbb{R}^2)$,

$$\mathrm{jac}(u) = \partial_1 u \wedge \partial_2 u = \frac{1}{2}\left(\partial_1 (u \wedge \partial_2 u) + \partial_2 (\partial_1 u \wedge u)\right)$$

$$= \frac{1}{2} \operatorname{curl} \underbrace{(u \wedge \nabla u)}_{\text{current}}, \quad u \wedge \nabla u = (u \wedge \partial_1 u, u \wedge \partial_2 u)$$

$\to u_\varepsilon \in W^{1,1} \cap L^\infty \Rightarrow \underbrace{u_\varepsilon}_{\in L^\infty} \wedge \underbrace{\nabla u_\varepsilon}_{\in L^1} \in L^1$

<u>Def</u>: $u \in W^{1,1} \cap L^\infty(\Omega, \mathbb{R}^2)$,

$$\mathrm{jac}(u) = \boxed{\frac{1}{2}} \operatorname{curl}(u \wedge \nabla u) \in \mathcal{D}', \text{ i.e.,}$$

$$\forall \zeta \in C_c^\infty(\Omega), \quad \langle \mathrm{jac}(u), \zeta \rangle_{\mathcal{D}', \mathcal{D}} = -\boxed{\frac{1}{2}} \int_\Omega \nabla^\perp \zeta \cdot \underbrace{u \wedge \nabla u}_{\in L^1} \, dx.$$

<u>Rk</u>: it can be extended $\zeta \in W^{1,\infty}(\Omega)$, $\zeta = 0$ $\partial\Omega$.

in $\left(W^{1,1}\cap L^\infty, \underbrace{\|\cdot\|_{W^{1,1}} + \|\cdot\|_{L^\infty}}_{=\,|\!|\!|\cdot|\!|\!|}\right)$, $u,v\in W^{1,1}\cap L^\infty$

$$\left|\langle \mathrm{jac}(u) - \mathrm{jac}(v), \zeta\rangle\right| = \left|\frac12\int_\Omega \nabla^\perp\zeta\cdot \underbrace{(u\wedge\nabla u - v\wedge \nabla v)}_{(u-v)\wedge\nabla u + v\wedge(\nabla u-\nabla v)}\right|$$

$$\lesssim \left(\|u-v\|_{L^\infty}\|\nabla u\|_{W^{1,1}} + \|v\|_{L^\infty}\|\nabla u-\nabla v\|_{W^{1,1}}\right)\|\nabla\zeta\|_{L^\infty}$$

$$\lesssim |\!|\!|u-v|\!|\!|\left(|\!|\!|u|\!|\!| + |\!|\!|v|\!|\!|\right)\|\nabla\zeta\|_{L^\infty}.$$

Examples: 1) $u\in C^1(\Omega, \underline{\underline{S^1}}) \Rightarrow \mathrm{jac}(u)=0$ (NO VORTICES)

$|u|^2=1 \Rightarrow \partial_1 u, \partial_2 u \perp u \Rightarrow \partial_1 u \parallel \partial_2 u$.

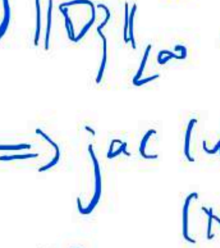
Also if $u\in H^1(\Omega,S^1)$, $\mathrm{jac}(u)=0$.

Another way: $u\in C^1(\Omega,S^1)$, locally $u=e^{i\varphi}$, $\varphi\in C^1$

$$\Rightarrow u\wedge\nabla u = \nabla\varphi \Rightarrow \mathrm{jac}(u) = \boxed{\tfrac12}\,\mathrm{curl}(\nabla\varphi)=0$$

2) $u_*(x) = \dfrac{x}{|x|} = e^{i\theta} \in \underline{\underline{W^{1,1}\cap L^\infty}}(B^2)$ $\left(|\nabla u_*| = \dfrac{1}{|x|}\right)$

(!) $\theta\notin H^1$, $\theta\in BV$



$S=[0,1]\times\{0\}$

$$\nabla\theta = \boxed{\frac1r \vec{\theta}} - 2\pi e_2\, \mathcal{H}^1 \llcorner S$$

$$\underline{\underline{\langle \mathrm{jac}(u_*),\zeta\rangle}}_{\mathcal{D}', \mathcal{D}} = -\frac12\int_{B^2}\nabla^\perp\zeta\cdot \underbrace{u_*\wedge\nabla u_*}_{\nabla^a\theta \,=\, \nabla\theta + 2\pi e_2 \mathcal{H}^1\llcorner S}$$

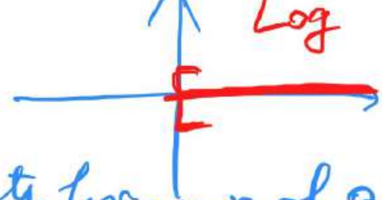
$$\nabla^\perp\zeta = (-\partial_2\zeta, \underline{\underline{\partial_1\zeta}})$$

$$= \underbrace{-\frac12\int\nabla^\perp\zeta\cdot\nabla\theta}_{0} - \pi\int_S \nabla^\perp\zeta\cdot e_2\, d\mathcal{H}^1 = -\pi\int_0^1\partial_1\zeta\,dx_1 = \underline{\underline{\pi\zeta(0)}}$$

$$\Rightarrow \boxed{\mathrm{jac}(u_*) = \pi\delta_0}$$

Another way: use the conjugate harmonic

$x = re^{i\theta} \Rightarrow \operatorname{Log} x = \underbrace{\log r}_{\Phi} + i\theta$

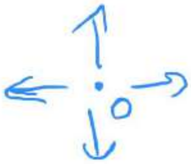


Φ is the conjugate harmonic of θ.

$\Rightarrow u_* \wedge \nabla u_* = \nabla^a \theta = \nabla^\perp \Phi$, $\Phi(x) = \log|x|$.

$\Rightarrow \langle \operatorname{jac}(u_*), \zeta \rangle_{\mathcal{D}', \mathcal{D}} = -\frac{1}{2} \int_\Omega \nabla^\perp \zeta \cdot \nabla^\perp \Phi = \frac{1}{2} \int_\Omega \underbrace{\Delta \Phi}_{2\pi\delta_0} \zeta$

$\Rightarrow \boxed{\operatorname{jac}(u_*) = \pi \delta_0}$



u_* 0 = vortex detected by $\operatorname{jac}(u_*)$.

3) $\operatorname{jac}\left(\left(\frac{x-a}{|x-a|}\right)^d\right) = \pi d \delta_a$.

4) $u_* = e^{i\varphi_*} \frac{x-a_1}{|x-a_1|} \cdots \frac{x-a_d}{|x-a_d|} \Rightarrow \operatorname{jac}(u_*) = \pi \sum_1^d \delta_{a_j}$.

φ_* smooth

Thm (Brezis–Mironescu–Ponce)

If $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ and $\operatorname{jac}(u)$ is a [finite measure] on Ω, then $\exists\, a_1, \dots, a_m \in \Omega$, $d_1, \dots, d_m \in \mathbb{Z}$ s.t. $\operatorname{jac}(u) = \pi \sum_1^m d_j \delta_{a_j}$. In particular, $\|\operatorname{jac}(u)\|_{\mathcal{M}} = \pi \sum_1^m |d_j|$.

Rk: In G-L, $\deg(g) = d > 0$, u_ε min of E_ε

$$E_\varepsilon(u_\varepsilon) = \pi d\,|\log\varepsilon| + \underbrace{W(a_1, \dots, a_d) + \gamma d}_{=O(1)} + o(1)$$

$\|jac(u_*)\| = \pi|d|$

$$\Rightarrow \boxed{\frac{1}{|\log\varepsilon|} E_\varepsilon(u_\varepsilon)} \to \|jac(u_*)\|_{\mathcal{M}}$$

Γ-convergence: Asymptotics of $\frac{1}{|\log\varepsilon|} E_\varepsilon$ in $W^{1,1}$ topology

Rk: $E_\varepsilon(u) < +\infty \iff u \in H^1(\Omega, \mathbb{R}^2)$.

Extend E_ε to $W^{1,1} \supset H^1$ by $E_\varepsilon(u) = +\infty$ if $u \in W^{1,1} \setminus H^1$. Now, $E_\varepsilon : W^{1,1}(\Omega, \mathbb{R}^2) \to [0, +\infty]$

Thm (Jerrard-Soner)

In $W^{1,1}$ top, define $E_0 : W^{1,1}(\Omega, \mathbb{R}^2) \to [0, +\infty]$

$$E_0(u) = \begin{cases} \|jac(u)\|_{\mathcal{M}} & \text{if } u \in W^{1,1}(\Omega, S^1),\ jac(u) \text{ finite measure} \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\frac{1}{|\log\varepsilon|} E_\varepsilon \to E_0$ Γ-conv in $W^{1,1}$, i.e.,

1) lower bound: $u_\varepsilon \to u$ $W^{1,1} \Rightarrow \liminf_{\varepsilon\to 0} \frac{E_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \geq E_0(u)$

2) upper bound: $u \in W^{1,1}(\Omega, S^1)$, $jac(u)$ finite measure $\Rightarrow \exists u_\varepsilon \to u$ $W^{1,1}$ s.t. $\frac{E_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \to E_0(u)$.

Rk: In general, no compactness in $W^{1,1}$, i.e, ⑥

$\underline{\frac{1}{|\log\varepsilon|} E_\varepsilon(u_\varepsilon) \leq C} \not\Rightarrow (u_\varepsilon)$ rel. compact in $W^{1,1}$,

BUT $\underbrace{\mathrm{jac}(u_\varepsilon)}_{\in L^1} \longrightarrow J = \pi \sum d_j \delta_{a_j}$ in $W^{-1,1}$ $\left(W_0^{1,\infty}\right)^*$

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} E_\varepsilon(u_\varepsilon) \geq \|J\|_{\mathcal{M}}$$

(!) the Jacobian is the appropriate object to play, not the map $\boxed{u}$.

Ex: $u_\varepsilon(x_1,x_2) = e^{i\underbrace{x_1\sqrt{|\log\varepsilon|}}_{=\varphi_\varepsilon}} \Rightarrow E_\varepsilon(u_\varepsilon) \sim |\log\varepsilon|$,

$\mathrm{jac}(u_\varepsilon) = \frac{1}{2}\,\mathrm{curl}\,(\nabla\varphi_\varepsilon) = 0$, $|Du_\varepsilon| \sim \sqrt{|\log\varepsilon|}$

$\Rightarrow \|u_\varepsilon\|_{W^{1,1}} \to +\infty$, $u_\varepsilon \to 0$ $(L^\infty)^*$.

Degree theory: (Brezis-Nirenberg, Degree theory of BMO)

1. Smooth maps: $g: \mathbb{S}^1 \to \mathbb{S}^1 \subset \mathbb{R}^2 \sim \mathbb{C}$ be C^1 map

Degree of g (winding number, index):

$$\deg(g) = \frac{1}{2\pi} \int_{\mathbb{S}^1} g \wedge g' \, d\mathcal{H}^1 = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \bar{g}\, g' \, d\mathcal{H}^1,$$

$\bar{g}$ = conjugate of g; g' = angular derivative of g

$A=(1,0)$ $\quad g = e^{i\varphi}$, $\varphi \in C^1(S^1\setminus\{A\}, \mathbb{R})$

$$\deg(g) = \frac{1}{2\pi i}\int_{S^1\setminus\{A\}} \cancel{e^{-i\varphi}}\cdot \cancel{i e^{i\varphi}}\,\varphi' = \frac{1}{2\pi}\big(\varphi(A+)-\varphi(A-)\big) \in \mathbb{Z}$$

since $g(A)=e^{i\varphi(A-)}=e^{i\varphi(A+)}$

Ex: $g(e^{i\theta}) = e^{id\theta}$, $d\in\mathbb{Z} \Rightarrow \deg(g)=d$.

Prop: if $z\in S^1$ is a regular value of g, ie, $g^{-1}(\{z\})=\{x_1,\dots,x_m\}$ and $g'(x_k)\neq 0$, then

$$\deg(g)=\sum_k \mathrm{sign}\big(g\wedge g'(x_k)\big).$$

Proof: on (x_k,x_{k+1}), $g=e^{i\varphi_k}$, $\varphi_k\in C^1(x_k,x_{k+1})$

$$2\pi\deg(g) = \int_{S^1} g\wedge g' = \sum\int_{(x_k,x_{k+1})}\varphi_k' = \sum\big(\varphi_k(x_{k+1})-\varphi_k(x_k)\big)$$

$$= \sum \pi\Big(\underbrace{\mathrm{sign}(\varphi_k'(x_k))}_{g\wedge g'(x_k)} + \mathrm{sign}(\varphi_k'(x_{k+1}))\Big). \quad \blacksquare$$

Prop: if $u: B^2 \to \mathbb{R}^2$ is a C^2 extension of g in B^2, then $\deg(g) = \frac{1}{\pi}\int_{B^2} \mathrm{jac}(u)\, dx$.

Proof:
$$\int_{B^2} \mathrm{jac}(u) = \frac{1}{2}\int_{B^2} \mathrm{curl}(u \wedge \nabla u) = \frac{1}{2}\int_{\partial B^2} \underbrace{u}_{g} \wedge \underbrace{\partial_\theta u}_{g'}\, d\mathcal{H}^1$$
$$= \pi \deg(g). \qquad \blacksquare$$

2. Continuous maps: $g: S^1 \to S^1$ continuous

1st way $\to$ $g = e^{i\varphi}$, $\varphi \in C^0(S^1 \setminus \{A\}, \mathbb{R})$ unique up to $2\pi\mathbb{Z}$

$$\deg g := \frac{1}{2\pi}\left(\varphi(A+) - \varphi(A-)\right)$$

(!) independent of A

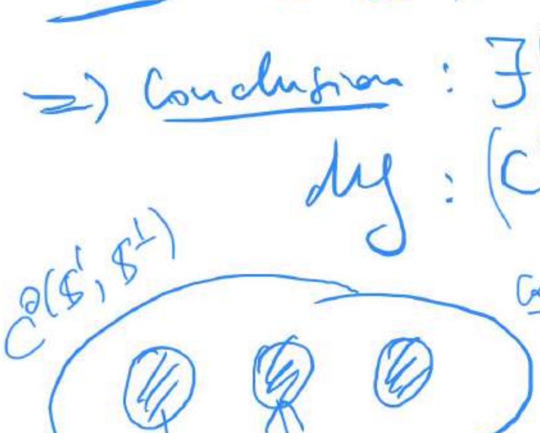
2nd way $\to$ define as continuous extension of $g \in C^1(S^1, S^1) \to \deg(g) \in \mathbb{Z}$ to $(C^0(S^1, S^1), \|\cdot\|_\infty)$.

Fact1: if $f,g \in C^1(S^1,S^1)$, $\|f-g\|_\infty \le \frac{1}{2}$, then $\deg(f)=\deg(g)$. ✓

Fact2: $C^1(S^1,S^1)$ is dense in $(C^0(S^1,S^1), \|\cdot\|_\infty)$

$\Rightarrow$ Conclusion: $\exists!$ continuous extension

$$\deg : (C^0(S^1,S^1), \|\cdot\|_\infty) \to \mathbb{Z}.$$



Corollary: The connected components of $C^0(S^1,S^1)$ are given by $\{\deg = k\}$, $k \in \mathbb{Z}$.

Proof (Fact1): Homotopy $H(t,x) = \dfrac{tf(x)+(1-t)g(x)}{|tf(x)+(1-t)g(x)|}$

$t\in[0,1]$, $x \in S^1$

* $H:[0,1]\times S^1 \to S^1$ well defined as $|tf(x)+(1-t)g(x)| \ge |g(x)| - t|f(x)-g(x)| \ge 1-\frac{1}{2} > 0$
* $H(0,\cdot)=g$, $H(1,\cdot)=f$
* $H(t,\cdot) \in C^1(S^1,S^1)$
* $t\in[0,1] \to \underbrace{\deg(H(t,\cdot))}_{\in\mathbb{Z}} = \dfrac{1}{2\pi}\displaystyle\int_{S^1} H(t,\cdot)\wedge \partial_\theta H(t,\cdot)$ is continuous in t

$\Rightarrow \deg(f)=\deg(g)$.

Proof (Fact 2): $g \in C^0(S^1, \underline{S^1})$, $g(e^{i\theta}) \sim g(\theta)$,

g 2π-per in $\mathbb{R}$; (ρ_ε) mollifiers,

$g_\varepsilon = g * \rho_\varepsilon \in C^1_{per}(\mathbb{R}, \mathbb{R}^2) \Rightarrow g_\varepsilon \to g$ unif

$\Rightarrow |g_\varepsilon| \to 1$ unif $\Rightarrow \tilde{g}_\varepsilon = \frac{g_\varepsilon}{|g_\varepsilon|} : \mathbb{R} \to S^1$ is C^1_{per}

for $\varepsilon > 0$ small

$\Rightarrow \tilde{g}_\varepsilon \to g$ unif

$\tilde{g}_\varepsilon(\theta) \simeq \tilde{g}_\varepsilon(e^{i\theta}) \in C^1(S^1, S^1) \to g$ unif. ∎

Rk: if $g \in C^k(S^1, S^1)$, $k \in \mathbb{N}$, then $\deg(g) = 0 \Leftrightarrow g = e^{i\varphi}$ with $\varphi \in C^k(S^1, \mathbb{R})$.

3. Degree for $H^{1/2}(S^1, S^1)$

Recall $g \in H^{1/2}(S^1, \underline{\mathbb{R}^2})$ if $g \in L^2$ and

$$\|g\|^2_{H^{1/2}} = \int_{S^1}\int_{S^1} \frac{|g(x) - g(y)|^2}{|x-y|^2}\,dx\,dy < +\infty.$$

Rk: $H^{1/2}(S^1, \mathbb{R}^2) = $ trace $H^1(\mathbb{D}^2, \mathbb{R}^2)$.

Def: $H^{1/2}(S^1, \underline{S^1}) = \{ g \in H^{1/2}(S^1, \mathbb{R}^2) : |g| = 1 \text{ a.e. in } S^1 \}$

How to define the degree in $H^{1/2}(S^1, S^1)$? (11)

$\rightarrow$ to extend continuously

$$\deg : C^1(S^1, S^1) \to \mathbb{Z}$$

to $\left(H^{1/2}(S^1, S^1), \|\cdot\|_{H^{1/2}}\right)$, $\|g\|_{H^{1/2}} = \|g\|_{L^2} + \|g\|_{\dot{H}^{1/2}}$

Fact 1: $\forall f, g \in C^1(S^1, S^1)$,

$$|\deg(f) - \deg(g)| \leq C \|f-g\|_{H^{1/2}} \left(\|f\|_{H^{1/2}} + \|g\|_{H^{1/2}}\right)$$

Fact 2: $C^1(S^1, S^1)$ is dense in $\left(H^{1/2}(S^1, S^1), \|\cdot\|_{H^{1/2}}\right)$

Proof (Fact 1): Let u, v harmonic extensions of f and g in B^2 = unit disk. Then $u, v \in C^2(B^2, \mathbb{R}^2)$ and $\underline{\|u\|_{H^1(B^2)} \leq C \|g\|_{H^{1/2}(S^1)}}$, $\|v\|_{H^1(B^2)} \leq C \|f\|_{H^{1/2}(S^1)}$ then

$$|\deg(f) - \deg(g)| = \frac{1}{\pi} \left| \int_{B^2} \left(\partial_1 u \wedge \partial_2 u - \partial_1 v \wedge \partial_2 v\right) \right|$$

$$= \frac{1}{\pi} \left| \int_{B^2} \partial_1 (u-v) \wedge \partial_2 u + \partial_1 v \wedge \partial_2 (u-v) \right|$$

$$\lesssim \|u-v\|_{H^1} \left(\|u\|_{H^1} + \|v\|_{H^1}\right)$$

$$\lesssim \|f-g\|_{H^{1/2}} \left(\|f\|_{H^{1/2}} + \|g\|_{H^{1/2}}\right).$$

(12)

<u>Rk</u>: 1) if $g \in H^{1/2}(\boxed{S^1}, S^1)$, $g' \in H^{-1/2}(S^1, \mathbb{R}^2)$
= dual of $H^{1/2}$; then

$$\deg(g) = \frac{1}{2\pi}\int_{S^1} \underbrace{g}_{\in H^{1/2}} \wedge \underbrace{g'}_{\in H^{-1/2}} = \text{duality } (H^{1/2}, H^{-1/2}).$$

2) The degree can be defined via Fourier:

$g \in \underline{\underline{C^1}}(S^1, S^1)$, $g(e^{i\theta}) = g(\theta) = \sum_{k \in \mathbb{Z}} a_k e^{ik\theta}$, $a_k \in \mathbb{C}$

Then $\|g\|^2_{\underline{\underline{H^{1/2}}}(S^1)} = 2\pi \sum_{k \in \mathbb{Z}} \underline{\underline{|k|}}\, |a_k|^2 < +\infty$

and $\deg(g) = \frac{1}{2\pi i}\int \bar{g}\, g'\, d\theta = \frac{1}{2\pi i}\int \left(\sum \bar{a}_k e^{-ik\theta}\right)$
$\cdot \left(\sum i l a_l e^{il\theta}\right) = \underbrace{\sum_{k \in \mathbb{Z}} k\, |a_k|^2}$ abs. conv. if $g \in H^{1/2}$

$\Longrightarrow$ well defined if $g \in H^{1/2}(S^1, S^1)$.