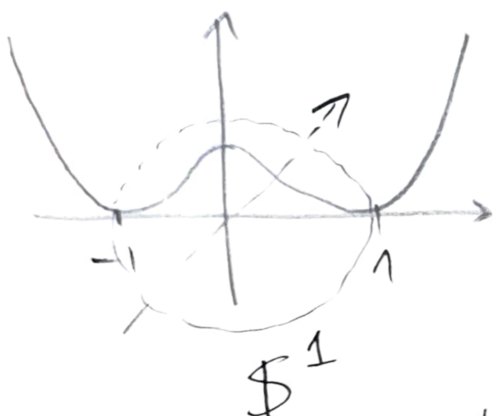
Vortices in Ginzburg-Landau problems

$\Omega \subset \mathbb{R}^2$, $u : \Omega \to \mathbb{R}^2 \simeq \mathbb{C}$, $\varepsilon > 0$

$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \, dx.$$



$W(t) = (1-t^2)^2 \geq 0$

$\{u \in \mathbb{R}^2 : W(|u|) = 0\} = \mathbb{S}^1$

As $\varepsilon \to 0$, limit configurations $u_* : \Omega \to \mathbb{S}^1$.

$\to$ toy model for energy concentration on point singularities

Motivation: 1) physics: Landau theories for phase transitions (superconductivity, Bose-Einstein condensates, liquid crystals, micromagnetics...)

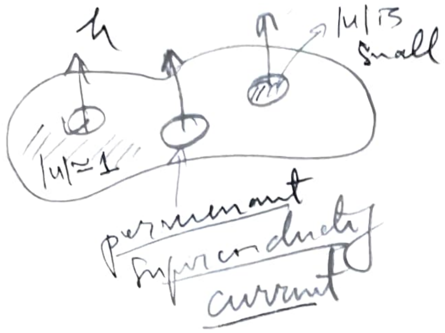
u = order parameter

- superconductivity: $|u|^2$ = density of Cooper pairs of superconducting electrons

  $|u| \simeq 1 \to$ superconducting phase

  $|u| \simeq 0 \to$ normal phase ($\to$ normal conductor)

  mixed phase: vortices ($\to$ toy model)



in a superconductor, for large enough applied magnetic field h, magnetic field penetrates through vortices

2) Geometry + topology: toy model = regularisation of the harmonic map problem

$$\text{"min"} \int_\Omega |\nabla u_*|^2 \quad : \quad \underbrace{\deg(g)}_{\text{winding number}} \neq 0 \Rightarrow \text{infinite energy (point singularities)}$$

$u_* : \Omega \to \mathbb{S}^1$

$u_*|_{\partial\Omega} = g : \partial\Omega \to \mathbb{S}^1$

→ relax the constraint $|u_*| = 1$

→ as $\varepsilon \to 0$, toy model induces canonical $\mathbb{S}^1$-valued harmonic maps with the least "infinite" energy. (optimal location of vortices)

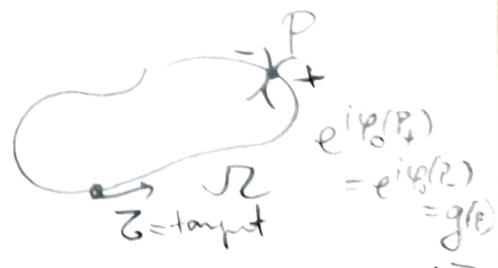
Goal: Analyse the asymptotic behaviour of E_ε as $\varepsilon \to 0$

I. Asymptotics of minimizers:

Strategy: Fix a boundary data $g : \partial\Omega \to \mathbb{S}^1$ smooth

Topological degree (winding number)

$$\deg(g) = \frac{1}{2\pi} \int_{\partial\Omega} g \wedge \partial_\tau g \, d\sigma \in \mathbb{Z}$$



$$\left[\left[g = e^{i\varphi_0} \text{ on } \partial\Omega \setminus \{P\} \Rightarrow \deg(g) = \frac{1}{2\pi} \int_{\partial\Omega} \partial_\tau \varphi_0 \, d\sigma = \frac{1}{2\pi} \left(\varphi_0(P+) - \varphi_0(P_-) \right) \right]\right]$$

Minimisation :

$$\min_{\substack{u:\Omega\to\mathbb{R}^2\\ u|_{\partial\Omega}=g}} E_\varepsilon(u)$$

: well defined → existence of minimizers u_ε (direct method) $\boxed{\text{Exo}}$ = Exercise

[→ $\exists u$ = extension of g with values in $\mathbb{R}^2$ $\left(\begin{array}{l}\Delta u=0\ \Omega\\ u=g\ \partial\Omega\end{array}\right)$

+ minimizing sequence $\Rightarrow 0\le E_\varepsilon(u_n)\le C \Rightarrow$ bdd in H^1

$\Rightarrow$ weak H^1-convergence of (u_n) + E_ε weakly l.s.c. in H^1

(→ limit satisfies B.C.) $\left[\begin{array}{l}\liminf \int|\nabla u_n|^2 \ge \int|\nabla u|^2\\ \text{Fatou } \liminf\int W(|u_n|)\ge\int W(|u|)\end{array}\right]$]

→ Euler-Lagrange equation: u_ε critical point of E_ε

i.e. $\displaystyle\frac{d}{dt}\Big|_{t=0} E_\varepsilon(u_\varepsilon+t\zeta)=0,\ \forall\zeta\in C_c^\infty(\Omega,\mathbb{R}^2).$

$$\Leftrightarrow\quad (GL)\left\{\begin{array}{l}-\Delta u_\varepsilon=\dfrac{1}{\varepsilon^2}u_\varepsilon(1-|u_\varepsilon|^2) \text{ in } \Omega\\ u_\varepsilon=g \quad \partial\Omega\end{array}\right.$$

$\boxed{\text{Exo}}$

→ $u_\varepsilon\in H^1(\Omega)\Longrightarrow u_\varepsilon\in C^\infty(\bar\Omega)$ by elliptic regularity

[$u_\varepsilon\in L^p(\Omega),\ \forall p<\infty \Rightarrow$ RHS $\in L^p_{p<\infty} \Rightarrow u_\varepsilon\in W^{2,p}\subset W^{1,\infty}$

$\Rightarrow$ RHS $\in W^{2,p}\Rightarrow \dots$]

→ $|u_\varepsilon|\le 1$ in Ω [→ Exercise]

Case 1: $\boxed{\deg(g)=0}\Rightarrow g=e^{i\varphi_0}$, $\varphi_0:\partial\Omega\to\mathbb{R}$ smooth and unique (up to $2\pi\mathbb{Z}$)

Rq: if ε small $\Rightarrow$ minimizer u_ε is unique (Ye-Zhou '96). Also, if ε large, E_ε is convex,

! E_ε convex $\Rightarrow$ uniqueness

Thus u_ε is unique critical point.

Thm (Bethuel-Brezis-Hélein) '93, Calc Var

as $\varepsilon \to 0$, $u_\varepsilon \to u_*$ in $H^1 \cap C^k_{loc} \cap C^{1,\alpha}(\Omega)$, $\forall k \in \mathbb{N}$, $\forall \alpha \in (0,1)$

where $u_* : \Omega \to S^1$ is the unique sol

(HMP) $\begin{cases} -\Delta u_* = |\nabla u_*|^2 u_* & \Omega \\ u_* = g & \partial\Omega \end{cases}$

More precisely, $u_* = e^{i\varphi_*}$, $\varphi_* : \Omega \to \mathbb{R}$ is the unique sol of $\begin{cases} \Delta \varphi_* = 0 & \text{in } \Omega \\ \varphi_* = \varphi_0 & \text{on } \partial\Omega. \end{cases}$

Also, $E_\varepsilon(u_\varepsilon) = \frac{1}{2}\int_\Omega |\nabla u_*|^2 + o(1)$ as $\varepsilon \to 0$.

RK: $|u_\varepsilon| \to |u_*| = 1$ unif $\Rightarrow$ u_ε NO zeros $\Rightarrow$ NO VORTICES

Proof: $\boxed{u_\varepsilon \to u_* \text{ in } H^1}$; indeed, u_* gives an upper bound $|u_*| = 1$, $u_* = g = e^{i\varphi_0}$ on $\partial\Omega$

* $E_\varepsilon(u_\varepsilon) \leq E_\varepsilon(u_*) = \frac{1}{2}\int |\nabla u_*|^2 = C$
* (u_ε) bdd in H^1 (because bdd in $\dot{H}^1$ + Poincaré-Wirtinger $\int |u_\varepsilon - \fint_{\partial\Omega} u_\varepsilon|^2 \leq C \int |\nabla u_\varepsilon|^2$) $u_\varepsilon = g$ $\partial\Omega$ see verso

$\Rightarrow u_{\varepsilon_n} \rightharpoonup u_0$ H^1 and strong in L^p, $p < \infty$ + a.e. in Ω

* $u_0 = g$ $\partial\Omega$
* $|u_0| = 1$: $\int (1 - |u_{\varepsilon_n}|^2)^2 \leq 4\varepsilon^2 C \xrightarrow[\varepsilon \to 0]{\text{Fatou}} \int (1 - |u_0|^2)^2 \leq 0$
* $u_{\varepsilon_n} \rightharpoonup u_0 \Rightarrow \varliminf E_\varepsilon(u_\varepsilon) \geq \varliminf \frac{1}{2}\int |\nabla u_\varepsilon|^2 \geq \frac{1}{2}\int |\nabla u_0|^2$

Fact: u_* is the unique minimizer

$$\min_{\substack{u:\Omega\to \mathbb{S}^1 \\ u|_{\partial\Omega}=g}} \int |\nabla u|^2$$

Idea: $u \in H^1(\Omega, \mathbb{S}^1) \overset{\text{Exo}}{\Longrightarrow} \exists \varphi \in H^1(\Omega, \mathbb{R})$, $u = e^{i\varphi}$
and $|\nabla u| = |\nabla \varphi|$.

Conclusion: $u_0 = u_*$ and $\int |\nabla u_\varepsilon|^2 \to \int |\nabla u_*|^2 \Rightarrow \boxed{u_\varepsilon \xrightarrow[H^1]{} u_*}$ ∎

Rk. $u_\varepsilon \not\to u_*$ $C^2(\Omega)$ if $g \neq$ cte

indeed, $-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2)$ Ω $\xRightarrow[\substack{x\to\partial\Omega \\ u_\varepsilon \in C^\infty(\bar\Omega)}]{|u_\varepsilon|=1\ \partial\Omega}$ $\Delta u_\varepsilon = 0$ $\partial\Omega$

and $-\Delta u_* = u_* |\nabla u_*|^2$ Ω $\xRightarrow[\substack{x\to\partial\Omega \\ u_* \in C^\infty(\bar\Omega)}]{}$ $|\Delta u_*| = |\nabla u_*|^2$.

if $\Delta u_\varepsilon \to \Delta u_*$ $L^\infty(\Omega)$

$\Rightarrow L^\infty(\partial\Omega) \Rightarrow \nabla u_* = 0 \Rightarrow u_* = \text{cte} \Rightarrow g = \text{cte}$ (↯).

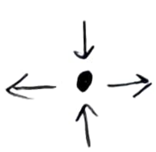
Case 2: $\deg(g) = d \neq 0$ (assume $d > 0$).

$\Rightarrow g = e^{i\varphi_0}$, φ_0 has jumps ($\Rightarrow \varphi_0 \in BV \setminus C^0$)

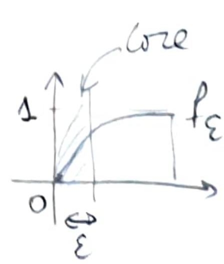
Vortex: 1) prototype in the limit $\varepsilon \to 0$: $\boxed{u_* = \frac{x}{|x|} = e^{i\theta}}$ $\in W^{1,p}$, $p<2$
$\notin H^1$

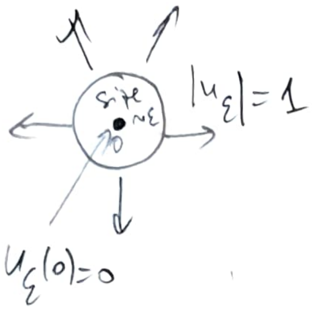
 degree $= +1$ $\Rightarrow$ singular $\mathbb{S}^1$-harmonic map

$-\Delta u_* = u_* |\nabla u_*|^2$

 degree $= -1$ $\left(\overset{\text{conjugate}}{\overline{u_*}} = e^{-i\theta}\right)$

2) prototype as $\varepsilon > 0$: $u_\varepsilon(x) = f_\varepsilon(|x|) e^{i\theta}$,





$x=0$ is a topological zero of u_ε of degree $+1$
$\to$ lies near the limit vortex.

Thm (Pacard–Rivière '00): Ω = unit disk, $g(x)=x$ on $\partial\Omega = \mathbb{S}^1$.
Then for ε small, $u_\varepsilon(x) = f_\varepsilon(|x|)e^{i\theta}$ is the unique min of E_ε under (B.C.) $u_\varepsilon = g$ $\partial\Omega$.

Blow-up of E_ε ~~for vortex of degree 1~~: $f_\varepsilon(r) \simeq \begin{cases} \frac{r}{\varepsilon}, & r<\varepsilon \\ 1, & r>\varepsilon \end{cases}$

$$|\nabla u_\varepsilon|^2 = f_\varepsilon^2 \underbrace{|\nabla(e^{i\theta})|^2}_{\frac{1}{r^2}} + (f_\varepsilon')^2$$

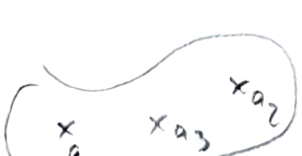
$$E_\varepsilon(u_\varepsilon, B_R) = \int_{B_R\setminus B_\varepsilon} \frac{1}{2} \underbrace{f_\varepsilon^2}_{=1} \frac{dx}{r^2} + \underbrace{\int_{B_\varepsilon} \frac{1}{2}\overbrace{|\nabla u_\varepsilon|^2}^{\sim\frac{1}{\varepsilon^2}} + \frac{1}{4\varepsilon^2}\overbrace{(1-f_\varepsilon^2)^2}^{\leq 1}}_{=\mathcal{O}(1)}$$

$$\sim \pi \log\frac{R}{\varepsilon} + \mathcal{O}(1).$$

Thm (BBH '94) if $\deg(g)=d>0$, then for a subsequence

$$u_\varepsilon \longrightarrow \boxed{u_* = e^{i\varphi_*}\frac{x-a_1}{|x-a_1|}\cdots\frac{x-a_d}{|x-a_d|}} \qquad \boxed{k\in\mathbb{N},\ \alpha\in(0,1)}$$

canonical harmonic map



in $W^{1,1}\cap C^k_{loc}(\Omega\setminus\{a_k\})\cap C^{1,\alpha}_{loc}(\bar\Omega\setminus\{a_k\})$

where $\begin{cases} \Delta\varphi_* = 0 & \Omega \\ \varphi_* = \tilde\varphi_0 & \partial\Omega \end{cases}$ and $e^{i\tilde\varphi_0} = g(x)\left(\overline{\frac{x-a_1}{|x-a_1|}}\right)\cdots\left(\overline{\frac{x-a_n}{|x-a_n|}}\right)$

and $a_1,\dots,a_M$ are distinct vortex points in Ω of $\boxed{\text{degree } 1}$.
Moreover, $E_\varepsilon(u_\varepsilon) = \pi d|\log\varepsilon| + \mathcal{O}(1)$ as $\varepsilon\to 0$.

<u>Rk</u>: 1) $\mathcal{O}(1) = \underbrace{W(a_1, \ldots, a_d)}_{\text{renormalized energy}} + d\underbrace{\gamma}_{\text{energy of a radial profile}} + o(1)$.

2) if $d \geq 2$, the minimizers are <u>not</u> unique in general if ε small

<u>Eg</u>: $\Omega = B_1$, $g(e^{i\theta}) = e^{2i\theta} \Rightarrow$ group action $\alpha \to e^{-2i\alpha} u(e^{i\alpha} x)$ keeps invariant g and $E_\varepsilon \Rightarrow$ infinite # minimizers