Asymptotic minimality of one-dimensional transition profiles in Aviles-Giga type models: an approach via 1-currents

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Abstract

For vector fields on a two-dimensional domain, we study the asymptotic behaviour of Modica-Mortola (or Allen-Cahn) type functionals under the assumption that the divergence converges to 0 at a certain rate, which effectively produces a model of Aviles-Giga type. This problem will typically give rise to transition layers, which degenerate into discontinuities in the limit. We analyse the energy concentration at these discontinuities and the corresponding transition profiles.

We derive an estimate for the energy concentration in terms of a novel geometric variational problem involving the notion of $\mathbb{R}^2$-valued 1-currents from geometric measure theory. This in turn leads to criteria, under which the energetically favourable transition profiles are essentially one-dimensional.

1 Introduction

1.1 The problem

Let $\Omega \subseteq \mathbb{R}^2$ be an open domain. Suppose that $W \colon \mathbb{R}^2 \to [0, \infty)$ is a locally Hölder continuous function. For $u \colon \Omega \to \mathbb{R}^2$ and for $\epsilon > 0$, consider a Modica-Mortola (or Allen-Cahn) type functional of the form

$$E_\epsilon(u; \Omega) = \frac{1}{2} \int_\Omega \left(\epsilon |Du|^2 + \frac{1}{\epsilon} W(u) \right) dx.$$

We are interested in the asymptotic behaviour of a family of vector fields u_ϵ such that

$$\limsup_{\epsilon \searrow 0} \left(E_\epsilon(u_\epsilon; \Omega) + \epsilon^{-2\tau} \| \operatorname{div} u_\epsilon \|^2_{L^s(\Omega)} \right) < \infty \tag{1}$$

for some $\tau > 0$ and some $s > 2$. This is relevant in the context of models that combine a Modica-Mortola type energy functional with a divergence penalisation of the above form, or even with the constraint $\operatorname{div} u = 0$.

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As in the classical theory of Modica and Mortola [37, 38, 36, 45], the above conditions typically imply convergence of a subsequence to a limit $u_0 \colon \Omega \to \mathbb{R}^2$ that takes values in $W^{-1}(\{0\})$ almost everywhere. (Also see the work of DeSimone, Kohn, Müller, and Otto [15].) In addition, the limit will satisfy $\operatorname{div} u_0 = 0$. Transitions between different zeroes of W are possible, but will require a certain amount of energy.

Depending on the exact structure of the potential function W, the vector field u_0 may belong to $\mathrm{BV}(\Omega; \mathbb{R}^2)$, or may belong to a larger space, but even then we typically have a countably 1-rectifiable jump set $J \subseteq \Omega$, where the values of u_0 jump from one value to another [13]. More precisely, this set is characterised by the behaviour of a blow-up around a point $x_0 \in J$: choose a sequence $r_k \searrow 0$ such that the functions $x \mapsto u_0(x_0 + r_k x)$ converge, say in $L^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, to the limit $v \colon \mathbb{R}^2 \to \mathbb{R}^2$. Let $\nu \in S^1$ be one of the approximate normal vectors to J at x_0 (which exist almost everywhere with respect to the 1-dimensional Hausdorff measure by the countable rectifiability). Then there exist $a^-, a^+ \in W^{-1}(\{0\})$ such that $v(x) = a^+$ when $\nu \cdot x > 0$ and $v(x) = a^-$ when $\nu \cdot x < 0$.

A blow-up is useful, too, when we want to understand how much energy will be concentrated at a point $x_0 \in J$ in the limit as $\epsilon \searrow 0$; or in other words, how much energy is required to generate a transition between a^- and a^+. Suppose that we rescale the vector fields u_ϵ similarly. Then we can expect that the limit $u_0 \colon \mathbb{R}^2 \to \mathbb{R}^2$ is already of the form

$$u_0(x) = \begin{cases} a^+ & \text{if } \nu \cdot x > 0, \\ a^- & \text{if } \nu \cdot x < 0. \end{cases} \tag{2}$$

Since u_0 must be divergence free under the conditions we are interested in, we expect that $\nu \perp (a^+ - a^-)$.

The density of the energy concentrated at a corresponding jump point is measured by the quantity

$$\liminf_{\epsilon \searrow 0} E_\epsilon(u_\epsilon; B_1(0)),$$

where $B_r(x)$ denotes the open disc of radius $r > 0$ centred at $x \in \mathbb{R}^2$. For $a^- \neq a^+$, we therefore consider the set $\mathcal{U}(a^-, a^+)$, comprising all families $(u_\epsilon)_{\epsilon > 0}$ of vector fields $u_\epsilon \in W^{1,2}(B_1(0); \mathbb{R}^2)$ such that $u_\epsilon \to u_0$ in $L^1(B_1(0); R^2)$ and such that there exist $\tau > 0$ and $s > 2$ with the property that

$$\lim_{\epsilon \searrow 0} \epsilon^{-\tau} \| \operatorname{div} u_\epsilon \|_{L^s(B_1(0))} = 0, \tag{3}$$

where u_0 is defined as in (2) with $\nu = (a^- - a^+)^\perp / |a^- - a^+|$. (We disregard the possibility that ν may point in the opposite direction, because that situation may be reduced to this one by applying reflections in the domain and codomain and adjusting W accordingly.) Then we define

$$\mathcal{E}(a^-, a^+) = \frac{1}{2} \inf \left\{ \liminf_{\epsilon \searrow 0} E_\epsilon(u_\epsilon; B_1(0)) \colon (u_\epsilon)_{\epsilon > 0} \in \mathcal{U}(a^-, a^+) \right\}.$$

One important question is whether the same infimum is obtained when we consider only one-dimensional, divergence-free transition profiles, i.e., vector fields of the form $u_\epsilon = a^- + w_\epsilon(x \cdot \nu)(a^+ - a^-)$ for some functions $w_\epsilon \colon \mathbb{R} \to \mathbb{R}$.

Under the typical assumptions on W, the resulting number is easy to compute with the methods from the Modica-Mortola theory and is

$$\int_{[a^-,a^+]} \sqrt{W}\, d\mathcal{H}^1,$$

where $[a^-, a^+]$ denotes the line segment connecting a^- with a^+ and $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure.

This question is the main focus of this paper, and it can be formulated as follows.

Question 1. *Under what conditions is*

$$\mathcal{E}(a^-, a^+) = \int_{[a^-,a^+]} \sqrt{W}\, d\mathcal{H}^1?$$

1.2 Main results

We now fix the points a^- and a^+. It is convenient to assume that $\nu = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, and thus $a_1^- = a_1^+$, as this will simplify the presentation of our results. The general situation can always be reduced to this case by a change of coordinates, so there is no loss of generality.

We also assume that W has a specific polynomial rate of growth as $|y| \to \infty$. More precisely, we assume that there exist certain constants $c_1, c_2 > 0$ and $\bar{p} > 0$ such that

$$c_1|y|^{2\bar{p}} - 1 \leq W(y) \leq c_2(|y|^{2\bar{p}} + 1) \tag{4}$$

for all $y \in \mathbb{R}^2$.

To formulate our first result, we need to introduce some tools, including the notion of $\mathbb{R}^2$-valued 1-currents. This is a variant of a standard concept from geometric measure theory. Its definition is normally given in terms of differential forms in $\mathbb{R}^2$, but for our purpose, the following, equivalent definition is just as convenient.

Definition 2. An $\mathbb{R}^2$*-valued* 1*-current on* $\mathbb{R}^2$ is an element of the dual space of $C_0^\infty(\mathbb{R}^2; \mathbb{R}^{2\times 2})$. If T is an $\mathbb{R}^2$-valued 1-current in $\mathbb{R}^2$, then its *boundary* ∂T is the $\mathbb{R}^2$-valued distribution such that $\partial T(\xi) = T(D\xi)$ for every $\xi \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$. We say that T is *normal* if there exists $C \geq 0$ such that

$$T(\zeta) + \partial T(\xi) \leq C \sup_{x\in\mathbb{R}^2} \big(|\zeta(x)| + |\xi(x)|\big)$$

for all $\zeta \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^{2\times 2})$ and all $\xi \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$.

We are particularly interested in normal $\mathbb{R}^2$-valued 1-currents T a with specific boundary, given by the condition that

$$\partial T(\xi) = \begin{pmatrix} \xi_1(a^+) - \xi_1(a^-) \\ 0 \end{pmatrix}$$

for all $\xi \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$. We write $\mathcal{C}^0_{2\times 2}$ for the set of all normal currents with this boundary.

Given any normal $\mathbb{R}^2$-valued 1-current T, there always exist a Radon measure $\|T\|$ on $\mathbb{R}^2$ and a $\|T\|$-measurable, matrix-valued function $\vec{T}\colon \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ with $|\vec{T}| = 1$ almost everywhere, such that

$$T(\zeta) = \int_{\mathbb{R}^2} \zeta : \vec{T}\, d\|T\|$$

for any $\zeta \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^{2\times 2})$. Here we use the notation $M : N$ for the Frobenius inner product between two matrices $M, N \in \mathbb{R}^{2\times 2}$. We further write $|M|$ for the corresponding norm of M.

The following is an example of a current with some relevance for our results. Define $T^0 \in \mathcal{C}^0_{2\times 2}$ by the condition that

$$T^0(\zeta) = \int_{[a^-,a^+]} \zeta : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} d\mathcal{H}^1$$

for $\zeta \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^{2\times 2})$. Then $\|T^0\| = \mathcal{H}^1 \llcorner [a^-, a^+]$ and $\vec{T}^0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ almost everywhere. (We can think of the first component of T^0 as a representation of the oriented line segment between a^- and a^+, whereas the second component vanishes.)

We now consider the function $F^*\colon \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{\infty\}$ such that

$$F^*(y, N) = \begin{cases} \frac{1}{4} W(y) \max\{|N|^2 - 2\det N, (n_{12} - n_{21})^2\} & \text{if } \operatorname{tr} N = 0, \\ \infty & \text{else,} \end{cases}$$

where we write $N = \left(\begin{smallmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{smallmatrix}\right)$. For any $T \in \mathcal{C}^0_{2\times 2}$, we define

$$\mathbf{M}_F(T) = \int_{\mathbb{R}^2} \sqrt{F^*(y, \vec{T}(y))}\, d\|T\|(y).$$

(This is a variant of the mass that is normally associated to a current. The connections will become more apparent in Section 3 below.)

We have the following results.

Theorem 3. *The inequality*

$$\mathcal{E}(a^-, a^+) \geq 2 \inf_{T \in \mathcal{C}^0_{2\times 2}} \mathbf{M}_F(T)$$

holds true.

Corollary 4. *If* $\mathbf{M}_F(T^0) \leq \mathbf{M}_F(T)$ *for every* $T \in \mathcal{C}^0_{2\times 2}$*, then*

$$\mathcal{E}(a^-, a^+) = \int_{[a^-,a^+]} \sqrt{W}\, d\mathcal{H}^1.$$

Thus we may be able to give an affirmative answer to the above question by solving a different variational problem involving currents. Since currents can be interpreted geometrically, that variational problem is geometric in nature. It is also rather unusual because of the structure of the above function F^*. It may be difficult to solve in general, but we can give some estimates that allow further conclusions.

For $j \in \mathbb{N}_0$, let $C^j_{\bar{p}}(\mathbb{R}^2)$ denote the space of all $\phi \in C^j(\mathbb{R}^2)$ such that there exists a constant $C \geq 0$ satisfying $|D^k \phi(y)| \leq C(|y|^{\bar{p}-k} + 1)$ for all $y \in \mathbb{R}^2$ and $k = 0, \ldots, j$. We now have the following result.

Corollary 5. *Suppose that $W = w^2$, and suppose that there exist Borel functions $\iota, \kappa, \lambda \colon \mathbb{R}^2 \to [-1, 1]$ with*

$$\iota^2 \leq \min\{1 - \lambda^2, (1 + \kappa)(1 - \lambda), (1 - \kappa)(1 + \lambda)\},$$

such that $\iota w, \kappa w, \lambda w \in C^2(\mathbb{R}^2) \cap C^1_{\bar{p}}(\mathbb{R}^2)$ and

$$\frac{\partial^2}{\partial y_1^2}(\kappa w) = \frac{\partial^2}{\partial y_2^2}(\lambda w) + 2\frac{\partial^2}{\partial y_1 \partial y_2}(\iota w). \tag{5}$$

Then

$$\mathcal{E}(a^-, a^+) \geq \int_{a_2^-}^{a_2^+} (\kappa w)(a_1^-, t)\, dt$$

for any $T \in \mathcal{C}^0_{2\times 2}(\mathbb{R}^2)$.

If in addition, we know that $\kappa(a_1^-, y_2) = 1$ for all $y \in [a_2^-, a_2^+]$, then it follows, of course, that

$$\mathcal{E}(a^-, a^+) = \int_{[a^-, a^+]} \sqrt{W}\, d\mathcal{H}^1.$$

Corollary 5 is a consequence of another, more general estimate, which may be more useful in certain situations. Since the statement is also more technical, however, we postpone the formulation to Section 5 (see Theorem 31).

1.3 Background

Problems like the above are relevant for a number of physical systems, including micromagnetics [19], smectic-A liquid crystals [33, 7], thin film blisters [39], or crystal surfaces [46]. Such models typically arise when a Ginzburg-Landau type energy functional is combined with a divergence penalisation, or is applied to a gradient vector field. Indeed, if we consider a quantity such as

$$\frac{1}{2}\int_\Omega \left(\epsilon |D^2\phi|^2 + \frac{1}{\epsilon} W(D\phi)\right) dx, \tag{6}$$

then the identification $u = \nabla^\perp \phi$ will give rise to $E_\epsilon(u; \Omega)$, and in this case, we even have the condition $\operatorname{div} u = 0$. The integral in (6) gives a variant of the Aviles-Giga functional [7].

Despite its importance, remarkably little is known about Question 1, let alone about how to determine $\mathcal{E}(a^-, a^+)$ in general, with the exception of some special cases. It can happen, of course, that the constructions from the vector-valued Modica-Mortola problem [45, 8] happen to be divergence free, in which case they also provide a solution to the above problem. Otherwise, only the case of the classical Aviles-Giga functional, which corresponds to $W(y) = (1 - |y|^2)^2$, has a reasonably comprehensive theory. One of the key contributions is of Jin and Kohn [28], who (among other things) determined the value of $\mathcal{E}(a^-, a^+)$ in this situation. Without attempting to give a complete list, we mention some other noteworthy contributions to this theory [2, 15, 26, 25].

More general potential functions have been studied by Ignat and Monteil [23]. In particular, they give some results similar to Corollary 5 (although weaker), which they prove with methods different from what we use here.

Theorem 3 and its corollaries (including Theorem 31 in Section 5 below) add a completely new tool to the study of these problems. The theorem provides an estimate for $\mathcal{E}(a^-, a^+)$ in terms of another variational problem, which is geometric in nature, and whose connection to the functionals E_ϵ is far from obvious. That variational problem is difficult to solve in general, but this novel connection is clearly of theoretical value, and we show in Section 6 that it can be used to answer Question 1 for some examples where the problem was previously open.

There is then the obvious question of how to determine $\mathcal{E}(a^-, a^+)$ when the equality from Question 1 is *not* satisfied. Almost nothing is known for this question in general, although for some specific problems of a similar nature, it can be answered [40, 41, 1, 24]. We provide no general results about this question here, but we give some examples in Section 6 which suggest that Theorem 3 may be useful in this context, too.

There are some aspects of the theory that we implicitly take for granted in the formulation of Question 1. If we were to fully analyse the problem with respect to Γ-convergence, we would have to prove that

- the limiting energy is really concentrated on a countably 1-rectifiable jump set, where we can perform an appropriate blow-up, and
- after the blow-up, we have convergence of a subsequence in $L^1(B_1(0); \mathbb{R}^n)$ to a limit u_0 as above.

That is, we would need some information about the structure of limit points and compactness of families $(u_\epsilon)_{\epsilon>0}$ satisfying (1). Such information is relatively easy to obtain when W has only isolated zeroes, and results of this type are available for the potential function $W(y) = (1-|y|^2)^2$ (the Aviles-Giga functional) [13, 15] and some generalisations thereof [11, 35]. Obviously, if we have such results for a potential function $\tilde{W}$ such that $\tilde{W} \leq CW$ for some constant $C > 0$, then the same follows for W. Nevertheless, these questions are open in general and are not studied here.

1.4 Strategy for the proofs and organisation of the paper

Theorem 3 may appear mysterious at first, as the connection between the energy E_ϵ and the F-mass $\mathbf{M}_F$ becomes apparent only when the ingredients for the proof are known. For this reason, we give an informal overview of the arguments here. At the same time, we explain how the paper is organised.

The first key idea in the proof is that of a 'calibration' (also called 'entropy' by some authors, because of some analogy with entropies for conservation laws). This idea goes back to the paper of Jin and Kohn [28], but has been refined by DeSimone, Kohn, Müller, and Otto [15] and subsequently studied by a number of authors [13, 22, 23]. The formulation that we use here is as follows. Let $\mathcal{L}(\mathbb{R}^2; \mathbb{R}^{2\times 2})$ denote the space of linear maps $\mathbb{R}^2 \to \mathbb{R}^{2\times 2}$. Suppose that there exist $\Phi \in C^1(\mathbb{R}^2; \mathbb{R}^2)$, $\alpha \in C^0(\mathbb{R}^2)$, and $a \in C^1(\mathbb{R}^2; \mathcal{L}(\mathbb{R}^2; \mathbb{R}^{2\times 2}))$ such that

$$\operatorname{div} \Phi(u) + \alpha(u) \operatorname{div} u \leq \frac{\epsilon}{2}|Du|^2 + \frac{1}{2\epsilon} W(u) + \epsilon \operatorname{div}(a(u)Du) \tag{7}$$

for all sufficiently regular vector fields $u \colon B_1(0) \to \mathbb{R}^2$. Then it is not difficult to see, when we integrate over $B_1(0)$ and integrate by parts, that we obtain an estimate of the form

$$\mathcal{E}(a^-, a^+) \geq \Phi_1(a^+) - \Phi_1(a^-)$$

under reasonable assumptions. Clearly, such an inequality is potentially useful for answering Question 1.

But it is not clear at all how to find Φ, α, and a in general, at least not such that they give rise to a *useful* estimate. (The choice $\Phi = 0$, $\alpha = 0$, and $a = 0$ will always work, but the resulting estimate is trivial.) Good calibrations have been constructed in special cases, most notably for the Aviles-Giga functional [28], but no general construction is known.

In Section 2 we derive a condition that is equivalent, for a given Φ, to the existence of α and a such that (7) holds true. If we define the function

$$f(M) = \frac{1}{2}\left(|M|^2 - \frac{1}{2}(\operatorname{tr} M)^2 + |m_{12} - m_{21}|\sqrt{|M|^2 - 2\det M}\right)$$

for $M = \left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$, this condition takes the form of the inequality

$$f(D\Phi) \leq W.$$

We use arguments inspired by the work of Ignat and Merlet [21] in this step, but we extend these ideas considerably.

This gives a convenient way to check whether a given function Φ gives rise to a calibration, but still does not tell us how to construct one. But suppose that we want to find the best possible calibration, which for our purposes means that $\Phi_1(a^+) - \Phi_1(a^-)$ should be as large as possible. Then the above inequality suggests that we determine

$$\eta_0 = \sup\left\{\Phi_1(a^+) - \Phi_1(a^-) \colon f(D\Phi) \leq W\right\}.$$

If we can solve this variational problem, then we have the best estimate that can be achieved with this approach.

It is convenient here to recast the problem in a different form. Define the function

$$F(y, M) = \frac{f(M)}{W(y)}$$

(assuming for the moment that $W(y) > 0$ for all $y \in \mathbb{R}^2$ and ignoring the fact that Question 1 is more interesting for a potential function with zeroes). Then we may instead try to determine

$$e_\infty^2 = \inf\left\{\|F(y, D\Phi)\|_{L^\infty(\mathbb{R}^2)} \colon \Phi_1(a^+) - \Phi_1(a^-) = 1\right\}.$$

It is easy to see that $\eta_0 = 1/e_\infty$. We thus obtain a variational problem involving the L^∞-norm.

Very little is known about problems of this sort. For similar problems involving a *scalar* function (in place of the vector-valued Φ), there is a body of literature going back to the work of Aronsson [3, 4, 5, 6] and including papers by many other authors. Once more we give an incomplete list [10, 27, 42, 16]. For vector-valued functions, this theory does not apply. There is some work by Katzourakis [29, 30, 31], but these results do not tell us much about the solutions to the above problem. Fortunately, we do not need to know anything about the structure of the solutions, we merely need to determine the number e_∞. For this purpose, the ideas of a recent paper by Katzourakis and Moser [32] are useful. This paper treats only the case of the function $F(y, M) = \frac{1}{2}|M|^2$, but the methods can be generalised, and this is what we do in Section 3. We can think of the results as a characterisation of the essential behaviour of the minimisers through a dual problem, in our case that of minimising $\mathbf{M}_F$ for $\mathbb{R}^2$-valued 1-currents. The analysis has to be carried out for a regularised version of F, but then we can prove that there exists a minimiser T of $\mathbf{M}_F$ in $\mathcal{C}^0_{2\times 2}$ such that $\Phi_1(a^+) - \Phi_1(a^-) = 2\mathbf{M}_F(T)$. This is where the inequality from Theorem 3 ultimately comes from.

Remarkably, even though calibrations are central to our approach, this result means that we do not need to construct any calibrations in the end. We only need to know $\Phi_1(a^+) - \Phi_1(a^-)$, and this information is encoded in T.

As already mentioned, these arguments require a regularisation of F, and we need to make sure that we can recover the relevant information when we relax the conditions on F again. This is the purpose of Section 4. At this point, the proof of Theorem 3 is complete. But to make use of it, we have to study the problem of minimising $\mathbf{M}_F$ in $\mathcal{C}^0_{2\times 2}$.

This is the problem that we study in Section 5. Superficially, it may look deceptively simple. After all, we may think of 1-currents as generalised curves in $\mathbb{R}^2$, and $\mathbf{M}_F$ resembles an anisotropic version of the length functional. That is, we have a variant of the problem of finding geodesics. (Incidentally, geodesics for a degenerate Riemannian metric appear in the solutions of the vector-valued Modica-Mortola problem as well [8].) There are, however, several complications. First, we have $\mathbb{R}^2$*-valued* 1-currents, so we should really think of a *pair* of curves linked through $\mathbf{M}_F$. Second, the function F is degenerate in some sense in both variables. Third, although T should be thought of as a one-dimensional object, it does not follow that it is supported on a one-dimensional set (and in general it is not; see Example 35 below). Because of all of this, the standard methods from geometric analysis do not apply here.

We do not have any general methods to solve the problem, but we can nevertheless give some estimates, which show that T^0 is a minimiser under certain conditions. One of the key tools we use for this purpose, is a result and Bonicatto and Gusev [12] (see also the work of Smirnov [44] and of Baratchart, Hardin, and Villalobos-Guillén [9]), which gives a decomposition of a normal 1-current into actual curves. This result applies to conventional 1-currents, not $\mathbb{R}^2$-valued ones, but at least we can apply it to the first component of $T \in \mathcal{C}^0_{2\times 2}$. We can then give some estimates relying on convexity and the structure of F to also take the second component into account. This first gives rise to a functional for Lipschitz curves, which is now really similar to an anisotropic version of the length functional and, in principle, can be analysed with standard methods involving ordinary differential equations. Unfortunately, it also involves some unknown functions, and therefore, the task is not so simple after all. Notwithstanding, with some further estimates, we finally prove Corollary 5 as a result.

We conclude the paper with some examples in Section 6. First, we discuss some potential functions W such that Corollary 5 applies, and the optimal transition layers therefore have one-dimensional profiles. This includes the well-known Aviles-Giga functionals, but also includes some new examples. Finally, we consider the question what Theorem 3 can tell us in situations where the equality from Question 1 does *not* hold true. We have no general results here, but we can compare the number $\mathbf{M}_F(T)$ for some specific currents with the energy density for certain known constructions for u_ϵ. If T minimises $\mathbf{M}_F$, then the former gives a bound for $\mathcal{E}(a^+, a^-)$ from below by Theorem 3, while the latter gives a bound from above by definition. If the two bounds match, then we know that the construction is optimal. We can achieve this for two different examples, assuming that the potential function W is such that the corresponding T is indeed a minimiser of $\mathbf{M}_F(T)$. This raises the question whether the estimate from Theorem 3 might be sharp in general. We have no evidence for this, however, beyond these two examples.

1.5 Notation

The following notation is used throughout the paper, with the exception of Section 3, where some adjustments are required due to a more general setting.

As mentioned previously, for $M, N \in \mathbb{R}^{2\times 2}$, we use the notation $M : N$ and $|M|$ for the Frobenius inner product and norm, respectively. (In Section 3, we will also use the corresponding notation for $(m \times n)$-matrices.) We write M^T for the transpose of M and I for the identity (2×2)-matrix.

Given two vector spaces X and Y, the space of linear maps $X \to Y$ is denoted by $\mathcal{L}(X, Y)$.

Although our problem is concerned with vector fields $u\colon \Omega \to \mathbb{R}^2$, much of our analysis will take place entirely in the codomain $\mathbb{R}^2$. We generally use the notation x for a generic point in the domain Ω, and y for a generic point in the codomain $\mathbb{R}^2$. (Section 3 is an exception here, too, as it is about an auxiliary problem independent of u.)

We will frequently work with convolutions with a standard mollifier. Therefore, we fix $\rho \in C_0^\infty(B_1(0))$ with $\rho \geq 0$ and $\int_{B_1(0)} \rho(y)\, dy = 1$. For $\delta > 0$, we set $\rho_\delta(y) = \delta^{-2}\rho(y/\delta)$.

2 Characterising calibrations through differential inequalities

In this section, we derive some conditions in the form of certain differential inequalities related to the inequality

$$\operatorname{div} \Phi(u) + \alpha(u) \operatorname{div} u \leq \frac{\epsilon}{2}|Du|^2 + \frac{1}{2\epsilon} W(u) + \epsilon \operatorname{div}(a(u) Du_\epsilon) \tag{8}$$

that characterises calibrations. These conditions will make it easier to study suitable calibrations later on. Some of the following arguments go back to the work of Ignat and Merlet [22], but we extend the theory significantly.

2.1 Pointwise conditions

For a given map $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ and a function $\alpha\colon \mathbb{R}^2 \to \mathbb{R}$, we want to understand the above inequality (8). First we show that it suffices to consider tensor fields a of a specific form.

Proposition 6. *Suppose that $\Phi \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ and $\alpha \in C^0(\mathbb{R}^2)$. Let $\epsilon > 0$. If $a\colon \mathbb{R}^2 \to \mathcal{L}(\mathbb{R}^{2\times 2}; \mathbb{R}^2)$ is continuously differentiable and satisfies* (8) *for all*

$u \in C^2(B_1(0);\mathbb{R}^2)$, then there exists a vector field $\omega \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ such that

$$a(y)M = -(M^T\omega(y))^\perp$$

for any $y \in \mathbb{R}^2$ and

$$\operatorname{div}(a(u)Du) = (\operatorname{curl}\omega)(u)\det Du$$

for any $u \in C^2(B_1(0);\mathbb{R}^2)$.

Proof. Let $a^i_{jk} \in C^1(\mathbb{R}^2)$, for $i,j,k = 1,2$, denote the coefficients of a, so that

$$a(y)M = \sum_{j,k=1}^{2} m_{jk}\begin{pmatrix} a^1_{jk}(y) \\ a^2_{jk}(y)\end{pmatrix}$$

for all $M = \left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ and all $y \in \mathbb{R}^2$.

Given an arbitrary point $y \in \mathbb{R}^2$ and two symmetric matrices

$$\Lambda^1 = \begin{pmatrix} \lambda^1_{11} & \lambda^1_{12} \\ \lambda^1_{21} & \lambda^1_{22}\end{pmatrix} \quad \text{and} \quad \Lambda^2 = \begin{pmatrix} \lambda^2_{11} & \lambda^2_{12} \\ \lambda^2_{21} & \lambda^2_{22}\end{pmatrix},$$

we can find $u \in C^2(B_1(0);\mathbb{R}^2)$ such that $u(0) = y$ and $Du(0) = 0$, while at the same time, $D^2u_k(0) = \Lambda^k$ for $k = 1,2$. Then

$$\operatorname{div}(a(u)Du)(0) = \sum_{i,j,k=1}^{2} a^k_{ij}(y)\lambda^i_{jk}.$$

Inequality (8), evaluated at 0, thus gives

$$0 \le \frac{1}{2\epsilon}W(y) + \epsilon\sum_{i,j,k=1}^{2} a^k_{ij}(y)\lambda^i_{jk}.$$

Since this also holds true for all real multiples of Λ^1 and Λ^2, it follows in fact that

$$\sum_{i,j,k=1}^{2} a^k_{ij}(y)\lambda^i_{jk} = 0$$

for any pair of symmetric matrices. Therefore, the coefficients a^1_{11}, a^2_{12}, a^1_{21}, and a^2_{22} must vanish, and

$$a^1_{12} + a^2_{11} = 0 \quad \text{and} \quad a^1_{22} + a^2_{21} = 0.$$

Set

$$\omega = \begin{pmatrix} a^1_{12} \\ a^1_{22}\end{pmatrix},$$

then the desired formulas follow by a direct calculation. □

We have the following characterisation of (8).

Proposition 7. *Suppose that $\Phi \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ and $\alpha \in C^0(\mathbb{R}^2)$. Set $\Xi = D\Phi + \alpha I$. Let $\omega \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ and $\sigma = \operatorname{curl}\omega$. Suppose that $\epsilon > 0$. Then the following statements are equivalent.*

(A) The inequality

$$\operatorname{div}\Phi(u) + \alpha(u)\operatorname{div} u \leq \frac{\epsilon}{2}|Du|^2 + \frac{1}{2\epsilon}W(u) - \epsilon \operatorname{div}\big((Du)^T\omega(u)\big)^{\perp} \tag{9}$$

is satisfied for all $u \in C^2(B_1(0);\mathbb{R}^2)$.

(B) The inequalities $|Du|^2 + 2\sigma(u)\det Du \geq 0$ *and*

$$\operatorname{div}\Phi(u) + \alpha(u)\operatorname{div} u \leq \Big(W(u)\big(|Du|^2 + 2\sigma(u)\det Du\big)\Big)^{1/2}$$

are satisfied for all $u \in C^2(B_1(0);\mathbb{R}^2)$.

(C) For all $y \in \mathbb{R}^2$ *and all* $M \in \mathbb{R}^{2\times 2}$,

$$(M : \Xi(y))^2 \leq W(y)\big(|M|^2 + 2\sigma(y)\det M\big)$$

and $|\sigma(y)| \leq 1$.

Proof. If we assume that (B) holds true, then (A) follows from the observation that

$$-\operatorname{div}\big((Du)^T\omega(u)\big)^{\perp} = \sigma(u)\det Du$$

and Young's inequality.

Now suppose that (A) holds true. We want to show that (C) follows. We note that

$$\Xi(u) : (Du)^T = \operatorname{div}\Phi(u) + \alpha(u)\operatorname{div} u \leq \frac{\epsilon}{2}|Du|^2 + \frac{1}{2\epsilon}W(u) + \epsilon\sigma(u)\det Du$$

for any $u \in C^2(B_1(0);\mathbb{R}^2)$. Consider an arbitrary point $y \in \mathbb{R}^2$. If $W(y) = 0$, then we choose an arbitrary matrix $M \in \mathbb{R}^{2\times 2}$ and consider $u \in C^2(B_1(0);\mathbb{R}^2)$ such that $u(0) = y$ and $Du(0) = M^T$. Then we conclude that

$$\Xi(y) : M \leq \frac{\epsilon}{2}|M|^2 + \epsilon\sigma(y)\det M \tag{10}$$

for any $M \in \mathbb{R}^{2\times 2}$. Since the left-hand side is linear in M and the right-hand side is quadratic, this can only hold true when $\Xi(y) = 0$. In this case, the first inequality in (C) is clear, and the second one follows from the fact that the right-hand side of (10) must be positive semi-definite in M.

Now suppose that $W(y) \neq 0$. Choose a matrix $M \in \mathbb{R}^{2\times 2}$ such that

$$|M|^2 = \frac{W(y)}{\epsilon^2}.$$

We can again choose u such that $u(0) = y$ and $Du(0) = M^T$. Thus

$$\Xi(y) : M \leq \frac{\epsilon}{2}|M|^2 + \frac{1}{2\epsilon}W(y) + \epsilon\sigma(y)\det M = \sqrt{W(y)}\left(|M| + \frac{\sigma(y)\det M}{|M|}\right).$$

Since the left-hand side and the right-hand side are both positive homogeneous in M of degree 1, it follows that in fact,

$$\Xi(y) : M \leq \sqrt{W(y)}\left(|M| + \frac{\sigma(y)\det M}{|M|}\right)$$

for *all* $M \in \mathbb{R}^{2\times 2} \setminus \{0\}$. If $\sigma(y) = 0$, then the desired inequalities hold at y.

If $\sigma(y) \neq 0$, then we fix a number $c \in [0, 1)$ such that $c|\sigma(y)| < 1$ and consider $M \in \mathbb{R}^{2\times 2}$ such that

$$|M|^2 + 2c\sigma(y) \det M = \frac{W(y)}{\epsilon^2}.$$

In this case, we obtain the inequality

$$\begin{aligned}
\Xi(y) : M &\leq \frac{\epsilon}{2}|M|^2 + \frac{1}{2\epsilon}W(y) + \epsilon\sigma(y) \det M \\
&= \frac{\epsilon}{2}\big(|M|^2 + 2c\sigma(y) \det M\big) + \frac{1}{2\epsilon}W(y) + \epsilon(1-c)\sigma(y) \det M \\
&= \sqrt{W(y)} \left(\sqrt{|M|^2 + 2c\sigma(y) \det M} + \frac{(1-c)\sigma(y) \det M}{\sqrt{|M|^2 + 2c\sigma(y) \det M}} \right).
\end{aligned}$$

Again we conclude that

$$\Xi(y) : M \leq \sqrt{W(y)} \left(\sqrt{|M|^2 + 2c\sigma(y) \det M} + \frac{(1-c)\sigma(y) \det M}{\sqrt{|M|^2 + 2c\sigma(y) \det M}} \right) \quad (11)$$

for all $M \in \mathbb{R}^{2\times 2} \setminus \{0\}$. If we replace M by $-M$, then the left-hand side changes its sign while the right-hand side stays the same. Therefore, the inequality

$$\sqrt{|M|^2 + 2c\sigma(y) \det M} + \frac{(1-c)\sigma(y) \det M}{\sqrt{|M|^2 + 2c\sigma(y) \det M}} \geq 0$$

must be satisfied for all $M \in \mathbb{R}^{2\times 2}$, which implies that

$$|M|^2 + (1+c)\sigma(y) \det M \geq 0.$$

We conclude that

$$(1+c)|\sigma(y)| \leq 2.$$

Since we have proved this inequality for any c such that

$$0 \leq c < \min\left\{1, \frac{1}{|\sigma(y)|}\right\},$$

it follows that $|\sigma(y)| \leq 1$.

It now follows that (11) is satisfied for any $c \in [0, 1)$. Letting $c \nearrow 1$, we derive the other inequality in (C) as well.

Now suppose that (C) is satisfied. Since

$$\operatorname{div} \Phi(u) + \alpha(u) = (Du)^T : \Xi(u),$$

statement (B) follows immediately. □

Next we examine inequalities as in statement (C) above. For this purpose, we require the function $g\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by

$$g(M) = \frac{1}{2}\left(|M|^2 + \sqrt{|M|^4 - 4(\det M)^2}\right).$$

We note that g is convex, which is most easily seen in different coordinates: let

$$q_1 = \frac{1}{\sqrt{2}}(m_{11} + m_{22}), \qquad q_2 = \frac{1}{\sqrt{2}}(m_{11} - m_{22}),$$
$$q_3 = \frac{1}{\sqrt{2}}(m_{12} + m_{21}), \qquad q_4 = \frac{1}{\sqrt{2}}(m_{12} - m_{21}).$$

Then

$$g(M) = \frac{1}{2}\left(\sqrt{q_1^2 + q_4^2} + \sqrt{q_2^2 + q_3^2}\right)^2,$$

which is clearly convex. Since g is also homogeneous of degree 2, it follows that for all $M, N \in \mathbb{R}^{2\times 2}$ and for $s, t \in (1, \infty)$ with $\frac{1}{s} + \frac{1}{t} = 1$,

$$g(M+N) = g\left(\frac{sM}{s} + \frac{tN}{t}\right) \leq \frac{g(sM)}{s} + \frac{g(tN)}{t} = sg(M) + tg(N). \tag{12}$$

Lemma 8. *Suppose that $\Lambda \in \mathbb{R}^{2\times 2} \setminus \{0\}$. Let*

$$s_0 = \frac{\det \Lambda}{g(\Lambda)}.$$

(i) The inequality $g(\Lambda) \leq 1$ holds true if, and only if, there exists $s \in [-1, 1]$ such that

$$(\Lambda : M)^2 \leq |M|^2 + 2s \det M \tag{13}$$

for all $M \in \mathbb{R}^{2\times 2}$.

(ii) If there is any $s \in [-1, 1]$ such that (13) is satisfied for all $M \in \mathbb{R}^{2\times 2}$, then the same holds true for $s = s_0$.

Proof. We first consider a matrix Λ such that $2 \det \Lambda = |\Lambda|^2$. Then $\Lambda = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ for some $a, b \in \mathbb{R}$. We then calculate $g(\Lambda) = \frac{1}{2}|\Lambda|^2$ and $s_0 = 1$. If (13) is satisfied for some $s \in [-1, 1]$, then inserting $M = \Lambda$ yields $|\Lambda|^2 \leq 2$, i.e., $g(\Lambda) \leq 1$. Conversely, if $g(\Lambda) \leq 1$, then $a^2 + b^2 \leq 1$. Hence

$$\begin{aligned}
(\Lambda : M)^2 &= \big(a(m_{11} + m_{22}) + b(m_{12} - m_{21})\big)^2 \\
&\leq (m_{11} + m_{22})^2 + (m_{12} - m_{21})^2 \\
&= |M|^2 + 2 \det M
\end{aligned}$$

by the Cauchy-Schwarz inequality. Both statements of the lemma follow immediately.

If $2 \det \Lambda = -|\Lambda|^2$, then we can use practically the same arguments, except that a few signs will change in the above calculations.

We now assume that $2|\det \Lambda| < |\Lambda|^2$. In this case, we first note that (13) cannot be satisfied for $s = \pm 1$. Indeed, if it did hold true for $s = 1$, then we could test it with the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ to find that $\Lambda = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ for some $a, b \in \mathbb{R}$. That is, we would find that we are in fact in the first case. For $s = -1$, the arguments are similar.

We therefore consider $s \in (-1, 1)$ now. Define the bilinear form

$$\langle M, N \rangle_s = M : N + s(m_{11}n_{22} - m_{12}n_{21} - m_{21}n_{12} + m_{22}n_{11}).$$

It is easy to see that this constitutes an inner product on $\mathbb{R}^{2\times 2}$. We also note that

$$|M|_s^2 := \langle M, M\rangle_s = |M|^2 + 2s \det M.$$

By the Riesz representation theorem, there exists $\Theta_s \in \mathbb{R}^{2\times 2}$ such that

$$\Lambda : M = \langle \Theta_s, M\rangle_s$$

for all $M \in \mathbb{R}^{2\times 2}$. By the Cauchy-Schwarz inequality, inequality (13) is satisfied if, and only if,

$$|\Theta_s|_s^2 \leq 1.$$

We can easily determine $\Theta_s = \left(\begin{smallmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{smallmatrix}\right)$ by solving a linear system of equations. We obtain

$$\theta_{11} = \frac{\lambda_{11} - s\lambda_{22}}{1 - s^2}, \qquad \theta_{12} = \frac{\lambda_{12} + s\lambda_{21}}{1 - s^2},$$
$$\theta_{21} = \frac{\lambda_{21} + s\lambda_{12}}{1 - s^2}, \qquad \theta_{22} = \frac{\lambda_{22} - s\lambda_{11}}{1 - s^2}.$$

Therefore,

$$|\Theta_s|_s^2 = \Lambda : \Theta_s = \frac{|\Lambda|^2 - 2s \det \Lambda}{1 - s^2}.$$

Thus we see that for $-1 < s < 1$, inequality (13) is satisfied if, and only if,

$$\frac{|\Lambda|^2 - 2s \det \Lambda}{1 - s^2} \leq 1.$$

We now define

$$\phi(s) = \frac{|\Lambda|^2 - 2s \det \Lambda}{1 - s^2}, \quad -1 < s < 1,$$

and minimise this function over $(-1, 1)$. Differentiating, we compute

$$\phi'(s) = -\frac{2s^2 \det \Lambda - 2s|\Lambda|^2 + 2 \det \Lambda}{(1 - s^2)^2}.$$

The derivative has a unique zero in $(-1, 1)$, which is at

$$\frac{|\Lambda|^2 - \sqrt{|\Lambda|^4 - 4(\det \Lambda)^2}}{2 \det \Lambda} = s_0 \tag{14}$$

(unless $\det \Lambda = 0$, in which case the left-hand side is meaningless but the unique zero is still at s_0). Moreover, we know that $\phi(s) \to \infty$ as $s \nearrow 1$ or $s \searrow -1$. It follows that ϕ has a unique minimum, which is attained at s_0. We further compute

$$\phi(s_0) = g(\Lambda).$$

(This is easier to calculate with the expression for s_0 on the left-hand side of (14) rather than in the definition of s_0.)

Hence if there is any $s \in [-1, 1]$ such that (13) holds true, then $\phi(s) \leq 1$, and it follows that $g(\Lambda) = \phi(s_0) \leq 1$. The number s_0 then also satisfies (13). Conversely, if $g(\Lambda) \leq 1$, then we still conclude that (13) holds true for $s = s_0$. $\square$

2.2 A regularity gap

The combination of Proposition 7 and Lemma 8 suggests that the functions Φ and α give rise to an inequality of the form (8) if $g(\Xi) \leq W$. Assuming that Φ is the quantity we are interested in primarily, we may also wish to minimise $g(D\Phi + \alpha I)$ over α, which will give $\alpha = -\frac{1}{2} \operatorname{div} \Phi$. (Then Ξ is the trace free part of $D\Phi$.) Lemma 8 then also gives a good idea of how to choose $\sigma = \operatorname{curl} \omega$.

In the following sections, we will indeed construct Φ such that $g(\Xi) \leq W$ for this definition of Ξ. Unfortunately, this function will not satisfy the regularity requirements of the preceding subsection. For this reason, we have to use a regularisation scheme, which eventually necessitates the construction of a family of vector fields ω_δ rather than a single ω as in Proposition 7.

Another technical difficulty arises from the fact that once we have σ, we need to invert the curl operator. It does not quite suffice to use standard results here, because we do not necessarily have decay at infinity for σ, but we still want to control the growth of ω. We use the following result here.

Lemma 9. *There exists a constant C such that the following holds true. Suppose that $\sigma \in C^{0,1/2}(\mathbb{R}^2)$ is bounded. Then there exists $\omega \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ such that* $\operatorname{curl} \omega = \sigma$ *and*

$$\sup_{y \in \mathbb{R}^2} \frac{|\omega(y)|}{1 + |y| \log |y|} \leq C \sup_{y \in \mathbb{R}^2} |\sigma(y)|.$$

Proof. We use a (one-sided) dyadic decomposition of σ in terms of a partition of unity

$$1 = \sum_{k=0}^{\infty} \eta_k,$$

where $\eta_k \in C_0^\infty(\mathbb{R}^2)$ are functions such that $0 \leq \eta_k \leq 1$ for all $k \in \mathbb{N}_0$ and $\operatorname{supp} \eta_0 \subseteq B_2(0)$, while $\operatorname{supp} \eta_k \subseteq B_{2^{k+1}}(0) \setminus B_{2^{k-1}}(0)$ for $k \geq 1$.

Let G denote the fundamental solution of the Laplace equation in $\mathbb{R}^2$. For $k \in \mathbb{N}_0$, define $\tilde{\phi}_k = G * (\eta_k \sigma)$. Then $\Delta \tilde{\phi}_k = \eta_k \sigma$. Standard Schauder estimates imply that $\tilde{\phi}_k \in C^{2,1/2}(\mathbb{R}^2)$. Furthermore, we know that $\tilde{\phi}_k$ is smooth in $B_{2^{k-1}}(0)$ for $k \geq 1$.

Set $S = \|\sigma\|_{L^\infty(\mathbb{R}^2)}$. For any $y \in \mathbb{R}^2$, we have the estimate

$$\begin{aligned} |D\tilde{\phi}_k(y)| &= \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \frac{y - z}{|y - z|^2} \eta_k(z) \sigma_k(z) \, dz \right| \\ &\leq \frac{S}{2\pi} \int_{B_{2^{k+1}}(0)} \frac{dz}{|y - z|} \\ &\leq \frac{S}{2\pi} \int_{B_{2^{k+1}}(0)} \frac{dz}{|z|} = 2^{k+1} S \end{aligned}$$

for all $k \in \mathbb{N}_0$. If $k \geq 1$, then we also estimate

$$|D^2 \tilde{\phi}_k(0)| \leq \frac{S}{2\pi} \int_{B_{2^{k+1}}(0) \setminus B_{2^{k-1}}(0)} \frac{dz}{|z|^2} = S \log 4.$$

Let $R \geq 1$. If $k \in \mathbb{N}$ is such that $2^{k-2} \geq R$, then with the same arguments, we find a universal constant $C_1 \geq 0$ such that

$$|D^3 \tilde{\phi}_k(y)| \leq 2^{-k} C_1 S$$

for all $y \in B_R(0)$.

Now define $\phi_0 = \tilde{\phi}_0$ and

$$\phi_k(y) = \tilde{\phi}_k(y) - \tilde{\phi}_k(0) - D\tilde{\phi}_k(0)y - \frac{1}{2}D^2\tilde{\phi}_k(0)(y,y), \quad k \geq 1.$$

Then $\phi_k(0) = 0$, $D\phi_k(0) = 0$, and $D^2\phi_k(0) = 0$ for $k \geq 1$. We still compute $\Delta\phi_k = \eta_k\sigma$ in $\mathbb{R}^2$. (For $k \geq 1$, this is because $\Delta\tilde{\phi}_k(0) = 0$.) Moreover, if $2^{k-2} \geq R$, then

$$|D^3\phi_k(y)| \leq 2^{-k}C_1S,$$

which implies that there exists a universal constant C_2 such that

$$R^{-3}|\phi_k| + R^{-2}|D\phi_k| + R^{-1}|D^2\phi_k| \leq 2^{-k}C_2S$$

uniformly in $B_R(0)$ when $2^{k-2} \geq R$. Therefore, the series

$$\phi = \sum_{k=0}^{\infty} \phi_k$$

converges in $C^2(\overline{B_R(0)})$ for any $R \geq 1$. In particular, the function ϕ is twice continuously differentiable.

Furthermore, for all $k \in \mathbb{N}_0$, we find that

$$|D\phi_k| \leq C_3(2^k + R)S$$

in $B_R(0)$ for another universal constant C_3. If we choose k_0 such that $2^{k_0-3} \leq R < 2^{k_0-2}$, then

$$\begin{aligned}|D\phi(y)| &\leq C_3S\sum_{k=0}^{k_0-1}(2^k + R) + C_2SR^2\sum_{k=k_0}^{\infty}2^{-k} \\ &\leq C_3S(2^{k_0} + k_0R) + 2^{1-k_0}C_2SR^2\end{aligned}$$

for all $y \in B_R(0)$. Thus we find a universal constant C_4 such that

$$|D\phi(y)| \leq C_4S(1 + |y|\log|y|).$$

Now we set

$$\omega = \begin{pmatrix} -\frac{\partial\phi}{\partial y_2} \\ \frac{\partial\phi}{\partial y_1} \end{pmatrix},$$

and then we have all the desired properties. □

We can now prove the following.

Proposition 10. *There exists a constant $C \geq 0$ with the following property. Let $\Phi \in \bigcap_{p<\infty} W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ and set $\alpha = -\frac{1}{2}\operatorname{div}\Phi$. Define $\Phi_\delta = \rho_\delta * \Phi$, $\alpha_\delta = \rho_\delta * \alpha + \delta$, and*

$$W_\delta = \frac{\rho_\delta * W}{1-\delta} + \delta.$$

If $g(D\Phi + \alpha I) \leq W$, then for every $\delta > 0$ there exists $\omega_\delta \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ such that

$$\operatorname{div}\Phi_\delta(u) + \alpha_\delta(u)\operatorname{div}u \leq \frac{\epsilon}{2}|Du|^2 + \frac{1}{2\epsilon}W_\delta(u) - \epsilon\operatorname{div}\big((Du)^T\omega_\delta(u)\big)^\perp \qquad (15)$$

for every $u \in C^2(B_1(0);\mathbb{R}^2)$ *and every* $\epsilon > 0$*. Furthermore,*

$$|\omega_\delta(y)| \leq C(1 + |y| \log |y|) \tag{16}$$

for every $y \in \mathbb{R}^2$*.*

Proof. In addition to the above quantities, define $\tilde{\alpha}_\delta = \rho_\delta * \alpha$ and $\tilde{W}_\delta = \rho_\delta * W$. Furthermore, set $\Xi = D\Phi + \alpha I$ and $\tilde{\Xi}_\delta = D\Phi_\delta + \tilde{\alpha}_\delta I$. Then by Jensen's inequality, the convexity of g implies that

$$\begin{aligned} g(\tilde{\Xi}_\delta(y)) &= g\left(\int_{B_\delta(y)} \rho_\delta(y-z)\Xi(z)\,dz\right) \\ &\leq \int_{B_\delta(y)} \rho_\delta(y-z) g(\Xi(z))\,dz \\ &\leq \int_{B_\delta(y)} \rho_\delta(y-z) W(z)\,dz = \tilde{W}_\delta(y). \end{aligned}$$

Now recall that $\alpha_\delta = \tilde{\alpha}_\delta + \delta$. Set $\Xi_\delta = D\Phi_\delta + \alpha_\delta I = \tilde{\Xi}_\delta + \delta I$. Then (12) gives

$$g(\Xi_\delta) \leq \frac{g(\tilde{\Xi}_\delta)}{1-\delta} + \frac{g(\delta I)}{\delta} \leq W_\delta.$$

Lemma 8 implies that

$$(\Xi_\delta(y) : M)^2 \leq W_\delta(y)\big(|M|^2 + 2\sigma_\delta \det M\big)$$

for all $M \in \mathbb{R}^{2\times 2}$, where

$$\sigma_\delta = \frac{\det \Xi_\delta}{g(\Xi_\delta)}.$$

Note that $g(\Xi_\delta) \geq \frac{1}{2}|\Xi_\delta|^2 \geq \frac{1}{4}(\operatorname{tr}\Xi_\delta)^2 = \delta^2$. As Ξ_δ is smooth, it follows that $\sigma_\delta \in C^{0,1/2}(\mathbb{R}^2)$. It is clear that $|\sigma_\delta| \leq 1$.

Lemma 9 provides vector fields $\omega_\delta \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ such that $\operatorname{curl}\omega_\delta = \sigma_\delta$ and such that (16) is satisfied for a universal constant C. Inequality (15) then follows from Proposition 7. □

We conclude this section with a brief discussion of how we proceed in the proofs of our main results.

Consider the function $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ given by

$$\begin{aligned} f(M) &= g\left(M - \frac{\operatorname{tr} M}{2} I\right) \\ &= \frac{1}{2}\left(|M|^2 - \frac{1}{2}(\operatorname{tr} M)^2 + |m_{12} - m_{21}|\sqrt{|M|^2 - 2\det M}\right). \end{aligned}$$

Owing to Proposition 10, one of the central questions of this paper is now whether we can satisfy the inequality

$$f(D\Phi) \leq W$$

while simultaneously keeping $\Phi_1(a^+) - \Phi_1(a^-)$ large enough to obtain a useful estimate. (The best possible value here is of course

$$\Phi_1(a^+) - \Phi_1(a^-) = \int_{[a^-,a^+]} \sqrt{W} d\mathcal{H}^1.)$$

Thus we want to find the number

$$\eta_0 = \sup\left\{\Phi_1(a^+) - \Phi_1(a^-) \colon \Phi \in C^{0,1}(\mathbb{R}^2;\mathbb{R}^2) \text{ with } f(D\Phi) \leq W\right\}.$$

Alternatively, assuming that $W > 0$, we can define

$$F(y, M) = \frac{f(M)}{W(y)}$$

and try to determine

$$e_\infty^2 = \inf\left\{\|F(y, D\Phi)\|_{L^\infty(\mathbb{R}^2)} \colon \Phi_1(a^+) - \Phi_1(a^-) = 1\right\}.$$

Then we note that $\eta_0 = 1/e_\infty$. (In general, we are interested in potential functions W that do have zeroes, but they can be approximated by positive functions.)

We therefore study problems of this nature in the next section. Since these results are potentially of independent interest, we formulate them more generally here.

3 An L^∞-minimisation problem

In this section, we assume that $n, m \in \mathbb{N}$ and study functions $\phi \colon \mathbb{R}^n \to \mathbb{R}^m$. Let $A \subseteq \mathbb{R}^n$ be a finite set and $\Omega = \mathbb{R}^n \setminus A$. Given $m_0 \in \{1, \ldots, m\}$, we fix a non-constant function $\phi^0 \colon A \to \mathbb{R}^{m_0}$. We further assume that $F \colon \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0, \infty)$ is a continuously differentiable function such that for every $x \in \mathbb{R}^n$, the function $F(x, \cdot)$ is homogeneous of degree 2, C^2-regular away from 0, and uniformly strictly convex in the sense that there exists a constant $c > 0$ such that

$$D_M^2 F(x, M)(N, N) \geq 2c|N|^2 \tag{17}$$

for all $x \in \mathbb{R}^n$ and all $M, N \in \mathbb{R}^{m\times n}$, where $D_M^2 F$ denotes the second derivative with respect to the second argument. (The function $F(x, \cdot)$ is not twice Fréchet differentiable at 0 in general, but we can always interpret the left-hand side of (17) in the Gâteaux sense even at 0.) We further write ∇F for the gradient of F with respect to the second argument only. We assume that there exists another constant $C > 0$ such that

$$|\nabla F(x, M)| \leq 2C|M| \tag{18}$$

for all $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{m\times n}$.

We also consider the Legendre transform

$$F^*(x, N) = \sup_{M\in\mathbb{R}^{m\times n}} (M : N - F(x, M))$$

of F with respect to the second argument.

Motivated by the previous section, we study functions that minimise the functional

$$E_\infty(\phi) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \sqrt{F(x, D\phi(x))}$$

subject to the condition $\phi_i = \phi_i^0$ on A for $i = 1, \ldots, m_0$.

Notation. In this section, we write $B_r(x)$ for an open ball in $\mathbb{R}^n$ with radius $r > 0$ and centre $x \in \mathbb{R}^n$. The symbol ρ now denotes a function $\rho \in C_0^\infty(B_1(0))$ with $\rho \geq 0$ and $\int_{B_1(0)} \rho\, dx = 1$. For $\delta > 0$, we then set $\rho_\delta(x) = \delta^{-n}\rho(x/\delta)$.

3.1 Summary of the results

Following the ideas from a paper of Katzourakis and Moser [32], we derive some properties of the minimisers of E_∞. They will be described in terms of an $\mathbb{R}^m$-valued 1-current on $\mathbb{R}^n$ and a mass functional depending on F. Although currents will normally be defined in terms of differential forms and the exterior derivative, if we only consider 1-currents, then we can work with matrix-valued functions and the Fréchet derivative instead.

The following is a more general version of Definition 2.

Definition 11. An $\mathbb{R}^m$*-valued* 1*-current on* $\mathbb{R}^n$ is an element of the dual space of $C_0^\infty(\mathbb{R}^n;\mathbb{R}^{m\times n})$. If T is an $\mathbb{R}^m$-valued 1-current, then its *boundary* ∂T is the $\mathbb{R}^m$-valued distribution such that $\partial T(\xi) = T(D\xi)$ for every $\xi \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^m)$. The F*-mass* of T is

$$\mathbf{M}_F(T) = \frac{1}{2}\sup\left\{T(\zeta)\colon \zeta \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^{m\times n}) \text{ with } \|F(x,\zeta)\|_{C^0(\mathbb{R}^n)} \le 1\right\}.$$

We say that T is *normal* if there exists $C \ge 0$ such that

$$|T(\zeta)| + |\partial T(\xi)| \le C\left(\sup_{x\in\mathbb{R}^n}|\zeta(x)| + \sup_{n\in\mathbb{R}^n}|\xi(x)|\right)$$

for all $\zeta \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^{m\times n})$ and all $\xi \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^m)$. We write $\mathcal{C}_{m\times n}(\mathbb{R}^n)$ for the space of all normal $\mathbb{R}^m$-valued 1-currents on $\mathbb{R}^n$.

Given $i \in \{1,\ldots,m\}$, we write T_i for the $\mathbb{R}$-valued 1-current such that $T_i(\xi) = T(e_i \otimes \xi)$ for $\xi \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$, where e_i denotes the i-th standard basis vector of $\mathbb{R}^m$. We can think of these as the components of T.

Many of the standard properties of currents, as described, e.g., in a book by Simon [43], also apply to this variant. In particular, if T has finite F-mass, then, by the properties of F, it automatically has finite mass in the standard sense. (In the above terminology, that means that $\mathbf{M}_{\tilde{F}}(T) < \infty$ for the function $\tilde{F}(x,M) = \frac{1}{4}|M|^2$.) It then follows that there exist a Radon measure $\|T\|$ and a $\|T\|$-measurable, matrix-valued function $\vec{T}\colon \mathbb{R}^n \to \mathbb{R}^{m\times n}$ with $|\vec{T}| = 1$ almost everywhere, such that

$$T(\zeta) = \int_{\mathbb{R}^n} \vec{T} : \zeta \, d\|T\|$$

for any $\zeta \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^{m\times n})$. In this situation, we can also make sense of the expression $T(\zeta)$ for $\zeta \in C_0^0(\mathbb{R}^n;\mathbb{R}^{m\times n})$. If T is normal, then we can make sense of $\partial T(\xi)$ for any $\xi \in C_0^0(\mathbb{R}^n;\mathbb{R}^m)$.

We write $W_*^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m)$ for the space of all $\phi \in W_{\mathrm{loc}}^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m)$ such that $\phi_i = \phi_i^0$ on A for $i = 1,\ldots,m_0$. We set

$$e_\infty = \inf\left\{E_\infty(\phi)\colon \phi \in W_*^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m)\right\}.$$

We will prove the following two statements.

Proposition 12. *There exists $\phi_\infty \in W_*^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m)$ such that $E_\infty(\phi_\infty) = e_\infty$.*

Theorem 13. *There exists $T \in \mathcal{C}_{m\times n}(\mathbb{R}^n) \setminus \{0\}$ with the following properties.*

(i) $\operatorname{supp}\partial T \subseteq A$*, and* $\partial T_i = 0$ *for* $i = m_0+1,\ldots,m$.

(ii) *If* $S \in \mathcal{C}_{m\times n}(\mathbb{R}^n)$ *satisfies* $\partial S = \partial T$*, then* $\mathbf{M}_F(T) \leq \mathbf{M}_F(S)$*.*

(iii) *Let* $\phi \in W^{1,\infty}_*(\mathbb{R}^n;\mathbb{R}^m)$ *be a minimiser of* E_∞ *in* $W^{1,\infty}_*(\mathbb{R}^n;\mathbb{R}^m)$ *and let* $e_\infty = E_\infty(\phi)$*. Then* $\partial T(\phi) = 2e_\infty \mathbf{M}_F(T)$ *and*

$$\lim_{\delta\searrow 0}\int_{\mathbb{R}^n}\left|\rho_\delta * D\phi - e_\infty\frac{\nabla F^*(x,\vec{T})}{\sqrt{F^*(x,\vec{T})}}\right|^2 d\|T\| = 0.$$

Note that the last statement gives a lot of information about the behaviour of ϕ on $\operatorname{supp} T$. Indeed, we interpret it as a generalised version of the equation

$$D\phi = e_\infty\frac{\nabla F^*(x,\vec{T})}{\sqrt{F^*(x,\vec{T})}},$$

but since $\operatorname{supp} T$ will be a Lebesgue null set in general, such an equation does not make sense pointwise.

The existence of a minimiser of E_∞ can be proved with the direct method, although, as one needs to work with weak* convergence in an L^∞-space, some of the details are not so obvious. We will use a different method for the proof of Proposition 12, because we will obtain the minimiser as a side product of the arguments for the proof of Theorem 13.

3.2 Properties of F and F^*

We first derive some properties of the function F and its Legendre transform F^* that follow from the above assumptions, in particular the strict uniform convexity (17).

The homogeneity of F implies that

$$\nabla F(x,M) : M = 2F(x,M). \tag{19}$$

Hence (18) gives rise to the inequality

$$F(x,M) \leq C|M|^2.$$

According to Taylor's theorem, for any $x \in \mathbb{R}^n$ and any $M, N \in \mathbb{R}^{m\times n}$, there exists $\theta \in (0,1)$ such that

$$F(x,N) = F(x,M) + \nabla F(x,M) : (N-M) + \frac{1}{2}D^2_M F(x,\theta M + (1-\theta)N)(N-M,N-M).$$

Therefore,

$$c|N-M|^2 \leq F(x,N) - F(x,M) - \nabla F(x,M) : (N-M). \tag{20}$$

Using (19) again, we can write the above inequality in the form

$$c|N-M|^2 \leq F(x,N) + F(x,M) - \nabla F(x,M) : N. \tag{21}$$

Inserting $N = 0$, we also see that

$$F(x,M) \geq c|M|^2.$$

The Legendre transform of F is automatically homogeneous of degree 2 and strictly convex again. It is a well-known property of the Legendre transform that $\nabla F^*(x,\cdot)$ is the inverse of $\nabla F(x,\cdot)$ regarded as a map $\mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$. That is, if $N = \nabla F(x,M)$, then $M = \nabla F^*(x,N)$, and vice versa. We then also find that

$$2F^*(x,N) = \nabla F^*(x,N) : N = M : \nabla F(x,M) = 2F(x,M).$$

That is,

$$F(x,M) = F^*(x, \nabla F(x,M)) \quad \text{and} \quad F^*(x,N) = F(x, \nabla F^*(x,N)).$$

In addition to the standard definition of the Legendre transform, we have the following characterisation.

Lemma 14. *For any $x \in \mathbb{R}^n$ and $N \in \mathbb{R}^{m\times n}$,*

$$F^*(x,N) = \frac{1}{4} \sup \left\{ (M:N)^2 \colon M \in \mathbb{R}^{m\times n} \text{ with } F(x,M) \le 1 \right\}.$$

Proof. Fix $x \in \mathbb{R}^n$. For every $M \in \mathbb{R}^{m\times n} \setminus \{0\}$ there exists $t > 0$ such that $F(x,tM) = 1$. Hence

$$\begin{aligned} F^*(x,N) &= \sup_{F(x,M)=1} \sup_{t\in\mathbb{R}} (tM : N - F(x,tM)) \\ &= \sup_{F(x,M)=1} \sup_{t\in\mathbb{R}} (tM : N - t^2). \end{aligned}$$

The function $t \mapsto tM : N - t^2$ attains its maximum at $t = \frac{1}{2} M : N$. Therefore,

$$F^*(x,N) = \sup_{F(x,M)=1} \frac{(M:N)^2}{4}.$$

Clearly this supremum is identical with the one in the lemma. □

As a consequence, we have an alternative representation of the F-mass M_F.

Proposition 15. *Let T be an $\mathbb{R}^m$-valued 1-current in $\mathbb{R}^n$ with finite F-mass. Then*

$$\mathbf{M}_F(T) = \int_{\mathbb{R}^n} \sqrt{F^*(x, \vec{T})}\, d\|T\|.$$

Proof. If $\zeta \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^{m\times n})$ with $\|F(x,\zeta)\|_{C^0(\mathbb{R}^n)} \le 1$, then

$$T(\zeta) = \int_{\mathbb{R}^n} \zeta : \vec{T}\, d\|T\| \le 2 \int_{\mathbb{R}^n} \sqrt{F^*(x,\vec{T})}\, d\|T\|$$

by Lemma 14. It follows that

$$\mathbf{M}_F(T) \le \int_{\mathbb{R}^n} \sqrt{F^*(x,\vec{T})}\, d\|T\|.$$

To prove the reverse inequality, we first note that there exists a sequence of uniformly bounded functions $\vec{T}_k \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^{m\times n})$ such that $\vec{T}_k \to \vec{T}$ almost everywhere with respect to $\|T\|$. Define

$$\zeta_k(x) = \frac{k \nabla F^*(x, \vec{T}_k(x))}{\sqrt{k^2 F^*(x, \vec{T}_k(x)) + 1}}, \quad k \in \mathbb{N}.$$

Then

$$\vec{T}_k(x) : \zeta_k(x) = \frac{2kF^*(x, \vec{T}_k(x))}{\sqrt{k^2 F^*(x, \vec{T}_k(x)) + 1}}$$

for every $x \in \mathbb{R}^n$. Hence

$$\vec{T}_k(x) : \zeta_k(x) \to 2\sqrt{F^*(x, \vec{T}(x))}$$

almost everywhere. We also compute

$$F(x, \zeta_k(x)) = \frac{k^2 F(x, \nabla F^*(x, \vec{T}_k(x)))}{k^2 F^*(x, \vec{T}_k(x)) + 1} = \frac{k^2 F^*(x, \vec{T}_k(x))}{k^2 F^*(x, \vec{T}_k(x)) + 1} \leq 1.$$

Therefore,

$$\int_{\mathbb{R}^n} \sqrt{F^*(x, \vec{T})}\, d\|T\| = \frac{1}{2} \lim_{k \to \infty} \int_{\mathbb{R}^2} \vec{T}_k : \zeta_k \, d\|T\| \leq \mathbf{M}_F(T).$$

This completes the proof. □

Lemma 16. *Let $K \subseteq \mathbb{R}^n$ be compact. Let $Q \in L^\infty(\mathbb{R}^n; \mathbb{R}^{m \times n})$, and let $U \subseteq \mathbb{R}^n$ be an open set with $K \subseteq U$. Then*

$$\limsup_{\delta \searrow 0} \sup_{x \in K} F(x, \rho_\delta * Q(x)) \leq \operatorname*{ess\,sup}_{x \in U} F(x, Q(x)).$$

Proof. Since F is continuous, it is uniformly continuous on the compact set

$$L = \left\{ (x, M) \in \mathbb{R}^n \times \mathbb{R}^{m \times n} \colon \operatorname{dist}(x, K) \leq 1 \text{ and } |M| \leq \|Q\|_{L^\infty(\mathbb{R}^n)} \right\}.$$

Let $\epsilon > 0$, and fix $\delta_0 \in (0, 1]$ such that $|F(x, M) - F(y, M)| \leq \epsilon$ for all $(x, M) \in L$ and $(y, M) \in L$ with $|x - y| \leq \delta_0$. Let $x \in K$. By the convexity of F and Jensen's inequality, we can estimate

$$\begin{aligned} F(x, \rho_\delta * Q) &\leq \int_{\mathbb{R}^n} \rho_\delta(x - y) F(x, Q(y))\, dy \\ &\leq \int_{\mathbb{R}^n} \rho_\delta(x - y) F(y, Q(y))\, dy + \epsilon \\ &\leq \operatorname*{ess\,sup}_{y \in U} F(y, Q(y)) + \epsilon \end{aligned}$$

when $\delta < \delta_0$ and $\delta < \operatorname{dist}(K, \mathbb{R}^n \setminus U)$. The claim follows. □

3.3 Approximation in L^p

In order to take advantage of the usual tools from the calculus of variations, we construct a minimiser of E_∞ as the limit of solutions to more conventional problems. To this end, we replace the L^∞-norm by L^p-norms.

Let $V \colon \mathbb{R}^n \to (0, \infty)$ be a smooth function such that

$$\int_{\mathbb{R}^n} V(x)\, dx = 1.$$

For $n < p < \infty$, define the functionals

$$E_p(\phi) = \left(\int_{\mathbb{R}^n} \big(F(x, D\phi)\big)^{p/2} V(x)\, dx \right)^{1/p}$$

on the Sobolev space $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$. Let $W^{1,p}_*(\mathbb{R}^n; \mathbb{R}^m)$ denote the space of all $\phi \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ such that $\phi_i = \phi_i^0$ on A for $i = 1, \ldots, m_0$. We consider the problem of minimising E_p in $W^{1,p}_*(\mathbb{R}^n; \mathbb{R}^m)$.

Since we now have a strictly convex functional given in terms of an integral, we can use standard methods from the calculus of variations to make a few statements immediately: there exists a unique minimiser $\phi_p \in W^{1,p}_*(\mathbb{R}^n; \mathbb{R}^m)$, which satisfies the Euler-Lagrange equation

$$\operatorname{div} \Big(V(x) \big(F(x, D\phi_p)\big)^{p/2-1} \nabla F(x, D\phi_p) \Big) = 0$$

weakly in Ω. (Here the divergence is applied row-wise, so that we actually have a system of m equations.) We use the notation $\nabla_i F$ for the i-th row of ∇F. Then we can write

$$\operatorname{div} \Big(V(x) \big(F(x, D\phi_p)\big)^{p/2-1} \nabla_i F(x, D\phi_p) \Big) = 0, \quad i = 1, \ldots, m. \tag{22}$$

We note that this is satisfied weakly in Ω for $i = 1, \ldots m_0$, and even weakly in $\mathbb{R}^n$ for $i = m_0 + 1, \ldots m$.

We obtain another necessary condition when we study inner variations of ϕ_p of the form $\phi_p^t(x) = \phi_p(x + t\chi(x))$ for a vector field $\chi \in C_0^\infty(\Omega; \mathbb{R}^2)$. A standard computation then gives

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|\, \big(E_p(\phi_p^t)\big)^p \\ &= \frac{p}{2} \int_{\mathbb{R}^n} V(x) \big(F(x, D\phi_p)\big)^{\frac{p-2}{2}} \left(\nabla F(x, D\phi_p) : (D\phi_p D\chi) - \frac{\partial F}{\partial x}(x, D\phi_p)\chi \right) dx \\ &\quad - \int_{\mathbb{R}^n} \big(F(x, D\phi_p)\big)^{p/2} \left(DV\chi + V \operatorname{div} \chi \right) dx, \end{aligned} \tag{23}$$

where $\frac{\partial F}{\partial x}$ denotes the derivative of F with respect to the first argument.

Proof of Proposition 12. Let $\psi \in W^{1,\infty}_*(\mathbb{R}^n; \mathbb{R}^m)$. By Hölder's inequality, for $n < q < p$, we have the inequalities

$$E_q(\phi_p) \le E_p(\phi_p) \le E_p(\psi) \le E_\infty(\psi). \tag{24}$$

Thus, the family $(\phi_p)_{q<p<\infty}$ is bounded in $W^{1,q}(B_R(0); \mathbb{R}^m)$ for any $q < \infty$ and any $R > 0$. We may therefore choose a sequence $p_k \to \infty$ such that we have the weak convergence $\phi_{p_k} \rightharpoonup \phi_\infty$ as $k \to \infty$ simultaneously in all of these spaces for some $\phi_\infty \in \bigcap_{q<\infty} W^{1,q}_*(\mathbb{R}^n; \mathbb{R}^m)$. We further see that

$$E_\infty(\phi_\infty) = \lim_{q\to\infty} E_q(\phi_\infty) \le \liminf_{q\to\infty} \liminf_{k\to\infty} E_q(\phi_{p_k}) \le \liminf_{k\to\infty} E_{p_k}(\phi_{p_k}) \le E_\infty(\psi) \tag{25}$$

by the lower semicontinuity of E_q with respect to weak convergence, Hölder's inequality again, and (24). In particular, we see that ϕ_∞ belongs to $W^{1,\infty}_*(\mathbb{R}^n; \mathbb{R}^m)$ and is a minimiser of E_∞ in this space. □

We continue to use the functions ϕ_p and ϕ_∞ for the rest of this section. Let

$$e_p = E_p(\phi_p), \quad n < p \leq \infty.$$

Because we have assumed that ϕ^0 is not constant, it is clear that $e_p \neq 0$. The inequalities in (24) and (25) imply that $e_\infty = \lim_{p\to\infty} e_p$. It is now convenient to introduce the measures

$$\mu_p = e_p^{2-p} V(x)\big(F(x, D\phi_p)\big)^{p/2-1}\mathcal{L}^n,$$

where $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$. This is normalised so that

$$\mu_p(\mathbb{R}^n) = e_p^{2-p}\int_{\mathbb{R}^n} V(x)\big(F(x, D\phi_p)\big)^{p/2-1}\,dx \leq e_p^{2-p}\big(E_p(\phi_p)\big)^{p-2} = 1$$

by Hölder's inequality. The Euler-Lagrange equation (22) implies that

$$\int_{\mathbb{R}^n} \nabla F(x, D\phi_p) : D\psi\, d\mu_p = 0 \tag{26}$$

for all $\psi \in C_0^\infty(\Omega)$. Note, however, that the identity also holds true if we merely know that $\psi \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ and $E_p(\psi) < \infty$ and $\psi_1 = \cdots = \psi_{m_0} = 0$ on A, because these conditions imply that the derivative $\frac{d}{dt}|_{t=0}E_p(\phi + t\psi)$ is given by the usual expression.

Equation (23) has the following representation in terms of μ_p:

$$\begin{aligned} 0 = \frac{p}{2}\int_{\mathbb{R}^n}\left(\nabla F(x, D\phi_p) : (D\phi_p D\chi) - \frac{\partial F}{\partial x}(x, D\phi_p)\chi\right)\,d\mu_p \\ - \int_{\mathbb{R}^n} F(x, \phi_p)\,(D(\log V)\chi + \operatorname{div}\chi)\,d\mu_p. \end{aligned} \tag{27}$$

The measure μ_p should be considered together with the function $D\phi_p$. These two objects form a measure-function pair $(\mu_p, D\phi_p)$ with values in $\mathbb{R}^{m\times n}$ in the sense of Hutchinson [20]. Since

$$\int_{\mathbb{R}^n} |D\phi_p|^2\,d\mu_p \leq \frac{1}{ce_p^{p-2}}\int_{\mathbb{R}^n}\big(F(x, D\phi_p)\big)^{p/2} V(x)\,dx \leq \frac{e_p^2}{c}$$

for every $p \in (n, \infty)$, we may assume that for the above sequence $p_k \to \infty$, we simultaneously have the weak convergence of $(\mu_{p_k}, D\phi_{p_k})$ to a measure-function pair (μ_∞, Z_∞) in the sense of Hutchinson. (This follows from [20, Theorem 4.4.2], but we may have to pass to a suitable subsequence.) Here μ_∞ is a Radon measure on $\mathbb{R}^n$ and $Z_\infty \in L^2(\mu_\infty;\mathbb{R}^{m\times n})$. Of course, we have a uniform bound for

$$\int_{\mathbb{R}^n} |\nabla F(x, D\phi_p)|^2\,d\mu_p$$

as well, and we may therefore assume at the same time that $(\mu_{p_k}, \nabla F(x, D\phi_{p_k}))$ converges weakly to (μ_∞, Y_∞) for some function $Y_\infty \in L^2(\mu_\infty;\mathbb{R}^{m\times n})$. Because of (26), we have the identity

$$\int_{\mathbb{R}^n} Y_\infty : D\psi\,d\mu_\infty = 0 \tag{28}$$

for all $\psi \in C_0^1(\mathbb{R}^n;\mathbb{R}^m)$ with $\psi_1 = \cdots = \psi_{m_0} = 0$ on A.

We now want to prove the following additional properties.

Proposition 17. *(i) Let $K \subseteq \Omega$ be a compact set. Then the convergence $(\mu_{p_k}, D\phi_{p_k}) \to (\mu_\infty, Z_\infty)$ is strong in K the sense of Hutchinson [20]. Equivalently,*

$$\int_K |Z_\infty|^2 \, d\mu_\infty = \lim_{k\to\infty} \int_K |D\phi_{p_k}|^2 \, d\mu_{p_k}.$$

(ii) The identity $F(x, Z_\infty) = e_\infty^2$ holds at μ_∞-almost every point $x \in \Omega$.

(iii) For any minimiser $\psi \in W_*^{1,\infty}(\mathbb{R}^n; \mathbb{R}^m)$ of E_∞,

$$\lim_{\delta\searrow 0} \int_\Omega |\rho_\delta * D\psi - Z_\infty|^2 \, d\mu_\infty = 0.$$

The proof follows the strategy of the aforementioned paper [32], which in turn makes use of some ideas of Evans and Yu [17] at this point. First, we require the following lemma.

Lemma 18. *Let $\xi \in C^0(\mathbb{R}^n)$ be a bounded function with $\xi \geq 0$. Then for any $p \in (n, \infty)$ and any $\beta \in (0,1)$,*

$$\int_{\mathbb{R}^n} \xi F(x, D\phi_p) \, d\mu_p \geq \beta^2 e_p^2 \int_{\mathbb{R}^n} \xi \, d\mu_p - \beta^p e_p^2 \sup_{\mathbb{R}^n} \xi.$$

Proof. Consider the sets

$$S_p = \left\{ x \in \mathbb{R}^n \colon F(x, D\phi_p(x)) \leq \beta^2 e_p^2 \right\}.$$

Then

$$\mu_p(S_p) = e_p^{2-p} \int_{S_p} V(x) \big(F(x, D\phi_p)\big)^{p/2-1} \, dx \leq \beta^{p-2}.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n} \xi F(x, D\phi_p) \, d\mu_p &\geq \int_{\mathbb{R}^n \setminus S_p} \xi F(x, D\phi_p) \, d\mu_p \\
&\geq \beta^2 e_p^2 \int_{\mathbb{R}^n \setminus S_p} \xi \, d\mu_p \\
&= \beta^2 e_p^2 \left(\int_{\mathbb{R}^n} \xi \, d\mu_p - \int_{S_p} \xi \, d\mu_p \right) \\
&\geq \beta^2 e_p^2 \int_{\mathbb{R}^n} \xi \, d\mu_p - \beta^p e_p^2 \sup_{R^n} \xi,
\end{aligned}$$

as claimed. □

Proof of Proposition 17. Fix an arbitrary minimiser $\psi \in W_*^{1,\infty}(\mathbb{R}^n; \mathbb{R}^m)$ of E_∞. Define $\psi_\delta = \rho_\delta * \psi$. Let $\xi \in C_0^\infty(\Omega)$ with $0 \leq \xi \leq 1$, and let $\beta \in (0,1)$. Then by

(20), (26), and Lemma 18, we find that

$$\begin{aligned}
c&\int_\Omega \xi|D\psi_\delta - D\phi_p|^2\,d\mu_p \\
&\le \int_\Omega \xi\big(F(x,D\psi_\delta) - F(x,D\phi_p) - \nabla F(x,D\phi_p) : (D\psi_\delta - D\phi_p)\big)\,d\mu_p \\
&= \int_\Omega \xi\big(F(x,D\psi_\delta) - F(x,D\phi_p)\big)\,d\mu_p + \int_\Omega \nabla F(x,D\phi_p) : (\psi_\delta - \phi_p)\otimes D\xi\,d\mu_p \\
&\le \int_\Omega \xi F(x,D\psi_\delta)\,d\mu_p - \beta^2 e_p^2 \int_\Omega \xi\,d\mu_p + \beta^p e_p^2 \\
&\quad + \int_\Omega \nabla F(x,D\phi_p) : (\psi_\delta - \phi_p)\otimes D\xi\,d\mu_p.
\end{aligned}$$

Applying this inequality to p_k and letting $k \to 0$, we obtain

$$\begin{aligned}
&c \limsup_{k\to\infty} \int_\Omega \xi|D\psi_\delta - D\phi_{p_k}|^2\,d\mu_{p_k} \\
&\quad \le \int_\Omega \xi F(x,D\psi_\delta)\,d\mu_\infty - \beta^2 e_\infty^2 \int_\Omega \xi\,d\mu_\infty + \int_\Omega Y_\infty : (\psi_\delta - \phi_\infty)\otimes D\xi\,d\mu_\infty.
\end{aligned}$$

For $r > 0$, define $B_r(A) = \bigcup_{x\in A} B_r(x)$. Suppose that r is so small that $B_r(a) \cap B_r(b) = \emptyset$ when $a, b \in A$ with $a \ne b$. Choose $\chi \in C_0^\infty(B_r(0))$ with $0 \le \chi \le 1$ and such that $\chi \equiv 1$ in $B_{r/2}(0)$ and $|D\chi| \le 4/r$ in $B_r(0)$. Furthermore, for $R \ge 1$ such that $B_r(A) \subseteq B_R(0)$, choose $\eta \in C_0^\infty(B_{2R}(0))$ with $0 \le \eta \le 1$ such that $\eta \equiv 1$ in $B_R(0)$ and $|D\eta| \le 2/R$. Set

$$\xi(x) = \eta(x) - \sum_{a\in A} \chi(x-a).$$

Note that $\psi_i(a) = \phi_{\infty i}(a)$ for $i = 1, \ldots, m_0$ and $a \in A$. Hence

$$\begin{aligned}
&\int_\Omega Y_\infty : (\psi_\delta - \phi_\infty)\otimes D\xi\,d\mu_\infty \\
&= \int_{B_{2R}(0)\setminus B_R(0)} Y_\infty : (\psi_\delta - \phi_\infty)\otimes D\eta\,d\mu_\infty \\
&\quad - \sum_{a\in A} \int_{B_r(a)} Y_\infty(x) : (\psi_\delta(x) - \phi_\infty(x))\otimes D\chi(x-a)\,d\mu_\infty(x) \\
&= \int_{B_{2R}(0)\setminus B_R(0)} Y_\infty : (\psi_\delta - \phi_\infty)\otimes D\eta\,d\mu_\infty \\
&\quad - \sum_{a\in A} \int_{B_r(a)} Y_\infty(x) : \big(\psi_\delta(x) - \psi(a) - \phi_\infty(x) + \phi_\infty(a)\big)\otimes D\chi(x-a)\,d\mu_\infty(x)
\end{aligned}$$

by (28). Therefore,

$$\begin{aligned}
c\limsup_{k\to\infty}\int_\Omega \xi|D\psi_\delta - D\phi_{p_k}|^2\,d\mu_{p_k} \\
&\le \int_\Omega \xi F(x, D\psi_\delta)\,d\mu_\infty - \beta^2 e_\infty^2 \int_\Omega \xi\,d\mu_\infty \\
&\quad + \int_{B_{2R}(0)\setminus B_R(0)} Y_\infty : (\psi_\delta - \phi_\infty)\otimes D\xi\,d\mu_\infty \\
&\quad + \sum_{a\in A}\int_{B_r(a)} Y_\infty : \big(\psi_\delta - \psi(a) - \phi_\infty + \phi_\infty(a)\big)\otimes D\xi\,d\mu_\infty.
\end{aligned}$$

Since ϕ_∞ is a minimiser of E_∞, it satisfies $F(x, D\phi_\infty) \le e_\infty^2$ almost everywhere. Hence

$$|D\phi_\infty| \le \left(\frac{1}{c}F(x, D\phi_\infty)\right)^{1/2} \le \frac{e_\infty}{\sqrt{c}}.$$

Similarly, we find that $|D\psi| \le e_\infty/\sqrt{c}$, as ψ is also a minimiser of E_∞. Hence

$$|\psi - \psi(a) - \phi_\infty + \phi_\infty(a)| \le \frac{2re_\infty}{\sqrt{c}}$$

in $B_r(a)$. Similarly, if $R \ge |\psi(0) - \phi_\infty(0)|$, then

$$|\psi - \phi_\infty| \le \left(\frac{4e_\infty}{\sqrt{c}} + 1\right) R$$

in $B_{2R}(0)$. If $\delta \le r$, the we also have the inequality

$$|\psi - \psi_\delta| \le \frac{re_\infty}{\sqrt{c}}.$$

Therefore,

$$\begin{aligned}
&\int_{B_{2R}(0)\setminus B_R(0)} Y_\infty : (\psi_\delta - \phi_\infty)\otimes D\xi\,d\mu_\infty \\
&\qquad + \sum_{a\in A}\int_{B_r(a)} Y_\infty : \big(\psi_\delta - \psi(a) - \phi_\infty + \phi_\infty(a)\big)\otimes D\xi\,d\mu_\infty \\
&\le \left(\frac{10e_\infty}{\sqrt{c}} + 2\right)\int_{B_{2R}(0)\setminus B_R(0)} |Y_\infty|\,d\mu_\infty + \frac{12e_\infty}{\sqrt{c}}\int_{B_r(A)\setminus B_{r/2}(A)} |Y_\infty|\,d\mu_\infty \\
&\le \left(\frac{12e_\infty}{\sqrt{c}} + 2\right)\left(\mu_\infty(G_{R,r})\int_{\mathbb{R}^n} |Y_\infty|^2\,d\mu_\infty\right)^{1/2},
\end{aligned}$$

where $G_{R,r} = (B_{2R}(0)\setminus B_R(0)) \cup (B_r(A)\setminus B_{r/2}(A))$. Given $\epsilon > 0$, we can choose r so small and R so large that

$$\left(\frac{12e_\infty}{\sqrt{c}} + 2\right)\left(\mu_\infty(G_{R,r})\int_{\mathbb{R}^n} |Y_\infty|^2\,d\mu_\infty\right)^{1/2} \le \epsilon.$$

Let $K \subseteq \Omega$ be a compact set. Then we can further assume that $K \subseteq B_R(0)$ and $K \cap B_r(A) = \emptyset$. With the help of Lemma 16, we conclude that

$$\begin{aligned}
c\limsup_{k\to\infty}\int_K |D\psi_\delta - D\phi_{p_k}|^2\,d\mu_{p_k} &\le \int_\Omega \xi\big(F(x, D\psi_\delta) - \beta^2 e_\infty^2\big)\,d\mu_\infty + \epsilon \\
&\le (1-\beta^2)e_\infty^2 + 2\epsilon
\end{aligned} \tag{29}$$

whenever δ is sufficiently small. By [20, Theorem 4.4.2], it follows that

$$c \int_K |D\psi_\delta - Z_\infty|^2 \, d\mu_\infty \leq (1 - \beta^2) e_\infty^2 + 2\epsilon.$$

This holds true for any $\beta \in (0, 1)$ and any $\epsilon > 0$, provided that δ is small enough (depending on ϵ). Hence

$$\lim_{\delta \searrow 0} \int_K |D\psi_\delta - Z_\infty|^2 \, d\mu_\infty = 0. \tag{30}$$

This L^2-convergence implies that there exists a sequence $\delta_\ell \searrow 0$ such that $D\psi_{\delta_\ell} \to Z_\infty$ almost everywhere in K (with respect to the measure μ_∞) as $\ell \to \infty$. Hence $F(x, D\psi_{\delta_\ell}) \to F(x, Z_\infty)$ almost everywhere in K. Recalling that $F(x, D\psi) \leq e_\infty^2$ almost everywhere (with respect to the Lebesgue measure) and using Lemma 16 again, we conclude that $F(x, Z_\infty) \leq e_\infty^2$ almost everywhere in K (with respect to μ_∞).

Choosing another compact set $K' \subseteq \Omega$ such that $\operatorname{supp} \xi \subseteq K'$, we similarly obtain the convergence $D\psi_\delta \to Z_\infty$ in $L^2(\mu_\infty \llcorner K'; \mathbb{R}^{m \times n})$. Hence

$$\int_\Omega \xi F(x, Z_\infty) \, d\mu_\infty = \lim_{\delta \searrow 0} \int_\Omega \xi F(x, D\psi_\delta) \, d\mu_\infty.$$

Inequality (29), on the other hand, implies that

$$\lim_{\delta \searrow 0} \int_\Omega \xi F(x, D\psi_\delta) \, d\mu_\infty \geq \beta^2 e_\infty^2 \int_\Omega \xi \, d\mu_\infty - \epsilon.$$

Again this holds true for any $\beta \in (0, 1)$ (and provided that r is sufficiently small and R is sufficiently large, depending on ϵ but not on β). Hence

$$\int_\Omega \xi F(x, Z_\infty) \, d\mu_\infty \geq e_\infty^2 \int_\Omega \xi \, d\mu_\infty - \epsilon.$$

When we let $r \searrow 0$ and $R \to \infty$ again, then the integrand on the left-hand side converges to $F(x, Z_\infty)$ pointwise in Ω. Since we know that it is bounded by e_∞^2, we can apply Lebesgue's dominated convergence theorem and obtain

$$\int_\Omega F(x, Z_\infty) \, d\mu_\infty \geq e_\infty^2 \mu_\infty(\Omega).$$

As we already know that $F(x, Z_\infty) \leq e_\infty^2$ almost everywhere, this implies statement (ii).

Furthermore, since the functions $D\psi_\delta$ are uniformly bounded and we now know that $Z_\infty \in L^\infty(\mu_\infty; \mathbb{R}^{m \times n})$, the local strong convergence (30) implies statement (iii).

For the proof of statement (i), we go back to (29) once more. For a given number $\gamma > 0$, we see that

$$\limsup_{k \to \infty} \int_K |D\psi_\delta - D\phi_{p_k}|^2 \, d\mu_{p_k} \leq \gamma$$

for any sufficiently small $\delta > 0$. But for a fixed δ, we also compute

$$\begin{aligned}
&\limsup_{k\to\infty} \int_K |D\psi_\delta - D\phi_{p_k}|^2 \, d\mu_{p_k} \\
&\quad = \limsup_{k\to\infty} \int_K \big(|D\psi_\delta|^2 - 2D\psi_\delta : D\phi_{p_k} + |D\phi_{p_k}|^2\big) \, d\mu_{p_k} \\
&\quad = \int_K \big(|D\psi_\delta|^2 - 2D\psi_\delta : Z_\infty\big) \, d\mu_\infty + \limsup_{k\to\infty} \int_K |D\phi_{p_k}|^2 \, d\mu_{p_k} \\
&\quad = \int_K |D\psi_\delta - Z_\infty|^2 \, d\mu_\infty - \int_K |Z_\infty|^2 \, d\mu_\infty + \limsup_{k\to\infty} \int_K |D\phi_{p_k}|^2 \, d\mu_{p_k}.
\end{aligned}$$

It follows that

$$\limsup_{k\to\infty} \int_K |D\phi_{p_k}|^2 \, d\mu_{p_k} \leq \int_K |Z_\infty|^2 \, d\mu_\infty,$$

and by [20, Theorem 4.4.2], we have strong L^2-convergence in K. □

We can improve the first statement in Proposition 17 if we test with functions that vanish on A.

Corollary 19. *Let $G\colon \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be a continuous function, and suppose that there exists $h \in C_0^0(\mathbb{R}^n; [0,\infty))$ such that $h(a) = 0$ for all $a \in A$ and $|G(x,M)| \leq h|M|^2$ for all $x \in \mathbb{R}^n$ and all $M \in \mathbb{R}^{m\times n}$. Then*

$$\int_{\mathbb{R}^n} G(x, Z_\infty) \, d\mu_\infty = \lim_{k\to\infty} \int_{\mathbb{R}^n} G(x, D\phi_{p_k}) \, d\mu_{p_k}.$$

Proof. Let $\epsilon > 0$. There exists $r > 0$ such that $|h| \leq c\epsilon$ in $B_r(A)$, which implies that

$$|G(x,M)| \leq \epsilon F(x,M)$$

for every $x \in B_r(A)$ and $M \in \mathbb{R}^{m\times n}$. Therefore,

$$\int_{B_r(A)} |G(x, D\phi_{p_k})| \, d\mu_{p_k} \leq \epsilon \int_{\mathbb{R}^n} F(x, D\phi_{p_k}) \, d\mu_{p_k} \leq e_{p_k}^2 \epsilon$$

and

$$\int_{B_r(A)} |G(x, Z_\infty)| \, d\mu_\infty \leq \epsilon \int_{\mathbb{R}^n} F(x, Z_\infty) \, d\mu_\infty \leq e_\infty^2 \epsilon.$$

Choose $\xi \in C_0^0(\mathbb{R}^n \setminus A)$ with $\xi \equiv 1$ in $(\operatorname{supp} h) \setminus B_r(A)$. By Proposition 17.(i), we have the convergence

$$\int_{\mathbb{R}^n} \xi G(x, Z_\infty) \, d\mu_\infty = \lim_{k\to\infty} \int_{\mathbb{R}^n} \xi G(x, D\phi_{p_k}) \, d\mu_{p_k}.$$

It now suffices to combine these facts. □

The following is also important, because it rules out that Proposition 17 is vacuous.

Proposition 20. *The measure μ_∞ does not vanish.*

Proof. Let $r > 0$ such that $B_r(a_1) \cap B_r(a_2) = \emptyset$ for $a_1, a_2 \in A$ with $a_1 \neq a_2$. Choose $\psi \in C_0^1(\mathbb{R}^n; \mathbb{R}^m)$ such that $\psi_i = \phi_i^0$ on A for $i = 1, \ldots, m_0$ and $D\psi = 0$ in $B_r(a)$ for every $a \in A$. Note that

$$\begin{aligned} e_p^2 &= \int_{\mathbb{R}^n} F(x, D\phi_p)\, d\mu_p \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \nabla F(x, D\phi_p) : D\phi_p \, d\mu_p \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \nabla F(x, D\phi_p) : D\psi \, d\mu_p \end{aligned}$$

for every $p \in (n, \infty)$ by the definition of μ_p and equations (19) and (26). It then follows from the local strong convergence in Proposition 17 that

$$e_\infty^2 = \lim_{k\to\infty} e_{p_k}^2 = \frac{1}{2} \int_{\mathbb{R}^n} \nabla F(x, Z_\infty) : D\psi \, d\mu_\infty.$$

But as the boundary data do not admit a constant function, we have $e_\infty \neq 0$. Hence μ_∞ cannot vanish. □

We can finally improve the convergence from Proposition 17 even more.

Proposition 21. *If $a \in A$, then $Z_\infty(a) = 0$ or $\mu_\infty(\{a\}) = 0$. Furthermore, for any compact set $K \subseteq \mathbb{R}^n$, the convergence $(\mu_{p_k}, D\phi_{p_k}) \to (\mu_\infty, Z_\infty)$ is strong in K.*

Proof. We use (27) with $\chi(x) = \eta(|x-a|)(x-a)$ for a function $\eta \in C_0^\infty((0, R))$, where $R > 0$ is so small that χ will vanish in a neighbourhood of A. We obtain

$$\begin{aligned} 0 = &\int_{\mathbb{R}^n} \eta(|x-a|) \left(\nabla F(x, D\phi_p) : D\phi_p - \frac{\partial F}{\partial x}(x, D\phi_p)(x-a) \right) d\mu_p \\ &+ \int_{\mathbb{R}^n} \frac{\eta'(|x-a|)}{|x-a|} \nabla F(x, D\phi_p) : \big(D\phi_p((x-a) \otimes (x-a))\big)\, d\mu_p \\ &- \frac{2}{p} \int_{\mathbb{R}^n} \eta(|x-a|) F(x, D\phi_p) \left(D(\log V)(x-a) + n\right)\, d\mu_p \\ &- \frac{2}{p} \int_{\mathbb{R}^n} |x-a| \eta'(|x-a|) F(x, D\phi_p)\, d\mu_p. \end{aligned} \tag{31}$$

Now consider $\eta \in C_0^\infty((-R, R))$ such that η' vanishes in a neighbourhood of 0, and define $\chi(x) = \eta(|x-a|)(x-a)$ again. We can find a sequence of functions $\eta_\ell \in C_0^\infty((0, R))$ such that $\eta_\ell(t) = \eta(t)$ for $t \geq 1/\ell$ and $|\eta_\ell'| \leq C_1 \ell$ for some constant C_1 independent of ℓ. Then the functions $\chi_\ell(x) = \eta_\ell(|x-a|)(x-a)$ converge uniformly to χ, and their derivatives $D\chi_\ell$ are uniformly bounded and converge to $D\chi$ at every point in $\mathbb{R}^n \setminus \{a\}$. Using Lebesgue's dominated convergence theorem, we therefore conclude that (31) holds in this situation as well.

Therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \eta(|x-a|)F(x, D\phi_p)\, d\mu_p \\
&\qquad = \frac{1}{2}\int_{\mathbb{R}^n} \eta(|x-a|)\nabla F(x, D\phi_p) : D\phi_p\, d\mu_p \\
&\qquad = \frac{1}{2}\int_{\mathbb{R}^n} \eta(|x-a|)\frac{\partial F}{\partial x}(x, D\phi_p)(x-a)\, d\mu_p \\
&\qquad\quad - \frac{1}{2}\int_{\mathbb{R}^n} \frac{\eta'(|x-a|)}{|x-a|}\nabla F(x, D\phi_p) : \big(D\phi_p((x-a)\otimes(x-a))\big)\, d\mu_p \\
&\qquad\quad + \frac{1}{p}\int_{\mathbb{R}^n} \eta(|x-a|)F(x, D\phi_p)\,(D(\log V)(x-a)+n)\ d\mu_p \\
&\qquad\quad + \frac{1}{p}\int_{\mathbb{R}^n} |x-a|\eta'(|x-a|)F(x, D\phi_p)\, d\mu_p.
\end{aligned}
$$

We now restrict the identity to p_k and let $k \to \infty$. Clearly, we have a constant C_2 such that

$$
\frac{1}{p}\int_{\mathbb{R}^n} |\eta(|x-a|)|F(x, D\phi_p)\,|D(\log V)(x-a)+n|\ d\mu_p \le C_2\frac{e_p^2}{p}
$$

and

$$
\frac{1}{p}\int_{\mathbb{R}^n} |x-a||\eta'(|x-a|)|F(x, D\phi_p)\, d\mu_p \le C_2\frac{e_p^2}{p},
$$

and the right-hand sides converge to 0 as $p \to \infty$. For the remaining terms, we can use Corollary 19. We finally find the identity

$$
\begin{aligned}
&\lim_{k\to\infty}\int_{\mathbb{R}^n} \eta(|x-a|)F(x, D\phi_{p_k})\, d\mu_{p_k} \\
&\qquad = \frac{1}{2}\int_{\mathbb{R}^n} \eta(|x-a|)\frac{\partial F}{\partial x}(x, Z_\infty)(x-a)\, d\mu_\infty \\
&\qquad\quad - \frac{1}{2}\int_{\mathbb{R}^n} \frac{\eta'(|x-a|)}{|x-a|}\nabla F(x, Z_\infty) : \big(Z_\infty((x-a)\otimes(x-a))\big)\, d\mu_\infty.
\end{aligned}
$$

Let $r \in (0, R/2]$. If we choose η such that $\eta \equiv 1$ in $[0, r]$ and $\eta \equiv 0$ in $[2r, \infty)$, and such that it satisfies $|\eta'| \le 2/r$ everywhere, then this inequality gives rise to a constant C_3, independent of r, such that

$$
\limsup_{k\to\infty}\int_{B_r(a)} F(x, D\phi_{p_k})\, d\mu_{p_k} \le C_3 r + C_3\mu_\infty(B_{2r}(a)\setminus B_r(0)).
$$

Because

$$
\sum_{\ell=1}^{\infty} \mu_\infty(B_{2^{1-\ell}}(a)\setminus B_{2^{-\ell}}(a)) < \infty,
$$

it is clear that

$$
\liminf_{r\searrow 0} \mu_\infty(B_{2r}(a)\setminus B_r(0)) = 0.
$$

It therefore follows that

$$
\lim_{r\searrow 0}\limsup_{k\to\infty}\int_{B_r(a)} F(x, D\phi_{p_k})\, d\mu_{p_k} = 0. \tag{32}
$$

By [20, Theorem 4.4.2] and Proposition 17,

$$\limsup_{k\to\infty} \int_{B_r(a)} F(x, D\phi_{p_k})\, d\mu_{p_k} \geq \int_{B_r(a)} F(x, Z_\infty)\, d\mu_\infty$$
$$= e_\infty^2 \mu_\infty(B_r(a) \setminus \{a\}) + F(a, Z_\infty(a))\mu_\infty(\{a\}).$$

Thus (32) implies that $Z_\infty(a) = 0$ or $\mu_\infty(\{a\}) = 0$.

Moreover, combining this information with (32) and the statement of Proposition 17.(i) in a way similar to the proof of Corollary 19, we obtain

$$\int_K |Z_\infty|^2\, d\mu_\infty = \lim_{k\to\infty} \int_K |D\phi_{p_k}|^2\, d\mu_{p_k},$$

which is equivalent to strong convergence in K by the results of Hutchinson [20]. □

3.4 Currents

The measure-function pair (μ_∞, Z_∞) constructed in the preceding subsection gives rise to the 1-current from Theorem 13. Indeed, we define the $\mathbb{R}^m$-valued 1-current T_∞ such that

$$T_\infty(\zeta) = \int_{\mathbb{R}^n} \nabla F(x, Z_\infty) : \zeta\, d\mu_\infty$$

for $\zeta \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^{m\times n})$. It then follows from (26) and Proposition 21 that

$$\partial T_\infty(\xi) = 0$$

for all $\xi \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^m)$ such that $\xi_1 = \cdots = \xi_{m_0} = 0$ on A. That is, we know that $\operatorname{supp} \partial T_\infty \subseteq A$ and $\partial T_{\infty i} = 0$ for $i = m_0 + 1, \ldots, m$.

To prove the remaining statements of Theorem 13, we require another proposition. This result also reveals a deeper connection between $\mathbb{R}^m$-valued 1-currents and the above variational problem.

Proposition 22. *Suppose that $T \in \mathcal{C}_{m\times n}(\mathbb{R}^n)$ satisfies $\operatorname{supp} \partial T \subseteq A$ and $\partial T_{m_0+1} = \cdots = \partial T_m = 0$. Then for any $\phi \in W_*^{1,\infty}(\mathbb{R}^n; \mathbb{R}^m)$ with $E_\infty(\phi) < \infty$, the inequality*

$$\partial T(\phi) \leq 2E_\infty(\phi)\mathbf{M}_F(T)$$

is satisfied. Equality holds if, and only if,

$$\lim_{\delta\searrow 0} \int_{\mathbb{R}^n} \left| \rho_\delta * D\phi - E_\infty(\phi) \frac{\nabla F^*(x, \vec{T})}{\sqrt{F^*(x, \vec{T})}} \right|^2 d\|T\| = 0. \tag{33}$$

Proof. We write $e_0 = E_\infty(\phi)$. Define $\psi_\delta = \rho_\delta * \phi$. Using (21), we estimate

$$c \int_{\mathbb{R}^n} \left| D\psi_\delta - e_0 \frac{\nabla F^*(x, \vec{T})}{\sqrt{F^*(x, \vec{T})}} \right|^2 \sqrt{F^*(x, \vec{T})}\, d\|T\|$$
$$\leq \int_{\mathbb{R}^n} \big(F(x, D\psi_\delta) + e_0^2\big) \sqrt{F^*(x, \vec{T})}\, d\|T\| - e_0 \int_{\mathbb{R}^n} D\psi_\delta : \vec{T}\, d\|T\|$$
$$= \int_{\mathbb{R}^n} \big(F(x, D\psi_\delta) + e_0^2\big) \sqrt{F^*(x, \vec{T})}\, d\|T\| - e_0 \partial T(\psi_\delta).$$

Because $\phi \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m)$, there exists a constant $C_1 > 0$ such that $F(x, D\psi_\delta) \leq C_1$ for all $x \in \mathbb{R}^n$ and all $\delta > 0$. Lemma 16 implies that

$$\limsup_{\delta\searrow 0} F(x, D\psi_\delta(x)) \leq e_0^2$$

for any $x \in \mathbb{R}^n$. Applying Fatou's lemma to $C_1 - F(x, D\psi_\delta)$, we find that

$$\limsup_{\delta\searrow 0} \int_{\mathbb{R}^n} F(x, D\psi_\delta)\sqrt{F^*(x,\vec{T})}\, d\|T\| \leq e_0^2 \int_{\mathbb{R}^n} \sqrt{F^*(x,\vec{T})}\, d\|T\| = e_0^2 \mathbf{M}_F(T).$$

Since ϕ is continuous, it is also clear that $\psi_\delta \to \phi$ locally uniformly in $\mathbb{R}^n$. As ∂T is represented by a measure, this implies that $\partial T(\psi_\delta) \to \partial T(\phi)$. Hence

$$c \limsup_{\delta\searrow 0} \int_{\mathbb{R}^n} \left| D\psi_\delta - e_0 \frac{\nabla F^*(x,\vec{T})}{\sqrt{F^*(x,\vec{T})}} \right|^2 \sqrt{F^*(x,\vec{T})}\, d\|T\| \leq 2e_0^2 \mathbf{M}_F(T) - e_0 \partial T(\phi).$$

It follows that $\partial T(\phi) \leq 2e_0 \mathbf{M}_F(T)$, and if we have equality, then (33) follows as well.

Now suppose that (33) holds true. Then

$$\begin{aligned}
\partial T(\phi) &= \lim_{\delta\searrow 0} \partial T(\psi_\delta) \\
&= \lim_{\delta\searrow 0} \int_{\mathbb{R}^n} \vec{T} : D\psi_\delta \, d\|T\| \\
&= e_0 \int_{\mathbb{R}^n} \frac{\vec{T} : \nabla F^*(x,\vec{T})}{\sqrt{F^*(x,\vec{T})}}\, d\|T\| \\
&= 2e_0 \int_{\mathbb{R}^n} \sqrt{F^*(x,\vec{T})}\, d\|T\| \\
&= 2e_0 \mathbf{M}_F(T).
\end{aligned}$$

This completes the proof. □

Proof of Theorem 13. As mentioned earlier, we consider the current T_∞ defined by the condition that

$$T_\infty(\zeta) = \int_{\mathbb{R}^n} \nabla F(x, Z_\infty) : \zeta \, d\mu_\infty$$

for $\zeta \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^{m\times n})$. We will show that T_∞ has the properties stated in Theorem 13.

It is clear that $\mathbf{M}_F(T_\infty) < \infty$. We have already seen at the beginning of this subsection that $\operatorname{supp} \partial T_\infty \subseteq A$ and $\partial T_{\infty i} = 0$ for $i = m_0 + 1, \ldots, m$. This makes ∂T_∞ a distribution supported on a finite set, which means that it is a finite linear combination of Dirac masses on A and their derivatives [18, Theorem 1.5.3]. But because $\|T\|(A) = 0$ by Proposition 21, it is easy to see that we have in fact just a sum of Dirac masses. It follows that $T_\infty \in \mathcal{C}_{m\times n}(\mathbb{R}^n)$.

By the definition of T_∞, we have

$$\vec{T}_\infty = \frac{\nabla F(x, Z_\infty)}{|\nabla F(x, Z_\infty)|}$$

at μ_∞-almost every point. It follows that

$$F^*(x,\vec{T}_\infty) = \frac{F^*(x,\nabla F(x,Z_\infty))}{|\nabla F(x,Z_\infty)|^2} = \frac{F(x,Z_\infty)}{|\nabla F(x,Z_\infty)|^2} = \frac{e_\infty^2}{|\nabla F(x,Z_\infty)|^2},$$

and thus

$$|\nabla F(x,Z_\infty)| = \frac{e_\infty}{\sqrt{F^*(x,\vec{T}_\infty)}}$$

almost everywhere. Hence

$$Z_\infty = e_\infty \frac{\nabla F^*(x,\vec{T})}{\sqrt{F^*(x,\vec{T})}}$$

almost everywhere. In view of Proposition 21, the measure $\|T\|$ is absolutely continuous with respect to $\mu_\infty \llcorner \Omega$. Moreover, if $\phi \in W^{1,\infty}_*(\mathbb{R}^n;\mathbb{R}^m)$ is a minimiser of E_∞, then Proposition 17 gives the convergence $\rho_\delta * D\phi \to Z_\infty$ in $L^2(\|T\|;\mathbb{R}^{m\times n})$, and that implies (33) for T_∞. Proposition 22 implies that $\partial T_\infty(\phi) = 2e_\infty \mathbf{M}_F(T_\infty)$, and thus we have proved the last statement of Theorem 13.

To prove the second statement, consider another $\mathbb{R}^m$-valued 1-current $S \in \mathcal{C}_{m\times n}(\mathbb{R}^n)$ with $\partial S = \partial T_\infty$. Then Proposition 22 gives

$$2e_\infty \mathbf{M}_F(T_\infty) = \partial T_\infty(\phi) = \partial S(\phi) \leq 2e_\infty \mathbf{M}_F(S).$$

Since our assumptions on ϕ^0 imply that $e_\infty > 0$, it follows that $\mathbf{M}_F(T_\infty) \leq \mathbf{M}_F(S)$. All the statements of the theorem are now verified. □

4 Regularisation

We return to the problem of constructing calibrations as in Section 2. Therefore, we consider the domain $\mathbb{R}^2$ again and we study functions $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$. Recall the function

$$f(M) = \frac{1}{2}\left(|M|^2 - \frac{1}{2}(\operatorname{tr} M)^2 + |m_{12} - m_{21}|\sqrt{|M|^2 - 2\det M}\right)$$

from Section 2. We would like to apply the results from Section 3 to

$$F(x,M) = \frac{f(M)}{W(x)}.$$

Unfortunately, this function does not have the required properties: it is convex, but not strictly convex in M and is Lipschitz regular at most. Unless W is bounded, uniformly positive, and of class C^1, it fails to satisfy other assumptions, too. But the potentials W we are most interested in, will have zeroes, certainly at $a^\pm$ and possibly elsewhere.

For this reason, we need to replace the above function F by regularised approximations, and we need to show that the relevant properties persist in the limit. We do this in two steps: first, we focus on the regularisation of f. We can improve some of the properties of W at the same time, but we still assume that it is uniformly positive. In the second step, we deal with the zeroes of W. Before we embark on this journey, however, we extend the definition of the F-mass from Definition 11 as follows. Suppose that $F\colon \mathbb{R}^2 \times \mathbb{R}^{2\times 2} \to [0,\infty]$ is a given function that is convex in the second argument. Assuming that the Legendre transform F^* with respect to the second argument is Borel measurable on $\mathbb{R}^2 \times \mathbb{R}^{2\times 2}$, we define

$$\mathbf{M}_F(T) = \int_{\mathbb{R}^2} \sqrt{F^*(x, \vec{T})}\, d\|T\|$$

for any $\mathbb{R}^2$-valued 1-current T on $\mathbb{R}^2$ with locally finite mass. This is consistent with the previous definition by Proposition 15.

4.1 A Korn type inequality

Our theory will naturally give rise to inequalities such as $f(D\Phi) \leq W$ in $\mathbb{R}^2$ for certain functions $\Phi \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ and for certain exponents $p \in (1,\infty)$. But because the function f controls only the trace free part of $D\Phi$, not the full Jacobian matrix, we need to have a closer look if we want to derive estimates in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$. Such estimates are available, and follow in fact quite easily from a local version of Korn's inequality (as stated and proved, e.g., by Kondrat′ev and Oleĭnik [34, §2, Theorem 8]). In this subsection, we formulate the appropriate inequality in the balls $B_R(0)$ and study how the corresponding constant depends on R.

First, however, we write down an observation that explains why Korn's inequality is useful here. Given $\Phi \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, consider $\Phi^\perp = \left(\begin{smallmatrix} -\Phi_2 \\ \Phi_1 \end{smallmatrix}\right)$ and its symmetrised derivative

$$(D\Phi^\perp)_{\mathrm{sym}} = \begin{pmatrix} -\frac{\partial \Phi_2}{\partial y_1} & \frac{1}{2}\left(\frac{\partial \Phi_1}{\partial y_1} - \frac{\partial \Phi_2}{\partial y_2}\right) \\ \frac{1}{2}\left(\frac{\partial \Phi_1}{\partial y_1} - \frac{\partial \Phi_2}{\partial y_2}\right) & \frac{\partial \Phi_1}{\partial y_2} \end{pmatrix}.$$

Then we note that

$$|(D\Phi^\perp)_{\mathrm{sym}}|^2 = |D\Phi|^2 - \frac{1}{2}(\operatorname{div}\Phi)^2 \leq 2f(D\Phi).$$

We can now prove the following lemma.

Lemma 23. *For every $p \in (n,\infty)$ there exists a constant $C \geq 0$ such that the following holds true. Let $h\colon [1,\infty) \to [0,\infty)$ be a non-decreasing function, and suppose that $\Phi \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ satisfies*

$$\left(\fint_{B_R(0)} \big(f(D\Phi)\big)^{p/2}\, dy\right)^{1/p} \leq h(R)$$

for every $R \geq 1$. Then there exists $b \in \mathbb{R}$ such that

$$\left(\fint_{B_R(0)} |D\Phi(y) - bI|^p\, dy\right)^{1/p} \leq CR^{2/p}h(R)$$

and

$$\sup_{y\in B_R(0)} |\Phi(y) - \Phi(0) - by| \leq CR^{1+2/p}h(R)$$

for every $R \geq 1$.

Proof. Given $R > 0$, we define $\Psi_R(x) = R^{-1}\Phi(Rx)$. Then

$$\left(\fint_{B_1(0)} \big(f(D\Psi_R)\big)^{p/2}\, dy\right)^{1/p} \leq h(R).$$

It follows immediately from the local version of Korn's inequality [34, §2, Theorem 8] that there exists $b_R \in \mathbb{R}$ such that

$$\left(\fint_{B_1(0)} |D\Psi_R(y) - b_R I|^p\, dy\right)^{1/p} \leq C_1 h(R)$$

for some constant C_1 that depends only on p. Morrey's inequality then gives a constant $C_2 = C_2(p)$ such that

$$\sup_{y\in B_1(0)} |\Psi_R(y) - \Psi_R(0) - b_R y| \leq C_2 h(R).$$

In terms of Φ, this means that

$$\left(\fint_{B_R(0)} |D\Phi(y) - b_R I|^p\, dy\right)^{1/p} \leq C_1 h(R)$$

and

$$\sup_{y\in B_R(0)} |\Phi(y) - \Phi(0) - b_R y| \leq C_2 R h(R).$$

The first inequality implies in particular that

$$\left(\fint_{B_1(0)} |D\Phi(y) - b_R I|^p\, dy\right)^{1/p} \leq C_1 R^{2/p} h(R).$$

But at the same time, we have the inequality

$$\left(\fint_{B_1(0)} |D\Phi(y) - b_1 I|^p\, dy\right)^{1/p} \leq C_1 h(1).$$

Hence there exists a constant C_3 such that $|b_R - b_1| \leq C_3 R^{2/p} h(R)$. Choosing $b = b_1$, we therefore obtain the desired inequalities. ☐

4.2 Relaxing the strict convexity

Suppose now that $W : \mathbb{R}^2 \to (0,\infty)$ is a continuous function such that $W(y) \to \infty$ as $|y| \to \infty$. Then we can clearly find a sequence of functions $W_k \in C^\infty(\mathbb{R}^2)$, for $k \in \mathbb{N}$, such that

- every W_k is bounded,
- there exists $c > 0$ such that $W_k \geq c$ in $\mathbb{R}^2$ for every $k \in \mathbb{N}$,
- $W_k \leq W_\ell$ when $k \leq \ell$, and
- $W(y) = \lim_{k\to\infty} W_k(y)$ for every $y \in \mathbb{R}^2$.

Recall that

$$f(M) = \frac{1}{2}\left(|M|^2 - \frac{1}{2}(\operatorname{tr} M)^2 + |m_{12} - m_{21}|\sqrt{|M|^2 - 2\det M}\right)^2.$$

In the coordinates

$$q_1 = \frac{1}{\sqrt{2}}(m_{11} + m_{22}), \qquad q_2 = \frac{1}{\sqrt{2}}(m_{11} - m_{22}),$$
$$q_3 = \frac{1}{\sqrt{2}}(m_{12} + m_{21}), \qquad q_4 = \frac{1}{\sqrt{2}}(m_{12} - m_{21}),$$

we can write

$$f(q) = \frac{1}{2}\left(|q_4| + \sqrt{q_2^2 + q_3^2}\right)^2.$$

We now consider the regularisation

$$f_k(q) = \frac{1}{2}\left(\sqrt{q_4^2 + \frac{|q|^2}{2k}} + \sqrt{q_2^2 + q_3^2 + \frac{|q|^2}{2k}}\right)^2.$$

In the original coordinates, this is

$$\begin{aligned} f_k(M) = \frac{1}{2}\Bigg(&\left(1 + \frac{1}{k}\right)|M|^2 - \frac{1}{2}(\operatorname{tr} M)^2 \\ &+ \left((m_{12} - m_{21})^2 + \frac{1}{k}|M|^2\right)^{1/2}\left(\left(1 + \frac{1}{k}\right)|M|^2 - 2\det M\right)^{1/2}\Bigg). \end{aligned}$$

This function is now strictly convex and smooth in $\mathbb{R}^{2\times 2} \setminus \{0\}$. Of course, it is still homogeneous of degree 2. We further note that $f_k \geq f$ and $f(M) = \lim_{k\to\infty} f_k(M)$ for every $M \in \mathbb{R}^{2\times 2}$, and this convergence is monotone. Define

$$F_k(y, M) = \frac{f_k(y)}{W_k(y)}$$

These functions then satisfy the assumptions from Section 3.

We also need to consider the Legendre transforms.

Lemma 24. *The Legendre transform of F with respect to the second argument is*

$$F^*(y, N) = \begin{cases} \frac{1}{4}W(y)\max\{|N|^2 - 2\det N, (n_{12} - n_{21})^2\} & \textit{if } \operatorname{tr} N = 0, \\ \infty & \textit{else.} \end{cases}$$

Proof. We can work in the coordinates q given above, as the transformation amounts to an isometry between $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^4$. Moreover, it suffices to consider

$$f(q) = \frac{1}{2}\left(|q_4| + \sqrt{q_2^2 + q_3^2}\right)^2$$

and its Legendre transform

$$f^*(p) = \sup_{q\in\mathbb{R}^4}\big(p \cdot q - f(q)\big).$$

It is clear that $f^*(p) = \infty$ if $p_1 \neq 0$, as f does not depend on q_1. Now assume that $p_1 = 0$. Then the supremum is attained at a point $q = (0, q_2, q_3, q_4) \in \mathbb{R}^4$ such that either

- $q_4 = 0$, or
- $q_2 = q_3 = 0$, or
- f is differentiable at q and

$$p_2 = \frac{\partial f}{\partial q_2}(q) = \left(|q_4| + \sqrt{q_2^2 + q_3^2}\right) \frac{q_2}{\sqrt{q_2^2 + q_3^2}},$$
$$p_3 = \frac{\partial f}{\partial q_3}(q) = \left(|q_4| + \sqrt{q_2^2 + q_3^2}\right) \frac{q_3}{\sqrt{q_2^2 + q_3^2}},$$
$$p_4 = \frac{\partial f}{\partial q_4}(q) = \left(|q_4| + \sqrt{q_2^2 + q_3^2}\right) \frac{q_4}{|q_4|}.$$

In the first case, we find that

$$f^*(p) = \sup_{q_2, q_3 \in \mathbb{R}} \left(p_2 q_2 + p_3 q_3 - \frac{q_2^2 + q_3^2}{2} \right) = \frac{p_2^2 + p_3^2}{2}.$$

Similarly, in the second case,

$$f^*(p) = \sup_{q_4 \in \mathbb{R}} \left(p_4 q_4 - \frac{q_4^2}{2} \right) = \frac{p_4^2}{2}.$$

In any case, $f^*(p)$ will be at least as large as either of these expressions, so

$$f^*(p) \geq \frac{1}{2} \max\{p_2^2 + p_3^2, p_4^2\} \tag{34}$$

for every $p \in \mathbb{R}^4$. Finally, in the third of the above cases, we conclude that $p_2^2 + p_3^2 = p_4^2$. Hence this case occurs only for points on this double cone.

To summarise, f^* is a convex function that satisfies (34) and such that $f^*(p) = (p_2^2 + p_3^2)/2$ or $f^*(p) = p_4^2/2$ whenever $p_2^2 + p_3^2 \neq p_4^2$. There exists only one function with these properties, which is

$$f^*(p) = \frac{1}{2} \max\{p_2^2 + p_3^2, p_4^2\}.$$

In terms of the original coordinates, we then have the expression

$$f^*(N) = \frac{1}{4} \max\{|N|^2 - 2 \det N, (n_{12} - n_{21})^2\},$$

and the claim follows. □

We do not need to compute the Legendre transforms of F_k explicitly, but we note that they are convex and homogeneous of degree 2 in the second argument. Furthermore,

$$\begin{aligned} F^*(y, N) &= \sup_{M \in \mathbb{R}^{2\times 2}} \left(M : N - \inf_{k \in \mathbb{N}} F_k(y, M) \right) \\ &= \sup_{k \in \mathbb{N}} \sup_{M \in \mathbb{R}^{2\times 2}} \big(M : N - F_k(y, N) \big) \\ &= \sup_{k \in \mathbb{N}} F_k^*(y, N) \end{aligned}$$

for any $y \in \mathbb{R}^2$ and $N \in \mathbb{R}^{2\times 2}$.

We now want to prove the following.

Proposition 25. *Suppose that $W \in C^0(\mathbb{R}^2; (0,\infty))$ satisfies $\lim_{|y|\to\infty} W(y) = \infty$. Then there exist $\Phi \in \bigcap_{p<\infty} W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ and $T \in \mathcal{C}^0_{2\times 2}$ such that*

(i) $f(D\Phi) \leq W$ almost everywhere,

(ii) $\mathbf{M}_F(T) \leq \mathbf{M}_F(S)$ for any $S \in \mathcal{C}^0_{2\times 2}$, and

(iii) $\partial T(\Phi) \geq 2\mathbf{M}_F(T)$.

Proof. We define the functions F_k as explained above. For any fixed $k \in \mathbb{N}$, we consider the functional

$$E^k_\infty(\Phi) = \operatorname*{ess\,sup}_{y\in\mathbb{R}^2} \sqrt{F_k(y, D\Phi(y))}.$$

By Proposition 12, there exists a minimiser $\tilde{\Phi}_k \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ of E^k_∞ subject to the conditions $\Phi_1(a^-) = 0$ and $\Phi_1(a^+) = 1$. Set

$$\Phi_k = \frac{\tilde{\Phi}_k}{E^k_\infty(\tilde{\Phi}_k)}.$$

Then $F_k(y, D\Phi_k) \leq 1$, i.e., $f_k(D\Phi_k) \leq W_k$, almost everywhere by construction.

Theorem 13 gives rise to a non-trivial current $T_k \in \mathcal{C}_{2\times 2}(\mathbb{R}^2)$ for every $k \in \mathbb{N}$ with $\operatorname{supp} \partial T_{k1} \subseteq \{a^\pm\}$ and $\partial T_{k2} = 0$, which minimises $\mathbf{M}_{F_k}$ for its boundary data and satisfies $\partial T_k(\Phi_k) = 2\mathbf{M}_{F_k}(T_k)$. Because all of these properties are invariant under multiplication with a positive constant, we can renormalise this current such that $T_k \in \mathcal{C}^0_{2\times 2}$.

Next we study the limit as $k \to \infty$. For any $R > 0$, the functions $W_k \leq W$ are uniformly bounded in $B_R(0)$ by the continuity of W. Hence

$$\sup_{k\in\mathbb{N}} \sup_{y\in B_R(0)} f(D\Phi_k) < \infty.$$

Lemma 23 implies that there exist $b_k \in \mathbb{R}$ such that the functions

$$\hat{\Phi}_k(y) = \Phi_k(y) - b_k y$$

are uniformly bounded in $W^{1,p}(B_R(0); \mathbb{R}^2)$ for all $p < \infty$ and all $R > 0$. Therefore, we may assume that we have weak convergence of $\hat{\Phi}_k$ in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ for any $p < \infty$ to some limit $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$. Since the set

$$\left\{\Psi \in W^{1,p}(B_R(0); \mathbb{R}^2) \colon F(x, D\Psi) \leq 1 \text{ almost everywhere}\right\}$$

is convex and closed in $W^{1,p}(B_R(0); \mathbb{R}^2)$, and every $\hat{\Phi}_k$ belongs to this set, it follows that $F(y, D\Phi) \leq 1$ almost everywhere.

Because ∂T_{k1} is supported on $\{a^\pm\}$, the functions $\hat{\Phi}_k$ still satisfy the condition $\partial T_k(\hat{\Phi}_k) = 2\mathbf{M}_{F_k}(T_k)$.

We can estimate $f_k(M) \leq 4|M|^2$ for all $M \in \mathbb{R}^{2\times 2}$ and all $k \in \mathbb{N}$. Hence $F_k^*(y, N) \geq \frac{1}{16} W(y)|N|^2$ for all $y \in \mathbb{R}^2$ and $N \in \mathbb{R}^{2\times 2}$. Since W is bounded below under the above assumptions, it follows that

$$\|T_k\|(\mathbb{R}^2) \leq C \int_{\mathbb{R}^2} \sqrt{F_k^*(y, \vec{T}_k)}\, d\|T_k\| = C\mathbf{M}_{F_k}(T_k) = \frac{C}{2} \partial T_k(\hat{\Phi}_k)$$

for some constant $C > 0$ that is independent of k, and the right-hand side is obviously bounded. Hence we may assume that T_k converges weakly* in the dual space of $C_0^0(\mathbb{R}^2; \mathbb{R}^{2\times 2})$ to some limit T, which will automatically belong to $\mathcal{C}^0_{2\times 2}$. From the definition of the F-mass in Definition 11, it follows easily that $\mathbf{M}_{F_\ell}$ is lower semicontinuous with respect to such convergence for any fixed $\ell \in \mathbb{N}$. Thus

$$\mathbf{M}_{F_\ell}(T) \leq \liminf_{k\to\infty} \mathbf{M}_{F_\ell}(T_k) \leq \liminf_{k\to\infty} \mathbf{M}_{F_k}(T_k).$$

Moreover, Beppo Levi's monotone convergence theorem gives

$$\mathbf{M}_F(T) = \int_{\mathbb{R}^2} \sqrt{F^*(y, \vec{T})}\, d\|T\| = \lim_{\ell\to\infty} \int_{\mathbb{R}^2} \sqrt{F^*_\ell(y, \vec{T})}\, d\|T\| = \lim_{\ell\to\infty} \mathbf{M}_{F_\ell}(T).$$

Therefore,

$$\mathbf{M}_F(T) \leq \liminf_{k\to\infty} \mathbf{M}_{F_k}(T_k).$$

Recall that the currents T_k all have the same boundary by construction. Since $\hat{\Phi}_k \to \Phi$ locally uniformly, it follows that

$$\partial T(\Phi) = \lim_{k\to\infty} \partial T_k(\hat{\Phi}_k) = 2 \lim_{k\to\infty} \mathbf{M}_{F_k}(T_k) \geq 2\mathbf{M}_F(T).$$

It remains to prove that T minimises the F-mass in $\mathcal{C}^0_{2\times 2}$. Let $S \in \mathcal{C}^0_{2\times 2}$. Then $\partial S = \partial T_k$ for every $k \in \mathbb{N}$, and we know that $\mathbf{M}_{F_k}(T_k) \leq \mathbf{M}_{F_k}(S)$. As above, we see that

$$\mathbf{M}_F(T) \leq \liminf_{k\to\infty} \mathbf{M}_{F_k}(T_k) \leq \liminf_{k\to\infty} \mathbf{M}_{F_k}(S) = \mathbf{M}_F(S).$$

This finally concludes the proof. □

4.3 Potentials with zeroes

We now want to remove the assumption that W is positive. While we do not obtain a specific current with the properties of Proposition 25 in this case, we can still prove the following.

Theorem 26. *Let $W\colon \mathbb{R}^2 \to [0, \infty)$ be a continuous function. Then there exists $\Phi \in \bigcap_{p<\infty} W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ such that $f(D\Phi) \leq W$ almost everywhere and*

$$\Phi_1(a^+) - \Phi_1(a^-) \geq 2 \inf_{T\in\mathcal{C}^0_{2\times 2}} M_F(T).$$

Proof. For $k \in \mathbb{N}$, define $W_k(y) = W(y) + \frac{1}{k}(1 + |y|^2)$, and then let $F_k(y, M) = f(M)/W_k(y)$. Then $\mathbf{M}_{F_k} \geq \mathbf{M}_F$.

For each $k \in \mathbb{N}$, Proposition 25 provides a function $\Phi_k\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(D\Phi_k) \leq W_k$ almost everywhere, and it also provides a current $T_k \in \mathcal{C}^0_{2\times 2}$ that minimises $\mathbf{M}_{F_k}$. Furthermore, Proposition 25 tells us that

$$\Phi_{k1}(a^+) - \Phi_{k1}(a^-) = \partial T_k(\Phi_k) \geq 2\mathbf{M}_{F_k}(T_k) \geq 2\mathbf{M}_F(T_k) \geq 2 \inf_{T\in\mathcal{C}^0_{2\times 2}} \mathbf{M}_F(T).$$

When we let $k \to \infty$, we see with the same arguments as in the proof of Proposition 25 that we can modify each Φ_k such that it still has the above properties, but such that we have weak convergence of some subsequence of $(\Phi_k)_{k\in\mathbb{N}}$, in the space $\bigcap_{p<\infty} W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, to a limit Φ that satisfies $f(D\Phi) \leq W$ almost everywhere. Since this also implies locally uniform convergence, it further follows that

$$\Phi_1(a^+) - \Phi_1(a^-) \geq 2 \inf_{T\in\mathcal{C}^0_{2\times 2}} M_F(T).$$

This concludes the proof. □

4.4 How calibrations give a lower bound

We expect that calibrations give rise to lower bounds for the energy, and this is indeed the reason why we consider them. Formal calculations give a good idea of the underlying estimates, but in order to obtain a rigorous proof, we need some control of the corresponding integrals when u is potentially unbounded. The purpose of this subsection is to justify the following statement, which depends on Proposition 10.

Once this is proved, we can proceed to prove Theorem 3 and Corollary 4, which we do at the end of the section.

Lemma 27. *Suppose that $W\colon \mathbb{R}^2 \to [0,\infty)$ is Hölder continuous and satisfies the growth condition (4). Suppose that $\Phi \in \bigcap_{p<\infty} W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ satisfies $f(D\Phi) \leq W$ almost everywhere. Then*

$$\mathcal{E}(a^-, a^+) \geq \Phi_1(a^-) - \Phi_1(a^+).$$

Proof. For any two constants $b, c \in \mathbb{R}$, the function $\Psi(y) = \Phi(y) + by + c$ satisfies $f(D\Psi) = f(D\Phi)$ and

$$\Psi_1(a^-) - \Psi_1(a^+) = \Phi_1(a^-) - \Phi_1(a^+).$$

Hence we may assume without loss of generality that $\Phi_2(a^+) = \Phi_2(a^-)$ and $\Phi(0) = 0$. Let $\alpha = -\frac{1}{2}\operatorname{div}\Phi$. Then the inequality $f(D\Phi) \leq W$ is equivalent to $g(D\Phi + \alpha I) \leq W$ for the function g from Section 2.

We regularise the calibration Φ and the potential function W the same way as in Proposition 10. That is, we define $\Phi_\delta = \rho_\delta * \Phi$ and $\alpha_\delta = \rho_\delta * \alpha + \delta$, and furthermore,

$$W_\delta = \frac{\rho_\delta * W}{1-\delta} + \delta.$$

According to Proposition 10, there exist vector fields $\omega_\delta \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ such that

$$\operatorname{div}\Phi_\delta(u) + \alpha_\delta(u)\operatorname{div} u \leq \frac{\epsilon}{2}|Du|^2 + \frac{1}{2\epsilon}W_\delta(u) - \epsilon\operatorname{div}\big((Du)^T\omega_\delta(u)\big)^\perp \tag{35}$$

for all $u \in C^2(B_1(0);\mathbb{R}^2)$ and all $\epsilon > 0$, and such that

$$|\omega_\delta(y)| \leq C_1(1 + |y|\log|y|)$$

for every $y \in \mathbb{R}^2$, where C_1 is a constant independent of δ or y.

Inequality (35), in its weak form, says that

$$\int_{B_1(0)} \big(\eta\alpha_\delta(u)\operatorname{div} u - \nabla\eta\cdot\Phi_\delta(u)\big)\,dx \le \int_{B_1(0)} \eta\left(\frac{\epsilon}{2}|Du|^2 + \frac{1}{2\epsilon}W_\delta(u)\right)dx - \epsilon\int_{B_1(0)} \omega_\delta(u)\cdot(Du\nabla^\perp\eta)\,dx \quad (36)$$

for all $u \in C^2(B_1(0);\mathbb{R}^2)$ and all $\eta \in C_0^\infty(B_1(0))$ with $\eta \ge 0$.

Fix $q > 2$. Using Lemma 23, we find a constant C_2 (depending on q) such that

$$\left(\fint_{B_R(0)} |D\Phi|^q\,dy\right)^{1/q} \le C_2 R^{\bar{p}+2/q} \quad (37)$$

for every $R \ge 1$ and

$$|\Phi(y)| \le C_2\big(|y|^{\bar{p}+1+2/q} + 1\big) \quad (38)$$

for every $y \in \mathbb{R}^2$. With the help of Hölder's inequality, we then also estimate

$$\begin{aligned} |\alpha_\delta(y)| &= \left|\delta + \int_{B_\delta(y)} \rho_\delta(y-z)\alpha(z)\,dz\right| \\ &\le \delta + \frac{1}{2}\|\rho_\delta\|_{L^{q/(q-1)}(\mathbb{R}^2)}\left(\int_{B_{|y|+1}(0)} |\operatorname{div}\Phi|^q\,dz\right)^{1/q} \\ &\le C_3\delta^{-2/q}\big(|y|^{\bar{p}+4/q} + 1\big), \end{aligned} \quad (39)$$

for some constant C_3, whenever $\delta \in (0,1]$.

Combining (37) with Morrey's inequality, we find a constant C_4 such that

$$|\Phi(y) - \Phi(z)| \le C_4 R^{\bar{p}+4/q}|y-z|^{1-2/q}$$

for all $y, z \in B_R(0)$. Hence there exists a constant C_5 such that

$$|\Phi_\delta(y) - \Phi(y)| = \left|\int_{B_\delta(y)} \rho_\delta(y-z)(\Phi(z) - \Phi(y))\,dz\right| \le C_5 R^{\bar{p}+4/q}\delta^{1-2/q} \quad (40)$$

when $y \in B_R(0)$ with $R \ge 1$ and $\delta \le 1$.

Since the functions Φ_δ and α_δ have at most polynomial growth by these estimates, and we know that the same applies to W_δ and ω_δ, a standard approximation argument now shows that (36) holds for all $u \in W^{1,2}(B_1(0);\mathbb{R}^2)$.

Recall that in the definition of $\mathcal{E}(a^-, a+)$, we consider $u_0\colon \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$u_0(x) = \begin{cases} a^+ & \text{if } x_1 > 0, \\ a^- & \text{if } x_1 < 0. \end{cases}$$

The set $\mathcal{U}(a^-, a^+)$ then comprises all families $(u_\epsilon)_{\epsilon>0}$ in $W^{1,2}(B_1(0);\mathbb{R}^2)$ such that $u_\epsilon \to u_0$ in $L^1(B_1(0); R^2)$ and such that there exist $\tau > 0$ and $s > 2$ with $\epsilon^{-\tau}\operatorname{div} u_\epsilon \to 0$ in $L^s(B_1(0))$ as $\epsilon \searrow 0$. Then

$$\mathcal{E}(a^-, a^+) = \frac{1}{2}\inf\left\{\liminf_{\epsilon\searrow 0} E_\epsilon(u_\epsilon; B_1(0))\colon (u_\epsilon)_{\epsilon>0} \in \mathcal{U}(a^-, a^+)\right\}.$$

We now fix $(u_\epsilon)_{\epsilon>0}$ from $\mathcal{U}(a^-, a^+)$. Choose a sequence $\epsilon_k \searrow 0$ such that

$$\lim_{k\to\infty} E_{\epsilon_k}(u_{\epsilon_k}; B_1(0)) = \liminf_{\epsilon\searrow 0} E_\epsilon(u_\epsilon; B_1(0)).$$

We may assume that this limit is finite.

By the growth condition (4), there exist constants $c, \theta > 0$ such that $W(y) \geq c|y|^{2\bar{p}}$ when $|y| \geq \theta$. For $u \in W^{1,2}(B_1(0); \mathbb{R}^2)$, we consider $v = \max\{|u|^{\bar{p}+1}, \theta\}$. Then we note that

$$\begin{aligned}\int_{B_1(0)} |Dv|\,dx &\leq (\bar{p}+1)\int_{\{|u|>\theta\}} |u|^{\bar{p}}|Du|\,dx\\ &\leq (\bar{p}+1)\int_{\{|u|>\theta\}} \left(\frac{\epsilon}{2}|Du|^2 + \frac{1}{2\epsilon}|u|^{2\bar{p}}\right)dx\end{aligned}$$

by Young's inequality. Hence the functions $\max\{|u_{\epsilon_k}|^{\bar{p}+1}, \theta\}$ are uniformly bounded in $W^{1,1}(B_1(0))$. By the Sobolev inequality, they are also bounded in $L^2(B_1(0))$. Thus $(u_{\epsilon_k})_{k\in\mathbb{N}}$ is bounded in $L^{2\bar{p}+2}(B_1(0); \mathbb{R}^2)$. Since it converges to u_0 in $L^1(B_1(0); \mathbb{R}^2)$, we conclude that this convergence holds in $L^r(B_1(0); \mathbb{R}^2)$ as well for any $r < 2\bar{p}+2$.

We now fix a number $\ell > 1$ and define $\delta_k = \epsilon_k^\ell$. Using (38) and (40), we see that

$$\int_{B_1(0)} \nabla\eta \cdot \Phi_{\delta_k}(u_{\epsilon_k})\,dx \to \int_{B_1(0)} \nabla\eta \cdot \Phi(u_0)\,dx$$

if q is chosen sufficiently large. Recall that there exist $\tau > 0$ and $s > 2$ with $\epsilon^{-\tau} \operatorname{div} u_\epsilon \to 0$ in $L^s(B_1(0))$ as $\epsilon \searrow 0$. Choose

$$q > \frac{4s}{\bar{p}(s-2) + 2s - 2}.$$

Then (39) implies that

$$\|\alpha_{\delta_k}(u_{\epsilon_k})\|_{L^{s/(s-1)}(B_1(0))} \leq C_4 \epsilon_k^{-2\ell/q}$$

for a constant C_4 that is independent of k. It follows that

$$\int_{B_1(0)} \eta\alpha_{\delta_k}(u_{\epsilon_k}) \operatorname{div} u_{\epsilon_k}\,dx \to 0$$

if we also choose $q > 2\ell/\tau$. It is not difficult to see that

$$\epsilon_k \int_{B_1(0)} \omega_{\delta_k}(u_{\epsilon_k}) \cdot Du_{\epsilon_k}\nabla^\perp\eta\,dx \to 0.$$

Because we assume that W is locally Hölder continuous, and because we have the growth condition (4), we have a number $\gamma > 0$ and a constant C_4 such that $W_\delta(y) \leq W(y) + C_4\delta^\gamma(1 + |y|^{2\bar{p}})$ for any $y \in \mathbb{R}^2$. If we choose $\ell > 1/\gamma$, then it follows that

$$\limsup_{k\to\infty} \int_{B_1(0)} \eta\left(\frac{\epsilon_k}{2}|Du_{\epsilon_k}|^2 + \frac{1}{2\epsilon_k} W_{\delta_k}(u_{\epsilon_k})\right)dx \leq \lim_{k\to\infty} E_{\epsilon_k}(u_{\epsilon_k}; B_1(0)).$$

Combining all the inequalities, we find that

$$\mathcal{E}(a^-, a^+) \geq -\frac{1}{2} \int_{B_1(0)} \nabla \eta \cdot \Phi(u_0)\, dx = -\frac{1}{2} \int_{B_1(0)} \frac{\partial \eta}{\partial x_1} \Phi_1(u_0)\, dx.$$

The integral on the right-hand side is easy to calculate because of the specific form of u_0: we conclude that

$$\mathcal{E}(a^-, a^+) \geq \frac{1}{2} \big(\Phi_1(a^+) - \Phi_1(a^-)\big) \int_{-1}^{1} \eta(0, x_2)\, dx_2.$$

If we approximate the characteristic function of $B_1(0)$ with η, we therefore obtain the desired inequality. □

We now have all the ingredients for the proof of our main result.

Proof of Theorem 3. The functional $\mathbf{M}_F$ defined in the introduction is identical to the F-mass defined in Section 4. Under the assumptions of Theorem 3, we can use Theorem 26 to obtain a suitable calibration. Lemma 27 then yields the desired inequality. □

Proof of Corollary 4. We compute

$$\mathbf{M}_F(T^0) = \frac{1}{2} \int_{[a^-, a^+]} \sqrt{W}\, d\mathcal{H}^1.$$

If T^0 minimises $\mathbf{M}_F$ in $\mathcal{C}^0_{2\times 2}$, then Theorem 3 therefore implies that

$$\mathcal{E}(a^-, a^+) \geq \int_{[a^-, a^+]} \sqrt{W}\, d\mathcal{H}^1.$$

The reverse inequality follows from a standard construction, which can be found, e.g., in a paper by Ignat and Monteil [23, Proposition 4.1]. □

5 The geometric problem

Theorem 26 suggests that we study the minimisers of

$$\mathbf{M}_F(T) = \int_{\mathbb{R}^2} \sqrt{F^*(x, \vec{T})}\, d\|T\|$$

for $T \in \mathcal{C}^0_{2\times 2}$. This now constitutes a geometric problem, which is similar in spirit to the problem of finding geodesics. But it is also a novel problem, because we have to consider *vector-valued* currents, the components of which interact in non-trivial ways with each other. This is the problem that we analyse in this section.

First recall that $F^*(x, N) = W(x) f^*(N)$, where

$$f^*(N) = \begin{cases} \frac{1}{4} \max\{|N|^2 - 2\det N, (n_{12} - n_{21})^2\} & \text{if } \operatorname{tr} N = 0, \\ \infty & \text{else.} \end{cases}$$

As in the introduction, we consider the current T^0, defined by

$$T^0(\zeta) = \int_{[a^-, a^+]} \zeta : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} d\mathcal{H}^1$$

for $\zeta \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^{2\times 2})$. Above all, we are interested in conditions that guarantee that T^0 minimises the functional.

5.1 An estimate for F^*

We will estimate $\mathbf{M}_F(T)$ in terms of the first component T_1 of T. Assuming that $\mathbf{M}_F(T) < \infty$, we first observe that $\vec{T}$ must be of the form

$$\vec{T} = \begin{pmatrix} r & s \\ t & -r \end{pmatrix}$$

for some $r, s, t \in \mathbb{R}$ almost everywhere. The numbers r and s will effectively be determined by T_1, but this does not apply to t. It turns out, however, that we can estimate $F^*(x, \left(\begin{smallmatrix} r & s \\ t & -r \end{smallmatrix}\right))$ in terms of the following functions: for $\lambda \in [-1, 1]$, we define

$$\Theta_\lambda(z) = \begin{cases} (1+\lambda)\frac{z_1^2}{|z_2|} + |z_2| & \text{if } z_2 > 0 \text{ and } (1+\lambda)z_1^2 < (1-\lambda)z_2^2, \\ (1-\lambda)\frac{z_1^2}{|z_2|} + |z_2| & \text{if } z_2 < 0 \text{ and } (1-\lambda)z_1^2 < (1+\lambda)z_2^2, \\ 2|z_1|\sqrt{1-\lambda^2} + \lambda z_2 & \text{else.} \end{cases}$$

We note that Θ_λ is positive homogeneous of degree 1 in z and that $\Theta_\lambda(z) \geq |z_2|$ for all $\lambda \in [-1, 1]$ and all $z \in \mathbb{R}^2$. Furthermore, we have the following inequality.

Lemma 28. *For any $\lambda \in [-1, 1]$ and any $r, s, t \in R$,*

$$\sqrt{f^* \begin{pmatrix} r & s \\ t & -r \end{pmatrix}} \geq \frac{1}{2}\big(\Theta_\lambda(r, s) + \lambda t\big).$$

Proof. We fix r and s and regard the left-hand side of the desired inequality as a function of t. Thus we define

$$\theta(t) = \sqrt{f^* \begin{pmatrix} r & s \\ t & -r \end{pmatrix}} = \frac{1}{2} \max\{\sqrt{4r^2 + (s+t)^2}, |s-t|\}.$$

We also consider $\theta_1(t) = \frac{1}{2}\sqrt{4r^2 + (s+t)^2}$ and $\theta_2(t) = \frac{1}{2}|s - t|$. These are convex functions, and hence $\theta = \max\{\theta_1, \theta_2\}$ is convex, too. Moreover, we see that θ is differentiable at every point with the exception of $t = -r^2/s$ (which is the unique point where $\theta_1(t) = \theta_2(t)$) if $s \neq 0$.

If $s > 0$ and $t > -r^2/s$ or if $s < 0$ and $t < -r^2/s$, then $\theta_1(t) > \theta_2(t)$, and we compute

$$\theta'(t) = \theta_1'(t) = \frac{s+t}{2\sqrt{4r^2 + (s+t)^2}}.$$

If $s > 0$ and $t < -r^2/s$ or if $s < 0$ and $t > -r^2/s$, then $\theta_1(t) < \theta_2(t)$, and

$$\theta'(t) = \theta_2'(t) = \frac{t-s}{2|t-s|}.$$

(If $s = 0$, then $\theta = \theta_1$.)

Now fix $\lambda \in [-1, 1]$. If $-1 < \lambda < 1$, then there exists a unique point $t_\lambda \in \mathbb{R}$ such that $\lambda/2$ is a subderivative of θ at t_λ. That point is $t_\lambda = -r^2/s$ if

$$s > 0 \quad \text{and} \quad (1+\lambda)r^2 \leq (1-\lambda)s^2$$

or if

$$s < 0 \quad \text{and} \quad (1-\lambda)r^2 \leq (1+\lambda)s^2.$$

Otherwise, it is the unique point where $\theta_1'(t) = \lambda/2$, namely

$$t_\lambda = -s + \frac{2\lambda|r|}{\sqrt{1-\lambda^2}}.$$

We now have the inequality

$$\theta(t) \geq \theta(t_\lambda) + \frac{\lambda}{2}(t - t_\lambda) = \theta_1(t_\lambda) + \frac{\lambda}{2}(t - t_\lambda).$$

If we compute the right-hand side, we obtain exactly $\frac{1}{2}(\Theta_\lambda(r,s) + \lambda t)$.

For $\lambda = -1$ and $\lambda = 1$, the inequality now follows by continuity. □

For our subsequent estimates, it will be useful to know more about the structure of Θ_λ. This turns out to be a convex function; in fact, the following is true.

Lemma 29. *For $-1 \leq \lambda \leq 1$, let*

$$H_\lambda(z) = \begin{cases} (1+\lambda)\frac{z_1^2}{z_2} + z_2 & \text{if } z_2 > 0, \\ -(1-\lambda)\frac{z_1^2}{z_2} - z_2 & \text{if } z_2 < 0, \\ \infty & \text{if } z_1 \neq 0 \text{ and } z_2 = 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Then Θ_λ is the convex envelope of H_λ.

Proof. Suppose first that $-1 < \lambda < 1$. Consider the sets

$$C_+ = \left\{z \in \mathbb{R}^2 \colon z_2 > 0 \text{ and } (1+\lambda)z_1^2 < (1-\lambda)z_2^2\right\},$$
$$C_- = \left\{z \in \mathbb{R}^2 \colon z_2 < 0 \text{ and } (1-\lambda)z_1^2 < (1+\lambda)z_2^2\right\},$$

and $D = \mathbb{R}^2 \setminus (C_+ \cup C_-)$.

If $z_2 > 0$, then we observe that

$$0 \leq (1+\lambda)z_2\left(\frac{|z_1|}{z_2} - \sqrt{\frac{1-\lambda}{1+\lambda}}\right)^2 = (1+\lambda)\frac{z_1^2}{z_2} - 2|z_1|\sqrt{1-\lambda^2} + (1-\lambda)z_2.$$

From this, we conclude that $H_\lambda(z) \geq 2|z_1|\sqrt{1-\lambda^2} + \lambda z_2$ when $z_2 > 0$, with equality on ∂C_+. Similarly, we show that $H_\lambda(z) \geq 2|z_1|\sqrt{1-\lambda^2} + \lambda z_2$ when $z_2 < 0$, with equality on ∂C_-.

Let

$$L = \left\{\ell \colon \mathbb{R}^2 \to \mathbb{R} \colon \ell \text{ is linear with } \ell \leq H_\lambda \text{ in } \mathbb{R}^2\right\},$$

and let $\check{H}_\lambda(z) = \sup_{\ell \in L} \ell(z)$ denote the convex envelope of H_λ. (Note that it suffices to consider linear rather than affine functions, because H_λ is positive homogeneous of degree 1.) Then the above observations imply that $\phi_+(z) = 2z_1\sqrt{1-\lambda^2} + \lambda z_2$ and $\phi_-(z) = -2z_1\sqrt{1-\lambda^2} + \lambda z_2$ belong to L. It follows that $\check{H}_\lambda(z) \geq 2|z_1|\sqrt{1-\lambda^2} + \lambda z_2$ for all $z \in \mathbb{R}^2$. Since $H_\lambda(z) = \phi_+(z)$ when $z \in \partial C_\pm$ and $z_1 \geq 0$, it also follows that $\check{H}_\lambda \leq \phi_+$ in $\{z \in D \colon z_1 \geq 0\}$ (the convex hull of $(\partial C_- \cup \partial C_+) \cap \{z_1 \geq 0\}$). Similarly, $\check{H}_\lambda \leq \phi_-$ in $\{z \in D \colon z_1 \leq 0\}$. Combining these inequalities, we see that $\check{H}_\lambda = \Theta_\lambda$ in D.

The restriction of H_λ to C_+ is smooth. It suffices to examine the Hessian to see that it is also convex. Thus for any $z_0 \in C_+$, there exists a linear function $\ell_0 \colon \mathbb{R}^2 \to \mathbb{R}$ such that $\ell_0(z_0) = H_\lambda(z_0)$ and $\ell_0(z) \leq H_\lambda(z)$ for all $z \in C_+$. Indeed, differentiating H_λ, we see that

$$\ell_0(z) = 2(1+\lambda)\frac{z_{01}}{z_{02}} z_1 + \left(1 - (1+\lambda)\frac{z_{01}^2}{z_{02}^2}\right) z_2$$

for $z \in \mathbb{R}^2$, where $z_0 = (z_{01}, z_{02})$. We claim that $\ell_0 \in L$. To see why, we note that

$$1 - (1+\lambda)\frac{z_{01}^2}{z_{02}^2} \geq \lambda$$

because $z_0 \in C_+$. For $\tilde{z} \in \partial C_+$, we already know that

$$\ell_0(\tilde{z}) \leq H_\lambda(\tilde{z}) = 2|\tilde{z}_1|\sqrt{1-\lambda^2} + \lambda \tilde{z}_2.$$

For $z \in D \cup C_-$, choose $\tilde{z} \in \partial C_+$ with $\tilde{z}_1 = z_1$. Then $z_2 \leq \tilde{z}_2$, and therefore,

$$\ell_0(z) \leq \ell_0(\tilde{z}) + \lambda(z_2 - \tilde{z}_2) \leq 2|z_1|\sqrt{1-\lambda^2} + \lambda z_2 \leq H_\lambda(z).$$

Hence $\ell_0 \in L$.

We conclude that $\check{H}_\lambda(z_0) = H_\lambda(z_0) = \Theta_\lambda(z_0)$. Similar arguments apply to C_- as well. Hence $\check{H}_\lambda = \Theta_\lambda$ everywhere.

It remains to study the cases $\lambda = 1$ and $\lambda = -1$. In both cases, we have the identity $\Theta_\lambda(z) = |z_2|$. Furthermore, in both cases, we compute $H_\lambda(0, z_2) = |z_2|$ and $H_\lambda(z) \geq |z_2|$ for all $z \in \mathbb{R}^2$. As $\liminf_{s\to\infty} H_\lambda(z_1, s) = 0$ for any $z_1 \in \mathbb{R}$, it is clear that Θ_λ is the convex envelope. □

The above information allows us to prove the following.

Lemma 30. *Suppose that $\kappa, \iota, \lambda \in [-1, 1]$ are three numbers such that*

$$\iota^2 \leq \min\{1-\lambda^2, (1+\kappa)(1-\lambda), (1-\kappa)(1+\lambda)\}.$$

Then

$$\Theta_\lambda(z) \geq 2\iota z_1 + \kappa z_2$$

for all $z \in \mathbb{R}^2$.

Proof. Suppose first that $-1 < \lambda < 1$. Since Θ_λ is positive 1-homogeneous, the convexity implied by Lemma 29 means that for any $z_0 \in \mathbb{R}^2$, if Θ_λ is differentiable at z_0, then

$$\Theta_\lambda(z) \geq D\Theta_\lambda(z_0)z$$

for every $z \in \mathbb{R}^2$. If $r, s \in \mathbb{R}$ such that $s > 0$ and $(1+\lambda)r^2 < (1-\lambda)s^2$, then we can differentiate Θ_λ at $z_0 = (r, s)$. We conclude that

$$\Theta_\lambda(z) \geq 2(1+\lambda)\frac{r}{s} z_1 + \left(1 - (1+\lambda)\frac{r^2}{s^2}\right) z_2.$$

By continuity, the inequality still holds true when $(1+\lambda)r^2 \leq (1-\lambda)s^2$.

Given a number $\iota \in [-1, 1]$ such that $\lambda^2 + \iota^2 \leq 1$, we can set $s = 1 + \lambda$ and $r = \iota$. Then

$$\left|\frac{r}{s}\right| = \frac{|\iota|}{1+\lambda} \leq \sqrt{\frac{1-\lambda}{1+\lambda}},$$

and the inequality applies. Thus

$$\Theta_\lambda(z) \geq 2\iota z_1 + \left(1 - \frac{\iota^2}{1+\lambda}\right) z_2. \tag{41}$$

Similarly, if $s < 0$ and $(1-\lambda)r^2 \leq (1+\lambda)s^2$, then

$$\Theta_\lambda(z) \geq -2(1-\lambda)\frac{r}{s} z_1 + \left((1-\lambda)\frac{r^2}{s^2} - 1\right) z_2.$$

If $\lambda^2 + \iota^2 \leq 1$, we consider $s = -(1-\lambda)$ and $r = \iota$. Thus we derive the inequality

$$\Theta_\lambda(z) \geq 2\iota z_1 + \left(\frac{\iota^2}{1-\lambda} - 1\right) z_2. \tag{42}$$

Finally, as we always have $\Theta_\lambda(z) \geq 2|z_1|\sqrt{1-\lambda^2} + \lambda z_2$, we find in particular that

$$\Theta_\lambda(z) \geq 2\iota z_1 + \lambda z_2. \tag{43}$$

The right-hand sides of (41)–(43) therefore represent subdifferentials of Θ_λ at 0. Since the space of subdifferentials is necessarily convex, the same applies to any convex combination. That is, whenever $\lambda^2 + \iota^2 \leq 1$ and

$$\frac{\iota^2}{1-\lambda} - 1 \leq \kappa \leq 1 - \frac{\iota^2}{1+\lambda},$$

then

$$\Theta_\lambda(z) \geq 2\iota z_1 + \kappa z_2$$

for all $z \in \mathbb{R}^2$. The above inequalities for κ, ι, and λ are clearly equivalent to the inequality from the statement of the lemma.

We also note that for $\lambda = \pm 1$, the condition of the lemma requires that $\iota = 0$. As $\Theta_\lambda(z) \geq |z_2|$ in any case, we still have the desired estimate. □

5.2 Decomposition into curves

According to a theory by Bonicatto and Gusev [12], any normal ($\mathbb{R}$-valued) 1-current on $\mathbb{R}^2$ has a decomposition into Lipschitz curves. We will apply this result to the first component of an $\mathbb{R}^2$-valued 1-current. To this end, we consider the space Γ, comprising all Lipschitz functions $\gamma\colon [0,1] \to \mathbb{R}^2$, equipped with the uniform norm. Given $\gamma \in \Gamma$, let $[\gamma]$ denote the 1-current induced by γ through the formula

$$[\gamma](\zeta) = \int_0^1 \zeta(\gamma(t))\dot{\gamma}(t)\, dt$$

for $\zeta \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^{1\times 2})$. We also write Γ_0 for the set of all $\gamma \in \Gamma$ with $\gamma(0) = \gamma(1)$, and Γ_1 for the set of all $\gamma \in \Gamma$ with $\gamma(0) = a^-$ and $\gamma(1) = a^+$. Let $\gamma^0 \in \Gamma_1$ denote the curve with $\gamma^0(t) = ta^+ + (1-t)a^-$ for $t \in [0,1]$ (parametrising the line segment between a^- and a^+).

Given a Borel measurable function $\lambda\colon \mathbb{R}^2 \to [-1,1]$ and a function $h \in C^2(\mathbb{R}^2)$, define the functional

$$Z_{\lambda,h}(\gamma) = \int_0^1 \left(\sqrt{W(\gamma(t))}\Theta_{\lambda(\gamma(t))}(\dot{\gamma}(t)) + \frac{\partial^2 h}{\partial y_2^2}(\gamma(t))\dot{\gamma}_2(t)\right) dt - \frac{\partial h}{\partial y_2}(\gamma(1)) + \frac{\partial h}{\partial y_2}(\gamma(0))$$

for $\gamma \in \Gamma$. Recall that $C^j_{\bar{p}}(\mathbb{R}^2)$ denotes the space of all $\phi \in C^2(\mathbb{R}^2)$ such that there exists a constant $C \geq 0$ satisfying $|D^k \phi(y)| \leq C(|y|^{\bar{p}-k} + 1)$ for all $y \in \mathbb{R}^2$ and for $k = 0, \ldots, j$.

Theorem 31. *Suppose that $W \colon \mathbb{R}^2 \to [0, \infty)$ is continuous and satisfies the growth condition* (4)*. Let $\lambda \colon \mathbb{R}^2 \to [-1, 1]$ be a Borel measurable function, and suppose that $h \in C^2_{2+\bar{p}}(\mathbb{R}^2)$ satisfies $\frac{\partial^2 h}{\partial y_1^2} = -\lambda\sqrt{W}$ in $\mathbb{R}^2$. Then for any $T \in \mathcal{C}^0_{2\times 2}(\mathbb{R}^2)$,*

$$\mathbf{M}_F(T) \geq \frac{1}{2} \inf_{\gamma \in \Gamma_1} Z_{\lambda,h}(\gamma).$$

Proof. Let

$$m_0 = \frac{1}{2} \inf_{\gamma \in \Gamma_1} Z_{\lambda,h}(\gamma).$$

If $m_0 < 0$, then there is nothing to prove, as $\mathbf{M}_F(T) \geq 0$ for all $T \in \mathcal{C}^0_{2\times 2}$. We therefore assume that $m_0 \geq 0$.

Let $T \in \mathcal{C}^0_{2\times 2}(\mathbb{R}^2)$ with $\mathbf{M}_F(T) < \infty$. We write

$$\vec{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

and we write T_1, T_2 for the components of T, i.e., for the $\mathbb{R}$-valued 1-currents such that

$$T_i(\zeta) = \int_{\mathbb{R}^2} \zeta \begin{pmatrix} T_{i1} \\ T_{i2} \end{pmatrix} \, d\|T\|$$

for $\zeta \in C^\infty_0(\mathbb{R}^2; \mathbb{R}^{1\times 2})$. We consider the Radon-Nikodym decomposition of the measure $\|T\|$ with respect to $\|T_1\|$. Thus we obtain two measures ν_1 and ν_2 with $\|T\| = \nu_1 + \nu_2$, such that $\nu_1 \ll \|T_1\|$ and $\nu_2 \perp \|T_1\|$. Since $F^*(y, \vec{T}(y)) < \infty$ at $\|T\|$-almost every $y \in \mathbb{R}^2$, it follows that $\vec{T} = \pm\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ at ν_2-almost every point. At ν_1-almost every point, on the other hand, we conclude that $\vec{T} = \left(\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & -T_{11} \end{smallmatrix}\right)$. According to Lemma 28, we now have the inequality

$$\begin{aligned}
\mathbf{M}_F(T) &= \int_{\mathbb{R}^2} \sqrt{W(y) f^*(\vec{T})} \, d\|T\| \\
&\geq \frac{1}{2} \int_{\mathbb{R}^2} \sqrt{W(y)} \big(\Theta_\lambda(T_{11}, T_{12}) + \lambda T_{21}\big) \, d\nu_1 + \frac{1}{2} \int_{\mathbb{R}^2} \sqrt{W(y)} \, d\nu_2 \\
&\geq \frac{1}{2} \int_{\mathbb{R}^2} \sqrt{W(y)} \Theta_\lambda(T_{11}, T_{12}) \, d\nu_1 + \frac{1}{2} \int_{\mathbb{R}^2} \sqrt{W(y)} \lambda T_{21} \, d\|T\| \\
&= \frac{1}{2} \int_{\mathbb{R}^2} \sqrt{W(y)} \Theta_\lambda(T_{11}, T_{12}) \, d\nu_1 - \frac{1}{2} \int_{\mathbb{R}^2} \frac{\partial^2 h}{\partial y_1^2} T_{21} \, d\|T\|.
\end{aligned} \tag{44}$$

We now want to test the condition $\partial T_2 = 0$ with the function $\frac{\partial h}{\partial y_1}$, but since it does not have compact support in general, we require an approximation procedure here. For $R > 0$, let $\chi_R \in C^\infty_0(B_{3R})$ with $\chi_R \equiv 1$ in $B_R(0)$ and $0 \leq \chi_R \leq 1$ everywhere, and such that $|D\chi_R| \leq 1/R$. Then

$$\begin{aligned}
0 &= \partial T_2 \left(\chi_R \frac{\partial h}{\partial y_1} \right) \\
&= \int_{\mathbb{R}^2} \chi_R \left(\frac{\partial^2 h}{\partial y_1^2} T_{21} + \frac{\partial^2 h}{\partial y_1 \partial y_2} T_{22} \right) \, d\|T\| \\
&\quad + \int_{\mathbb{R}^2} \frac{\partial h}{\partial y_1} \left(\frac{\partial \chi_R}{\partial y_1} T_{21} + \frac{\partial \chi_R}{\partial y_2} T_{22} \right) \, d\|T\|.
\end{aligned}$$

Because of the assumptions on h, there is a constant $C_1 > 0$ such that $|D^2 h| \leq C_1(\sqrt{W} + 1)$ and $|Dh||D\chi_R| \leq C_1(\sqrt{W} + 1)$ in $\mathbb{R}^2$ for all $R > 0$. Since $\|T\|$ is a Radon measure and $\mathbf{M}_F(T) < \infty$, we know that $\sqrt{W} + 1$ is integrable with respect to $\|T\|$. Hence we can use Lebesgue's dominated convergence theorem when we take the limit $R \to \infty$. It follows that

$$0 = \int_{\mathbb{R}^2} \left(\frac{\partial^2 h}{\partial y_1^2} T_{21} + \frac{\partial^2 h}{\partial y_1 \partial y_2} T_{22} \right) d\|T\|.$$

Similarly, since $\partial T_1 = \partial T_1^0$, we can show that

$$\int_{\mathbb{R}^2} \left(\frac{\partial^2 h}{\partial y_1 \partial y_2} T_{11} + \frac{\partial^2 h}{\partial y_2^2} T_{12} \right) d\|T\| = \frac{\partial h}{\partial y_2}(a^+) - \frac{\partial h}{\partial y_2}(a^-).$$

Using also the fact that $T_{11} + T_{22} = 0$ almost everywhere, and combining these formulas with (44), we obtain

$$\mathbf{M}_F(T) \geq \frac{1}{2} \int_{\mathbb{R}^2} \left(\sqrt{W(y)} \Theta_\lambda(T_{11}, T_{12}) + \frac{\partial^2 h}{\partial y_2^2} T_{12} \right) d\|T\| - \frac{\partial h}{\partial y_2}(a^+) + \frac{\partial h}{\partial y_2}(a^-). \tag{45}$$

The results of Bonicatto and Gusev [12] give rise to a Borel measure μ on Γ such that

$$T_1 = \int_\Gamma [\gamma] \, d\mu(\gamma), \tag{46}$$

in the sense that

$$T_1(\zeta) = \int_\Gamma [\gamma](\zeta) \, d\mu(\gamma)$$

for any $\zeta \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^{1\times 2})$. Moreover, this measure also satisfies

$$\|T_1\| = \int_\Gamma \|[\gamma]\| \, d\mu(\gamma) \tag{47}$$

and

$$\|\partial T_1\| = \int_\Gamma \|\partial[\gamma]\| \, d\mu(\gamma) \tag{48}$$

(which is to be interpreted similarly).

For any $\gamma \in \Gamma$, we clearly have $\partial[\gamma] = 0$ if $\gamma \in \Gamma_0$ and $\|\partial[\gamma]\|(\mathbb{R}^2) = 2$ otherwise. Thus (48) implies that $\mu(\Gamma_1) = 1$ and $\mu(\Gamma \setminus (\Gamma_0 \cup \Gamma_1)) = 0$, while (45)–(47) imply that

$$\mathbf{M}_F(T) \geq \frac{1}{2} \int_\Gamma Z_{\lambda,h}(\gamma) \, d\mu(\gamma).$$

Since $Z_{\lambda,h}(\gamma) \geq 2m_0$ for all $\gamma \in \Gamma_1$, we automatically have the inequality $Z_{\lambda,h}(\gamma) \geq 0$ for all $\gamma \in \Gamma_0$. (If we had $\hat{\gamma} \in \Gamma_0$ with $Z_{\lambda,h}(\hat{\gamma}) < 0$, then we could construct a sequence of curves $\gamma_k \in \Gamma_1$ with $\lim_{k\to\infty} Z_{\lambda,h}(\gamma_k) = -\infty$ as follows: choose $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \Gamma$ with $\tilde{\gamma}_1(0) = a^-$, $\tilde{\gamma}_2(1) = a^+$, and $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(0) = \hat{\gamma}(0)$. Concatenate $\tilde{\gamma}_1$ with k copies of $\hat{\gamma}$ and then $\tilde{\gamma}_2$, and reparametrise appropriately. Note that $Z_{\lambda,h}$ is invariant under reparametrisation.)

It therefore follows that

$$\mathbf{M}_F(T) \geq \frac{1}{2} \int_{\Gamma_1} Z_{\lambda,h}(\gamma) \, d\mu(\gamma) \geq m_0.$$

This concludes the proof. □

5.3 Proof of Corollary 5

Given suitable functions λ and h, the minimisers of the functional $Z_{\lambda,h}$ can in principle be determined with the conventional tools from the calculus of variations. There are some difficulties coming from the fact that Θ_λ has linear growth, but nevertheless, an analysis of certain ordinary differential equations can then potentially reveal some information about the central question of this paper. In practice, however, it is difficult to make any specific statements this way. The proof of Corollary 5, however, also relies on Theorem 31.

Proof of Corollary 5. We first observe that we may assume without loss of generality that $w \geq 0$ in $\mathbb{R}^2$. If this condition does not hold true, then we can replace w by $|w|$ and replace ι, κ, and λ by $w\iota/|w|$, $w\kappa/|w|$, and $w\lambda/|w|$, respectively. This will change neither the inequalities in the statement nor equation (5).

Let $b \in \mathbb{R}$. Define

$$h(y) = -\int_{a_1^-}^{y_1} (y_1 - s)(\lambda w)(s, y_2)\, ds + (y_1 - a_1^-) \int_b^{y_2} \left(2(\iota w)(a_1^-, t) - (y_2 - t) \frac{\partial}{\partial y_1}(\kappa w)(a_1^-, t) \right) dt.$$

Then

$$\frac{\partial^2 h}{\partial y_1^2} = -\lambda w$$

and

$$\frac{\partial^2 h}{\partial y_1 \partial y_2}(y) = -\int_{a_1^-}^{y_1} \frac{\partial}{\partial y_2}(\lambda w)(s, y_2)\, ds + 2(\iota w)(a_1^-, y_2) - \int_b^{y_2} \frac{\partial}{\partial y_1}(\kappa w)(a_1^-, t)\, dt.$$

Moreover,

$$\begin{aligned}
\frac{\partial^2 h}{\partial y_2^2}(y) &= -\int_{a_1^-}^{y_1} (y_1 - s) \frac{\partial^2}{\partial y_2^2}(\lambda w)(s, y_2)\, ds \\
&\quad + (y_1 - a_1^-) \left(2\frac{\partial}{\partial y_2}(\iota w)(a_1^-, y_2) - \frac{\partial}{\partial y_1}(\kappa w)(a_1^-, y_2) \right) \\
&= \int_{a_1^-}^{y_1} (y_1 - s) \left(2\frac{\partial^2}{\partial y_1 \partial y_2}(\iota w)(s, y_2) - \frac{\partial^2}{\partial y_1^2}(\kappa w)(s, y_2) \right) ds \\
&\quad + (y_1 - a_1^-) \left(2\frac{\partial}{\partial y_2}(\iota w)(a_1^-, y_2) - \frac{\partial}{\partial y_1}(\kappa w)(a_1^-, y_2) \right) \\
&= \int_{a_1^-}^{y_1} \left(2\frac{\partial}{\partial y_2}(\iota w)(s, y_2) - \frac{\partial}{\partial y_1}(\kappa w)(s, y_2) \right) ds \\
&= 2\int_{a_1^-}^{y_1} \frac{\partial}{\partial y_2}(\iota w)(s, y_2)\, ds - (\kappa w)(y) + (\kappa w)(a_1^-, y_2)
\end{aligned}$$

by (5) and an integration by parts. From this, we see that $h \in C^2_{2+\bar{p}}(\mathbb{R}^2)$.

Also consider the function

$$\phi(y) = \frac{\partial h}{\partial y_2}(y) + \int_b^{y_2} (\kappa w)(y_1, t)\, dt - 2y_1(\iota w)(a_1^-, b) + \int_{a_1^-}^{y_1} \left((y_1 - s)\frac{\partial}{\partial y_2}(\lambda w)(s, b) + 2(\iota w)(s, b) \right) ds.$$

Then

$$\begin{aligned}\frac{\partial \phi}{\partial y_1}(y) &= \frac{\partial^2 h}{\partial y_1 \partial y_2}(y) + \int_b^{y_2} \frac{\partial}{\partial y_1}(\kappa w)(y_1, t)\, dt - 2(\iota w)(a_1^-, b) \\ &\quad + \int_{a_1^-}^{y_1} \frac{\partial}{\partial y_2}(\lambda w)(s, b)\, ds + 2(\iota w)(y_1, b) \\ &= -\int_{a_1^-}^{y_1} \frac{\partial}{\partial y_2}(\lambda w)(s, y_2)\, ds - \int_b^{y_2} \frac{\partial}{\partial y_1}(\kappa w)(a_1^-, t)\, dt \\ &\quad + \int_b^{y_2} \frac{\partial}{\partial y_1}(\kappa w)(y_1, t)\, dt + \int_{a_1^-}^{y_1} \frac{\partial}{\partial y_2}(\lambda w)(s, b)\, ds \\ &\quad + 2(\iota w)(a_1^-, y_2) + 2(\iota w)(y_1, b) - 2(\iota w)(a_1^-, b).\end{aligned} \tag{49}$$

Because of equation (5), we compute

$$\begin{aligned}0 &= \int_{a_1^-}^{y_1} \int_b^{y_2} \left(\frac{\partial^2}{\partial y_1^2}(\kappa w)(s, t) - \frac{\partial^2}{\partial y_2^2}(\lambda w)(s, t) - 2\frac{\partial^2}{\partial y_1 \partial y_2}(\iota w)(s, t) \right) dt\, ds \\ &= \int_b^{y_2} \left(\frac{\partial}{\partial y_1}(\kappa w)(y_1, t) - \frac{\partial}{\partial y_1}(\kappa w)(a_1^-, t) \right) dt \\ &\quad - \int_{a_1^-}^{y_1} \left(\frac{\partial}{\partial y_2}(\lambda w)(s, y_2) - \frac{\partial}{\partial y_2}(\lambda w)(s, b) \right) ds \\ &\quad - 2\int_{a_1^-}^{y_1} \left(\frac{\partial}{\partial y_1}(\iota w)(s, y_2) - \frac{\partial}{\partial y_1}(\iota w)(s, b) \right) ds \\ &= \int_b^{y_2} \frac{\partial}{\partial y_1}(\kappa w)(y_1, t)\, dt - \int_b^{y_2} \frac{\partial}{\partial y_1}(\kappa w)(a_1^-, t)\, dt \\ &\quad - \int_{a_1^-}^{y_1} \frac{\partial}{\partial y_2}(\lambda w)(s, y_2)\, ds + \int_{a_1^-}^{y_1} \frac{\partial}{\partial y_2}(\lambda w)(s, b)\, ds \\ &\quad - 2(\iota w)(y) + 2(\iota w)(a_1^-, y_2) + 2(\iota w)(y_1, b) - 2(\iota w)(a_1^-, b).\end{aligned}$$

Comparing with (49), we conclude that

$$\frac{\partial \phi}{\partial y_1} = 2\iota w.$$

Furthermore, we compute

$$\frac{\partial \phi}{\partial y_2} = \kappa w + \frac{\partial^2 h}{\partial y_2^2}.$$

Let $\gamma \in \Gamma_1$. By Lemma 30, we can now estimate

$$\begin{aligned} Z_{\lambda,h}(\gamma) &= \int_0^1 \left(w(\gamma)\Theta_{\lambda(\gamma)}(\dot{\gamma}) + \frac{\partial^2 h}{\partial y_2^2}(\gamma)\dot{\gamma}_2 \right) dt - \frac{\partial h}{\partial y_2}(a^+) + \frac{\partial h}{\partial y_2}(a^-) \\ &\geq \int_0^1 D\phi(\gamma)\dot{\gamma}\, dt - \frac{\partial h}{\partial y_2}(a^+) + \frac{\partial h}{\partial y_2}(a^-) \\ &= \phi(a^+) - \phi(a^-) - \frac{\partial h}{\partial y_2}(a^+) + \frac{\partial h}{\partial y_2}(a^-) \\ &= \int_{a_2^-}^{a_2^+} (\kappa w)(a_1^-, t)\, dt. \end{aligned}$$

The claim then follows from Theorem 31. □

6 Examples

6.1 Variants of the Aviles-Giga functional

A singular perturbation problem involving the quantity

$$\frac{1}{2}\int_\Omega \left(\epsilon |D^2\phi|^2 + \frac{1}{\epsilon}(1 - |D\phi|^2)^2 \right) dx,$$

was studied by Aviles and Giga [7], and subsequently by many other authors, including, e.g., Ambrosio, De Lellis, and Mantegazza [2] and DeSimone, Kohn, Müller, and Otto [15]. A key contribution by Jin and Kohn [28] determined the energy required for a jump of $D\phi$, with tools similar to what we use in this paper.

If we define $u = \nabla^\perp \phi$, then we have the functional $E_\epsilon(u; \Omega)$ from the introduction with the constraint $\operatorname{div} u = 0$. Our theory therefore applies in principle (but will of course not give anything new, as the problem is well understood, at least in relation to the question that we study here).

Indeed, the function $w(y) = 1 - |y|^2$ (corresponding to $W(y) = (w(y))^2 = (1 - |y|^2)^2$) is a solution of the wave equation

$$\frac{\partial^2 w}{\partial y_1^2} - \frac{\partial^2 w}{\partial y_2^2} = 0$$

(a fact that was also observed by Ignat and Monteil [23]), thus it satisfies the hypothesis of Corollary 5 with $\kappa = \lambda = 1$ and $\iota = 0$. For $a^-, a^+ \in S^1 = \left\{ y \in \mathbb{R}^2 : |y| = 1 \right\}$, we therefore obtain

$$\mathcal{E}(a^-, a^+) = \int_{[a^-, a^+]} w\, d\mathcal{H}^1 = \frac{1}{6}|a^+ - a^-|^3.$$

We now consider potentials that are different, but similar in structure, including

$$w(y) = |y|^{2n}(1 - |y|^2) \quad \text{and} \quad w(y) = 1 - |y|^{2n}$$

for some $n \in \mathbb{N}$, and

$$w(y) = (1 - |y|^2)^\beta \tag{50}$$

for a number $\beta \in (0,1)$. We first note that for the last of these, when $\beta > 1$, the optimal transitions between two points $a^-, a^+ \in S^1$ are *not* expected to be one-dimensional by the results of Ambrosio, De Lellis, and Mantegazza [2] (see also the discussion by Ignat and Merlet [22]).

We restrict our attention to transitions between the points $a^- = (0,-1)$ and $a^+ = (0,1)$ here, because we make use of the resulting symmetry. It is an open problem whether the corresponding statements hold true in general, but for (50), the work of Ignat and Merlet [22] at least gives some results supporting the conjecture that the optimal transition profile will be one-dimensional when $\beta \in (0,1)$ and $|a^+ - a^-|$ is small.

We wish to make use of Corollary 5, but it suffices to consider the case $\iota = 0$. Thus we study the question whether there exist two functions $\kappa, \lambda \colon \mathbb{R}^2 \to [-1,1]$ such that

$$\frac{\partial^2}{\partial y_1^2}(\kappa w) = \frac{\partial^2}{\partial y_2^2}(\lambda w)$$

in $\mathbb{R}^2$ and $\kappa = 1$ on $[a^-, a^+]$. Note that these conditions are satisfied if there exists $\phi \in C^4(\mathbb{R}^2)$ such that

$$\left|\frac{\partial^2 \phi}{\partial y_1^2}\right| \leq w \quad \text{and} \quad \left|\frac{\partial^2 \phi}{\partial y_2^2}\right| \leq w \tag{51}$$

and

$$\frac{\partial^2 \phi}{\partial y_2^2} = w \quad \text{on } [a^-, a^+]. \tag{52}$$

Indeed, in this case, we can set

$$\kappa = w^{-1}\frac{\partial^2 \phi}{\partial y_2^2} \quad \text{and} \quad \lambda = w^{-1}\frac{\partial^2 \phi}{\partial y_1^2}.$$

We therefore consider the set

$$\mathcal{W}(a^-, a^+) = \left\{ w \in C^0(\mathbb{R}^2) \colon \text{there exists } \phi \in C^4(\mathbb{R}^2) \text{ satisfying (51) and (52)} \right\}.$$

It is easy to see that $\mathcal{W}(a^-, a^+)$ has the following properties.

(i) If $w \in \mathcal{W}(a^-, a^+)$ and $t \geq 0$, them $tw \in \mathcal{W}(a^-, a^+)$.

(ii) If $w_1, w_2 \in \mathcal{W}(a^-, a^+)$, then $w_1 + w_2 \in \mathcal{W}(a^-, a^+)$.

Thus $\mathcal{W}(a^-, a^+)$ is a convex cone.

Proposition 32. *For any $n \in \mathbb{N}_0$, there exists a polynomial $P \colon \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\left|\frac{\partial^2 P}{\partial y_1^2}(y)\right| \leq |y|^{2n}\big|1 - |y|^2\big| \quad \text{and} \quad \left|\frac{\partial^2 P}{\partial y_2^2}(y)\right| \leq |y|^{2n}\big|1 - |y|^2\big| \tag{53}$$

for every $y \in \mathbb{R}^2$ and and

$$\frac{\partial^2 P}{\partial y_2^2}(0, y_2) = y_2^{2n}(1 - y_2^2) \tag{54}$$

for every $y_2 \in \mathbb{R}$.

Proof. For $n = 0$, a suitable polynomial is

$$P(y) = \frac{|y|^2}{2} - \frac{|y|^4}{12} - \frac{y_1^2 y_2^2}{3}.$$

We now assume that $n \geq 1$. We look for a polynomial such that

$$\frac{\partial^2 P}{\partial y_2^2}(y) = (1 - |y|^2)\left(y_2^{2n} + \sum_{k=0}^{n-1} c_k y_1^{2n-2k} y_2^{2k}\right)$$

for certain coefficients $c_0, \ldots, c_{n-1}$. Setting $c_n = 1$, we can write

$$\begin{aligned}\frac{\partial^2 P}{\partial y_2^2}(y) &= (1 - |y|^2) \sum_{k=0}^{n} c_k y_1^{2n-2k} y_2^{2k} \\ &= (1 - y_1^2) \sum_{k=0}^{n} c_k y_1^{2n-2k} y_2^{2k} - \sum_{k=0}^{n} c_k y_1^{2n-2k} y_2^{2k+2}.\end{aligned}$$

A possible solution is

$$\begin{aligned}P(y) = (1 - y_1^2) &\sum_{k=0}^{n} \frac{c_k}{(2k+2)(2k+1)} y_1^{2n-2k} y_2^{2k+2} \\ &- \sum_{k=0}^{n} \frac{c_k}{(2k+4)(2k+3)} y_1^{2n-2k} y_2^{2k+4} \\ &+ \frac{y_1^{2n+2}}{(2n+2)(2n+1)} - \frac{y_1^{2n+4}}{(2n+4)(2n+3)} \\ = &\sum_{k=0}^{n-1} \frac{c_k}{(2k+2)(2k+1)} y_1^{2n-2k} y_2^{2k+2} + \frac{y_1^{2n+2} + y_2^{2n+2}}{(2n+2)(2n+1)} \\ &- \sum_{k=1}^{n-1} \frac{c_k + c_{k-1}}{(2k+2)(2k+1)} y_1^{2n-2k+2} y_2^{2k+2} - \frac{c_0}{2} y_1^{2n+2} y_2^2 \\ &- \frac{1 + c_{n-1}}{(2n+2)(2n+1)} y_1^2 y_2^{2n+2} - \frac{y_1^{2n+4} + y_2^{2n+4}}{(2n+4)(2n+3)}.\end{aligned}$$

We want to impose the symmetry condition $P(y_1, y_2) = P(y_2, y_1)$ (so that the above condition on $\frac{\partial^2 P}{\partial y_2^2}$ automatically gives a similar condition for $\frac{\partial^2 P}{\partial y_1^2}$). This requires that

$$\frac{c_0}{2} = \frac{1 + c_{n-1}}{(2n+2)(2n+1)} \tag{55}$$

and

$$\frac{c_k}{(2k+2)(2k+1)} = \frac{c_{n-k-1}}{(2n-2k)(2n-2k-1)}, \quad k = 0, \ldots, n-1, \tag{56}$$

and

$$\frac{c_k + c_{k-1}}{(2k+2)(2k+1)} = \frac{c_{n-k} + c_{n-k-1}}{(2n-2k+2)(2n-2k+1)}, \quad k = 1, \ldots, n-1. \tag{57}$$

(It may appear at first that there are too many equations for the n variables $c_0, \ldots, c_{n-1}$, but there is some repetition here. Once the redundant equations are discarded, it is easy to see that there is a unique solution.)

The combination of (56) and (57) gives

$$\begin{aligned} c_k + c_{k-1} &= \frac{(2k+2)(2k+1)}{(2n-2k+2)(2n-2k+1)}(c_{n-k} + c_{n-k-1}) \\ &= \frac{(2n-2k)(2n-2k-1)}{(2n-2k+2)(2n-2k+1)}c_k + \frac{(2k+2)(2k+1)}{2k(2k-1)}c_{k-1} \end{aligned}$$

for $k = 1, \ldots, n-1$. Thus

$$\left(1 - \frac{(2n-2k)(2n-2k-1)}{(2n-2k+2)(2n-2k+1)}\right) c_k = \left(\frac{(2k+2)(2k+1)}{2k(2k-1)} - 1\right) c_{k-1}.$$

We compute

$$1 - \frac{(2n-2k)(2n-2k-1)}{(2n-2k+2)(2n-2k+1)} = \frac{8(n-k)+2}{(2n-2k+2)(2n-2k+1)}$$

and

$$\frac{(2k+2)(2k+1)}{2k(2k-1)} - 1 = \frac{8k+2}{2k(2k-1)}.$$

Therefore, we obtain the equation

$$\frac{c_k}{c_{k-1}} = \frac{(2n-2k+2)(2n-2k+1)(4k+1)}{2k(2k-1)(4(n-k)+1)}$$

for $k = 1, \ldots, n-1$.

Define

$$b_k = \frac{c_k}{(2k+1)\binom{n}{k}}, \quad k = 0, \ldots, n-1.$$

Then

$$\begin{aligned} \frac{b_k}{b_{k-1}} &= \frac{k(2k-1)}{(n-k+1)(2k+1)} \frac{c_k}{c_{k-1}} \\ &= \frac{(2n-2k+1)(4k+1)}{(2k+1)(4(n-k)+1)} \\ &= \frac{8nk - 8k^2 + 2n + 2k + 1}{8nk - 8k^2 + 4n - 2k + 1}. \end{aligned}$$

We note that $b_k/b_{k-1} \geq 1$ if, and only if, $k \geq n/2$; and $b_k/b_{k-1} \leq 1$ if, and only if, $k \leq n/2$. Hence b_k, as a function of k, is first decreasing, may possibly be constant for one step, and is then increasing. (If $n = 1$, then this is a vacuous statement, as we have only b_0 in this case.)

From (55) and (56) for $k = 0$, we also conclude that

$$\frac{1 + c_{n-1}}{(2n+2)(2n+1)} = \frac{c_{n-1}}{2n(2n-1)}.$$

Solving this equation, we obtain

$$c_{n-1} = \frac{2n^2 - n}{4n+1}.$$

It then also follows from (56) that

$$c_0 = \frac{c_{n-1}}{2n^2 - n} = \frac{1}{4n+1}.$$

This means that

$$b_0 = c_0 = \frac{1}{4n+1} \quad \text{and} \quad b_{n-1} = \frac{c_{n-1}}{2n^2 - n} = \frac{1}{4n+1}.$$

We conclude that $b_k \leq \frac{1}{4n+1}$ for all $k = 0, \dots, n-1$, i.e.,

$$c_k \leq \frac{2k+1}{4n+1} \binom{n}{k}.$$

Because $c_k / c_{k-1} > 0$ for every $k = 1, \dots, n-1$, it is clear that $c_k > 0$ for every $k = 1, \dots, n$. Therefore,

$$0 \leq \sum_{k=0}^{n} c_k y_1^{2n-2k} y_2^{2k} \leq \sum_{k=0}^{n} \binom{n}{k} y_1^{2n-2k} y_2^{2k} = |y|^{2n}.$$

The inequalities in (53) follow. We also have identity (54) by construction. □

Recall that we consider the points $a^- = (0, -1)$ and $a^+ = (0, 1)$ here. It follows from Proposition 32 that $|y|^{2n} \big| 1 - |y|^2 \big| \in \mathcal{W}(a^-, a^+)$ for every $n \in \mathbb{N}$. Since

$$|y|^{2n} \big| 1 - |y|^{2m} \big| = (|y|^{2n} + \cdots + |y|^{2n+2m-2}) \big| 1 - |y|^2 \big|,$$

these potentials belong to $\mathcal{W}(a^-, a^+)$, too. Now for $\beta \in (0, 1)$, we consider

$$w(y) = \big| 1 - |y|^2 \big|^{\beta}.$$

We define the function

$$\psi(t) = (1-t)^{\beta - 1}, \quad -1 < t < 1.$$

We compute

$$\psi^{(n)}(t) = (1-\beta) \cdots (n-\beta)(1-t)^{\beta - n - 1}.$$

The function is analytic in $(-1, 1)$ and we have the Taylor expansion

$$\psi(t) = \sum_{n=0}^{\infty} a_n t^n,$$

where $a_n = \frac{\psi^{(n)}(0)}{n!} > 0$ for every $n \in \mathbb{N}_0$. We therefore have the formula

$$\big| 1 - |y|^2 \big|^{\beta} = (1 - |y|^2) \psi(|y|^2) = (1 - |y|^2) \sum_{n=0}^{\infty} a_n |y|^{2n}$$

in $B_1(0)$.

It is not clear if the space $\mathcal{W}(a^-, a^+)$ is closed in a suitable topology, but examining the coefficients of the polynomials from Proposition 32, it is not difficult to see that w satisfies a condition like (51) and (52) in the unit ball.

But it is still not clear how to extend this observation to $\mathbb{R}^2$. Therefore, rather than using the series, we use an approximation by

$$w_n(y) = \left|1 - |y|^2\right| \sum_{k=0}^{n} a_k |y|^{2k}.$$

The limit, as $n \to \infty$, is

$$w_\infty(y) = \begin{cases} (1 - |y|^2)^\beta & \text{if } |y| \leq 1, \\ \infty & \text{if } |y| > 1. \end{cases}$$

This function does of course not fit into the above theory, as it is not continuous. Nevertheless, we can prove the following.

Corollary 33. *Let $a^- = (0, -1)$ and $a^+ = (0, 1)$.*

(i) If $W(y) = |y|^{4n}(1 - |y|^{2m})^2$ for some $n, m \in \mathbb{N}$, then

$$\mathcal{E}(a^-, a^+) = \frac{4m}{(2n+1)(2n+2m+1)}.$$

(ii) Suppose that $W(y) = (1 - |y|^2)^{2\beta}$ for some $\beta \in (0, 1)$. If $(u_\epsilon)_{\epsilon>0}$ is a family of vector fields from $\mathcal{U}(a^-, a^+)$ such that $|u_\epsilon| \leq 1$ for every $\epsilon > 0$, then

$$\liminf_{\epsilon \searrow 0} E_\epsilon(u_\epsilon; B_1(0)) \geq 2 \int_0^1 (1 - t^2)^\beta \, dt.$$

Proof. The first statement follows immediately from Proposition 32 and Corollary 5 by the above observations.

To prove the second statement, we consider the potentials

$$w_n(y) = \left|1 - |y|^2\right| \sum_{k=0}^{n} a_k |y|^{2k},$$

as explained above. We know that $w_n \in \mathcal{W}(a^-, a^+)$. Hence by Corollary 5,

$$\frac{1}{4} \liminf_{\epsilon \searrow 0} \int_{B_1(0)} \left(\epsilon |Du_\epsilon|^2 + \frac{1}{\epsilon} (w_n(u_\epsilon))^2 \right) dx \geq \int_{[a^-, a^+]} w_n \, d\mathcal{H}^1.$$

Since $|u_\epsilon| \leq 1$ for every $\epsilon > 0$, it follows that

$$\frac{1}{4} \liminf_{\epsilon \searrow 0} \int_{B_1(0)} \left(\epsilon |Du_\epsilon|^2 + \frac{1}{\epsilon} \left(1 - |u_\epsilon|^2\right)^{2\beta} \right) dx \geq \int_{[a^-, a^+]} w_n \, d\mathcal{H}^1$$

as well. Letting $n \to \infty$, we conclude that

$$\frac{1}{4} \liminf_{\epsilon \searrow 0} \int_{B_1(0)} \left(\epsilon |Du_\epsilon|^2 + \frac{1}{\epsilon} \left(1 - |u_\epsilon|^2\right)^{2\beta} \right) dx \geq \int_{[a^-, a^+]} w \, d\mathcal{H}^1.$$

This is the inequality from the statement in a different form. □

6.2 Other candidates for minimisers

We cannot expect that T^0 will always be a minimiser of $\mathbf{M}_F$. In this section, we therefore have a look at some other elements of $\mathcal{C}^0_{2\times 2}$ that may be minimisers for certain potential functions W. We do not have any specific results here, but we do have some examples indicating that there may be a deeper relationship between the elements of $\mathcal{C}^0_{2\times 2}$ and possible transition profiles.

In Section 5, we decomposed the first component of an $\mathbb{R}^2$-valued current into curves from a^- to a^+ (and possibly some closed curves). Conversely, given such a curve, we may wish to consider the corresponding $\mathbb{R}$-valued 1-current and complement it with a second component to obtain an element of $T^0_{2\times 2}$. Since we require that $\mathbf{M}_F(T) < \infty$, however, we will need to make sure that $\operatorname{tr} \vec{T} = 0$ away from $W^{-1}(\{0\})$. This condition, on the other hand, gives rise to significant restrictions on what is possible. For most curves from a^- to a^+, there is no second component with the required properties. But we can instead consider a *pair* of curves that are symmetric with respect to reflection on $[a^-, a^+]$.

We still assume that $a = (0, -1)$ and $a^+ = (0, 1)$. Suppose now that $\gamma \colon [0, 1] \to \mathbb{R}^2$ is Lipschitz continuous with $\gamma(0) = a^-$ and $\gamma(1) = a^+$, such that $\dot\gamma_2(t) \neq 0$ at almost every $t \in [0, 1]$ and such that the function $\psi = \dot\gamma_1/\dot\gamma_2$ is of bounded variation in $[0, 1]$. Its derivative, denoted by $\dot\psi$, is therefore given by a measure on $[0, 1]$. We write $|\dot\psi|$ for its total variation measure.

Define $T \in \mathcal{C}_{2\times 2}$ by

$$\begin{aligned} T(\zeta) &= \frac{1}{2} \int_0^1 \frac{1}{\dot\gamma_2(t)} \begin{pmatrix} \dot\gamma_1(t)\dot\gamma_2(t) & (\dot\gamma_2(t))^2 \\ -(\dot\gamma_1(t))^2 & -\dot\gamma_1(t)\dot\gamma_2(t) \end{pmatrix} : \zeta(\gamma(t))\, dt \\ &\quad + \frac{1}{2} \int_0^1 \frac{1}{\dot\gamma_2(t)} \begin{pmatrix} -\dot\gamma_1(t)\dot\gamma_2(t) & (\dot\gamma_2(t))^2 \\ -(\dot\gamma_1(t))^2 & \dot\gamma_1(t)\dot\gamma_2(t) \end{pmatrix} : \zeta(-\gamma_1(t), \gamma_2(t))\, dt \\ &\quad - \frac{1}{2} \int_0^1 \int_{-\gamma_1(t)}^{\gamma_1(t)} \zeta_{21}(s, \gamma_2(t))\, ds\, d\dot\psi(t). \end{aligned}$$

We note that $T_1 = \frac{1}{2}([\gamma] + [\gamma^\dagger])$, where $\gamma^\dagger(t) = (-\gamma_1(t), \gamma_2(t))$. For any $\xi \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$, we compute

$$\begin{aligned} T(D\xi) &= \frac{1}{2} \int_0^1 \big(D\xi_1(\gamma(t))\dot\gamma(t) - \psi(t) D\xi_2(\gamma(t))\dot\gamma(t)\big)\, dt \\ &\quad + \frac{1}{2} \int_0^1 \big(D\xi_1(\gamma(t))\dot\gamma^\dagger(t) + \psi(t) D\xi_2(\gamma(t))\dot\gamma^\dagger(t)\big)\, dt \\ &\quad - \frac{1}{2} \int_0^1 \int_{-\gamma_1(t)}^{\gamma_1(t)} \frac{\partial \xi_2}{\partial y_1}(s, \gamma_2(t))\, ds\, d\dot\psi(t) \\ &= \xi_1(a^+) - \xi_1(a^-) - \frac{1}{2} \int_0^1 \psi(t) \frac{d}{dt}\big(\xi_2(\gamma(t)) - \xi_2(\gamma^\dagger(t))\big)\, dt \\ &\quad - \frac{1}{2} \int_0^1 \big(\xi_2(\gamma(t)) - \xi_2(\gamma^\dagger(t))\big)\, d\dot\psi(t) \\ &= \xi_1(a^+) - \xi_1(a^-). \end{aligned}$$

Hence $T \in \mathcal{C}^0_{2\times 2}$. We further compute

$$\mathbf{M}_F(T) = \frac{1}{4}\int_0^1 \left(\sqrt{W(\gamma(t))} + \sqrt{W(\gamma^\dagger(t))}\right) \frac{|\dot\gamma(t)|^2}{|\dot\gamma_2(t)|}\, dt + \frac{1}{4}\int_0^1 \int_{-\gamma_1(t)}^{\gamma_1(t)} \sqrt{W(s, \gamma_2(t))}\, ds\, d|\dot\psi|(t).$$

We now look at two specific examples of this type.

Example 34. Let $b_1 > 0$, and consider the points $b^+ = (b_1, 0)$ and $b^- = (-b_1, 0)$. Suppose that

$$\gamma(t) = \begin{cases} (2tb_1, 2t-1) & \text{if } 0 \le t \le \frac{1}{2}, \\ ((2-2t)b_1, 2t-1) & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

(a curve consisting of a line segment from a^- to b^+ and a line segment from b^+ to a^+). Then

$$\psi(t) = \begin{cases} b_1 & \text{if } 0 \le t \le \frac{1}{2}, \\ -b_1 & \text{if } \frac{1}{2} < t \le 1, \end{cases}$$

and we compute

$$\mathbf{M}_F(T) = \frac{1}{4}\sqrt{b_1^2 + 1}\int_\Diamond \sqrt{W}\, d\mathcal{H}^1 + \frac{b_1}{2}\int_{[b^-, b^+]} \sqrt{W}\, d\mathcal{H}^1,$$

where we use the abbreviation $\Diamond = [a^-, b^+] \cup [b^+, a^+] \cup [a^-, b^-] \cup [b^-, a^+]$.

Compare this with the construction by Jin and Kohn [28, Section 4] of a non-one-dimensional transition profile between a^- and a^+. This is a two-scale construction, where the coarser scale is given by

$$\tilde u_0(x) = \begin{cases} (0, -1) & \text{if } x_1 \le -b_1|x_2|, \\ (0, 1) & \text{if } x_1 \ge b_1|x_2|, \\ (-b_1, 0) & \text{if } |x_1| < b_1 x_2, \\ (b_1, 0) & \text{if } |x_1| < -b_1 x_2, \end{cases}$$

for $-1 < x_2 \le 1$. This is extended periodically in x_2, with period 2, to the whole of $\mathbb{R}^2$. Thus $\tilde u_0$ is piecewise constant, with a jump set as illustrated in Figure 1.

Next, we construct $\tilde u_\epsilon \colon \mathbb{R}^2 \to \mathbb{R}^2$ as follows: we replace the jumps in $\tilde u_0$ by the standard one-dimensional transitions with a width of order ϵ (as explained, e.g., in a paper by Ignat and Monteil [23, Proposition 4.1]). This requires some smoothing near the corners, which can be done with an insignificant gain of energy and a small change of the divergence. (The details are tedious and are omitted here.) We then compute

$$\lim_{\epsilon \searrow 0} E_\epsilon(\tilde u_\epsilon; \mathbb{R} \times [s-1, s+1]) = \frac{1}{2}\sqrt{b_1^2 + 1}\int_\Diamond \sqrt{W}\, d\mathcal{H}^1 + b_1 \int_{[b^-, b^+]} \sqrt{W}\, d\mathcal{H}^1$$

for any $s \in \mathbb{R} \setminus \mathbb{Z}$.

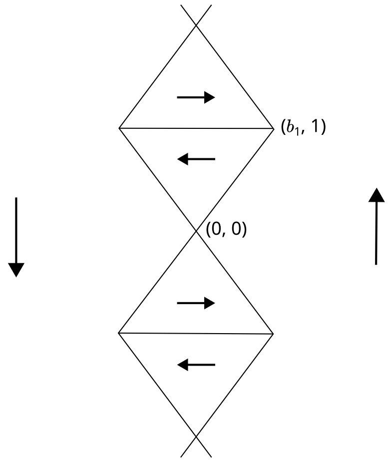

Figure 1: A construction of Jin and Kohn [28]

But we want convergence to u_0, not $\tilde{u}_0$, as $\epsilon \searrow 0$. Therefore, we now rescale by a parameter $\eta_\epsilon > 0$, which converges to 0 when ϵ does, but at a slower rate; i.e., we assume that $\eta_\epsilon \to 0$ and $\epsilon/\eta_\epsilon \to 0$. We set $u_\epsilon(x) = \tilde{u}_{\epsilon/\eta_\epsilon}(x/\eta_\epsilon)$. Then

$$\lim_{\epsilon \searrow 0} E_\epsilon(u_\epsilon; B_1(0)) = \frac{1}{2}\sqrt{b_1^2 + 1} \int_\Diamond \sqrt{W}\, d\mathcal{H}^1 + b_1 \int_{[b^-, b^+]} \sqrt{W}\, d\mathcal{H}^1.$$

Therefore,

$$\mathcal{E}(a^-, a^+) \leq \frac{1}{2}\sqrt{b_1^2 + 1} \int_\Diamond \sqrt{W}\, d\mathcal{H}^1 + b_1 \int_{[b^-, b^+]} \sqrt{W}\, d\mathcal{H}^1.$$

If the current T happens to minimise $\mathbf{M}_T$ in $\mathcal{C}^0_{2\times 2}$, then we have equality by Theorem 3. Therefore, in this case, the above construction gives the optimal energy asymptotically.

Example 35. Now suppose that $W(y) = 0$ for all $y \in S^1$. Choose $b_1, b_2 \in [0, 1]$ with $b_1^2 + b_2^2 = 1$, and define $b^{(1)} = (b_1, -b_2)$, $b^{(2)} = (b_1, b_2)$, $b^{(3)} = (-b_1, b_2)$, and $b^{(4)} = (-b_1, -b_2)$. Let $\theta = \arcsin b_1$ and consider the curve

$$\gamma(t) = \begin{cases} (\sin(\pi t), -\cos(\pi t)) & \text{if } 0 \leq t \leq \frac{\theta}{\pi} \text{ or } 1 - \frac{\theta}{\pi} \leq t \leq 1, \\ (b_1, \frac{(2t-1)\pi}{\pi - 2\theta} b_2) & \text{if } \frac{\theta}{\pi} < t < 1 - \frac{\theta}{\pi} \end{cases}$$

(consisting of a circular arc from a^- to $b^{(1)}$, a line segment from $b^{(1)}$ to $b^{(2)}$, and another circular arc from $b^{(2)}$ to a^+). Then

$$\psi(t) = \begin{cases} \cot(\pi t) & \text{if } 0 \leq t \leq \frac{\theta}{\pi} \text{ or } 1 - \frac{\theta}{\pi} \leq t \leq 1, \\ 0 & \text{if } \frac{\theta}{\pi} < t < 1 - \frac{\theta}{\pi}. \end{cases}$$

This function is not of bounded variation (not even bounded), and so, strictly speaking, the above calculations do not apply. We ignore this problem for the sake of simplicity. We will obtain an $\mathbb{R}^2$-valued current on $\mathbb{R}^2 \setminus \{a^-, a^+\}$ instead of $\mathbb{R}^2$, which can, however, be approximated with elements of $\mathcal{C}^0_{2\times 2}$.

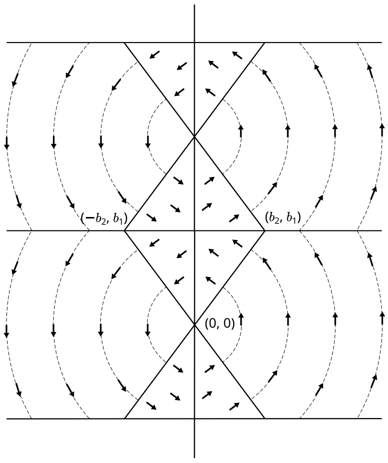

Figure 2: The cross-tie wall profile

Define $V = [b^{(1)}, b^{(2)}] \cup [b^{(4)}, b^{(3)}]$ and $H = [b^{(4)}, b^{(1)}] \cup [b^{(3)}, b^{(2)}]$, and also define $D = B_1(0) \setminus (\mathbb{R} \times [-b_2, b_2])$. Then we compute

$$\begin{aligned}\mathbf{M}_F(T) &= \frac{1}{4}\int_V \sqrt{W}\, d\mathcal{H}^1 + \frac{b_2}{4b_1}\int_H \sqrt{W}\, d\mathcal{H}^1 \\ &\quad + \frac{\pi}{4}\int_{[0,\theta]\cup[1-\theta,1]} \frac{1}{\sin^2(\pi t)} \int_{-\sin(\pi t)}^{\sin(\pi t)} \sqrt{W(s, -\cos(\pi t))}\, ds\, dt \\ &= \frac{1}{4}\int_V \sqrt{W}\, d\mathcal{H}^1 + \frac{b_2}{4b_1}\int_H \sqrt{W}\, d\mathcal{H}^1 + \frac{1}{4}\int_D \frac{\sqrt{W(y)}}{(1-y_2^2)^{3/2}}\, dy.\end{aligned}$$

(This may be infinite, unless we impose additional conditions on W at $a^\pm$.)

Compare this with the following transition profile, which is called a cross-tie wall in the theory of micromagnetics [14, 1]. This is a two-scale construction again, and the coarser scale is given by

$$\tilde{u}_0(x) = \begin{cases} b^{(1)} & \text{if } x_1 < 0,\ x_2 < 0, \text{ and } -b_1 x_1 + b_2 x_2 < 0, \\ b^{(2)} & \text{if } x_1 > 0,\ x_2 < 0, \text{ and } b_1 x_1 + b_2 x_2 < 0, \\ b^{(3)} & \text{if } x_1 > 0,\ x_2 > 0, \text{ and } -b_1 x_1 + b_2 x_2 > 0, \\ b^{(4)} & \text{if } x_1 < 0,\ x_2 > 0, \text{ and } b_1 x_1 + b_2 x_2 > 0, \\ \frac{x^\perp}{|x|} & \text{else,} \end{cases}$$

when $-b_1 < x_2 \leq b_1$. This is extended periodically in x_2, with period $2b_1$, so that $\tilde{u}_0$ is defined on all of $\mathbb{R}^2$. The result is illustrated in Figure 2.

The vector field $\tilde{u}_0$ has discontinuities along the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{(1 + 2k)b_1\}$ for every $k \in \mathbb{Z}$. On the finer scale, these have to be replaced by smooth transitions again. For $\epsilon > 0$, we choose the standard one-dimensional transitions, with a width of order ϵ, along the line segments $\{0\} \times [-b_1, 0]$ (where we have a transition between $b^{(1)}$ and $b^{(2)}$) and $\{0\} \times [0, b_1]$ (with a transition between $b^{(3)}$ and $b^{(4)}$), as well as $[-b_2, 0] \times \{b_1\}$ (with a transition between $b^{(4)}$ and $b^{(1)}$) and $[0, b_2] \times \{b_1\}$ (with a transition between $b^{(3)}$ and $b^{(2)}$). This part of the construction is similar to Example 34.

Along $(-\infty, -b_2] \times \{b_1\}$ and $[b_2, \infty) \times \{b_1\}$, we still use a similar construction, but since the jump in $\tilde{u}_0$ depends on the position, the result will not truly be one-dimensional here. At the point (x_1, b_1) with $|x_1| \geq b_2$, we have a jump between the points

$$\frac{(-b_1, x_1)}{\sqrt{b_1^2 + x_1^2}} \quad \text{and} \quad \frac{(b_1, x_1)}{\sqrt{b_1^2 + x_1^2}}.$$

We replace this with a smooth transition, again with width of order ϵ, along the horizontal line segment between these two points in $\mathbb{R}^2$. Once more this requires some smoothing at the corners, and in the end everything is extended periodically to $\mathbb{R}^2$.

We will have some divergence at the corners, and also near $(-\infty, -b_2] \times \{b_1\}$ and $[b_2, \infty) \times \{b_1\}$, with this construction. We can, however, achieve that a condition similar to (3) holds in any compact subset of $\mathbb{R}^2$. Once more, the details are omitted.

Finally, we rescale at a rate $\eta_\epsilon > 0$ as $\epsilon \searrow 0$, where $\eta_\epsilon \to 0$, but sufficiently slowly that (3) remains true.

Calculating the energy of the cross-tie wall per unit wall length, we find that

$$\mathcal{E}(a^+, a^-) \leq \frac{1}{2} \int_V \sqrt{W} \, d\mathcal{H}^1 + \frac{b_2}{2b_1} \int_H \sqrt{W} \, d\mathcal{H}^1 + \frac{1}{2} \int_D \frac{\sqrt{W(y)}}{(1 - y_2^2)^{3/2}} \, dy.$$

If suitable approximations of T give rise to a minimising sequence of $\mathbf{M}_F$ in $\mathcal{C}^0_{2\times 2}$, then Theorem 3 implies that we have equality, and the cross-tie wall is energetically optimal.

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