A Γ-convergence result for Néel walls in micromagnetics

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Abstract

Three reduced models are considered for Néel walls which are dominant transition layers in thin-film micromagnetics. Each model comes as a nonlocal and nonconvex variational principle for one-dimensional magnetizations and it depends on a small parameter $\varepsilon > 0$. Our aim is to study the $\Gamma-$convergence of these models as $\varepsilon \downarrow 0$. We prove that the limiting magnetization patterns are piecewise constant functions that correspond to a finite number of walls of the same angle. The $\Gamma-$limit energy is proportional to the number of walls of these configurations and the energetic cost of each wall is quartic for small wall angles.

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1 Introduction

The Néel wall is a dominant transition layer in thin ferromagnetic films. It is characterized by a one-dimensional in-plane rotation connecting two (opposite) directions of the magnetization. It has two length scales: a small core with fast varying rotation and two logarithmically decaying tails. In order for the Néel wall to exist, the tails are to be contained. There are three confining mechanisms for the Néel wall tails: the anisotropy of the material, the steric interaction with the sample edges and the steric interaction with the tails of neighboring Néel walls. In the following, we describe these models that correspond to three nonconvex and nonlocal variational problems depending on a small parameter:

Model 1. Confinement of Néel wall tails by anisotropy. The admissible configurations are functions satisfying the following conditions:

$$m = (m_1, m_2) : \mathbb{R} \to S^1 \text{ and } m(\pm\infty) = \begin{pmatrix} \alpha \\ \pm\sqrt{1-\alpha^2} \end{pmatrix}, \tag{1}$$

where $\alpha \in [0, 1)$. Denoting $\theta = \arccos \alpha$, then 2θ is called the wall angle (see Figure 1). The energy is defined as follows:

$$m \mapsto \delta \|m\|^2_{\dot{H}^1} + \|m_1\|^2_{\dot{H}^{1/2}} + \|m_1 - \alpha\|^2_{L^2} \tag{2}$$

with $\delta > 0$ a small parameter.

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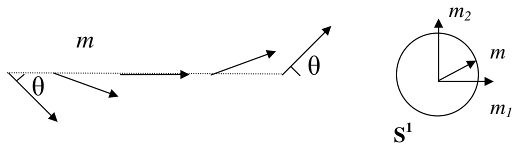

Figure 1: Néel wall of angle 2θ.

It is a model for one-dimensional magnetization in infinite ferromagnetic layers of small thickness (see Melcher [11]). The first term in (2) is called the exchange energy and is due to short-range spin interactions; it favors parallel alignment of neighboring spins. The second term stands for the stray field energy and is due to long-range spin interactions modelled by the static Maxwell equation (see Proposition 4). The last term in (2) comes from the crystalline anisotropy and favors the direction $(\alpha, \pm\sqrt{1-\alpha^2})$ of the magnetization. The energy (2) is invariant under translation. Since configurations m of finite energy are continuous, the limit conditions in (1) enforce a transition (*wall domain*) for the magnetization. One can fix the center of the wall at the origin by setting $m(0) = (1, 0)$. Under these restrictions, a Néel wall corresponds to a minimizer of the energy (2).

The variational problem is nonconvex because of the saturation constraint $|m| = 1$ and nonlocal due to the stray-field interaction. It is a nondegenerate problem since the anisotropy term prevents a Néel wall to spread over the complete domain $\mathbb{R}$; therefore, the Néel wall tails are forced to be limited and the energy cannot reach arbitrary small levels (see Proposition 1). The main feature of the variational problem is that energy (2) only gives uniform bound of m_1 in $\dot{H}^{1/2}(\mathbb{R})$ that barely fails to control the $L^\infty(\mathbb{R})$-norm $\|m_1\|_{L^\infty(\mathbb{R})} = 1$. This suggests a logarithmic decay of the energy. The prediction of the logarithmic scaling for minimal energies (2) was formally proved by Riedel and Seeger [13]; a detailed mathematical discussion of their results was carried out by Garcia-Cervera [4] by means of a perturbation argument. The exact leading order term of the minimal energy was finally deduced by DeSimone, Kohn, Müller and Otto [7, 9] by matching upper and lower bounds in the case of a 180° Néel wall (when $\alpha = 0$):

$$\min_{\substack{(1)\\ \alpha=0}} \delta\|m\|^2_{\dot{H}^1} + \|m_1\|^2_{\dot{H}^{1/2}} + \|m_1\|^2_{L^2} = \frac{\pi + o(1)}{|\log \delta|} \quad \text{as} \quad \delta \downarrow 0. \tag{3}$$

The analysis of the structure of a minimizer of (3) is rather subtle due to the different scaling behavior of the energy terms in (2). Remark that omitting the $\dot{H}^{1/2}-$norm, the formulation of (3) in terms of $v := m_2$ corresponds to a variational problem associated to the Cahn-Hilliard model (see Cahn and Hilliard [3]):

$$\min_{\substack{v:\mathbb{R}\to[-1,1]\\ v(0)=0, v(\pm\infty)=\pm1}} \int_{\mathbb{R}} \left(\frac{\delta}{1-v^2} \Big|\frac{dv}{dt}\Big|^2 + 1 - v^2 \right) dt. \tag{4}$$

The minimizer v of (4) satisfies the Cauchy problem associated to the first order ODE:

$$\frac{dv}{dt} = \frac{1}{\sqrt{\delta}}(1 - v^2), \quad v(0) = 0.$$

Therefore, it is a transition layer with a single length scale $\sqrt{\delta}$, i.e., $v(t) = \tanh(x/\sqrt{\delta})$ and satisfies $v(\pm\infty) = \pm 1$. The first component of the magnetization m_1 would correspond in (4) to $\operatorname{sech}(x/\sqrt{\delta})$ and the minimal energy is equal to $4\sqrt{\delta}$.

Coming back to our variational problem (3), the presence of the nonlocal term as an homogeneous $\dot{H}^{1/2}(\mathbb{R})-$seminorm in competition with the energy (4) creates a second length scale of the transition layer. The Néel wall is divided in two regions: a core ($|t| \lesssim w_{core}$) and two tails ($w_{core} \lesssim |t| \lesssim w_{tail}$). This particular structure enables the magnetization to decrease the energy by a logarithmic factor (3). Melcher [11, 12] rigorously established the optimal profile of the Néel wall, i.e., a minimizer m of (3) with $m_1(0) = 1$ exhibits two uniformly logarithmic tails beyond a core region of order δ close to the origin (see Figures 2 and 3):

$$m_1(t) \sim \frac{|\log|t||}{|\log\delta|} \quad \text{for} \quad \delta < |t| < 1/e.$$

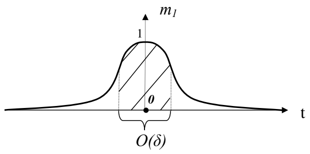

Figure 2: First component of a 180° Néel wall.

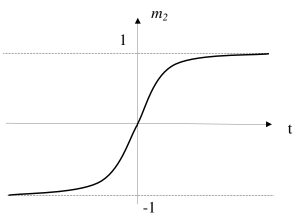

Figure 3: Second component of a 180° Néel wall.

We are interested in the asymptotics of the energy (2) as $\delta \downarrow 0$. Due to the logarithmic decay (3), we consider a new length scale $\varepsilon > 0$ such that $\delta = \varepsilon/|\log\varepsilon|$ and we renormalize the energy (2) by a factor $|\log\varepsilon|$ in order that the minimal energy become of order $O(1)$:

$$E_\varepsilon(m) = \varepsilon\|m\|^2_{\dot{H}^1} + |\log\varepsilon| \left(\|m_1\|^2_{\dot{H}^{1/2}} + \|m_1 - \alpha\|^2_{L^2}\right). \tag{5}$$

Our goal is to study the $\Gamma-$convergence of energies $\{E_\varepsilon\}$ as $\varepsilon \downarrow 0$ and to characterize the limiting configurations of the magnetization. We will prove that the limiting configurations are piecewise constant functions of bounded total variation that can take two values $\{(\alpha, \pm\sqrt{1-\alpha^2})\}$. The $\Gamma-$limit energy is proportional to the number of jumps of these configurations and the energetic cost of each jump is $\pi(1-|\alpha|)^2$.

Model 2. Confinement of Néel wall tails by the finite size of the sample. The constraints are given by:

$$m = (m_1, m_2) : \mathbb{R} \to S^1 \text{ and } m(\pm t) = \begin{pmatrix} \cos\theta \\ \pm\sin\theta \end{pmatrix} \text{ for } \pm t \geq 1, \tag{6}$$

with $\theta \in [0, 2\pi)$ (see Figure 4), whereas the energy functional is:

$$m \mapsto \delta \|m\|^2_{\dot{H}^1} + \|m_1\|^2_{\dot{H}^{1/2}} \tag{7}$$

with $\delta > 0$ a small parameter.

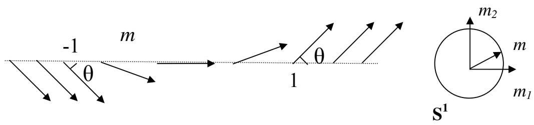

Figure 4: Néel wall of angle 2θ confined in $[-1, 1]$.

It models a one-dimensional magnetization in a thin-ferromagnetic film of finite width where the effect of cristalline anisotropy and external magnetic fields is neglected. The corresponding variational problem was considered by DeSimone, Kohn, Müller and Otto [8], DeSimone, Knüpfer and Otto [6] and Ignat and Otto [10]. The main difference with respect to Model 1 consists in the confinement of Néel wall tails by the interaction with the sample edges played by -1 and 1 in our framework. However, the properties of the transition layer in Model 1 naturally transfer to the structure of a minimizer of (7) that satisfies $m(0) = (1, 0)$. It is a two length scale object with a small core of order δ and two logarithmically decaying tails contained in $[-1, 1]$ and it attains the same level of minimal energy $\frac{\pi + o(1)}{|\log \delta|}$ as $\delta \downarrow 0$. As before, by rescaling and renormalization of (7), the corresponding energy writes:

$$F_\varepsilon(m) = \varepsilon \|m\|^2_{\dot{H}^1} + |\log \varepsilon| \|m_1\|^2_{\dot{H}^{1/2}}$$

for a small parameter $\varepsilon > 0$. We will analyse the asymptotics of F_ε by the $\Gamma-$convergence method as $\varepsilon \downarrow 0$. We expect to have a similar behavior for the limiting configurations and the $\Gamma-$limit energy as in Model 1. The difference will consist in having all the walls confined in the interval $[-1, 1]$.

Model 3. Confinement of Néel wall tails by the neighboring Néel walls. The magnetizations are periodic functions such that:

$$m = e^{i\varphi}, \ \varphi : \mathbb{R} \to \mathbb{R} \text{ with } \varphi(t + 2) = \varphi(t) \text{ and } \varphi(t + 1) = \varphi(t) + \pi \tag{8}$$

(see Figure 5). The energy is given by:

$$m \mapsto \delta \|m\|^2_{\dot{H}^1_{per}} + \|m_1\|^2_{\dot{H}^{1/2}_{per}}, \tag{9}$$

for a small parameter $\delta > 0$.

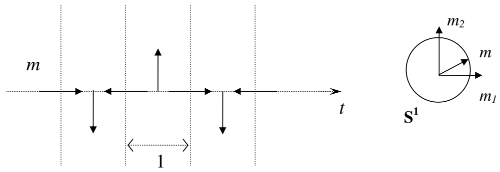

Figure 5: Periodic array of winding walls.

This model was investigated by DeSimone, Kohn, Müller and Otto [9] in order to quantify the repulsive interaction of Néel walls. It consists in considering a periodic array of winding walls at a renormalized distance $w = 2$ in the absence of anisotropy and external magnetic fields. A transition of 180° is enforced in the middle of each period by the constraint (8). Therefore, the tails of a Néel wall are limited by the tails of the neighboring walls at a distance 1 and we expect that this model generates only 180° Néel walls. As before, we will analyse the following rescaled and renormalized energy associated to (9):

$$G_\varepsilon(m) = \varepsilon \|m\|^2_{\dot{H}^1_{per}} + |\log \varepsilon| \|m_1\|^2_{\dot{H}^{1/2}_{per}}$$

in the asymptotic $\varepsilon \downarrow 0$.

2 Main results

First we show a compactness result for magnetizations with uniformly bounded energies and we deduce the pattern of the limiting configurations for all three models. Then we present our main result: we compute the $\Gamma-$limit of the three families of energies $\{E_\varepsilon\}_{\varepsilon\downarrow 0}$, $\{F_\varepsilon\}_{\varepsilon\downarrow 0}$ and $\{G_\varepsilon\}_{\varepsilon\downarrow 0}$ and we prove the corresponding $\Gamma-$convergence result for each of these models.

Model 1. We start with the compactness result in Model 1. It is related with a result for 1−d magnetizations proved by Ignat and Otto [10] (see Theorem 2 in [10]) where Model 2 is treated for fixed boundary data. In our context, the anisotropy replaces the role of interaction of Néel wall tails with the edges of a finite sample and no fixed boundary conditions are imposed.

Theorem 1 *Consider a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \downarrow 0$ as $k \uparrow \infty$. For $k \in \mathbb{N}$, let $\alpha_k \in [-1,1]$, E_{ε_k} be the energy functional* (5) *associated to ε_k and α_k and let $m_k = (m_{1,k}, m_{2,k}) : \mathbb{R} \to S^1$. Suppose that*

$$\limsup_{k\uparrow\infty} E_{\varepsilon_k}(m_k) < +\infty. \tag{10}$$

Then $\{m_k\}_{k\uparrow\infty}$ is relatively compact in $L^1_{loc}(\mathbb{R}, S^1)$. Moreover, any accumulation point $m : \mathbb{R} \to S^1$ is of bounded total variation, takes values in the set $\{(\cos\theta, \pm\sin\theta)\}$ for some $\theta \in [0, 2\pi)$ and can be written as:

$$m = \sum_{n=1}^{N+1} \begin{pmatrix} \cos\theta \\ (-1)^n \sin\theta \end{pmatrix} 1_{(t_{n-1},t_n)}, \tag{11}$$

where $N \geq 0$ and $-\infty = t_0 < t_1 < \cdots < t_N < t_{N+1} = +\infty$.

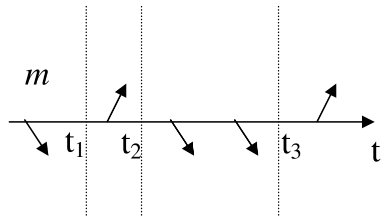

Figure 6: A limit configuration with 3 walls.

Remark 1 i) The walls of a limiting configuration have the same angle 2θ, i.e., m_1 is a constant function in $\mathbb{R}$ equal to $\cos\theta$ and m_2 has a finite number of jumps of size $2|\sin\theta|$. The angle θ represents an accumulation point of the set $\{\text{Arccos}\,\alpha_k\}_{k\in\mathbb{N}}$ in $[0, 2\pi)$.

ii) The number of walls may vanish in the limit, i.e., $N = 0$. Indeed, one can imagine a transition layer m_k of angle 2θ centered in k so that the center of the wall tends to infinity and m_k converges to the constant function $(\cos\theta, -\sin\theta)$ as $k \uparrow \infty$. Moreover, one can easily check that a limiting configuration (11) is constant if and only if $N = 0$ or $\sin\theta = 0$ (i.e., $|\alpha_k| \to 1$). Also notice that N can be both an odd or an even integer. In fact, condition (1) is not imposed for configurations $\{m_k\}$ in Theorem 1, so that m_k may exhibit an even number of wall transitions and $m_{2,k}$ may have the same limit at $\pm\infty$.

iii) A configuration m_k of finite energy E_{ε_k} is continuous and the first component $m_{1,k}$ has the limit α_k at $\pm\infty$. Indeed, if $E_{\varepsilon_k}(m_k) < \infty$, then $m_{1,k} - \alpha_k \in H^1(\mathbb{R})$ which entails the latter statement (see e.g., Brezis [2]). We highlight the fact that we do not impose a fixed boundary condition for the sequence $\{m_k\}_{k\in\mathbb{N}}$ at infinity in Theorem 1, i.e., $\{\alpha_k\}_{k\in\mathbb{N}}$ may vary with k. For an accumulation point m as in (11), we have that $\alpha_k \to \cos\theta$ as $k \uparrow \infty$.

iv) The compactness result is also valid in the $L^p_{loc}(\mathbb{R}, S^1)-$topology for every $p \in [1, \infty)$ since the values of limiting configurations are of length 1. Due to the constraint $|m| = 1$ which implies $m \notin L^p(\mathbb{R}, \mathbb{R}^2)$, we always work in the framework of $L^p_{loc}(\mathbb{R}, \mathbb{R}^2)$-spaces.

Let us denote by $\mathcal{A}$ the set of all limiting configurations given by (11). For such a configuration $m \in \mathcal{A}$, we define the following energy:

$$E_0(m) = \pi(1 - |m_1|)^2 \cdot \Big(\text{ number of jumps of } m\Big), \tag{12}$$

where the number N of jumps of m corresponds to the number of walls of the limiting magnetization m. Our main result shows that E_0 represents the $\Gamma-$limit of energies E_k:

Theorem 2 *Consider a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \infty)$ with $\varepsilon_k \downarrow 0$ as $k \uparrow \infty$. For $k \in \mathbb{N}$, let $\alpha_k \in [-1, 1]$ and E_{ε_k} be the energy functional (5) associated to ε_k and α_k. Then*

$$E_{\varepsilon_k} \xrightarrow{\Gamma} E_0 \text{ under the } L^1_{loc}(\mathbb{R}, S^1)-\text{topology as } k \uparrow \infty, \text{ i.e.,}$$

(i) If the sequence $\{m_k : \mathbb{R} \to S^1\}_{k\in\mathbb{N}}$ *satisfies* (10) *and* $m_k \overset{k\uparrow\infty}{\to} m$ *in* $L^1_{loc}(\mathbb{R}, S^1)$*, then* $m \in \mathcal{A}$ *and*

$$\liminf_{k\uparrow\infty} E_{\varepsilon_k}(m_k) \geq E_0(m); \tag{13}$$

(ii) For every $m \in \mathcal{A}$*, there exists a sequence of smooth functions* $\{m_k : \mathbb{R} \to S^1\}_{k\in\mathbb{N}}$ *such that* $m_k - m$ *has compact support in* $\mathbb{R}$ *for all* $k \in \mathbb{N}$*,* $m_k - m \overset{k\uparrow\infty}{\to} 0$ *in* $L^1(\mathbb{R}, \mathbb{R}^2)$ *and*

$$\lim_{k\uparrow\infty} E_{\varepsilon_k}(m_k) = E_0(m).$$

Remark 2 If $\{E_\varepsilon\}_{\varepsilon\downarrow 0}$ is the family of energies (5) associated to a fixed $\alpha := \cos\theta$, then Theorem 2 yields that the energy of a Néel wall of angle 2θ is quartic in θ for small angles θ:

$$\min_{(1)} E_\varepsilon \overset{\varepsilon\downarrow 0}{=} \pi(1 - |\cos\theta|)^2 \approx \frac{\pi}{4}\theta^4 \quad \text{as} \quad \theta \downarrow 0.$$

Observe the importance of the anisotropy in Model 1: it is the confining mechanism that prevents the tails to spread over $\mathbb{R}$. In the absence of the anisotropy, the variational problem (3) becomes degenerate. It is stated in the following proposition:

Proposition 1 *Let* $\alpha \in [0, 1)$*. We have that*

$$\min_{\substack{(1)\\ m(0)=(1,0)}} \delta\|m\|^2_{\dot{H}^1} + \|m_1\|^2_{\dot{H}^{1/2}} = 0,$$

for every $\delta > 0$*.*

Model 2. Now we present the corresponding results for Model 2 when the anisotropy effect is replaced by a confinement of Néel wall tails in a finite interval. The compactness result of configurations of uniformly bounded energy F_ε is given in the following:

Theorem 3 *Consider a sequence* $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \infty)$ *with* $\varepsilon_k \downarrow 0$ *as* $k \uparrow \infty$*. For* $k \in \mathbb{N}$*, let* $\theta_k \in [0, 2\pi)$ *and* $m_k = (m_{1,k}, m_{2,k}) : \mathbb{R} \to S^1$ *be such that* (6) *holds. Suppose that*

$$\limsup_{k\uparrow\infty} F_{\varepsilon_k}(m_k) < +\infty. \tag{14}$$

Then $\{m_k\}_{k\uparrow\infty}$ *is relatively compact in* $L^1_{loc}(\mathbb{R}, S^1)$*. Any accumulation point* $m : \mathbb{R} \to S^1$ *is of bounded total variation and can be written as*

$$m = \sum_{n=1}^{N+1} \begin{pmatrix} \cos\theta \\ (-1)^n \sin\theta \end{pmatrix} 1_{(t_{n-1}, t_n)}, \tag{15}$$

where $\theta \in [0, 2\pi)$*,* $N \geq 0$ *and* $-\infty = t_0 < -1 \leq t_1 < \cdots < t_N \leq 1 < t_{N+1} = +\infty$*. Moreover, if* $\sin\theta \neq 0$*, then* N *is an odd integer and* m *satisfies* (6)*.*

Remark 3 i) The limiting configurations in Model 2 have the same feature as in Model 1: they exhibit a finite number of walls of identical angle 2θ. The difference consists in having all these walls confined in $[-1, 1]$ because of the boundary condition (6).

ii) The compactness result fails in general under the strict convergence in BV_{loc} even if the limiting configurations are of bounded variation in $\mathbb{R}$. In fact, it is constructed in [10] a sequence of magnetizations $\{m_k\}$ with (6) and of uniformly bounded energies $F_{\varepsilon_k}(m_k) \leq C$ such that the sequence of total variations $\{\int |\frac{dm_{1,k}}{dt}|\}$ blows-up (see Theorem 3 in [10]).

The fading $\dot{H}^1-$control of the magnetization is essential for the compactness result: in the absence of it, we can construct a sequence of magnetizations that does not converge in L^1_{loc}. These magnetizations will asymptotically have an infinite number of transition walls and the sequence of their homogeneous $\dot{H}^{1/2}-$seminorm converges to zero. Obviously, the condition (14) fails for these configurations.

Proposition 2 *There exists a sequence of smooth magnetizations $\{m_k : \mathbb{R} \to S^1\}_{k\in\mathbb{N}}$ with* (6) *such that*

$$\|m_{1,k}\|^2_{\dot{H}^{1/2}} \leq \frac{C}{|\log \frac{1}{k}|} \quad \text{and} \quad \{m_k\} \text{ is not relatively compact in } L^1_{loc}(\mathbb{R}, \mathbb{R}^2).$$

Let us denote by $\mathcal{B}$ the set of all limiting configurations given by (15). We have a similar $\Gamma-$convergence result of energies $\{F_\varepsilon\}_{\varepsilon\downarrow 0}$ to the same $\Gamma-$limit E_0 as in Model 1:

Theorem 4 *Consider a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \downarrow 0$ as $k \uparrow \infty$. Then*

$$F_{\varepsilon_k} \xrightarrow{\Gamma} E_0 \text{ under the } L^1_{loc}(\mathbb{R}, S^1)-\text{topology as } k \uparrow \infty, \text{ i.e.,}$$

(i) If for any $k \in \mathbb{N}$, $\theta_k \in [0, 2\pi)$ and $m_k : \mathbb{R} \to S^1$ are such that (6) *holds, then the condition* (14) *together with $m_k \overset{k\uparrow\infty}{\to} m$ in $L^1_{loc}(\mathbb{R}, S^1)$ imply that $m \in \mathcal{B}$ and*

$$\liminf_{k\uparrow\infty} F_{\varepsilon_k}(m_k) \geq E_0(m);$$

(ii) For every $m \in \mathcal{B}$, there exists a sequence of smooth functions $\{m_k : \mathbb{R} \to S^1\}_{k\in\mathbb{N}}$ such that $m_k = m$ in $\mathbb{R} \setminus [-1, 1]$, $m_k - m \overset{k\uparrow\infty}{\to} 0$ in $L^1(\mathbb{R}, \mathbb{R}^2)$ and

$$\lim_{k\uparrow\infty} F_{\varepsilon_k}(m_k) = E_0(m).$$

Observe that for the upper bound in the $\Gamma-$convergence result, we construct a sequence of magnetizations that coincide with the limit configuration outside the sample $[-1, 1]$ and they asymptotically have the same energy.

Model 3. Finally, we discuss the third model when the Néel wall tails are confined by interaction with neighboring walls. The setting consists in periodic configurations where rotations of 180° are enforced. Therefore, we expect that the limit magnetization points in opposite directions across each wall.

Theorem 5 *Consider a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \downarrow 0$ as $k \uparrow \infty$. For $k \in \mathbb{N}$, let $m_k = (m_{1,k}, m_{2,k}) : \mathbb{R} \to S^1$ be such that* (8) *holds. Suppose that*

$$\limsup_{k\uparrow\infty} G_{\varepsilon_k}(m_k) < +\infty. \tag{16}$$

Then $\{m_k\}_{k\uparrow\infty}$ *is relatively compact in* $L^1_{loc}(\mathbb{R}, S^1)$. *Any accumulation point* $m : \mathbb{R} \to S^1$ *is a* $2-$*periodic function of bounded total variation on* $[-1, 1)$ *that takes exactly two values* $\begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$ *and satisfies* $m(t) = -m(t + 1)$ *for every* $t \in \mathbb{R}$.

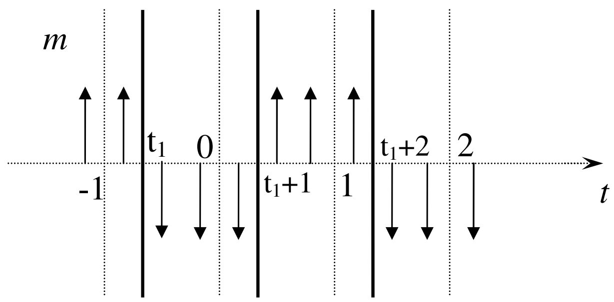

Figure 7: A periodic limit configuration having 2 walls per period.

Remark 4 The constraint (8) imposes for the limiting configuration an even number $2N$ of walls of 180° in $[-1, 1)$ with $N \geq 1$. Suppose that these N pairs of walls are placed in $t_n \in [-1, 0)$ and $t_n + 1 \in [0, 1)$ respectively, for $n = 1, \dots, N$ where the orientation of the magnetization is reversed. Then the limit magnetization m can be written on the interval $[-1, 1)$ as

$$m = \pm \sum_{n=1}^{N+1} \begin{pmatrix} 0 \\ (-1)^n \end{pmatrix} \left(1_{(t_{n-1}, t_n)} - 1_{(t_{n-1}+1, t_n+1)} \right) \quad \text{in } [-1, 1), \tag{17}$$

where $N \geq 1$ and $-1 = t_0 \leq t_1 < \cdots < t_N < t_{N+1} = 0$.

Let us denote by $\mathcal{C}$ the set of all limiting configurations in Model 3 given by Theorem 5. For every $m \in \mathcal{C}$, we denote

$$G_0(m) = \pi \cdot \Bigg(\text{number of jumps of } m \text{ in } [-1, 1) \Bigg).$$

We have the same $\Gamma-$convergence result of energies $\{G_\varepsilon\}$ as in Models 1 and 2:

Theorem 6 *Consider a sequence* $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \infty)$ *with* $\varepsilon_k \downarrow 0$ *as* $k \uparrow \infty$. *Then*

$$G_{\varepsilon_k} \xrightarrow{\Gamma} G_0 \text{ under the } L^1_{loc}(\mathbb{R}, S^1)-\text{topology as } k \uparrow \infty, \text{ i.e.,}$$

(i) If $\{m_k : \mathbb{R} \to S^1\}_{k\in\mathbb{N}}$ *is a sequence of* $2-$*periodic functions such that* (8) *and* (16) *hold true and* $m_k \overset{k\uparrow\infty}{\to} m$ *in* $L^1_{loc}(\mathbb{R}, S^1)$, *then* $m \in \mathcal{C}$ *and*

$$\liminf_{k\uparrow\infty} G_{\varepsilon_k}(m_k) \geq G_0(m);$$

(ii) *For every $m \in \mathcal{C}$, there exists a sequence $\{m_k : \mathbb{R} \to S^1\}_{k\in\mathbb{N}}$ of smooth 2−periodic functions that satisfy* (8)*, $m_k \overset{k\uparrow\infty}{\to} m$ in $L^1_{loc}(\mathbb{R}, S^1)$ and*

$$\lim_{k\uparrow\infty} G_{\varepsilon_k}(m_k) = G_0(m).$$

The sequel of the paper is organized as follows: In Section 3, we review some properties of functions in homogeneous Sobolev spaces and we recall a logarithmically failing Gagliardo-Nirenberg inequality that we use in the proof of our results. In Section 4, we prove compactness of configurations with uniformly bounded energies in all three models as stated in Theorems 1, 3 and 5. In Section 5, we prove the lower bound in the Γ−convergence results given at point *(i)* of Theorems 2, 4 and 6. In Section 6, we conclude with the proof of Theorems 2, 4 and 6 by constructing appropriate sequences for any limiting configuration; we also present the proof of Propositions 1 and 2. We end with an Appendix where we show some characterizations of the homogeneous $\dot{H}^{1/2}$−seminorm.

3 Preliminaries

First, we recall several definitions and properties of functions in some homogeneous Sobolev spaces. Let $s \in \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$ be a tempered distribution in $\mathcal{S}'(\mathbb{R})$. We denote the homogeneous $\dot{H}^s$-seminorm of u by

$$\|u\|^2_{\dot{H}^s} := \int_{\mathbb{R}} |\xi|^{2s} |\hat{u}|^2(\xi)\, d\xi,$$

where $\hat{u} \in \mathcal{S}'(\mathbb{R})$ stands for the Fourier transform of u (as a tempered distribution), i.e.,

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} u(x)\, dx, \quad \forall \xi \in \mathbb{R}.$$

If $s \in \{\frac{1}{2}, 1\}$ we have the following properties: $\|u\|_{\dot{H}^1} = \|\frac{du}{dt}\|_{L^2}$ and

$$\|u\|^2_{\dot{H}^{1/2}} = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(s) - u(t)|^2}{|s-t|^2}\, ds dt \tag{18}$$

(see Proposition 7 in Appendix).

If u is periodic (assume that the period is equal to 2), then the periodic $\dot{H}^s$-seminorm of u is given by

$$\|u\|^2_{\dot{H}^s_{per}} := \sum_{\beta\in\pi\mathbb{Z}} |\beta|^{2s} |\hat{u}(\beta)|^2,$$

where the sequence $\{\hat{u}(\beta)\}_{\beta\in\pi\mathbb{Z}}$ stands for the Fourier coefficients of u, i.e.,

$$\widehat{u}(\beta) = \frac{1}{\sqrt{2}} \int_{[-1,1)} e^{-i\beta t} u(t)\, dt, \; \forall \beta \in \pi\mathbb{Z}. \tag{19}$$

The formula (18) has an equivalent in the periodic case. Indeed, regarding u as being defined on the unit circle S^1 by the convention $u(e^{i\pi t}) := u(t)$ for $t \in [-1, 1)$, we have that:

$$\|u\|^2_{\dot{H}^{1/2}_{per}} = \frac{1}{2\pi} \int_{S^1} \int_{S^1} \frac{|u(z_1) - u(z_2)|^2}{|z_1 - z_2|^2}\, dz_1 dz_2$$

(see Proposition 7 in Appendix).

Another characterization of the $\dot{H}^{1/2}$-seminorm of u can be expressed as the minimal $\dot{H}^1-$seminorm of functions $U : \mathbb{R}^2 \to \mathbb{R}$ that have u as trace on the horizontal line $\{(x_1, x_2) \in \mathbb{R}^2 \, : \, x_2 = 0\}$. In other words, the homogeneous $\dot{H}^{1/2}$-seminorm is given by the Dirichlet energy of the harmonic extension of u in $\mathbb{R}^2$. For the sake of completeness, we give the proof of this property in Appendix.

Proposition 3 *Let $u : \mathbb{R} \to \mathbb{R}$.*
(i) If $u \in \dot{H}^{1/2}(\mathbb{R})$, then we have the following trace estimate:

$$\|u\|^2_{\dot{H}^{1/2}} = \frac{1}{2} \min \left\{ \int_{\mathbb{R}^2} |\nabla U|^2 \, dx_1 dx_2 \, : \, U(x_1, 0) = u(x_1) \right\}. \tag{20}$$

(ii) If u is $2-$periodic belonging to $\dot{H}^{1/2}_{per}(\mathbb{R})$, then

$$\|u\|^2_{\dot{H}^{1/2}_{per}} = \frac{1}{2} \min \left\{ \int_{[-1,1)\times\mathbb{R}} |\nabla U|^2 \, dx_1 dx_2 \, : \, U \text{ is } 2-\text{periodic in } x_1 \text{ and } U(x_1, 0) = u(x_1) \right\}. \tag{21}$$

As mentioned in the introduction, the $\dot{H}^{1/2}-$seminorm of the first component of the magnetization represents the stray field energy. Let us discuss in details this property. We call a stray field associated to the magnetization $m = (m_1, m_2) : \mathbb{R} \to S^1$, every two-dimensional vector field $h : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies

$$\int_{\mathbb{R}^2} h \cdot \nabla\zeta \, dx = \int_{\mathbb{R}} \frac{dm_1}{dx_1} \zeta(\cdot, 0) \, dx_1 \text{ for every } \zeta \in C^\infty_c(\mathbb{R}^2). \tag{22}$$

The form of the stray field energy that appears in all three models is justified by the following proposition. Its proof is presented in Appendix.

Proposition 4 *Let $m_1 : \mathbb{R} \to \mathbb{R}$.*
(i) If $m_1 \in \dot{H}^{1/2}(\mathbb{R})$, we have that

$$\min_{(22)} \int_{\mathbb{R}^2} |h|^2 \, dx = \frac{1}{2} \|m_1\|^2_{\dot{H}^{1/2}}. \tag{23}$$

The minimizer $H = (H_1, H_2)$ of (23) is unique in $L^2(\mathbb{R}^2, \mathbb{R}^2)$ and it is a gradient field. Moreover, H satisfies the static Maxwell equations:

$$\begin{cases} \nabla \times H = 0 & \text{in } \mathbb{R}^2, \\ \nabla \cdot H = 0 & \text{in } \{x_2 \neq 0\}, \\ [H_2] = -\frac{dm_1}{dx_1} & \text{on } \{x_2 = 0\}, \end{cases} \tag{24}$$

where $[\cdot]$ stands for the size of the jump over the horizontal line $\{x_2 = 0\}$.
(ii) If m_1 is a $2-$periodic function in $\dot{H}^{1/2}_{per}(\mathbb{R})$, then

$$\min \left\{ \int_{[-1,1)\times\mathbb{R}} |h|^2 \, dx \, : \, h : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is } 2-\text{periodic in } x_1 \text{ with } (22) \right\} = \frac{1}{2} \|m_1\|^2_{\dot{H}^{1/2}_{per}}. \tag{25}$$

The minimizer H of (25) is a $2-$periodic (in x_1) gradient field that is unique in $L^2_{per}(\mathbb{R}^2, \mathbb{R}^2)$ and satisfies the static Maxwell equations (24).

For the compactness of magnetizations $\{m_k\}$ in Theorems 1, 3 and 5, we need to control the number of their large variations which consists in studying the derivatives of the first components $\{\sigma_k := \frac{dm_{1,k}}{dt}\}$. The stray field energy controls the homogeneous $\dot{H}^{-1/2}$-seminorm of σ_k in the regime $O(\frac{1}{|\log \varepsilon_k|})$. The idea is to use a duality argument by estimating the product

$$< \chi_k, \sigma_k >_{\dot{H}^{1/2}, \dot{H}^{-1/2}} \tag{26}$$

for a trial function χ_k that counts the large variations of $m_{1,k}$. Therefore, it is enough to analyse the rate of the failing interpolation embedding

$$BV \cap L^\infty(\mathbb{R}) \not\subseteq \dot{H}^{1/2}(\mathbb{R})$$

that corresponds to the failing Gagliardo-Nirenberg type inequality:

$$\|\chi_k\|^2_{\dot{H}^{1/2}} \not\lesssim \sup |\chi_k| \int |\frac{d\chi_k}{dt}|. \tag{27}$$

By a duality argument, the failing inequality (27) entails via (26) that

$$\left| \int \chi_k \sigma_k \right|^2 \not\lesssim \sup |\chi_k| \int |\frac{d\chi_k}{dt}| \, \|\sigma_k\|^2_{\dot{H}^{-1/2}}. \tag{28}$$

Typically, the trial function χ_k has jumps so that $\chi_k \notin \dot{H}^{1/2}(\mathbb{R})$. That can be corrected by regularizing the homogeneous $\dot{H}^{1/2}-$seminorm. This perturbation gives a weaker seminorm that is controlled by the RHS term in (27) with a logarithmically slow rate having the optimal prefactor $\frac{2}{\pi}$:

Proposition 5 (DeSimone, Knüpfer and Otto [6]) *For $\varepsilon \ll w$ and for any $\chi : \mathbb{R} \to \mathbb{R}$, we have that*

$$\int_{\mathbb{R}} \min\{\frac{1}{\varepsilon}, |\xi|, w|\xi|^2\} |\hat{\chi}|^2 \, d\xi \lessapprox \frac{2}{\pi} \left(\log \frac{w}{\varepsilon}\right) \sup_{\mathbb{R}} |\chi| \int_{\mathbb{R}} \left|\frac{d\chi}{dt}\right|. \tag{29}$$

If χ is $2-$periodic then

$$\sum_{\beta \in \pi\mathbb{Z}} \min\{\frac{1}{\varepsilon}, |\beta|, w|\beta|^2\} |\hat{\chi}(\beta)|^2 \lessapprox \frac{2}{\pi} \left(\log \frac{w}{\varepsilon}\right) \sup_{[-1,1)} |\chi| \int_{[-1,1)} \left|\frac{d\chi}{dt}\right|. \tag{30}$$

Coming back to (28), the logarithmically failing rate of (27) in Proposition 5 matches well with the control of the $\dot{H}^{-1/2}$-seminorm of σ_k by the stray field energy of order of $O(\frac{1}{|\log \varepsilon_k|})$. Notice that cutting off the large wave length $\geq 1/\varepsilon_k$ in the weaker norm in Proposition 5 must be compensated by the fading L^2-control of σ_k that corresponds to the exchange energy. On the other hand, the $\dot{H}^{-1/2}$-seminorm of σ_k can be replaced by the energy of a stray field as in Proposition 4: there exists a stray field $h_k : \mathbb{R}^2 \to \mathbb{R}^2$ associated to m_k by (22) such that

$$\|\sigma_k\|^2_{\dot{H}^{-1/2}} = 2 \int_{\mathbb{R}^2} |h_k|^2 \, dx.$$

These arguments suggest the following localized version of the failing inequality (28). The reason of using a cut-off function η is that the localized duality term

$$\int \eta^2 \chi_k \sigma_k$$

controls the local number of variations $\int \eta^2 \left|\frac{d\chi_k}{dx_1}\right|$ of $m_{1,k}$.

Proposition 6 (Ignat and Otto [10]) *Let* $R, L > 0$ *and* $(x_1^0, 0) \in \mathbb{R}^2$. *Let* $h : \mathbb{R}^2 \to \mathbb{R}^2$ *and* $m_1 : \mathbb{R} \to \mathbb{R}$ *be related by* (22) *and let* $\chi : \mathbb{R} \to \mathbb{R}$ *be a bounded function of locally bounded total variation.*
(i) If $\eta \in C_c^\infty(\mathbb{R}^2)$ *such that*

$$\operatorname{supp} \eta \subset B((x_1^0, 0), R) \subset \mathbb{R}^2,$$

then there exists a universal constant $C = C(R) > 0$ *depending only on* R *such that for all* $\varepsilon \in (0, R]$,

$$\begin{aligned}\Big| \int_{\mathbb{R}} \eta^2(\cdot, 0)\chi \frac{dm_1}{dx_1}\, dx_1 \Big| \le & \Big(\frac{4}{\pi} |\log \varepsilon| \sup_{\mathbb{R}} |\chi| \int_{\mathbb{R}} \eta^2(\cdot, 0) \big|\frac{d\chi}{dx_1}\big| \int_{\mathbb{R}^2} \eta^2 |h|^2\, dx \Big)^{1/2} \\ & + C(R) \Big(\varepsilon \int_{x_1^0 - R}^{x_1^0 + R} \big|\frac{dm_1}{dx_1}\big|^2\, dx_1 + \int_{B((x_1^0, 0), R)} |h|^2\, dx \Big)^{1/2} \\ & \times \Big(\sup_{\mathbb{R}^2} |\eta| + \sup_{\mathbb{R}^2} |\nabla \eta| \Big) \Big(\sup_{\mathbb{R}^2} |\eta| \sup_{\mathbb{R}} |\chi| + \int_{\mathbb{R}} |\eta(\cdot, 0)| \big|\frac{d\chi}{dx_1}\big| \Big).\end{aligned}$$

(ii) If $\eta \in C^\infty(\mathbb{R}^2)$ *with* $\operatorname{supp} \eta \subset \mathbb{R} \times (-1, 1)$ *and*

$$h,\ m_1,\ \chi \text{ and } \eta \text{ are } L\text{-periodic in } x_1,$$

then there exists a universal constant $C = C(L) > 0$ *depending only on* L *such that for all* $\varepsilon \in (0, L]$,

$$\begin{aligned}\Big| \int_{[0,L)} \eta^2(\cdot, 0)\chi \frac{dm_1}{dx_1}\, dx_1 \Big| \le & \Big(\frac{4}{\pi} |\log \varepsilon| \sup_{\mathbb{R}} |\chi| \int_{[0,L)} \eta^2(\cdot, 0) \big|\frac{d\chi}{dx_1}\big| \int_{[0,L)\times\mathbb{R}} \eta^2 |h|^2\, dx \Big)^{1/2} \\ & + C(L) \Big(\varepsilon \int_{[0,L)} \big|\frac{dm_1}{dx_1}\big|^2\, dx_1 + \int_{[0,L)\times\mathbb{R}} |h|^2\, dx \Big)^{1/2} \\ & \times \Big(\sup_{\mathbb{R}^2} |\eta| + \sup_{\mathbb{R}^2} |\nabla \eta| \Big) \Big(\sup_{\mathbb{R}^2} |\eta| \sup_{\mathbb{R}} |\chi| + \int_{[0,L)} |\eta(\cdot, 0)| \big|\frac{d\chi}{dx_1}\big| \Big).\end{aligned}$$

In this paper, we consider several energy functionals $\{E_\varepsilon\}$, $\{F_\varepsilon\}$ and $\{G_\varepsilon\}$ depending on a small parameter $\varepsilon > 0$ and we are interested in their limiting behavior as $\varepsilon \downarrow 0$. The limit we are looking for is not the usual pointwise limit, but a limit adapted for the convergence of minimizers. This appropriate notion of convergence is called the $\Gamma-$convergence (see e.g., Attouch [1], Dal Maso [5]). Let us recall its definition in the particular case of a sequence of energies $\{E_{\varepsilon_k}\}_{k\uparrow\infty}$ given by (5) where $\varepsilon_k \downarrow 0$ and $\{\alpha_k\} \subset [-1, 1]$. Let $X = L^1_{loc}(\mathbb{R}, S^1)$ be endowed with the metric

$$d(f, g) = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \int_{-n}^{n} |f - g|\, dt.$$

Then (X, d) is a complete metric space. We can naturally extend (5) to a functional $E_{\varepsilon_k} : X \to [0, \infty]$ defined in the whole space X, i.e., $E_{\varepsilon_k}(f) < \infty$ if $f - \alpha_k \in H^1(\mathbb{R}, S^1)$ and $E_{\varepsilon_k}(f) = \infty$ otherwise. We say that $E_0 : X \to [0, \infty]$ is the $\Gamma-$limit of $\{E_{\varepsilon_k}\}_{k\uparrow\infty}$, or equivalently, E_{ε_k} is $\Gamma-$convergent to E_0 under the $X-$topology if the following two conditions are satisfied:

(i) (lower bound) for every sequence $f_k \to f$ in X, then $\liminf_{k\uparrow\infty} E_{\varepsilon_k}(f_k) \ge E_0(f)$;

(ii) (upper bound) for every $f \in X$, there exists a recovery sequence $f_k \in X$ such that $f_k \to f$ in X and $\lim_{k\uparrow\infty} E_{\varepsilon_k}(f_k) = E_0(f)$.

By Theorem 2, the expression of E_0 is given by (12) and we have that $E_0(f) < \infty$ if and only if $f \in \mathcal{A}$.

4 Compactness

We start by proving the compactness results for the three models. For configurations $\{m_k\}$ of uniformly bounded energy, the control of the stray field energy entails relative convergence of the first components $\{m_{1,k}\}$ to a limit function m_1 that is a constant. Since $|m_k| = 1$, we deduce that the second components $\{m_{2,k}\}$ will asymptotically take two possible values $\pm\sqrt{1-m_1^2}$. In order to prove compactness, it is enough to bound uniformly the number of large variations of $\{m_{2,k}\}$. The number of such variations can be estimated by the total energy through a duality argument based on the logarithmically failing interpolation inequality presented in Proposition 5. These ideas are also used in [10].

For the simplicity of notation, every convergence of a sequence should be considered up to a subsequence in this Section.

Proof of Theorem 1. We proceed in several steps.

Step 1. *We show that there exists a constant $m_{1,\infty} \in [-1,1]$ such that*

$$m_{1,k} \to m_{1,\infty} \quad \text{in} \quad L^1_{loc}(\mathbb{R}) \text{ as } k \uparrow \infty.$$

Indeed, there exists $m_{1,\infty} \in [-1,1]$ such that

$$\alpha_k \to m_{1,\infty} \text{ as } k \uparrow \infty. \tag{31}$$

Since (10) implies that

$$\|m_{1,k} - \alpha_k\|_{L^2} \to 0,$$

we deduce immediately the conclusion of Step 1. Moreover,

$$|m_{2,k}|^2 = 1 - |m_{1,k}|^2 \to 1 - |m_{1,\infty}|^2 \quad \text{in} \quad L^1_{loc}(\mathbb{R}).$$

Remark that if $m_{1,\infty} \in \{-1,1\}$, then $m_{2,k} \to 0$ in $L^1_{loc}(\mathbb{R})$ and the limit function is $m = (\pm 1, 0)$. Therefore, in the rest of the proof, we analyze the remaining case $m_{1,\infty} \in (-1,1)$.

Step 2. *The location of large variations of $m_{1,k}$ and $m_{2,k}$.* Let k be fixed here. The function $m_{1,k} - \alpha_k$ belongs to $H^1(\mathbb{R})$, therefore $m_{1,k}$ is continuous in $\mathbb{R}$ and has the limit α_k at $\pm\infty$. We deduce that $m_{2,k}$ is continuous in $\mathbb{R}$ and $|m_{2,k}|$ has the limit $\sqrt{1-\alpha_k^2}$ at $\pm\infty$. By (31), $\sqrt{1-\alpha_k^2} \to \sqrt{1-{m_{1,\infty}}^2}$ as $k \uparrow \infty$; thus, for k large enough, there exists a bounded interval $I_k \subset \mathbb{R}$ such that

$$|m_{2,k}| \geq \frac{\sqrt{1-{m_{1,\infty}}^2}}{2} \qquad \text{outside} \quad I_k$$

and $m_{2,k}$ does not change sign on the left and on the right of I_k, respectively. Hence, one can detect a finite number N_k of intervals $(a_n^k, b_n^k) \subset I_k$, $n = 1, \dots, N_k$ where $m_{2,k}$ varies between $-\frac{\sqrt{1-{m_{1,\infty}}^2}}{2}$ and $\frac{\sqrt{1-{m_{1,\infty}}^2}}{2}$ (see Figure 8):

$$a_1^k < b_1^k \leq a_2^k < b_2^k \leq \dots \leq a_{N_k}^k < b_{N_k}^k$$

and for each $n = 1, \dots, N_k$,

$$|m_{2,k}(a_n^k)| = |m_{2,k}(b_n^k)| = \frac{\sqrt{1-{m_{1,\infty}}^2}}{2} \quad \text{and} \quad |m_{2,k}(t)| < \frac{\sqrt{1-{m_{1,\infty}}^2}}{2} \text{ if } t \in (a_n^k, b_n^k). \tag{32}$$

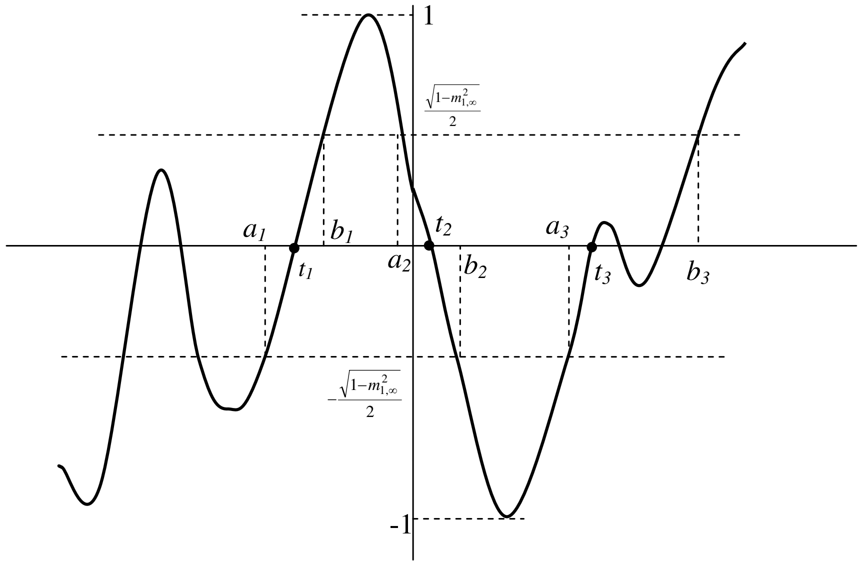

Figure 8: The variations of m_2.

Since m_k is a continuous function on I_k, the construction of these intervals makes sense. It could happen that $N_k = 0$ (i.e., no large variation exists for $m_{2,k}$). We notice that $m_{2,k}(a_n^k)$ has a different sign than both $m_{2,k}(b_n^k)$ and $m_{2,k}(a_{n+1}^k)$ for all n. Moreover,

$$\operatorname{sgn}(m_{2,k}(b_n^k))\, m_{2,k} \leq \frac{\sqrt{1 - {m_{1,\infty}}^2}}{2} \quad \text{in} \quad (b_{n-1}^k, b_n^k) \tag{33}$$

for $n = 1, \ldots, N_k$, where we set $b_0^k = -\infty$. We can estimate the number N_k of large variations; more precisely, we prove that

$$N_k \leq \frac{|I_k|}{1 - {m_{1,\infty}}^2} \int_{\mathbb{R}} \left|\frac{dm_{2,k}}{dt}\right|^2 dt.$$

Indeed, the Cauchy-Schwarz inequality yields

$$1 - {m_{1,\infty}}^2 = \left(\int_{a_n^k}^{b_n^k} \frac{dm_{2,k}}{dt}\, dt \right)^2 \leq (b_n^k - a_n^k) \int_{a_n^k}^{b_n^k} \left|\frac{dm_{2,k}}{dt}\right|^2 dt \leq (b_n^k - a_n^k) \int_{\mathbb{R}} \left|\frac{dm_{2,k}}{dt}\right|^2 dt$$

and we deduce that

$$N_k \leq \frac{\sum_{n=1}^{N_k} (b_n^k - a_n^k)}{1 - {m_{1,\infty}}^2} \int_{\mathbb{R}} \left|\frac{dm_{2,k}}{dt}\right|^2 dt \leq \frac{|I_k|}{1 - {m_{1,\infty}}^2} \int_{\mathbb{R}} \left|\frac{dm_{2,k}}{dt}\right|^2 dt.$$

Step 3. *We prove that the sequence $\{N_k\}_{k\uparrow\infty}$ is uniformly bounded.* The idea is to apply Proposition 5 for a step function χ_k that is adapted to m_k in order to count the large variations of $m_{2,k}$. If $N_k > 0$, we denote by $t_n^k \in [a_n^k, b_n^k]$ the smallest number such that $m_{2,k}(t_n^k) = 0$, $n = 1, \ldots, N_k$. By (32), $m_{1,k}$ does not change sign in (a_n^k, t_n^k). Therefore, we consider

$$\chi_k = \begin{cases} \operatorname{sgn}(m_{1,k}) & \text{in} \quad (a_n^k, t_n^k) \subset I_k, n = 1, \ldots, N_k, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, we obtain:

$$\int_{\mathbb{R}} \Big|\frac{d\chi_k}{dt}\Big| = 2N_k, \tag{34}$$

$$\begin{aligned}\int_{\mathbb{R}} \chi_k \frac{dm_{1,k}}{dt}\,dt &= \sum_{n=1}^{N_k} \int_{a_n^k}^{t_n^k} \operatorname{sgn}(m_{1,k}) \frac{dm_{1,k}}{dt}\,dt \\ &= \sum_{n=1}^{N_k} \Big(|m_{1,k}|(t_n^k) - |m_{1,k}|(a_n^k) \Big) = N_k \Big(1 - \frac{\sqrt{3 + {m_{1,\infty}}^2}}{2} \Big). \end{aligned} \tag{35}$$

As we mentioned in Section 3, the duality term (26) controls the number N_k of large variations of $m_{2,k}$. On the other hand, by Proposition 5, we expect that the duality term (at power 2) is controlled by the energy E_{ε_k} and N_k. More precisely, setting $w_k := |\log \varepsilon_k|^2$, Proposition 5 applied for the parameter $\delta_k := \frac{\varepsilon_k}{|\log \varepsilon_k|} \ll w_k$ and the step function χ_k yields for k large enough:

$$\begin{aligned}\Big| \int_{\mathbb{R}} \chi_k \frac{dm_{1,k}}{dt}\,dt \Big| &= \Big| \int_{\mathbb{R}} \widehat{\chi_k}\, \overline{\widehat{\frac{dm_{1,k}}{dt}}}\, d\xi \Big| \\ &\le \Big(\int_{\mathbb{R}} (\delta_k + \frac{1}{|\xi|} + \frac{1}{w_k|\xi|^2})^{-1} |\widehat{\chi_k}|^2\, d\xi \Big)^{1/2} \\ &\quad \times \Big(\int_{\mathbb{R}} (\delta_k + \frac{1}{|\xi|} + \frac{1}{w_k|\xi|^2}) |\widehat{\frac{dm_{1,k}}{dt}}|^2\, d\xi \Big)^{1/2} \\ &\le \Big(\int_{\mathbb{R}} \min\{\frac{1}{\delta_k}, |\xi|, w_k|\xi|^2\} |\widehat{\chi_k}|^2\, d\xi \Big)^{1/2} \\ &\quad \times \Big(\int_{\mathbb{R}} (\delta_k + \frac{1}{|\xi|} + \frac{1}{w_k|\xi|^2}) |\xi|^2\, |\widehat{m_{1,k} - \alpha_k}|^2\, d\xi \Big)^{1/2} \\ &\overset{(29)}{\le} C \Big(\log \frac{w_k}{\delta_k} \sup_{\mathbb{R}} |\chi_k| \int_{\mathbb{R}} \Big|\frac{d\chi_k}{dt}\Big| \Big)^{1/2} \\ &\quad \times \Big(\delta_k \|m_{1,k}\|_{\dot{H}^1}^2 + \|m_{1,k}\|_{\dot{H}^{1/2}}^2 + \frac{1}{w_k} \|m_{1,k} - \alpha_k\|_{L^2}^2 \Big)^{1/2} \\ &\le C \Big(\sup_{\mathbb{R}} |\chi_k| \int_{\mathbb{R}} \Big|\frac{d\chi_k}{dt}\Big| \Big)^{1/2} \\ &\quad \times \Big(\varepsilon_k \|m_{1,k}\|_{\dot{H}^1}^2 + |\log \varepsilon_k| \|m_{1,k}\|_{\dot{H}^{1/2}}^2 + \frac{1}{|\log \varepsilon_k|} \|m_{1,k} - \alpha_k\|_{L^2}^2 \Big)^{1/2} \end{aligned}$$

that is,

$$\Big| \int_{\mathbb{R}} \chi_k \frac{dm_{1,k}}{dt}\,dt \Big| \le C \Big(E_{\varepsilon_k}(m_k) \int_{\mathbb{R}} \Big|\frac{d\chi_k}{dt}\Big| \Big)^{1/2}.$$

(This duality argument was already used in [6].) Therefore, by (10), (34) and (35), we deduce that

$$N_k \le C N_k^{1/2},$$

which yields that $N_k \le C$ for every k where $C > 0$ is some absolute constant.

Step 4. *We show that the sequence $\{m_{2,k}\}_{k\uparrow\infty}$ is relatively compact in $L^1_{loc}(\mathbb{R})$.* First, we construct a good approximating step function $\psi_k : \mathbb{R} \to \{\pm\sqrt{1-m_{1,\infty}{}^2}\}$ for $m_{2,k}$. If $N_k > 0$, then we choose

$$\psi_k = \sum_{n=1}^{N_k} 2m_{2,k}(a_n^k)\ 1_{(t_{n-1}^k, t_n^k)} + 2m_{2,k}(b_{N_k}^k)\ 1_{(t_{N_k}^k, +\infty)},$$

where $t_0^k = -\infty$ and t_n^k are given in Step 3 for $n = 1, \ldots, N_k$. If $N_k = 0$, then $m_{2,k}$ stays either above $-\frac{\sqrt{1-m_{1,\infty}{}^2}}{2}$ or below $\frac{\sqrt{1-m_{1,\infty}{}^2}}{2}$ in $\mathbb{R}$ and it has the same limit at $\pm\infty$, i.e., $m_{2,k}(-\infty) = m_{2,k}(+\infty) \in \{\pm\sqrt{1-\alpha_k^2}\}$; in this case, set

$$\psi_k \equiv \operatorname{sgn}\Big(m_{2,k}(+\infty)\Big)\sqrt{1-m_{1,\infty}{}^2} \quad \text{in} \quad \mathbb{R}.$$

It is obvious that

$$\int_{\mathbb{R}} |\frac{d\psi_k}{dt}| = 2N_k\sqrt{1-m_{1,\infty}{}^2}.$$

It follows by Step 3 that the sequence $\{\psi_k\}$ has uniformly bounded total variation. Therefore, any accumulation point $\psi : \mathbb{R} \to \{\pm\sqrt{1-m_{1,\infty}{}^2}\}$ of $\{\psi_k\}_{k\uparrow\infty}$ in $L^1_{loc}(\mathbb{R})$ is of bounded total variation and has the form

$$\psi = \sum_{n=1}^{N+1} (-1)^n \sin\theta\ 1_{(t_{n-1}, t_n)},$$

where $N \in \mathbb{N}$, $\cos\theta = m_{1,\infty}$, $\theta \in [0, 2\pi)$ and $-\infty = t_0 < t_1 < \cdots < t_N < t_{N+1} = +\infty$. Finally, (33) leads to

$$|\psi_k + m_{2,k}| \geq \frac{\sqrt{1-m_{1,\infty}{}^2}}{2} \quad \text{in } \mathbb{R}, \tag{36}$$

and by Step 1, we have for every bounded interval $I \subset \mathbb{R}$,

$$\begin{aligned}
\int_I |\psi_k - m_{2,k}|^2\, dt &\overset{(36)}{\leq} \frac{4}{1-m_{1,\infty}{}^2}\int_I |\psi_k^2 - m_{2,k}^2|^2\, dt \\
&\leq \frac{4}{1-m_{1,\infty}{}^2}\int_I |(1-m_{1,\infty}{}^2) - m_{2,k}^2|^2\, dt \\
&\leq \frac{16}{1-m_{1,\infty}{}^2}\int_I |m_{1,k} - m_{1,\infty}|^2\, dt \overset{\textit{Step 1}}{\to} 0 \quad \text{as } k\uparrow\infty.
\end{aligned}$$

Since $\psi_k \to \psi$ in L^1_{loc}, it follows that $m_{2,k} \to \psi$ in $L^1_{loc}(\mathbb{R})$, i.e.,

$$m_k \to \begin{pmatrix} \cos\theta \\ \psi \end{pmatrix} \quad \text{in } L^1_{loc}(\mathbb{R}, S^1) \text{ as } k\uparrow\infty.$$

□

Now we prove the compactness result for Model 2 (with the same convention that every convergence should be considered up to a subsequence):

Proof of Theorem 3. The same steps as in the proof of Theorem 1 are to be followed. The only difference concerns the compactness of $\{m_{1,k}\}$. The L^2-norm of $m_{1,k} - \cos\theta_k$ present in the energy E_{ε_k} in Model 1 does not appear anymore in the energy F_{ε_k}. However, this term is penalized by F_{ε_k}

via the compact embedding $\dot{H}^{1/2}(-1,1) \hookrightarrow L^2(-1,1)$. More precisely, we show that there exists a constant $m_{1,\infty} \in [-1,1]$ such that

$$m_{1,k} \to m_{1,\infty} \quad \text{in} \quad L^1_{loc}(\mathbb{R}) \text{ as } k \uparrow \infty.$$

Indeed, let $m_{1,\infty} \in [-1,1]$ be such that $\cos\theta_k \to m_{1,\infty}$ as $k \uparrow \infty$. We have

$$\begin{aligned}
\int_{\mathbb{R}} |m_{1,k} - \cos\theta_k|^2 \, dt &\overset{(6)}{=} \int_{-1}^{1} |m_{1,k} - \cos\theta_k|^2 \, dt \\
&\overset{(6)}{=} \int_{-1}^{1}\int_{2}^{3} |m_{1,k}(t) - m_{1,k}(t+s)|^2 \, dt\, ds \\
&\leq 9 \int_{-1}^{1}\int_{2}^{3} \frac{|m_{1,k}(t) - m_{1,k}(t+s)|^2}{s^2} \, dt\, ds \\
&\leq 9 \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|m_{1,k}(t) - m_{1,k}(\tilde{t})|^2}{|t-\tilde{t}|^2} \, dt\, d\tilde{t} \overset{(18)}{\leq} 18\pi \frac{F_{\varepsilon_k}(m_k)}{|\log\varepsilon_k|} \overset{(14)}{\to} 0 \quad \text{as} \quad k \uparrow \infty.
\end{aligned}$$

The conclusion of Step 1 now is straightforward. As before, if $m_{1,\infty} \in \{-1,1\}$, then

$$m_{2,k} \to 0 \quad \text{in} \quad L^1_{loc}(\mathbb{R}).$$

In this case, the limit m is a constant function equal to $(m_{1,\infty}, 0)$.

If $m_{1,\infty} \in (-1,1)$, the rest of the proof comes by repeating the arguments at Steps 2, 3 and 4 in the proof of Theorem 1. In this case, condition (6) implies that for k large enough, the number N_k of intervals $\{(a_k^n, b_k^n)\}_{1\leq n\leq N_k}$ of large variations of $m_{1,k}$ and $m_{2,k}$ is an odd number. Moreover, any accumulation point $m = (m_1, m_2) : \mathbb{R} \to S^1$ satisfies (6), m is constant on the intervals $(-\infty,-1)$ and $(1,+\infty)$, respectively and m_2 has different sign on these intervals; therefore, m has an odd number N of walls. □

The proof for the compactness result in Model 3 can be adapted as follows:

Proof of Theorem 5. The same ideas presented in the proof of Theorems 1 and 3 are to be repeated. We only develop the arguments that are different with respect to Models 1 and 2. The main issue is that (8) imposes that the normal component of the limit magnetization across the walls vanishes. More precisely, we show that

$$m_{1,k} \to 0 \quad \text{in} \quad L^1_{loc}(\mathbb{R}) \text{ as } k \uparrow \infty.$$

Indeed, let us denote the mean value of $m_{1,k}$ on $(-1,1)$ by

$$\alpha_k := \frac{\widehat{m_{1,k}}(0)}{\sqrt{2}} = \fint_{-1}^{1} m_{1,k}(t)\, dt \in [-1,1].$$

There exists $m_{1,\infty} \in [-1,1]$ such that $\alpha_k \to m_{1,\infty}$ as $k \uparrow \infty$. By Plancherel's identity, we have

$$\begin{aligned}
\int_{-1}^{1} |m_{1,k} - \alpha_k|^2 \, dt &= \sum_{\beta\in\pi\mathbb{Z}\setminus\{0\}} |\widehat{m_{1,k}}(\beta)|^2 \\
&\leq \sum_{\beta\in\pi\mathbb{Z}\setminus\{0\}} |\beta||\widehat{m_{1,k}}(\beta)|^2 \leq \frac{G_{\varepsilon_k}(m_k)}{|\log\varepsilon_k|} \overset{(16)}{\to} 0 \quad \text{as} \quad k \uparrow \infty.
\end{aligned}$$

Thus, we deduce that $m_{1,k} \to m_{1,\infty}$ in L^2_{per} as $k \uparrow \infty$. On the other hand, we know by (8) that

$$m_{1,k}(t) = -m_{1,k}(t+1) \quad \text{for every } t \in \mathbb{R}. \tag{37}$$

Passing to the limit $k \uparrow \infty$, we deduce that $m_{1,\infty} = 0$. As a consequence, $|m_{2,k}| \to 1$ in $L^2(-1,1)$ as $k \to \infty$.

Then we consider all the intervals $(a_n^k, b_n^k) \subset \mathbb{R}$, $n \in \mathbb{Z}$ where $m_{2,k}$ varies between $-\frac{1}{2}$ and $\frac{1}{2}$ as at Step 2 in the proof of Theorem 1. Since $m_{2,k}$ is 2−periodic, these intervals are periodically distributed. As before, the number N_k of such intervals included in $[-1,1)$ is finite.

Now we prove that $\{N_k\}$ is uniformly bounded. One can construct a 2−periodic step function χ_k that is adapted to the large variations of $m_{2,k}$ as at Step 3 in the proof of Theorem 1. Then we apply inequality (30) for the parameters $w_k := |\log \varepsilon_k|^2$ and $\delta_k := \frac{\varepsilon_k}{|\log \varepsilon_k|} \ll w_k$ and the step function χ_k. For the sake of completeness, we rewrite the duality argument in the periodic case. We have for k large enough:

$$\begin{aligned}
\left| \int_{[-1,1)} \chi_k \frac{dm_{1,k}}{dt}\, dt \right| &= \left| \sum_{\beta \in \pi\mathbb{Z}} \widehat{\chi_k}(\beta)\, \overline{\widehat{\frac{dm_{1,k}}{dt}}(\beta)} \right| \\
&\le \Bigg(\sum_{\beta \in \pi\mathbb{Z}} \min\{\frac{1}{\delta_k}, |\beta|, w_k|\beta|^2\} |\widehat{\chi_k}(\beta)|^2 \Bigg)^{1/2} \\
&\quad \times \Bigg(\sum_{\beta \in \pi\mathbb{Z}} (\delta_k + \frac{1}{|\beta|} + \frac{1}{w_k|\beta|^2})|\beta|^2\, |\widehat{m_{1,k}}(\beta)|^2 \Bigg)^{1/2} \\
&\overset{(30)}{\le} C\Bigg(\log \frac{w_k}{\delta_k} \sup_{[-1,1)} |\chi_k| \int_{[-1,1)} \left| \frac{d\chi_k}{dt} \right| \Bigg)^{1/2} \\
&\quad \times \Bigg(\delta_k \int_{[-1,1)} \left| \frac{dm_{1,k}}{dt} \right|^2 dt + \sum_{\beta \in \pi\mathbb{Z}} |\beta|\, |\widehat{m_{1,k}}(\beta)|^2 + \frac{1}{w_k} \int_{[-1,1)} |m_{1,k} - \alpha_k|^2\, dt \Bigg)^{1/2} \\
&\le C\Bigg((G_{\varepsilon_k}(m_k) + 1) \int_{[-1,1)} |\frac{d\chi_k}{dt}| \Bigg)^{1/2}.
\end{aligned}$$

As before, one concludes that $\{N_k\}$ is uniformly bounded.

As at Step 4 in the proof of Theorem 1, we construct a 2−periodic step function ψ_k with values in $\{\pm 1\}$ that approximates $m_{2,k}$. The sequence $\{\psi_k\}$ has uniformly bounded total variation in $[-1,1)$ and the set of its accumulation points coincides with the one of $\{m_{2,k}\}$. Such a limit m_2 is a 2−periodic step function belonging to $BV_{loc}(\mathbb{R}, \{\pm 1\})$. Moreover, by (8), we have that $m_2(t) = -m_2(t+1)$ for all $t \in \mathbb{R}$. Therefore, m_2 has an even number $2N$ of walls on $[-1,1)$ with $N \ge 1$ and writes as

$$m_2 = \pm \sum_{n=1}^{N+1} (-1)^n \Big(1_{(t_{n-1}, t_n)} - 1_{(t_{n-1}+1, t_n+1)} \Big) \quad \text{in } [-1,1),$$

where $-1 = t_0 \le t_1 < \cdots < t_N < t_{N+1} = 0$.

□

5 Lower bound

We prove the first assertion in Theorem 2 for the lower bound of the energy E_ε in Model 1:

Proof of *(i)* in Theorem 2. By Theorem 1, we know that $m \in \mathcal{A}$, i.e.,

$$m = \sum_{n=1}^{N+1} \begin{pmatrix} \cos\theta \\ (-1)^n \sin\theta \end{pmatrix} 1_{(t_{n-1},t_n)},$$

where $\theta \in [0, 2\pi)$ and $-\infty = t_0 < t_1 < \cdots < t_N < t_{N+1} = +\infty$. Notice that if $\sin\theta = 0$ (i.e., $|m_1| = 1$) or $N = 0$, then $E_0(m) = 0$ and the inequality (13) is trivial. Therefore, we assume that $N \geq 1$ and $|\sin\theta| > 0$. Set the interval

$$I := (t_1 - 1, t_N + 1) = (x_1^0 - R, x_1^0 + R)$$

with $x_1^0 := \frac{t_1+t_N}{2}$ and $R := 1 + \frac{t_N - t_1}{2}$. Let

$$\mu := \begin{cases} \frac{1}{5} \min_{2\leq n\leq N} \{|t_n - t_{n-1}|, 1\} & \text{if } N \geq 2, \\ \frac{1}{5} & \text{if } N = 1. \end{cases} \tag{38}$$

Since $m_{2,k} \to m_2$ in $L^1(I)$, there exists $k_\mu \in \mathbb{N}$ such that for every $k \geq k_\mu$ we have

$$\int_I |m_{2,k} - m_2|\, dt \leq \frac{\mu |\sin\theta|}{4}. \tag{39}$$

This condition implies that for every $n = 1, \ldots, N$ and every $k \geq k_\mu$, $m_{2,k}$ changes sign on $(t_n - \frac{\mu}{2}, t_n + \frac{\mu}{2})$. Suppose that this is not the case. W.l.o.g., we may assume that $m_{2,k} \geq 0$ in $(t_n - \frac{\mu}{2}, t_n + \frac{\mu}{2})$. We know that the second component m_2 of the limit configuration is negative either in $(t_n - \frac{\mu}{2}, t_n)$ or in $(t_n, t_n + \frac{\mu}{2})$. On that interval of length $\mu/2$, we have $m_2 = -|\sin\theta|$ and $|m_{2,k} - m_2| \geq |\sin\theta|$. It would mean that

$$\int_{t_n - \frac{\mu}{2}}^{t_n + \frac{\mu}{2}} |m_{2,k} - m_2|\, dt \geq \frac{\mu |\sin\theta|}{2}$$

which contradicts with (39).

By (39), the continuity of $m_{2,k}$ yields the existence of $t_n^k \in (t_n - \frac{\mu}{2}, t_n + \frac{\mu}{2})$ with

$$m_{2,k}(t_n^k) = 0, \text{ i.e., } |m_{1,k}(t_n^k)| = 1, \ n = 1, \ldots, N. \tag{40}$$

For every $k \geq k_\mu$, we define the step functions $\chi_k : \mathbb{R} \to \{-1, 0, 1\}$,

$$\chi_k = \begin{cases} \operatorname{sgn}(m_{1,k}(t_n^k)) & \text{in } (t_n - 2\mu, t_n^k), n = 1, \ldots, N, \\ -\operatorname{sgn}(m_{1,k}(t_n^k)) & \text{in } (t_n^k, t_n + 2\mu), n = 1, \ldots, N, \\ 0 & \text{elsewhere.} \end{cases} \tag{41}$$

We also consider the cut-off function $\eta_k \in C_c^\infty(B((x_1^0, 0), R) \subset \mathbb{R}^2)$ be such that

$$\eta_k(t_n^k, 0) = 1 \text{ and } \operatorname{supp} \eta_k(\cdot, 0) \subset \cup_{n=1}^N (t_n - \mu, t_n + \mu) \text{ and } |\eta_k| \leq 1, |\nabla \eta_k| \leq \frac{C}{\mu} \text{ in } B((x_1^0, 0), R). \tag{42}$$

We have that

$$\int_{\mathbb{R}} \eta_k^2(\cdot,0)|\frac{d\chi_k}{dt}| \overset{(42)}{=} \sum_{n=1}^{N} \int_{t_n-\mu}^{t_n+\mu} \eta_k^2(\cdot,0)|\frac{d\chi_k}{dt}| \overset{(41),(42)}{=} 2N. \tag{43}$$

The localized duality term (26) controls the number N of large variations of $m_{1,k}$ and $m_{2,k}$. Indeed, integration by parts leads to

$$\begin{aligned}
\int_{\mathbb{R}} \eta_k^2(\cdot,0)\chi_k \frac{dm_{1,k}}{dx_1} &\overset{(41),(42)}{=} \sum_{n=1}^{N} \mathrm{sgn}(m_{1,k}(t_n^k))\Big(\int_{t_n-\mu}^{t_n^k} \eta_k^2(\cdot,0)\frac{dm_{1,k}}{dx_1} - \int_{t_n^k}^{t_n+\mu} \eta_k^2(\cdot,0)\frac{dm_{1,k}}{dx_1}\Big) \\
&\overset{(42)}{=} \sum_{n=1}^{N} \mathrm{sgn}(m_{1,k}(t_n^k))\Big(2m_{1,k}(t_n^k) - \int_{t_n-\mu}^{t_n^k} m_{1,k}\frac{d\eta_k^2(\cdot,0)}{dx_1} + \int_{t_n^k}^{t_n+\mu} m_{1,k}\frac{d\eta_k^2(\cdot,0)}{dx_1}\Big) \\
&\overset{(42)}{\geq} \sum_{n=1}^{N} \mathrm{sgn}(m_{1,k}(t_n^k))\Big(2m_{1,k}(t_n^k) - 2\cos\theta - \int_{t_n-\mu}^{t_n+\mu} |m_{1,k}-\cos\theta||\frac{d\eta_k^2(\cdot,0)}{dx_1}|\Big) \\
&\overset{(40),(42)}{\geq} N(2-2|\cos\theta|) - \frac{C}{\mu}\int_I |m_{1,k}-\cos\theta|\,dx_1.
\end{aligned} \tag{44}$$

Set $\delta_k = \frac{\varepsilon_k}{|\log \varepsilon_k|}$. Let $h_k : \mathbb{R}^2 \to \mathbb{R}^2$ be any stray field associated to $m_{1,k}$ by (22). We apply Proposition 6 for the parameter $\delta_k \ll R$ and the functions χ_k and η_k (with the support lying inside $B((x_1^0,0),R) \subset \mathbb{R}^2$):

$$\begin{aligned}
\int_{\mathbb{R}} \eta_k^2(x_1,0)\chi_k \frac{dm_{1,k}}{dx_1}\,dx_1 \leq{}& \Big(\frac{4}{\pi}|\log\delta_k| \int_{\mathbb{R}} \eta_k^2(\cdot,0)|\frac{d\chi_k}{dt}| \int_{\mathbb{R}^2} |h_k|^2\,dx\Big)^{1/2} \\
&+ C(R)\Big(\delta_k \int_{\mathbb{R}} |\frac{dm_{1,k}}{dx_1}|^2\,dx_1 + \int_{\mathbb{R}^2} |h_k|^2\,dx\Big)^{1/2} \\
&\times (1+\frac{C}{\mu})\Big(1 + \int_{\mathbb{R}} |\eta_k(\cdot,0)|\,|\frac{d\chi_k}{dt}|\Big).
\end{aligned}$$

We minimize over all stray fields h_k with (22). By (23), (43) and (44), it implies that

$$\begin{aligned}
2N(1-|\cos\theta|) &- \frac{C}{\mu}\int_I |m_{1,k}-\cos\theta| \\
&\leq \Big(\frac{4N}{\pi}|\log\delta_k|\,\|m_{1,k}\|^2_{\dot{H}^{1/2}}\Big)^{1/2} + \frac{C(R)N}{\mu}|\log\varepsilon_k|^{-1/2}E_{\varepsilon_k}(m_k)^{1/2}.
\end{aligned}$$

Since $m_{1,k} \to \cos\theta$ in $L^1(I)$, we conclude by passing to liminf as $k \uparrow \infty$ that

$$\pi N(1-|\cos\theta|)^2 \leq \liminf_{k\to\infty} |\log\varepsilon_k|\,\|m_{1,k}\|^2_{\dot{H}^{1/2}}$$

which leads to (13). □

Proof of *(i)* in Theorem 4. The arguments presented in the proof of *(i)* of Theorem 2 for Model 1 are to be repeated for Model 2. □

Proof of *(i)* of Theorem 6. For the sake of completeness, we adapt the arguments presented above to the periodic case in Model 3.
(i) Let $m \in \mathcal{C}$, i.e.,

$$m = \sum_{n\in\mathbb{Z}} \begin{pmatrix} 0 \\ (-1)^n \end{pmatrix} 1_{(t_{n-1},t_n)}$$

where we denote by $\{t_n\}_{n\in\mathbb{Z}}$ the sequence of walls of m that is 1−periodically distributed in $\mathbb{R}$ such that $m(t) = -m(t+1)$ for every $t \in \mathbb{R}$. Let N be the number of jumps of m in $[-1, 1)$ and $N \geq 2$ is an even number. Set

$$\mu = \frac{1}{5} \min_{n\in\mathbb{Z}} \{|t_n - t_{n-1}|\}.$$

For k large enough, using the same argument as in the proof of (i) of Theorem 2, one detects a sequence $\{t_n^k\}_{n\in\mathbb{Z}}$ of zeros of $m_{2,k}$ that is 1−periodically distributed in $\mathbb{R}$ and contains exactly N terms in $[-1, 1)$. Then we construct a similar step function χ_k to (41) and a cut-off function η_k corresponding to (42) that are both 2−periodic. We have

$$\int_{[-1,1)} \eta_k^2(\cdot, 0)|\frac{d\chi_k}{dt}|=2N \quad \text{and} \quad \int_{[-1,1)} \eta_k^2(\cdot, 0)\chi_k \frac{dm_{1,k}}{dx_1} \geq 2N - \frac{C}{\mu}\int_{[-1,1)} |m_{1,k}|\, dx_1. \tag{45}$$

We apply the periodic version of inequality in (ii) of Proposition 6 for $\delta_k = \frac{\varepsilon_k}{|\log \varepsilon_k|}$ small enough and any stray field $h_k : \mathbb{R}^2 \to \mathbb{R}^2$ associated to $m_{1,k}$ by (22) that is 2−periodic in x_1. It leads to

$$\begin{aligned}\int_{[-1,1)} \eta_k^2(x_1, 0)\chi_k \frac{dm_{1,k}}{dx_1}\, dx_1 \leq& \Big(\frac{4}{\pi}|\log \delta_k| \int_{[-1,1)} \eta_k^2(\cdot, 0)|\frac{d\chi_k}{dt}| \int_{[-1,1)\times\mathbb{R}} |h_k|^2\, dx\Big)^{1/2}\\ &+ C\Big(\delta_k \int_{[-1,1)} |\frac{dm_{1,k}}{dx_1}|^2\, dx_1 + \int_{[-1,1)\times\mathbb{R}} |h_k|^2\, dx\Big)^{1/2}\\ &\times (1+\frac{C}{\mu})\Big(1+\int_{[-1,1)} |\eta_k(\cdot, 0)|\,|\frac{d\chi_k}{dt}|\Big).\end{aligned}$$

We minimize over all 2−periodic (in x_1−direction) stray fields h_k that satisfy (22). Using the assumption that $m_{1,k} \to 0$ in $L^1_{loc}(\mathbb{R})$, it implies by (25) and (45),

$$2N - \frac{C}{\mu}\int_{[-1,1)} |m_{1,k}|\, dx_1 \leq \Big(\frac{4N}{\pi}|\log \delta_k| \|m_{1,k}\|^2_{\dot{H}^{1/2}_{per}}\Big)^{1/2} + \frac{CN}{\mu}|\log \varepsilon_k|^{-1/2} G_{\varepsilon_k}(m_k)^{1/2}.$$

We conclude that

$$\pi N \leq \liminf_{k\to\infty} |\log \varepsilon_k| \|m_{1,k}\|^2_{\dot{H}^{1/2}_{per}}.$$

□

6 Upper bound

Now we prove the second assertion in Theorem 2 for the attainment of the lower bound of the energy E_ε in Model 1:

Proof of *(ii)* in Theorem 2. Let $m \in \mathcal{A}$, i.e.,

$$m = \sum_{n=1}^{N+1} \begin{pmatrix} \cos\theta \\ (-1)^n \sin\theta \end{pmatrix} 1_{(t_{n-1}, t_n)},$$

where $\theta \in [0, 2\pi)$ and $-\infty = t_0 < t_1 < \cdots < t_N < t_{N+1} = +\infty$. We want to construct smooth transition layers m_k such that $m_k - m$ has compact support in $\mathbb{R}$, $m_k \to m$ in $L^1_{loc}(\mathbb{R}, S^1)$ and

$$\limsup_{k\uparrow\infty} E_{\varepsilon_k}(m_k) \leq E_0(m). \tag{46}$$

In the case when $N = 0$ or $\sin\theta = 0$, i.e., m is constant, then $E_0(m) = E_\varepsilon(m) = 0$ and hence, we may consider $m_k := m$. Otherwise, $N \geq 1$ and $\sin\theta \neq 0$. W.l.o.g., we assume that $\cos\theta \geq 0$. (For the case $\cos\theta < 0$, one should consider the sequence $m_k = (-m_{1,k}, m_{2,k})$.)

Let $\varepsilon > 0$ and set $\delta := \varepsilon|\log\varepsilon|$. We consider the following transition layer $(u_\varepsilon, v_\varepsilon) : \mathbb{R} \to S^1$ that approximates a wall of angle 2θ centered at the origin (see Figure 9):

$$u_\varepsilon(t) = \begin{cases} \cos\theta + (1-\cos\theta)\frac{|\log\sqrt{t^2+\delta^2}|}{|\log\delta|} & \text{if } |t| \leq \sqrt{1-\delta^2}, \\ \cos\theta & \text{elsewhere,} \end{cases} \tag{47}$$

and

$$v_\varepsilon(t) = \begin{cases} -\operatorname{sgn}(\sin\theta)\sqrt{1-u_\varepsilon^2(t)} & \text{if } t \leq 0, \\ \operatorname{sgn}(\sin\theta)\sqrt{1-u_\varepsilon^2(t)} & \text{if } t \geq 0. \end{cases} \tag{48}$$

Then $(u_\varepsilon, v_\varepsilon) \in \dot{H}^1 \cap C^0(\mathbb{R}, S^1)$ and

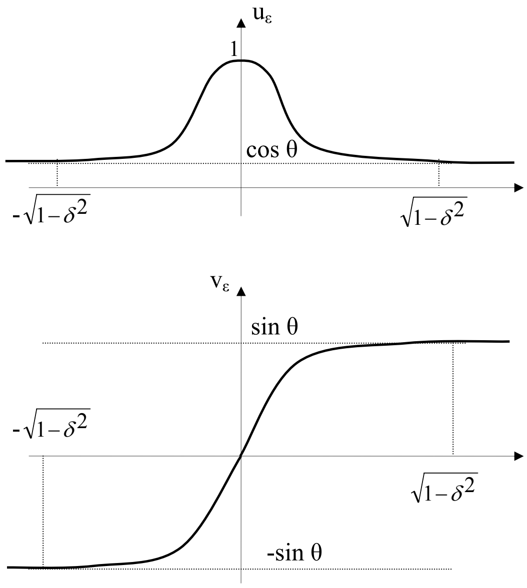

Figure 9: Transition layer $(u_\varepsilon, v_\varepsilon)$ of angle 2θ with a core of size $\delta = \varepsilon|\log\varepsilon|$.

$$(u_\varepsilon, v_\varepsilon) \to (u_0, v_0) := \begin{cases} (\cos\theta, -\sin\theta) & \text{if } t \leq 0, \\ (\cos\theta, \sin\theta) & \text{if } t \geq 0, \end{cases} \quad \text{in } L^1_{loc}(\mathbb{R}, S^1) \text{ as } \varepsilon \downarrow 0.$$

We will prove that

$$\limsup_{\varepsilon\downarrow 0} E_\varepsilon((u_\varepsilon, v_\varepsilon)) \leq E_0((u_0, v_0)) = \pi(1-\cos\theta)^2. \tag{49}$$

First, we estimate the exchange energy corresponding to the transition layer:

$$
\begin{aligned}
\varepsilon \int_{\mathbb{R}} |\frac{du_\varepsilon}{dt}|^2 + |\frac{dv_\varepsilon}{dt}|^2 \, dt &= \varepsilon \int_{\mathbb{R}} \frac{1}{1-u_\varepsilon^2} |\frac{du_\varepsilon}{dt}|^2 \, dt \\
&\le \varepsilon \int_{\mathbb{R}} \frac{1}{1-u_\varepsilon} |\frac{du_\varepsilon}{dt}|^2 \, dt \\
&= \frac{2\varepsilon(1-\cos\theta)}{|\log\delta|} \int_{-\sqrt{1-\delta^2}}^{\sqrt{1-\delta^2}} \frac{t^2}{(t^2+\delta^2)^2 \log\frac{t^2+\delta^2}{\delta^2}} \, dt \\
&\le \frac{4\varepsilon(1-\cos\theta)}{\delta|\log\delta|} \int_0^{1/\delta} \frac{s^2}{(s^2+1)^2\log(s^2+1)} \, ds \\
&= O\Big(\frac{1}{|\log\varepsilon||\log\delta|}\Big) = O\Big(\frac{1}{|\log\varepsilon|^2}\Big). \qquad (50)
\end{aligned}
$$

For the anisotropy term, integration by parts leads to the following estimate:

$$
\begin{aligned}
\int_{\mathbb{R}} |u_\varepsilon - \cos\theta|^2 \, dt &= \frac{(1-\cos\theta)^2}{2|\log\delta|^2} \int_0^{\sqrt{1-\delta^2}} \log^2(t^2+\delta^2) \, dt \\
&= \frac{2(1-\cos\theta)^2}{|\log\delta|^2} \int_0^{\sqrt{1-\delta^2}} \frac{t^2}{t^2+\delta^2} \log\frac{1}{t^2+\delta^2} \, dt \\
&\le \frac{2(1-\cos\theta)^2}{|\log\delta|^2} \int_0^{\sqrt{1-\delta^2}} \log\frac{1}{t^2+\delta^2} \, dt \\
&= \frac{4(1-\cos\theta)^2}{|\log\delta|^2} \int_0^{\sqrt{1-\delta^2}} \frac{t^2}{t^2+1} \, dt \qquad (51) \\
&= O\Big(\frac{1}{|\log\delta|^2}\Big) = O\Big(\frac{1}{|\log\varepsilon|^2}\Big). \qquad (52)
\end{aligned}
$$

In order to estimate the stray-field energy, let U_ε be the radial extension of u_ε in $\mathbb{R}^2$:

$$
U_\varepsilon(x_1,x_2) = u_\varepsilon(\sqrt{x_1^2+x_2^2}\,).
$$

By (20), it follows that

$$
\begin{aligned}
\|u_\varepsilon\|^2_{\dot H^{1/2}} &\le \frac{1}{2} \int_{\mathbb{R}^2} |\nabla U_\varepsilon|^2 \, dx \\
&\le \pi \int_0^1 r |\frac{du_\varepsilon}{dr}|^2 \, dr \\
&\le \frac{\pi(1-\cos\theta)^2}{|\log\delta|^2} \int_0^1 \frac{r^3}{(r^2+\delta^2)^2} \, dr \\
&= \frac{\pi(1-\cos\theta)^2}{|\log\delta|^2} \int_0^{1/\delta} \frac{s^3}{(s^2+1)^2} \, ds \\
&\le \frac{\pi(1-\cos\theta)^2}{|\log\delta|^2}(1+|\log\delta|) = \frac{\pi(1-\cos\theta)^2}{|\log\varepsilon|} + O\Big(\frac{\log|\log\varepsilon|}{|\log\varepsilon|^2}\Big). \qquad (53)
\end{aligned}
$$

Hence, (50), (52) and (53) yield that

$$
E_\varepsilon((u_\varepsilon, v_\varepsilon)) \le \pi(1-\cos\theta)^2 + O\Big(\frac{\log|\log\varepsilon|}{|\log\varepsilon|}\Big),
$$

and (49) immediately follows.

We adapt this transition layer for the walls of the limit magnetization m. Let $T_p f(\cdot) = f(\cdot - p)$ be the translation operator and $R_l f(\cdot) = f\big(\frac{\cdot}{l}\big)$ be the dilation operator. Let μ be given by (38). For every $k \in \mathbb{N}$, we consider

$$\delta_k := \varepsilon_k |\log \varepsilon_k|$$

and $m_k := (m_{1,k}, m_{2,k})$ with

$$m_{1,k}(t) = \begin{cases} R_\mu T_{t_n} u_{\varepsilon_k}(t) & \text{if } t \in (t_n - \mu, t_n + \mu), n = 1, \dots, N, \\ \cos\theta & \text{elsewhere,} \end{cases}$$

and

$$m_{2,k}(t) = \begin{cases} (-1)^{n-1} R_\mu T_{t_n} v_{\varepsilon_k}(t) & \text{if } t \in (t_n - \mu, t_n + \mu), n = 1, \dots, N, \\ (-1)^n \sin\theta & \text{elsewhere in } (t_{n-1}, t_n), n = 1, \dots, N+1. \end{cases}$$

Then

$$m_k - m \to 0 \quad \text{in } L^1(\mathbb{R}, \mathbb{R}^2) \text{ as } k \uparrow \infty$$

and (46) holds. Indeed, the exchange energy and the anisotropy estimate like:

$$\begin{aligned} \varepsilon_k \int_{\mathbb{R}} \Big|\frac{dm_k}{dt}\Big|^2 dt = \varepsilon_k \sum_{n=1}^{N} \int_{t_n-\mu}^{t_n+\mu} \Big|\frac{d}{dt} R_\mu T_{t_n}(u_{\varepsilon_k}, v_{\varepsilon_k})\Big|^2 dt = \frac{N\varepsilon_k}{\mu} \int_{\mathbb{R}} \frac{1}{1-u_{\varepsilon_k}^2} \Big|\frac{du_{\varepsilon_k}}{ds}\Big|^2 ds \\ \overset{(50)}{=} O\Big(\frac{N}{\mu |\log \varepsilon_k|^2}\Big), \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}} |m_{1,k} - \cos\theta|^2 dt = \sum_{n=1}^{N} \int_{t_n-\mu}^{t_n+\mu} \big|R_\mu T_{t_n} u_{\varepsilon_k} - \cos\theta\big|^2 dt = \mu N \int_{-1}^{1} |u_{\varepsilon_k} - \cos\theta|^2 ds \\ \overset{(52)}{=} O\Big(\frac{\mu N}{|\log \varepsilon_k|^2}\Big). \end{aligned}$$

In order to estimate the stray-field energy, we introduce the following extension M_k of $m_{1,k}$ in $\mathbb{R}^2$:

$$M_k(x_1, x_2) = \begin{cases} R_\mu T_{(t_n,0)} U_{\varepsilon_k}(x_1, x_2) & \text{if } x_1 \in (t_n - \mu, t_n + \mu), n = 1, \dots, N, \\ \cos\theta & \text{elsewhere in } \mathbb{R}^2, \end{cases}$$

where U_{ε_k} is the radial extension of u_{ε_k} in $\mathbb{R}^2$. Then it follows by (20),

$$\begin{aligned} \|m_{1,k}\|^2_{\dot{H}^{1/2}} &\le \frac{1}{2} \int_{\mathbb{R}^2} |\nabla M_k|^2 dx \\ &= \frac{1}{2} \sum_{n=1}^{N} \int_{(t_n-\mu, t_n+\mu)\times\mathbb{R}} |\nabla R_\mu T_{(t_n,0)} U_{\varepsilon_k}|^2 dx \\ &= \frac{N}{2} \int_{I\times\mathbb{R}} |\nabla U_{\varepsilon_k}|^2 dy \\ &\overset{(53)}{\le} \frac{\pi(1-\cos\theta)^2 N}{|\log \delta_k|^2} (1 + |\log \delta_k|) = \frac{\pi N (1-\cos\theta)^2}{|\log \varepsilon_k|} + O\Big(\frac{\log|\log \varepsilon_k|}{|\log \varepsilon_k|^2}\Big). \end{aligned}$$

From here, (46) follows. Observe that $m_k - m$ has a compact support in $(t_1 - 1, t_N + 1)$. For each $k \in \mathbb{N}$, the function m_k only belongs to $\dot{H}^1 \cap C^0(\mathbb{R}, S^1)$ and it is not a C^1 function. However, one can approximate m_k in $H^1_{loc}(\mathbb{R}, S^1)$ by a sequence of smooth functions $\{m_k^n : \mathbb{R} \to S^1\}_{n\in\mathbb{N}}$ that coincide with m outside the interval $(t_1 - 1, t_N + 1)$. Then

$$E_{\varepsilon_k}(m_k^n) \to E_{\varepsilon_k}(m_k) \text{ as } n \uparrow \infty.$$

Therefore, by a diagonal selection argument, m can be approximated by a smooth sequence, still denoted by $\{m_k\}$, for which (46) holds. Moreover, since $m_k \to m$ in $L^1_{loc}(\mathbb{R}, S^1)$, by *(i)* in Theorem 2, (13) holds and now the conclusion is straightforward. □

Proof of *(ii)* in Theorem 4. For the construction of the recovery sequence in Model 2, the only difference with respect to Model 1 is the following: the approximating sequence should satisfy $m_k = m$ in $\mathbb{R} \setminus [-1, 1]$. This condition is not satisfied by the sequence built for Model 1 if m has a wall at the boundary of the sample $[-1, 1]$, i.e., $t_1 = -1$ or $t_N = 1$. Let us explain the way to fix this problem when $t_1 = -1$ (the case $t_N = 1$ is similar). Let μ be given by (38). For $0 < \gamma < \mu$, set $t_1^{(\gamma)} = -1 + \gamma$ and $m^{(\gamma)}$ be the modified limit function m that has the first wall present in $t_1^{(\gamma)}$ (and not in t_1) and the other walls remain in t_2, ..., t_N. Then we consider the sequence $\{m_k^{(\gamma)}\}_{k\in\mathbb{N}}$ constructed in the proof of (ii) of Theorem 2 for a new $\mu_\gamma := \frac{\gamma}{5}$. This sequence satisfies the conditions in (ii) of Theorem 2 for $m^{(\gamma)}$. Now letting $\gamma \downarrow 0$, by a diagonal selection procedure, we can extract the desired sequence $\{m_k\}$ for which conditions in (ii) in Theorem 4 hold true for m. □

Proof of *(ii)* in Theorem 6. Let $m \in \mathcal{C}$ and we consider the set of walls $\{t_n\}_{n\in\mathbb{Z}}$ of m that are 1−periodically distributed in $\mathbb{R}$. Set $\mu = \frac{1}{5}\min_{n\in\mathbb{Z}}\{|t_n - t_{n-1}|\}$. Using translation and scaling by μ of the transition layer (47) & (48) corresponding to a wall $(0, \pm 1)$, the same construction as in the proof of (ii) of Theorem 2 gives a sequence of smooth 2−periodic functions $\{m_k\}$ that converges to m and satisfies $\limsup_{k\uparrow\infty} G_{\varepsilon_k}(m_k) = \pi N$. □

Now we show Proposition 1: the anisotropy term is essential in (3) in order that the variational problem is nondegenerate.

Proof of Proposition 1 . We construct a sequence of functions $\{m_k = (m_{1,k}, m_{2,k}) : \mathbb{R} \to S^1\}_{k\geq 2}$ that satisfies the limit conditions in (1), the wall domain is centered in the origin $m_k(0) = (1, 0)$ and

$$\|m_k\|_{\dot{H}^1} \to 0, \ \|m_{1,k}\|_{\dot{H}^{1/2}} \to 0 \quad \text{as} \quad k \uparrow \infty.$$

Let

$$m_{1,k}(t) = \begin{cases} \alpha + (1-\alpha)^{\frac{\log \frac{k}{\sqrt{t^2+1}}}{\log k}} & \text{if } |t| \leq \sqrt{k^2 - 1} \\ \alpha & \text{elsewhere,} \end{cases}$$

and

$$m_{2,k}(t) = \begin{cases} -\sqrt{1 - m_{1,k}^2(t)} & \text{if } t \leq 0, \\ \sqrt{1 - m_{1,k}^2(t)} & \text{if } t \geq 0. \end{cases}$$

The difference with respect to the transition layer (47) & (48) consists in the fact that the tails will spread over the entire $\mathbb{R}$ as $k \uparrow \infty$.

We first estimate the $\dot{H}^1-$norm of the transition layer:

$$\begin{aligned}
\int_{\mathbb{R}} \Big|\frac{dm_{1,k}}{dt}\Big|^2 + \Big|\frac{dm_{2,k}}{dt}\Big|^2 \, dt &= \int_{\mathbb{R}} \frac{1}{1-m_{1,k}^2}\Big|\frac{dm_{1,k}}{dt}\Big|^2 \, dt \\
&\leq \int_{\mathbb{R}} \frac{1}{1-m_{1,k}}\Big|\frac{dm_{1,k}}{dt}\Big|^2 \, dt \\
&= \frac{2(1-\alpha)}{\log k} \int_{-\sqrt{k^2-1}}^{\sqrt{k^2-1}} \frac{t^2}{(t^2+1)^2 \log(t^2+1)} \, dt \\
&\leq \frac{4(1-\alpha)}{\log k} \int_0^{\infty} \frac{t^2}{(t^2+1)^2 \log(t^2+1)} \, dt \\
&= O\Big(\frac{1}{\log k}\Big). \qquad (54)
\end{aligned}$$

In order to estimate the $\dot{H}^{1/2}-$norm of the transition layer, we consider U_k be the radial extension of $m_{1,k}$ in $\mathbb{R}^2$:

$$U_k(x_1, x_2) = m_{1,k}(\sqrt{x_1^2 + x_2^2}\,).$$

Applying (20), we obtain that

$$\begin{aligned}
\|m_{1,k}\|_{\dot{H}^{1/2}}^2 &\leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla U_k|^2 \, dx \\
&= \pi \int_0^k r\Big|\frac{dm_{1,k}}{dr}\Big|^2 \, dr \\
&\leq \frac{\pi(1-\alpha)^2}{\log^2 k} \int_0^k \frac{r^3}{(r^2+1)^2} \, dr \\
&\leq \frac{\pi(1-\alpha)^2}{\log^2 k}(1 + \log k) = O\Big(\frac{1}{\log k}\Big). \qquad (55)
\end{aligned}$$

For arbitrary $\delta > 0$, we conclude by (54) and (55) that

$$\delta \|m_k\|_{\dot{H}^1}^2 + \|m_{1,k}\|_{\dot{H}^{1/2}}^2 \to 0 \quad \text{as } k \uparrow \infty.$$

We notice that the anisotropy for the sequence $\{m_k\}$ blows-up. Indeed, integration by parts leads to:

$$\begin{aligned}
\int_{\mathbb{R}} |m_{1,k} - \alpha|^2 \, dt &= \frac{(1-\alpha)^2}{2\log^2 k} \int_0^{\sqrt{k^2-1}} \log^2 \frac{k^2}{t^2+1} \, dt \\
&= \frac{k(1-\alpha)^2}{2\log^2 k} \int_0^{\sqrt{1-1/k^2}} \log^2(y^2 + 1/k^2) \, dy \\
&= \frac{2k(1-\alpha)^2}{\log^2 k} \int_0^{\sqrt{1-1/k^2}} \frac{y^2}{y^2+1/k^2} \log \frac{1}{y^2+1/k^2} \, dy \\
&\geq \frac{2k(1-\alpha)^2}{\log^2 k} \int_{\sqrt{1/3-1/k^2}}^{\sqrt{1/2-1/k^2}} \frac{y^2}{y^2+1/k^2} \log \frac{1}{y^2+1/k^2} \, dy \\
&\geq \frac{4k(1-\alpha)^2 \log 2}{\log^2 k} \int_{\sqrt{1/3-1/k^2}}^{\sqrt{1/2-1/k^2}} y^2 \, dy \\
&\sim \frac{k}{\log^2 k} \to \infty \quad \text{as} \quad k \uparrow \infty,
\end{aligned}$$

where we used that the function $x \to x \log x$ is increasing in $[1, \infty)$ and $\dfrac{1}{y^2 + 1/k^2} \geq 2$ if $y \in (\sqrt{1/3 - 1/k^2}, \sqrt{1/2 - 1/k^2})$. $\square$

Finally, we highlight in Proposition 2 the importance of the fading H^1-control in the energy expression in order for the compactness result in the previous theorems to hold true. In general, the only control in the $H^{1/2}-$norm of the magnetization is not sufficient to enforce a compactness result:

Proof of Proposition 2. Let us consider the transition layer (47) & (48) corresponding to a 180° Néel wall for a small $\varepsilon \in (0, 1/3)$, i.e.,

$$u_\varepsilon(t) = \begin{cases} \frac{|\log \sqrt{t^2+\varepsilon^2}|}{|\log \varepsilon|} & \text{if } |t| \leq \sqrt{1-\varepsilon^2} \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$v_\varepsilon(t) = \begin{cases} -\sqrt{1 - u_\varepsilon^2(t)} & \text{if } t \leq 0, \\ \sqrt{1 - u_\varepsilon^2(t)} & \text{if } t \geq 0 \end{cases}$$

(see Figures 2 and 3). By (53), it follows that

$$\|u_\varepsilon\|^2_{\dot{H}^{1/2}} \leq \frac{2\pi}{|\log \varepsilon|}.$$

For every $n \in \mathbb{N}$, we construct a function $m_n = (m_{1,n}, m_{2,n}) : \mathbb{R} \to S^1$ that has $2n+1$ transitions of 180° and $\operatorname{supp} m_{1,n} \subset [0, 1]$. Set $\varepsilon_n = \frac{1}{n^{4n}}$, $\mu_n = \frac{1}{16n}$, $t_0^n = -\infty$, $t_k^n = \frac{2k-1}{4n+2}$ for $k = 1, \ldots, 2n+1$ and $t_{2n+2}^n = \infty$. As in the proof of *(ii)* of Theorem 2, let T_p and R_l be the translation and the dilation operator, respectively. We define

$$m_{1,n}(t) = \begin{cases} R_{\mu_n} T_{t_k^n} u_{\varepsilon_n}(t) & \text{if } t \in (t_k^n - \mu_n, t_k^n + \mu_n), k = 1, \ldots, 2n+1, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$m_{2,n}(t) = \begin{cases} (-1)^{k+1} R_{\mu_n} T_{t_k^n} v_{\varepsilon_n}(t) & \text{if } t \in (t_k^n - \mu_n, t_k^n + \mu_n), k = 1, \ldots, 2n+1, \\ (-1)^k & \text{elsewhere in } (t_{k-1}^n, t_k^n), k = 1, \ldots, 2n+2. \end{cases}$$

Then

$$m_{1,n} \to 0 \quad \text{in } L^2(\mathbb{R}) \text{ as } n \uparrow \infty. \tag{56}$$

Indeed,

$$\|m_{1,n}\|^2_{L^2} = (2n+1)\mu_n \int_{-1}^{1} u^2_{\varepsilon_n}(s)\, ds \leq \frac{C}{|\log \varepsilon_n|^2} \int_0^{\sqrt{1-\varepsilon_n^2}} \log^2(s^2 + \varepsilon_n^2)\, ds = O\left(\frac{1}{|\log \varepsilon_n|^2}\right).$$

We now prove that

$$\|m_{1,n}\|^2_{\dot{H}^{1/2}} \leq \frac{2\pi}{|\log n|}.$$

Indeed, we consider U_{ε_n} be the radial extension of u_{ε_n} in $\mathbb{R}^2$:

$$U_{\varepsilon_n}(x_1, x_2) = u_{\varepsilon_n}(\sqrt{x_1^2 + x_2^2}).$$

Then let M_n be the following extension of $m_{1,n}$ in $\mathbb{R}^2$:

$$M_n(x_1,x_2) = \begin{cases} R_{\mu_n} T_{(t_k^n,0)} U_{\varepsilon_n}(x_1,x_2) & \text{if } x_1 \in (t_k-\mu, t_k+\mu), k=1,\ldots,2n+1, \\ 0 & \text{elsewhere in } \mathbb{R}^2. \end{cases}$$

Applying (20) to the extension M_n of $m_{1,n}$, it follows

$$\begin{aligned} \|m_{1,n}\|^2_{\dot{H}^{1/2}} &\le \frac{1}{2}\int_{\mathbb{R}^2} |\nabla M_n|^2\,dx \\ &= \frac{1}{2}\sum_{k=1}^{2n+1}\int_{(t_k^n-\mu_n,t_k^n+\mu_n)\times\mathbb{R}} |\nabla R_{\mu_n} T_{(t_k^n,0)} U_{\varepsilon_n}|^2\,dx \\ &= \frac{2n+1}{2}\int_{(-1,1)\times\mathbb{R}} |\nabla U_{\varepsilon_n}|^2\,dy \\ &\overset{(53)}{\le} \frac{2\pi(2n+1)}{|\log\varepsilon_n|} \le \frac{2\pi}{|\log n|} \quad \text{for } n \text{ large enough.} \end{aligned}$$

It remains to prove that $\{m_{2,n}\}$ is not relatively compact in L^1_{loc}. Assume by contradiction that this would be the case. Without loss of generality, we may then assume that $m_{2,n} \to m_2$ in L^1 and a.e. in $(-1,1)$. Then there exists $n_0 > 0$ such that

$$\int_{-1}^{1} |m_{2,n} - m_{2,n_0}|\,dt \le \frac{1}{100}, \text{ for all } n \ge n_0. \tag{57}$$

By construction, the set $\{t : m_{2,n_0}(t) = 1\}$ contains disjoint intervals $I_k = (t^{n_0}_{2k-1} + \mu_{n_0}, t^{n_0}_{2k} - \mu_{n_0})$, $k = 1,\ldots,n_0$ of total length larger than $1/4$. On each such interval I_k, a function $m_{2,n}$ takes the value -1 at least on a subset of measure $|I_k|/5$, for n large enough. Therefore,

$$\int_{-1}^{1} |m_{2,n} - m_{2,n_0}|\,dt \ge \sum_{k=1}^{n_0}\int_{I_k} |m_{2,n} - 1|\,dt \ge \frac{2}{5}\sum_{k=1}^{n_0} |I_k| \ge \frac{1}{10},$$

which is a contradiction with (57). □

7 Appendix

We prove some known characterizations of the homogeneous $\dot{H}^{1/2}-$seminorm that we used in the previous sections.

Proposition 7 *Let $u : \mathbb{R} \to \mathbb{R}$. Then*

(i)

$$\|u\|^2_{\dot{H}^{1/2}} = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(s)-u(t)|^2}{|s-t|^2}\,dsdt.$$

(ii) If u is $2-$periodic, we have

$$\|u\|^2_{\dot{H}^{1/2}_{per}} = \frac{\pi}{2}\int_{[-1,1)}\int_{[-1,1)} \frac{|u(s)-u(t)|^2}{|e^{i\pi s} - e^{i\pi t}|^2}\,dsdt.$$

Proof. *(i)* By Plancherel's identity, we have that

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(s)-u(t)|^2}{|s-t|^2}\,dsdt &\overset{l:=s-t}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(t+l)-u(t)|^2}{l^2}\,dldt \\ &= \int_{\mathbb{R}} \frac{1}{l^2}\int_{\mathbb{R}} |e^{il\xi}-1|^2|\hat{u}(\xi)|^2\,dld\xi \\ &= 4\int_{\mathbb{R}} |\hat{u}(\xi)|^2 \int_{\mathbb{R}} \frac{1}{l^2}\sin^2\left(\frac{l|\xi|}{2}\right)\,d\xi dl \\ &\overset{y:=l|\xi|/2}{=} 4\int_{\mathbb{R}} |\xi|\,|\hat{u}(\xi)|^2\,d\xi \cdot \int_0^\infty \frac{\sin^2 y}{y^2}\,dy.\end{aligned}$$

In order to conclude, it is enough to prove that

$$\int_0^\infty \frac{\sin^2 y}{y^2}\,dy = \frac{\pi}{2}.$$

Indeed, integration by parts leads to

$$\int_0^\infty \frac{\sin^2 y}{y^2}\,dy = \int_0^\infty \frac{\sin(2y)}{y}\,dy \overset{s=2y}{=} \int_0^\infty \frac{\sin s}{s}\,ds.$$

To compute the last integral, we use the Laplace transform of the function $s \mapsto \frac{\sin s}{s}$, i.e.,

$$L(p) = \int_0^\infty e^{-ps}\frac{\sin s}{s}\,ds, \quad p \geq 0.$$

We have that

$$\frac{dL}{dp}(p) = -\frac{1}{1+p^2} \quad \text{and} \quad \lim_{p\to\infty} L(p) = 0.$$

Therefore, $L(0) = \pi/2$.

(ii) We compute:

$$\begin{aligned}\int_{[-1,1)}\int_{[-1,1)} \frac{|u(s)-u(t)|^2}{|e^{i\pi s}-e^{i\pi t}|^2}\,dsdt &\overset{h:=s-t}{=} \int_{[-1,1)} dt \int_{[-1-t,1-t)} \frac{|u(t+h)-u(t)|^2}{|e^{i\pi h}-1|^2}\,dh \\ &= \int_{[-1,1)} dt \int_{[-1,1)} \frac{|u(t+h)-u(t)|^2}{|e^{i\pi h}-1|^2}\,dh.\end{aligned}$$

By the Fourier representation, we know that

$$u(t+h)-u(t) = \sum_{\beta\in\pi\mathbb{Z}} \hat{u}(\beta)(e^{i\beta h}-1)\frac{e^{i\beta t}}{\sqrt{2}}.$$

Then Parseval's identity leads to

$$\begin{aligned}\int_{[-1,1)} \frac{dh}{|e^{i\pi h}-1|^2}\int_{[-1,1)} |u(t+h)-u(t)|^2\,dt &= \int_{[-1,1)} \sum_{\beta\in\pi\mathbb{Z}} |\hat{u}(\beta)|^2 \frac{|e^{i\beta h}-1|^2}{|e^{i\pi h}-1|^2}\,dh \\ &= \sum_{\beta\in\pi\mathbb{Z}} |\hat{u}(\beta)|^2 \int_{[-1,1)} \frac{\sin^2\frac{\beta h}{2}}{\sin^2\frac{\pi h}{2}}\,dh \\ &= \sum_{\beta\in\pi\mathbb{Z}^*} |\beta|\,|\hat{u}(\beta)|^2 \int_{[-1,1)} J_\beta(h)\,dh,\end{aligned}$$

where $J_\beta(h) = \dfrac{1}{|\beta|}\dfrac{\sin^2 \frac{\beta h}{2}}{\sin^2 \frac{\pi h}{2}}$, $\beta \in \pi\mathbb{Z}^*$ stands for the Fejér kernel. In order to conclude it is enough to show that

$$\int_{[-1,1)} J_\beta(h)\, dh = \frac{2}{\pi}.$$

For that, we introduce the Dirichlet kernel:

$$D_n(h) = \sum_{k=-n}^{n} e^{ikh} = 1 + 2\sum_{k=1}^{n} \cos(kh)$$

and we compute that

$$\int_{[-1,1)} J_\beta(h)\, dh = \frac{1}{|\beta|}\int_{[-1,1)} \sum_{n=0}^{\frac{|\beta|}{\pi}-1} D_n(h)\, dh = \frac{1}{|\beta|}\sum_{n=0}^{\frac{|\beta|}{\pi}-1} 2 = \frac{2}{\pi}.$$

□

Finally we prove Proposition 4 and then Proposition 3:

Proof of Proposition 4. *(i)* First we solve the problem (24). We search the solution H as a gradient field, i.e., $H = \nabla U$ with $U : \mathbb{R}^2 \to \mathbb{R}$. In terms of U, (24) turns into a Neumann type problem for Laplace's equation:

$$\begin{cases} \Delta U = 0 & \text{in} \quad \{x_2 \neq 0\}, \\ [U] = 0,\ \left[\frac{\partial U}{\partial x_2}\right] = -\frac{dm_1}{dx_1} & \text{on} \quad \{x_2 = 0\}. \end{cases} \tag{58}$$

In the sequel, we will denote by $\hat{U}$ the Fourier transform of U with respect to the x_1-direction. Then $\hat{U}$ solves a second order ODE in x_2 having the Fourier variable ξ as a parameter that is obtained via (58):

$$\begin{cases} \frac{\partial^2}{\partial x_2^2}\hat{U}(\xi,\cdot) - \xi^2\hat{U}(\xi,\cdot) = 0 & \text{if } x_2 \neq 0, \\ \left[\hat{U}(\xi,\cdot)\right] = 0,\ \left[\frac{\partial}{\partial x_2}\hat{U}(\xi,\cdot)\right] = -i\xi\widehat{m_1}(\xi) & \text{if } x_2 = 0. \end{cases}$$

Solving explicitly the ODE, we obtain

$$\hat{U}(\xi, x_2) = \frac{i\xi}{2|\xi|}e^{-|\xi||x_2|}\widehat{m_1}(\xi),\ \xi \neq 0, x_2 \in \mathbb{R}. \tag{59}$$

How to deduce regularity results for U starting from the Fourier expression (59), for arbitrary functions $m_1 \in \dot{H}^{1/2}(\mathbb{R})$? We will proceed as follows. We start with the *a-priori* formula of $H := \nabla U$ in the Fourier transform in x_1-direction and we prove the properties of the stray field stated in Proposition 4. In particular, we will deduce that $U \in \dot{H}^1(\mathbb{R}^2) \cap L^1_{loc}(\mathbb{R}^2)$.

We set $H \in L^2(\mathbb{R}^2, \mathbb{R}^2)$ be given by its Fourier transform in x_1:

$$\hat{H}(\xi, x_2) = \left(i\xi\hat{U}(\xi, x_2), \frac{\partial \hat{U}}{\partial x_2}(\xi, x_2)\right) := e^{-|\xi||x_2|}\widehat{\frac{dm_1}{dt}}(\xi)\left(\frac{i\xi}{2|\xi|}, -\frac{\operatorname{sgn}(x_2)}{2}\right), \quad \xi \neq 0, x_2 \neq 0.$$

Let us check that H belongs to $L^2(\mathbb{R}^2, \mathbb{R}^2)$. Indeed, by Plancherel's identity, it follows that:

$$\begin{aligned}\int_{\mathbb{R}^2} |H|^2\, dx &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \xi^2 e^{-2|\xi||x_2|}|\widehat{m_1}(\xi)|^2\, d\xi dx_2 \\ &= \frac{1}{2}\int_{\mathbb{R}} |\xi|\, |\widehat{m_1}(\xi)|^2\, d\xi = \frac{1}{2}\|m_1\|^2_{\dot{H}^{1/2}} < +\infty.\end{aligned}$$

We will rigourously prove that H is a gradient field (formally, H is the gradient ∇U). For that, we check that

$$\nabla \times H = 0, \text{ i.e., } \frac{\partial H_1}{\partial x_2} = \frac{\partial H_2}{\partial x_1} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^2).$$

Indeed, integration by parts and Parseval's identity lead to

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{\partial H_2}{\partial x_1} \zeta \, dx &= -\int_{\mathbb{R}^2} \hat{H}_2 \overline{\widehat{\frac{\partial \zeta}{\partial x_1}}} \, d\xi dx_2 \\
&= \int_{\mathbb{R}^2} i\xi \hat{H}_2 \overline{\hat{\zeta}} \, d\xi dx_2 \\
&= \int_{\mathbb{R}^2} \frac{\partial (i\xi \hat{U})}{\partial x_2} \overline{\hat{\zeta}} \, d\xi dx_2 \\
&= -\int_{\mathbb{R}^2} i\xi \hat{U} \frac{\partial}{\partial x_2} \overline{\hat{\zeta}} \, d\xi dx_2 \\
&= -\int_{\mathbb{R}^2} \widehat{H_1} \overline{\widehat{\frac{\partial \zeta}{\partial x_2}}} \, dx \\
&= \int_{\mathbb{R}^2} \frac{\partial H_1}{\partial x_2} \zeta \, dx, \, \forall \zeta \in \mathcal{S}(\mathbb{R}^2).
\end{aligned}$$

Therefore, by Poincaré's lemma, there exists $\tilde{U} \in \dot{H}^1(\mathbb{R}^2) \cap L^1_{loc}(\mathbb{R}^2)$ such that $\nabla \tilde{U} = H$. Obviously, up to a constant, $\tilde{U}$ coincides with U a.e. in $\mathbb{R}^2$.

We now check that H is a stray field; indeed, Parseval's identity and integration by parts yield that

$$\begin{aligned}
\int_{\mathbb{R}^2} H \cdot \nabla \zeta \, dx &= \int_{\mathbb{R}^2} \hat{H}(\xi, x_2) \cdot \left(\overline{i\xi \hat{\zeta}(\xi, x_2), \frac{\partial \hat{\zeta}}{\partial x_2}(\xi, x_2)} \right) d\xi dx_2 \\
&= \int_{\mathbb{R}^2} \xi^2 \hat{U} \cdot \overline{\hat{\zeta}} \, d\xi dx_2 + \int_{\mathbb{R}} \widehat{\frac{dm_1}{dt}}(\xi) \left(\int_{\mathbb{R}} \frac{-\operatorname{sgn}(x_2)}{2} e^{-|\xi||x_2|} \overline{\frac{\partial \hat{\zeta}}{\partial x_2}(\xi, x_2) \, dx_2} \right) d\xi \\
&= \int_{\mathbb{R}^2} \xi^2 \hat{U} \cdot \overline{\hat{\zeta}} \, d\xi dx_2 + \int_{\mathbb{R}} \widehat{\frac{dm_1}{dt}}(\xi) \left(\overline{\hat{\zeta}}(\xi, 0) - \int_{\mathbb{R}} \frac{|\xi|}{2} e^{-|\xi||x_2|} \overline{\hat{\zeta}} \, dx_2 \right) d\xi \\
&= \int_{\mathbb{R}} \frac{dm_1}{dx_1} \zeta(\cdot, 0) \, dx_1, \, \forall \zeta \in C_c^\infty(\mathbb{R}^2).
\end{aligned}$$

Then (24) follows by (22) and from the fact that H is a gradient field.

We want to prove that H is a minimizer of (23). First, we notice that for every stray field $h \in L^2(\mathbb{R}^2, \mathbb{R}^2)$, (22) makes sense for every test function $\zeta \in \dot{H}^1(\mathbb{R}^2) \cap L^1_{loc}(\mathbb{R}^2)$. Indeed, we regularize ζ by a sequence $\{\zeta_n\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^2)$ that converges to ζ in $\dot{H}^1(\mathbb{R}^2)$. For example, one can take $\zeta_n = \chi_n \rho_n \star (\zeta - a_n)$ where $\{\rho_n\}$ is a mollifying sequence, $a_n = \fint_{n<|x|<2n} \rho_n \star \zeta \, dx$ and $\chi_n(x) = \chi(\frac{|x|}{n})$ where χ is a smooth cut-off function such that $\chi = 1$ in $(-1, 1)$ and $\chi = 0$ outside $(-2, 2)$. The fact that the function $f_n := \rho_n \star (\zeta - a_n)$ is of vanishing mean value in the annulus $\{n < |x| < 2n\}$ is used for showing that $f_n \nabla \chi_n \to 0$ in $L^2(\mathbb{R}^2)$ as $n \uparrow \infty$. More precisely, by Poincaré's inequality, we have that:

$$\int_{\mathbb{R}^2} |f_n|^2 |\nabla \chi_n|^2 \, dx \leq \frac{C}{n^2} \int_{n<|x|<2n} |f_n|^2 \, dx \leq C \int_{n<|x|<2n} |\nabla f_n|^2 \, dx \to 0 \text{ as } n \uparrow \infty,$$

since $f_n \to \zeta$ in $\dot{H}^1$ (here, C stands for a universal constant). Therefore, $\zeta_n \to \zeta$ in $\dot{H}^1(\mathbb{R}^2)$. Then Parseval's identity leads to

$$\begin{aligned}\int_{\mathbb{R}^2} h \cdot \nabla \zeta_n \, dx &= \int_{\mathbb{R}} \frac{dm_1}{dx_1} \zeta_n(\cdot, 0) \, dx_1 \\ &= \int_{\mathbb{R}} \widehat{\frac{dm_1}{dx_1}} \, \overline{\widehat{\zeta_n(\cdot, 0)}} \, d\xi \\ &\le \|m_1\|^2_{\dot{H}^{1/2}} \|\zeta_n(\cdot, 0)\|^2_{\dot{H}^{1/2}} \\ &\overset{(20)}{\le} \frac{1}{\sqrt{2}} \|m_1\|^2_{\dot{H}^{1/2}} \|\zeta_n\|_{\dot{H}^1(\mathbb{R}^2)}\end{aligned}$$

and we conclude that (22) holds for ζ by passing to the limit $n \uparrow \infty$ (here, $\zeta(\cdot, 0) \in \dot{H}^{1/2}(\mathbb{R})$ is the trace of ζ on the horizontal line $\{x_2 = 0\}$). The stray field H is a minimizer of (23); indeed, for every stray field $h \in L^2(\mathbb{R}^2, \mathbb{R}^2)$, we have that

$$\int_{\mathbb{R}^2} |H|^2 \, dx = \int_{\mathbb{R}^2} |\nabla U|^2 \, dx \overset{(22)}{=} \int_{\mathbb{R}} \frac{dm_1}{dx_1} U \, dx_1 \overset{(22)}{=} \int_{\mathbb{R}^2} h \cdot \nabla U \le \|h\|_{L^2} \|\nabla U\|_{L^2} = \|h\|_{L^2} \|H\|_{L^2}.$$

Therefore, $\|h\|_{L^2} \ge \|H\|_{L^2}$.

Moreover, H is the unique minimizer in L^2; indeed, if h is another minimizing stray field, then $h + t(H - h)$ is also a stray field associated to m_1 and satisfies

$$\|h\|_{L^2} \le \|h + t(H - h)\|_{L^2}, \ \forall t \in \mathbb{R}.$$

That implies

$$\int_{\mathbb{R}^2} h \cdot (H - h) \, dx = 0.$$

Interchanging h by H, we get

$$\int_{\mathbb{R}^2} H \cdot (h - H) \, dx = 0.$$

Adding the last two identities, we obtain $\|H - h\|_{L^2} = 0$, i.e., $H = h$ a.e. in $\mathbb{R}^2$.
(ii) The same argument as in *(i)* leads to the conclusion in the periodic case, too. □

Proof of Proposition 3. *(i)* We solve the Laplace equation with Dirichlet boundary data:

$$\begin{cases} \Delta U = 0 & \text{in} \quad \{x_2 \neq 0\}, \\ U(x_1, \cdot) = u(x_1) & \text{on} \quad \{x_2 = 0\}. \end{cases}$$

As in the proof of Proposition 4, the Fourier transform in x_1-direction turns this problem into an ordinary differential equation in x_2 having the Fourier variable ξ as a parameter:

$$\begin{cases} \frac{\partial^2}{\partial x_2^2} \hat{U}(\xi, \cdot) - |\xi|^2 \hat{U}(\xi, \cdot) = 0 & \text{if } x_2 \neq 0, \\ \hat{U}(\xi, \cdot) = \hat{u}(\xi) & \text{if } x_2 = 0. \end{cases}$$

The solution of the ODE is given by

$$\hat{U}(\xi, x_2) = e^{-|\xi||x_2|} \hat{u}(\xi).$$

Starting from the *a-priori* formula of the gradient field $H := \nabla U$, one can repeat the procedure presented in the proof of Proposition 4. It follows that $U \in \dot{H}^1(\mathbb{R}^2) \cap L^1_{loc}(\mathbb{R}^2)$ with

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla U|^2\, dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(|\xi|^2 |\hat{U}|^2 + \left|\frac{\partial \hat{U}}{\partial x_2}\right|^2 \right) d\xi dx_2 \\ &= 2\int_{\mathbb{R}}\int_{\mathbb{R}} |\xi|^2 e^{-2|\xi||x_2|} |\hat{u}(\xi)|^2\, d\xi dx_2 \\ &= 2\int_{\mathbb{R}} |\xi|\, |\hat{u}(\xi)|^2\, d\xi.\end{aligned}$$

It also satisfies the condition:

$$\int_{\mathbb{R}^2} \nabla U \cdot \nabla \zeta\, dx = -\int_{\mathbb{R}} \left[\frac{\partial U}{\partial x_2}\right] \zeta\, dx_1, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^2). \tag{60}$$

Here, the jump of the normal derivative of U across the line $\{x_2 = 0\}$ is given by the Fourier transform in x_1-direction:

$$\left[\widehat{\frac{\partial U}{\partial x_2}}\right] = -2|\xi|\hat{u} \in \dot{H}^{-1/2}(\mathbb{R}).$$

Notice that (60) stands true for every $\zeta \in \dot{H}^1(\mathbb{R}^2)$; it comes by regularizing ζ in $\dot{H}^1(\mathbb{R}^2)$ by smooth functions of compact support as in the proof of Proposition 4. Then we show that U is a minimizer of (20): for every $V \in \dot{H}^1(\mathbb{R}^2)$ with $V(\cdot, 0) = u$, (60) leads to

$$\int_{\mathbb{R}^2} \nabla U \cdot \nabla V\, dx = -\int_{\mathbb{R}} \left[\frac{\partial U}{\partial x_2}\right] u\, dx_1 = \int_{\mathbb{R}^2} |\nabla U|^2\, dx.$$

Therefore, $\|V\|_{\dot{H}^1(\mathbb{R}^2)} \geq \|U\|_{\dot{H}^1(\mathbb{R}^2)}$. Moreover, U is the unique minimizer of (20); indeed, if V is another minimizer of (20), then $\|V\|_{\dot{H}^1(\mathbb{R}^2)} \leq \|V + t(U - V)\|_{\dot{H}^1(\mathbb{R}^2)}$ for all $t \in \mathbb{R}$. That implies $\int_{\mathbb{R}^2} \nabla V \cdot \nabla(U - V)\, dx = 0$. Interchanging U and V, it follows $\int_{\mathbb{R}^2} \nabla U \cdot \nabla(V - U)\, dx = 0$. Adding the last two identities, we obtain $\|U - V\|_{\dot{H}^1(\mathbb{R}^2)} = 0$, i.e., $U - V$ is a constant. Since U and V have the same trace in $\dot{H}^{1/2}$ on the horizontal line $\{x_2 = 0\}$ as $\dot{H}^1(\mathbb{R}^2)-$functions, we conclude that $U = V$ a.e. in $\mathbb{R}^2$.
(ii) The periodic case follows similarly. □

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