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Two-dimensional unit-length vector fields of vanishing divergence

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Abstract

We prove the following regularity result: any two-dimensional unit-length divergence-free vector field belonging to $W^{1/p,p}$ ($p \in [1,2]$) is locally Lipschitz except at a locally finite number of vortex-point singularities. We also prove approximation results for such vector fields: the dense sets are formed either by unit-length divergence-free vector fields that are smooth except at a finite number of points and the approximation result holds in the $W^{1,q}_{loc}$-topology ($1 \leqslant q < 2$), or by everywhere smooth unit-length vector fields (not necessarily divergence-free) and the approximation result holds in a weaker topology.

Keywords: Sobolev spaces; Eikonal equation; Regularity; Approximation; Vortices; Kinetic formulation; Entropy

1. Introduction

Let $\Omega \subset \mathbf{R}^2$ be an open bounded set. We will focus on measurable vector fields $m : \Omega \to \mathbf{R}^2$ that satisfy

$$|m| = 1 \quad \text{a.e. in } \Omega \quad \text{and} \quad \nabla \cdot m = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{1}$$

One can equivalently consider measurable vector fields $v : \Omega \to \mathbf{R}^2$ such that

$$|v| = 1 \quad \text{a.e. in } \Omega \quad \text{and} \quad \nabla \times v = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{2}$$

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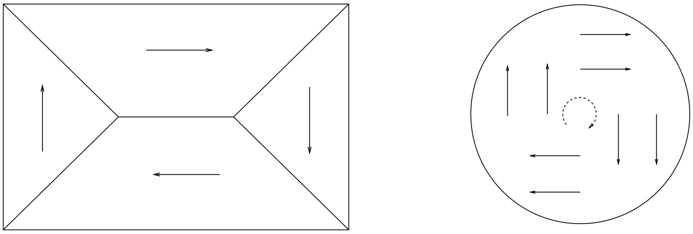

Fig. 1. Landau states in a rectangle and a disk.

(The passage from (1) to (2) is done via $v = m^\perp = (-m_2, m_1)$.) Locally, m (resp. v) can be written in terms of a stream function ψ, i.e., $m = \nabla^\perp \psi$ (resp. $v = -\nabla\psi$) so that we get to the eikonal equation through ψ:

$$|\nabla\psi| = 1. \tag{3}$$

Typically, one can construct such vector fields by considering stream functions of the form $\psi = \operatorname{dist}(\cdot, K)$ for some closed set $K \subset \mathbf{R}^2$; these vector fields are called Landau states in micromagnetic jargon (see Fig. 1). However, not every stream function can be written as a distance function (up to a sign ± 1 and an additive constant); for example, if $\psi(x) = \max\{\operatorname{dist}(x, P_1), \operatorname{dist}(x, P_2)\}$ for two different points $P_1, P_2 \in \mathbf{R}^2$, then (3) holds even if ψ is not a distance function.

2. Main results

For $p \geqslant 1$ and $s > 0$, we denote by

$$W^{s,p}_{div}(\Omega, S^1) = \{m \in W^{s,p}(\Omega, \mathbf{R}^2)\colon m \text{ satisfies (1)}\}.$$

2.1. Regularity results

The first goal is to prove the following regularity result:

Theorem 1. *If $m \in W^{1/p,p}_{div}(\Omega, S^1)$ for some $p \in [1,2]$ then m is locally Lipschitz continuous inside Ω except at a locally finite number of singular points. Moreover, every singular point P of m corresponds to a vortex singularity of degree 1 of m, i.e., there exists a sign $\alpha = \pm 1$ such that*

$$m(x) = \alpha \frac{(x-P)^\perp}{|x-P|} \quad \textit{for every } x \neq P \textit{ in any convex neighborhood of } P \textit{ in } \Omega.$$

In particular, if $m \in H^1_{div}(\Omega, S^1)$ then m is locally Lipschitz.

Remark 1. The above result was proved by Jabin, Otto, and Perthame [25] in the particular case of zero-energy states of a line-energy Ginzburg–Landau model. More precisely, for $\varepsilon > 0$, one defines the functional $E_\varepsilon : H^1(\Omega, \mathbf{R}^2) \to \mathbf{R}_+$ by

$$E_\varepsilon(m_\varepsilon) = \varepsilon \int_\Omega |\nabla m_\varepsilon|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega (1 - |m|^2)^2 \, dx + \frac{1}{\varepsilon} \|\nabla \cdot m_\varepsilon\|^2_{\dot{H}^{-1}(\Omega)}, \quad m_\varepsilon \in H^1(\Omega, \mathbf{R}^2)$$

(we refer to [1,3,27,15,26,32,25] for the analysis of this model). A vector field $m : \Omega \to \mathbf{R}^2$ is called zero-energy state if there exists a family $\{m_\varepsilon \in H^1(\Omega, \mathbf{R}^2)\}_{\varepsilon \to 0}$ satisfying

$$m_\varepsilon \to m \quad \text{in } L^1(\Omega) \quad \text{and} \quad E_\varepsilon(m_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

Then m satisfies (1) and shares the structure stated in Theorem 1 (see [25]). Conversely, as we shall see later at Remark 9, any vector field sharing the structure in Theorem 1 (that in addition, has H^1 regularity near the boundary $\partial\Omega$) is a zero-energy state.

The hypothesis $m \in W^{1/p,p}$ in Theorem 1 is a critical regularity assumption in order to avoid line-singularities for vector fields m satisfying (1) (see Proposition 8 in Appendix A). As consequence of Theorem 1, one has the following equality

$$\left\{m \in W^{1,1}_{loc}\left(\Omega, \mathbf{R}^2\right)\colon m \text{ satisfies } (1)\right\} = \left\{m \in H^{1/2}_{loc}\left(\Omega, \mathbf{R}^2\right)\colon m \text{ satisfies } (1)\right\}.$$

Let us now discuss the optimality of the result in Theorem 1: Firstly, observe that Lipschitz regularity of m cannot be improved.

Proposition 1. *There exist Lipschitz vector fields $m : \Omega \to \mathbf{R}^2$ that satisfy* (1) *and are not C^1 in Ω.*

In general, a vector field $m \in W^{1/p,p}_{div}(\Omega, S^1)$ ($p \in [1,2]$) (without interior vortex singularities) is only locally Lipschitz, and not necessary globally Lipschitz in Ω. This is the case of a "boundary vortex" vector field, e.g., $m(x) = \frac{(x-P)^\perp}{|x-P|}$ for every $x \in \Omega$ where P is some point on $\partial\Omega$. If the domain Ω has a cusp in $P \in \partial\Omega$, the "boundary vortex" vector field could belong even to $H^1(\Omega, \mathbf{R}^2)$; moreover, there exist convex domains Ω and $m \in H^1_{div}(\Omega, S^1)$ such that m is not globally Lipschitz in Ω (see Section 4.2).

The geometry of Ω influences the number of vortex singularities of $W^{1/p,p}$-vector fields satisfying (1). For example, if Ω is convex, then every vector field $m \in W^{1/p,p}_{div}(\Omega, S^1)$ (with $p \in [1,2]$) is either a "vortex" vector field (i.e., $m(x) = \pm\frac{(x-P)^\perp}{|x-P|}$ for every $x \in \Omega$ where P is some point in Ω), or locally Lipschitz in Ω (i.e. no interior vortex singularity); therefore, convex domains do not allow for more than one interior vortex singularity (see Remark 6). However, we prove that there are nonconvex domains where configurations with arbitrary number of vortex-point singularities do exist: vector fields with infinitely many vortex singularities can be constructed in some nonconvex *piecewise Lipschitz* domains Ω (i.e., $\partial\Omega = \bigcup_{j=1}^k \{\gamma_j\}$ where $\gamma_j : [0,1] \to \partial\Omega$ are parametrized Lipschitz curves, each two curves having disjoint interiors).

Proposition 2. *There exist an open simply-connected nonconvex piecewise Lipschitz domain Ω and a vector field $m \in W^{1,q}_{div}(\Omega, S^1)$ for every $q \in [1,2)$ that has infinitely many vortex-point singularities $\{P_1, P_2, \ldots\}$.*

Observe that the following embedding holds: $W^{1,q}_{loc}(\Omega, S^1) \subset W^{1/p,p}_{loc}(\Omega, S^1)$ for $q > 1$ and $p \geqslant 1$, and the embedding fails for $q = 1$ (see Proposition 9 in Appendix A and [6, Lemma D.1]). Also notice that configurations with infinitely many (interior) vortex-point singularities can occur only in a non-Lipschitz domain Ω; indeed, if $\partial\Omega$ is Lipschitz, then a configuration

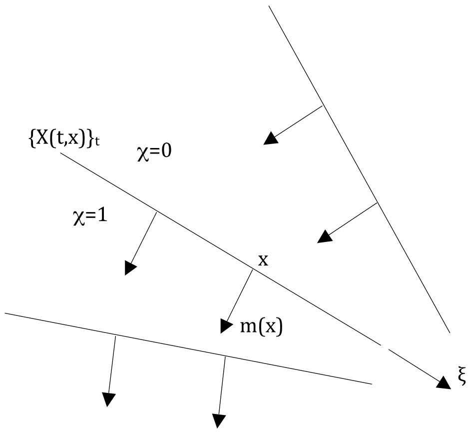

Fig. 2. Characteristics of m.

$m \in W^{1/p,p}_{div}(\Omega, S^1)$ (with $p \in [1,2]$) has only a finite number of interior vortex singularities (see Proposition 7).

The main ingredient of the proof of Theorem 1 resides in the following kinetic formulation. It is a generalization to the case of $W^{1/p,p}_{div}(\Omega, S^1)$ vector fields (with $p \in [1,2]$) of the result in [25] for zero-energy states of E_ε (given in Remark 1):

Proposition 3 *(Kinetic formulation). Let $m \in W^{1/p,p}_{div}(\Omega, S^1)$ (with $p \in [1,2]$). For every direction $\xi \in S^1$, we define $\chi(\cdot,\xi) : \Omega \to \{0,1\}$ (resp. $\tilde{\chi}(\cdot,\xi) : S^1 \to \{0,1\}$) by*

$$\chi(x,\xi) = \tilde{\chi}\big(m(x),\xi\big) = \begin{cases} 1 & \textit{for } m(x)\cdot\xi > 0, \\ 0 & \textit{for } m(x)\cdot\xi \leqslant 0. \end{cases}$$

Then the following kinetic equation holds for every $\xi \in S^1$:

$$\xi \cdot \nabla\chi(\cdot,\xi) = 0 \quad \textit{in } \mathcal{D}'(\Omega). \tag{4}$$

Here, χ corresponds to the concept of characteristic of a weak solution m satisfying (1). Indeed, if m is smooth around a point $x \in \Omega$, then the characteristic of m at x (by means of the eikonal equation (3) with $m = \nabla^\perp\psi$ around x) is given by $\dot{X}(t,x) = m^\perp(X(t,x))$ with the initial condition $X(0,x) = x$; then the orbit $\{X(t,x)\}_t$ is a straight line (i.e., $X(t,x) = x + tm^\perp(x)$ for t in some interval around 0) along which m is perpendicular and constant. Therefore, in the direction $\xi := m^\perp(x)$, either $\nabla\chi(\cdot,\xi)$ locally vanishes (if m is constant in a neighborhood of x), or it concentrates on $\{X(t,x)\}_t$ and is oriented by $\xi^\perp$ (see Fig. 2). The knowledge of $\chi(\cdot,\xi)$ in every direction $\xi \in S^1$ determines completely the vector field m due to the straightforward formula

$$m(x) = \frac{1}{2}\int_{S^1} \xi\chi(x,\xi)\,d\xi \quad \text{for a.e. } x \in \Omega. \tag{5}$$

Remark 2. Classical kinetic averaging lemma (see e.g. Golse, Lions, Perthame, and Sentis [17]) shows that a measurable vector-field $m : \Omega \to S^1$ satisfying (4) belongs to $H^{1/2}_{loc}$ (due to (5)). (This property could be read as the inverse of Proposition 3 for the case $m \in H^{1/2}(\Omega, S^1)$.) Moreover, Jabin–Otto–Perthame (see Theorem 1.3 in [25]) proved that such a vector field has stronger regularity, i.e., it shares the structure described in Theorem 1. Therefore, the proof of Theorem 1 strongly relies on Jabin–Otto–Perthame's result [25] via Proposition 3.

Remark 3. The proof of Proposition 3 strongly relies on the structure of lifting of vector fields $m \in W^{1/p,p}(\Omega, S^1)$ (with $p \in [1,2]$) and an appropriate chain rule. More precisely, if $m \in W^{1/p,p}(\Omega, S^1)$, then there exists a lifting $\Theta = \Theta_1 + \Theta_2$ with $\Theta_1 \in W^{1/p,p}$, $\Theta_2 \in SBV$ and $e^{i\Theta_2} \in W^{1/p,p} \cap W^{1,1}$ (see [7] and [30]). Recall that $SBV(\Omega, \mathbf{R}^d)$ is the subspace of vector fields $m \in BV(\Omega, \mathbf{R}^d)$ whose differential Dm has vanishing Cantor part $D^c m$ (i.e., $D^c m \equiv 0$ as a measure in Ω).

We conjecture that Proposition 3 also holds for $p > 2$ so that Theorem 1 is expected to be valid for $p > 2$, too.

A natural question concerns higher dimensions $N \geqslant 3$ in the same context of the eikonal equation (3). We mention that our technics seem to be typical for the two-dimensional case and do not adapt to the case $N \geqslant 3$. Indeed, if $N = 3$, the system of scalar conservation laws associated to (3) admits only the trivial entropies. Moreover, the regularity result in Theorem 1 is based on a certain order relation between the characteristics of m. Obviously, such an order relation does not exist in higher dimensions. However, a positive answer to this question is given in a recent paper of Caffarelli and Crandall [9] under the stronger assumption that $v = \nabla\psi$ is pointwise differentiable away from a set of zero Hausdorff 1-measure.

We also address the following open problem:

Open Problem 1. *Is it true that every $m \in BV(\Omega, \mathbf{R}^2)$ with* (1) *satisfies $m \in SBV$?*

This question is related with a recent work of Bianchini, DeLellis, and Robyr [5]: they show that the viscosity solution ψ of a Hamilton–Jacobi equation $H(\nabla\psi) = 0$ in Ω (with a uniformly convex hamiltonian H) satisfies $\nabla\psi \in SBV$. Open Problem 1 asks whether for the particular case of the eikonal equation (3), the result in [5] still holds when replacing the assumption of viscosity solution with the hypothesis of a general solution ψ of (3) with $\nabla\psi \in BV$.

2.2. Density results

The second goal of the paper is to present approximation results for the class of vector fields $W^{1/p,p}_{div}(\Omega, S^1)$ (with $p \in [1,2]$): our subsets are formed either by divergence-free vector fields that are smooth except at a finite number of points and the approximation result holds in the $W^{1/p,p}$-topology, or by everywhere smooth vector fields (not necessarily divergence-free) and the approximation result holds in a weaker topology. We start by extending Bethuel–Zheng's density result (see [4]) for $W^{1,1}(\Omega, S^1)$ vector fields, respectively Rivière's density result (see [31]) for $H^{1/2}(\Omega, S^1)$ vector fields to the case of divergence-free vector fields:

Theorem 2. *Let Ω be a Lipschitz bounded simply-connected domain and $m \in W^{1/p,p}_{div}(\Omega, S^1)$ (with $p \in [1,2]$). Then m has a finite number $k \geqslant 0$ of vortex-point singularities $\{P_1, \ldots, P_k\}$*

and m can be approximated in $W^{1,q}_{loc}(\Omega)$ (for any $q \in [1,2)$) by divergence-free vector fields $m_n \in C^\infty(\Omega \setminus \{P_{1,n}, \dots, P_{k,n}\}, S^1)$ that are smooth except at the k vortex singularities of m_n. In particular, if $m \in H^1_{div}(\Omega, S^1)$, the sequence $\{m_n\}$ can be chosen to be smooth everywhere in Ω and the approximation result holds in $H^1_{loc}(\Omega)$.

In various applications (see e.g. Remarks 1 and 4), we need to approximate vector fields m (with the structure given in Theorem 1) by $H^1(\Omega, S^1)$ vector fields. But $H^1(\Omega, S^1)$-vector fields cannot allow for vortex-point singularities. Therefore, an approximation result by everywhere smooth vector fields is needed in some weak topology. What is the optimal weak topology where such a density result holds? The following result shows that L^1-topology is too strong for having density of smooth vector fields of vanishing divergence and values in S^1.

Proposition 4. *Let $m : B^2 \to S^1$ be the vortex vector field $m(x) = \frac{x^\perp}{|x|}$ in the unit disk B^2. Then there exists no sequence of vector fields $m_n \in C^\infty(\Omega, S^1)$ of vanishing divergence such that $m_n \to m$ a.e. in B^2.*

We now generalize this property: the density result still fails if we relax the divergence-free constraint on the approximated smooth vector fields, but we impose this restriction in the limit in L^1-topology (or H^{-s} weak topology for some $s \in [0, \frac{1}{2})$).

Proposition 5. *Let $m : B^2 \to S^1$ be the vortex vector field $m(x) = \frac{x^\perp}{|x|}$ in B^2. Then there exists no sequence of vector fields $m_n \in C^\infty(\Omega, S^1)$ such that $m_n \to m$ a.e. in B^2 and one of the following two conditions holds*:

(a) $\nabla \cdot m_n \to 0$ *in* $L^1(B^2)$;
(b) $\nabla \cdot m_n \rightharpoonup 0$ *weakly in* $H^{-s}(B^2)$ *for some* $s \in [0, \frac{1}{2})$.

Finally, we prove an approximation result in L^1-topology by smooth vector fields with values in S^1 (not necessary divergence-free), but the divergence-free constraint holds in the limit in the $H^{-1/2}$ topology. This topology is optimal due Proposition 5(b).

Theorem 3. *Let Ω be a Lipschitz bounded simply-connected domain and $m \in W^{1/p,p}_{div}(\Omega, S^1)$ (with $p \in [1,2]$). Then there exists a sequence of vector fields $m_n \in C^\infty(\Omega, S^1)$ such that $m_n \to m$ a.e. in Ω and $(\nabla \cdot m_n)\mathbf{1}_\Omega \to 0$ in $\dot{H}^{-1/2}(\mathbf{R}^2)$.*

Remark 4. The motivation of Theorem 3 comes from thin-film micromagnetics. The following $2D$ energy (see [14]) is considered as an approximation of the full $3D$ micromagnetic model for thin films: for $\varepsilon > 0$, one defines the functional $F_\varepsilon : H^1(\Omega, S^1) \to \mathbf{R}_+$ by

$$F_\varepsilon(m_\varepsilon) = \varepsilon \int_\Omega |\nabla m_\varepsilon|^2 \, dx + \left\| (\nabla \cdot m_\varepsilon)\mathbf{1}_\Omega \right\|^2_{\dot{H}^{-1/2}(\mathbf{R}^2)}, \quad m_\varepsilon \in H^1\big(\Omega, S^1\big).$$

This model was analyzed in [13,24,21]. In particular, it is proved in [21] that a vortex configuration $m_0(x) = \frac{x^\perp}{|x|}$ in $\Omega := B^2$ is a zero-energy state, i.e., there exists a family $\{m_\varepsilon \in H^1(B^2, S^1)\}$

such that $m_\varepsilon \to m_0$ a.e. in B^2 and $F_\varepsilon(m_\varepsilon) \to 0$ as $\varepsilon \to 0$. The role of Theorem 3 is to generalize this approximation result for every vector field $m \in W^{1/p,p}$ ($p \in [1,2]$) satisfying (1).

Most of the results of the paper have been announced in [20].

The outline of the paper is the following: in Section 3, we define a class of entropies that is used in the proof of Proposition 3 and Theorem 1. In Section 4, we present several examples of vector fields satisfying (1); in particular, we prove Propositions 1 and 2. Section 5 deals with the proof of non-density results in Propositions 5 and 4, while in Section 6, we present the proofs of density results in Theorems 2 and 3. We finish with Appendix A where we recall some properties of Sobolev spaces.

3. Entropies. Proof of Theorem 1

The starting point consists in regarding the structure (1) of our configurations as a scalar conservation law. Indeed, writing $m = (u, h(u))$ for the flux $h(u) = \pm\sqrt{1-u^2}$, the vanishing divergence condition in m turns into

$$\partial_t u + \partial_s h(u) = 0, \tag{6}$$

where $(t,s) := (x_1, x_2)$ correspond to (time, space) variables. Let us recall some definitions from the theory of scalar conservation laws. Since the flux h is nonlinear, there is in general no smooth solution of the Cauchy problem associated to (6). Therefore, the solutions of (6) are to be understood in the sense of distributions and in general, there are infinitely many weak solutions for the Cauchy problem. The concept of entropy solution has been formulated in order to have uniqueness (see Kružkov [28]). To introduce this notion, the pair (entropy, entropy-flux) is defined as a couple of scalar functions (η, q) such that $\frac{dq}{ds} = \frac{dh}{ds}\frac{d\eta}{ds}$; then for every smooth solution u of (6), the entropy production vanishes, i.e.,

$$\partial_t\big[\eta(u)\big] + \partial_s\big[q(u)\big] = 0.$$

A solution u of (6) (in the sense of distributions) is called entropy solution if for every convex entropy η, the entropy production $\partial_t[\eta(u)] + \partial_s[q(u)] \leqslant 0$ is a nonpositive measure. Such solutions u have the property that for every pair (η, q), the entropy production concentrates on lines (corresponding to "shocks" of u). It suggests the interest of using "global" quantities (η, q) to detect "local" line-singularities of u. This idea has been used by Jin and Kohn [27], Aviles and Giga [3], DeSimone, Kohn, Müller, and Otto [15], Ambrosio, DeLellis, and Mantegazza [1], Ignat and Merlet [23,22].

In the sequel we will always use the following notion of entropy introduced in [15] (see also [12,22]). It corresponds to the pair (entropy, entropy-flux) from the scalar conservation laws, but the pair is defined in terms of the couple $(u, h(u))$.

Definition 1. (See [15].) We will say that $\Phi \in C^\infty(S^1, \mathbf{R}^2)$ is an entropy if

$$\frac{d}{d\theta}\Phi(z)\cdot z = 0, \quad \text{for every } z = e^{i\theta} = (\cos\theta, \sin\theta) \in S^1. \tag{7}$$

Here, $\frac{d}{d\theta}\Phi(z) = \frac{d}{d\theta}[\Phi(e^{i\theta})]$ stands for the angular derivative of Φ. The set of all entropies is denoted by ENT.

Remark 5. A way to construct entropies is given by the following equivalent definition: if $\Phi \in C^\infty(S^1, \mathbf{R}^2)$, then $\Phi \in ENT$ is an entropy if and only if there exists a (unique) 2π-periodic $\varphi \in C^\infty(\mathbf{R})$ such that for every $z = e^{i\theta} \in S^1$,

$$\Phi(z) = \varphi(\theta)z + \frac{d\varphi}{d\theta}(\theta)z^\perp; \tag{8}$$

therefore, one has

$$\frac{d}{d\theta}\Phi(z) \cdot z^\perp = \varphi(\theta) + \frac{d^2\varphi}{d\theta^2}(\theta) =: \gamma(z) \quad \text{for every } z \in S^1 \tag{9}$$

for some smooth function $\gamma \in C^\infty(S^1)$ (see details in [15,22]).

This notion is coherent with the property that a smooth vector field m satisfying (1) induces vanishing entropy production $\nabla \cdot [\Phi(m)] = 0$. In fact, it is equivalent to Definition 1 as stated in the following property:

Proposition 6. *Let $\Phi \in C^\infty(S^1, \mathbf{R}^2)$. Then Φ is an entropy if and only if for every $m \in W^{1/p,p}_{div}(\Omega, S^1)$ (with $p \in [1,2]$), the following identity holds*:

$$\nabla \cdot \big[\Phi(m)\big] = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{10}$$

Proof. We divide the proof in several steps:

Step 1. If $\Phi \in ENT$ and $m \in W^{1,1}_{div}(\Omega, S^1)$, then (10) *holds.* Indeed, let us consider an *SBV* lifting Θ of m in Ω, i.e., $m = e^{i\Theta}$ in Ω (see e.g. [16,8,11,19]). For the *SBV*-function Θ, the measure $D\Theta$ splits into two terms

$$D\Theta = D^a\Theta + \big(\Theta^+ - \Theta^-\big)\nu\mathcal{H}^1 \llcorner J(\Theta) \tag{11}$$

where $D^a\Theta = \nabla^a\Theta\mathcal{H}^2$ is the absolutely continuous part of the measure $D\Theta$ with respect to two-dimensional Lebesgue measure $\mathcal{H}^2$ and the last term stands for the jump part concentrated on the $\mathcal{H}^1$-rectifiable set $J(\Theta)$ oriented by the unit normal vector ν and the traces of Θ on $J(\Theta)$ with respect to ν are denoted by $\Theta^\pm(x) = \lim_{\varepsilon\downarrow 0}\Theta(x \pm \varepsilon\nu(x))$ in $L^1_{loc}(J(\Theta))$. (Recall that an *SBV* function Θ has vanishing Cantor part of the measure $D\Theta$.) Since $m \in W^{1,1}$, the chain rule applied to $m = e^{i\Theta}$ implies that $\Theta^+ - \Theta^- \in L^1(J(\Theta), 2\pi\mathbb{Z})$ and $\nabla^a\Theta = m \wedge \nabla m \in L^1(\Omega)$. Moreover, since $\nabla \cdot m = 0$, the chain rule also yields $m^\perp \cdot \nabla^a\Theta = 0$ in $L^1(\Omega)$. Therefore, there exists a function $\lambda := m \cdot \nabla^a\Theta \in L^1(\Omega)$ such that $\nabla^a\Theta = \lambda m$. Applying now the chain rule for $\Phi(e^{i\Theta}) \in W^{1,1}$, we deduce

$$\nabla \cdot \big[\Phi(m)\big] = \nabla \cdot \big[\Phi\big(e^{i\Theta}\big)\big] = \frac{d}{d\theta}\Phi(m) \cdot \nabla^a\Theta = \frac{d}{d\theta}\Phi(m) \cdot m\lambda \overset{(7)}{=} 0 \quad \text{in } L^1(\Omega), \tag{12}$$

i.e., (10) holds.

Step 2. If $\Phi \in ENT$ and $m \in H^{1/2}_{div}(\Omega, S^1)$, then (10) holds. Indeed, let $B \subset \Omega$ be an arbitrary ball and let us consider a lifting $\Theta = \Theta_1 + \Theta_2$ of m in B with $\Theta_1 \in H^{1/2}(B)$, $\Theta_2 \in SBV(B)$ and $h = e^{i\Theta_2} \in W^{1,1} \cap H^{1/2}(B, S^1)$ (see Brezis, Bourgain, and Mironescu [7, Theorem 5]).

Therefore, the corresponding decomposition (11) of Θ_2 satisfies $\Theta_2^+ - \Theta_2^- \in L^1(J(\Theta_2), 2\pi\mathbb{Z})$ and $\nabla^a \Theta_2 = h \wedge \nabla h \in L^1(B)$.

Claim. *For every $\Psi \in C^2(S^1, \mathbf{R}^2)$, then $\Psi(m) \in H^{1/2}(B)$ and*

$$\nabla \cdot \left[\Psi(m)\right] = \frac{d}{d\theta}\Psi(m) \cdot \left(\nabla\Theta_1 + \nabla^a\Theta_2\right) \quad \textit{in } \mathbf{H}^{-1/2}(B). \tag{13}$$

Here, $\mathbf{H}^{-1/2}(B)$ is the dual space of $H^{1/2}_{00}(B)$ (see Appendix A *for more details).*

Proof of Claim. The fact that if $m \in H^{1/2}(B)$ then $\Psi(m) \in H^{1/2}(B)$ is standard and follows from

$$\int_B\int_B \frac{|\Psi(m(x)) - \Psi(m(y))|^2}{|x-y|^3}\,dx\,dy \leqslant \|\nabla\Psi\|^2_{L^\infty} \int_B\int_B \frac{|m(x)-m(y)|^2}{|x-y|^3}\,dx\,dy < \infty.$$

Therefore, $\nabla\cdot[\Psi(m)] \in \mathbf{H}^{-1/2}(B)$ (since the differential operator is continuous from $H^{1/2}(B)$ to $\mathbf{H}^{-1/2}(B)$, see e.g. [18]). Before proving (13), let us observe that the RHS of (13) is a distribution (a-priori, it doesn't belong to $\mathbf{H}^{-1/2}(B)$). Indeed, one has that $\frac{d}{d\theta}\Psi(m) \in H^{1/2} \cap L^\infty(B)$ (here we use that $\Psi \in C^2(S^1)$ so that $\frac{d}{d\theta}\Psi \in C^1(S^1)$); therefore, $\frac{d}{d\theta}\Psi(m) \cdot \nabla^a\Theta_2 \in \mathcal{D}'(B)$ as a duality product between L^∞ and L^1, while $\frac{d}{d\theta}\Psi(m) \cdot \nabla\Theta_1 \in \mathcal{D}'(B)$ since for every test function $\zeta \in C_c^\infty(B)$, one has

$$\left\langle \frac{d}{d\theta}\Psi(m) \cdot \nabla\Theta_1, \zeta \right\rangle_{(\mathcal{D}'(B), C_c^\infty(B))} = \left\langle \nabla\Theta_1, \zeta\frac{d}{d\theta}\Psi(m) \right\rangle_{(\mathbf{H}^{-1/2}(B), H^{1/2}_{00}(B))}. \tag{14}$$

In order to prove (13), we consider an approximating sequence $\Theta_{1,n} \in C^1(\bar{B})$ such that $\Theta_{1,n} \to \Theta_1$ in $H^{1/2}(B)$. We set $m_n = e^{i(\Theta_{1,n}+\Theta_2)} \in W^{1,1} \cap H^{1/2}(B, S^1)$. Applying the chain rule as in (12) for $\Psi(m_n) \in W^{1,1}$, we obtain

$$\nabla \cdot \left[\Psi(m_n)\right] = \frac{d}{d\theta}\Psi(m_n) \cdot \left(\nabla\Theta_{1,n} + \nabla^a\Theta_2\right) \quad \text{in } L^1 \cap \mathbf{H}^{-1/2}(B).$$

We want to pass to the limit $n \to \infty$ in order to get to (13). For that, we have $e^{i\Theta_{1,n}} \to e^{i\Theta_1}$ in $H^{1/2}(B)$ and we deduce that $m_n \to m$ in $H^{1/2}(B)$ by Proposition 10 (see Appendix A) and also, a.e. in B (up to a subsequence). The continuity of the differential operator from $H^{1/2}$ to $\mathbf{H}^{-1/2}$ combined with Proposition 10 lead to $\nabla\cdot[\Psi(m_n)] \to \nabla\cdot[\Psi(m)]$ in $\mathbf{H}^{-1/2}(B)$. On the other hand, the same arguments lead to $\frac{d}{d\theta}\Psi(m_n) \to \frac{d}{d\theta}\Psi(m)$ in $H^{1/2}(B)$ and a.e. in B (up to a subsequence); thus, by duality as in (14), one has $\frac{d}{d\theta}\Psi(m_n) \cdot \nabla\Theta_{1,n} \to \frac{d}{d\theta}\Psi(m) \cdot \nabla\Theta_1$ in $\mathcal{D}'(B)$ and by dominated convergence theorem, one also deduces that $\frac{d}{d\theta}\Psi(m_n) \cdot \nabla^a\Theta_2 \to \frac{d}{d\theta}\Psi(m) \cdot \nabla^a\Theta_2$ in $L^1(B)$, which yields (13). □

Coming back to Step 2, (13) applied for $\Psi(z) = z$ for $z \in S^1$ yields $m^\perp \cdot (\nabla\Theta_1 + \nabla^a\Theta_2) = 0$ in $\mathbf{H}^{-1/2}(\Omega)$ since $\nabla\cdot m = 0$. Idem, defining $\Psi(z) = -z^\perp$ for $z \in S^1$, (13) leads to $\lambda := m \cdot (\nabla\Theta_1 + \nabla^a\Theta_2) = \nabla\cdot[\Psi(m)] = \nabla \times m \in \mathbf{H}^{-1/2}(B)$. Formally, one writes (as at Step 1)

$\nabla\Theta_1 + \nabla^a\Theta_2 = m\lambda$ which belongs to $\mathcal{D}'(B)$ by duality as in (14) and as in (12), we apply (13) to obtain

$$\nabla \cdot \big[\Phi(m)\big] = \frac{d}{d\theta}\Phi(m) \cdot m\lambda = 0 \quad \text{in } \mathcal{D}'(B).$$

Let us explain rigorously this argument. Using the same approximation as in the proof of Claim, we have the following decomposition into the frame $(m_n, m_n^\perp)$:

$$\begin{aligned}\nabla\Theta_{1,n} + \nabla^a\Theta_2 &= \big(\big(\nabla\Theta_{1,n} + \nabla^a\Theta_2\big)\cdot m_n\big)m_n + \big(\big(\nabla\Theta_{1,n} + \nabla^a\Theta_2\big)\cdot m_n^\perp\big)m_n^\perp \\ &= \nabla \times m_n m_n + \nabla \cdot m_n m_n^\perp \quad \text{in } L^1,\end{aligned}$$

where we used the chain rule as at Step 1. Therefore, as in (12), we deduce by (7):

$$\begin{aligned}\nabla \cdot \big[\Phi(m_n)\big] &= \frac{d}{d\theta}\Phi(m_n) \cdot \big(\nabla\Theta_{1,n} + \nabla^a\Theta_2\big) \\ &= \nabla \cdot m_n \frac{d}{d\theta}\Phi(m_n) \cdot m_n^\perp \\ &\overset{(9)}{=} \gamma(m_n)\nabla \cdot m_n \quad \text{in } L^1. \end{aligned} \tag{15}$$

On the one hand, $\nabla \cdot [\Phi(m_n)] \to \nabla \cdot [\Phi(m)]$ in $\mathbf{H}^{-1/2}(B)$. On the other hand, $\nabla \cdot m_n \to \nabla \cdot m = 0$ in $\mathbf{H}^{-1/2}(B)$ and $\gamma(m_n) \to \gamma(m)$ in $H^{1/2}(B)$. By duality (as in (14)), we deduce that $\nabla \cdot [\Phi(m)] = 0$ in $\mathcal{D}'(B)$. Since B is an arbitrary ball in Ω, using a partition of unity, we conclude that (10) holds in $\mathcal{D}'(\Omega)$.

Step 3. If $\Phi \in ENT$ and $m \in W^{1/p,p}_{div}(\Omega, S^1)$ with $p \in (1,2)$, then (10) *holds*. Indeed, one uses the following Gagliardo–Nirenberg embedding: $L^\infty \cap W^{1/p,p} \subset H^{1/2}$ (see [6, Lemma D.1]) and concludes by Step 2.

Step 4. Conversely, let $\Phi \in C^\infty(S^1, \mathbf{R}^2)$ such that (10) holds for every $m \in W^{1/p,p}_{div}(\Omega, S^1)$ (with $p \in [1,2]$). Set $z \in S^1$. We prove that (7) holds for z using the same argument as in [23,22]. Up to translations, we may assume that Ω contains the origin 0. Motivated by (12), we consider a map m given by the vortex structure centered at $z^\perp$, i.e.,

$$m(x) := \left(\frac{x - z^\perp}{|x - z^\perp|}\right)^\perp \quad \text{for } x \in \Omega.$$

Then $m \in W^{1,q}_{div}(\Omega, S^1)$ for every $q \in [1,2)$ (in particular, $m \in W^{1/p,p}_{div}(\Omega, S^1)$ with $p \in [1,2]$) and $m(0) = z$. Moreover, since m is smooth around the origin 0, m has a smooth lifting Θ around 0 (unique up to a constant) that satisfies $\nabla\Theta(0) = z$. Then by (10) we know that $\nabla \cdot [\Phi(m)](0) = 0$. Therefore, as in (12), we obtain

$$\frac{d}{d\theta}\Phi(z) \cdot z = \frac{d}{d\theta}\Phi\big(m(0)\big) \cdot \nabla\Theta(0) = \nabla \cdot \big[\Phi(m)\big](0) = 0. \qquad \square$$

As a consequence, we prove the kinetic formulation (4):

Proof of Proposition 3. For every $\xi \in S^1$, we define "elementary entropies" $\Phi^\xi : S^1 \to \mathbf{R}^2$ given by

$$\Phi^\xi(z) := \xi \tilde{\chi}(z,\xi) = \begin{cases} \xi & \text{for } z \cdot \xi > 0, \\ 0 & \text{for } z \cdot \xi \leqslant 0. \end{cases}$$

Although Φ^ξ is not a smooth entropy (in fact, Φ^ξ has a jump at the points $\pm\xi^\perp \in S^1$), the equality (7) trivially holds in $\mathcal{D}'(S^1)$. That's why Φ^ξ is a generalized entropy. Moreover, as shown in [15], there exists a sequence of smooth entropies $\{\Phi_k\} \subset ENT$ such that $\{\Phi_k\}$ is uniformly bounded and $\lim_k \Phi_k(z) = \Phi^\xi(z)$ for every $z \in S^1$. Indeed, this smoothing result follows by (8): if one writes $\xi = e^{i\theta_0}$ with $\theta_0 \in (-\pi, \pi]$, then the unique 2π-periodic function $\varphi \in C(\mathbf{R})$ satisfying (8) for Φ^ξ is given by:

$$\varphi(\theta) = \xi \cdot z \mathbf{1}_{\{z\cdot\xi>0\}} = \cos(\theta - \theta_0)\mathbf{1}_{\{\theta-\theta_0\in(-\pi/2,\pi/2)\}} \quad \text{for } z = e^{i\theta},\ \theta \in (-\pi + \theta_0, \pi + \theta_0).$$

By (8) for Φ^ξ, the choice of the derivative φ' is fixed at the jump points $\pm\xi^\perp \in S^1$:

$$\varphi'(\theta) = -\sin(\theta - \theta_0)\mathbf{1}_{\{\theta-\theta_0\in(-\pi/2,\pi/2)\}} \quad \text{for } \theta \in (-\pi + \theta_0, \pi + \theta_0).$$

Now, one regularizes φ by 2π-periodic functions $\varphi_k \in C^\infty(\mathbf{R})$ that are uniformly bounded in $W^{1,\infty}(\mathbf{R})$ and $\lim_k \varphi_k(\theta) = \varphi(\theta)$ as well as $\lim_k \varphi_k'(\theta) = \varphi'(\theta)$ for every $\theta \in \mathbf{R}$. Thus, the desired (smooth) approximating entropies Φ_k are given by φ_k via (8). Therefore, Proposition 6 implies that for every $m \in W^{1/p,p}_{div}(\Omega, S^1)$ (with $p \in [1,2]$), one has $\int_\Omega \Phi_k(m) \cdot \nabla\zeta\, dx = 0$ for every $\zeta \in C_c^\infty(\Omega)$ and by the dominated convergence theorem, one concludes that

$$0 = \nabla \cdot \left[\Phi^\xi(m)\right] = \nabla \cdot \left[\xi \tilde{\chi}(m,\xi)\right] = \nabla \cdot \left[\xi \chi(\cdot,\xi)\right] = \xi \cdot \nabla\chi(\cdot,\xi). \qquad \square$$

Proof of Theorem 1. It is a consequence of Proposition 3 combined with the strategy of Jabin, Otto, and Perthame (see Theorem 1.3 in [25]). For completeness of the writing, let us recall the main steps of that argument: let $m : \Omega \to S^1$ be a measurable function that satisfies (4) for every $\xi \in S^1$. Notice that the divergence-free condition is automatically satisfied (in $\mathcal{D}'(\Omega)$) because of (5). The first step consists in defining an L^∞-trace of m on each segment $\Sigma \subset \Omega$. More precisely, if $\Sigma := \{0\} \times [-1,1] \subset \Omega$, then there exists a trace $\tilde{m} \in L^\infty(\Sigma, S^1)$ such that

$$\lim_{r\to 0} \frac{1}{r} \int_{-r}^{r} \int_{-1}^{1} \left| m(x_1,x_2) - \tilde{m}(x_2) \right| dx_2\, dx_1 = 0$$

and for each Lebesgue point $(0,x_2) \in \Sigma$ of m, one has $m(0,x_2) = \tilde{m}(x_2)$. Observe that this step is straightforward in the case of $m \in W^{1,1}_{div}(\Omega, S^1)$; however, it is essential for example in the case of $m \in H^{1/2}_{div}(\Omega, S^1)$. The second step is to prove that the trace $\tilde{m}$ of m on Σ is almost everywhere orthogonal at Σ (which coincides with the classical principle of characteristics for smooth vector fields m). The key point for that resides in a relation of order of characteristics of m, i.e., for every two Lebesgue points $x, y \in \Omega$ of m with the segment $[x,y] \subset \Omega$, the following implication holds:

$$m(x) \cdot (y - x) > 0 \quad \Rightarrow \quad m(y) \cdot (y - x) > 0.$$

The final step consists is proving that on any open convex subset $\omega \subset \Omega$ with $d = \operatorname{dist}(\omega, \partial\Omega) > 0$, only two situations may occur: either two characteristics of m intersect at $P \in \Omega$ with $\operatorname{dist}(P, \omega) < d$ and $m(x) = \pm\frac{(x-P)^\perp}{|x-P|}$ for $x \in \omega \setminus \{P\}$, or m is $1/d$-Lipschitz in ω, i.e.,

$$\left|m(x) - m(y)\right| \leqslant \frac{1}{d}|x - y|, \quad \text{for every } x, y \in \omega$$

(in this case, every two characteristics passing through ω may intersect only at distances $\geqslant d$ outside ω). Notice that m may have infinitely many vortex points P_k and any vortex point has degree one, but the orientation α_k of the vortex point P_k could change or not in Ω (see Section 4). □

4. Several examples

4.1. Lipschitz vector fields (1) that are not C^1

Proof of Proposition 1. Let $\Theta : (0, 1) \to (\frac{\pi}{4}, \frac{\pi}{3})$ be a Lipschitz function that is not in $C^1(0, 1)$. On the "space" axis s, we define m as given by

$$m(s, 0) = \big(\cos\Theta(s), \sin\Theta(s)\big) = e^{i\Theta(s)} \in S^1 \quad \text{for every } s \in (0, 1).$$

Then m has a unique Lipschitz extension satisfying (1): The initial value (at "time" $t = 0$) of m determines the characteristics along which m remains constant. More precisely, we define the flow of characteristics $F : Dom(F) = (0, 1) \times (-\frac{1}{3\|\frac{d}{ds}\Theta\|_{L^\infty}}, \frac{1}{3\|\frac{d}{ds}\Theta\|_{L^\infty}}) \to \mathbf{R}^2$ as

$$F(s, t) = (s, 0) + tm(s, 0)^\perp = s + ite^{i\Theta(s)} \in \mathbf{C} \quad \text{for every } (s, t) \in Dom(F).$$

The choice of "time" range is done in order that F is a bi-Lipschitz homeomorphism onto its open range, denoted Ω (which implies that characteristics of m do not intersect in the domain Ω). Indeed, one computes that

$$\nabla F(s, t) = \left(\frac{\partial}{\partial s}F \frac{\partial}{\partial t}F\right) = \begin{pmatrix} 1 - t\cos\Theta(s)\frac{d}{ds}\Theta(s) & -\sin\Theta(s) \\ -t\sin\Theta(s)\frac{d}{ds}\Theta(s) & \cos\Theta(s) \end{pmatrix}$$

with

$$\det\nabla F(s, t) = \cos\Theta(s) - t\frac{d}{ds}\Theta(s) \geqslant \frac{1}{2} - \left|t\frac{d}{ds}\Theta(s)\right| \geqslant \frac{1}{6} \quad \text{in } Dom(F).$$

Here, we used that

$$|t| \leqslant \frac{1}{3\|\frac{d}{ds}\Theta\|_{L^\infty}} \quad \text{and} \quad \cos\Theta(s) \geqslant 1/2. \tag{16}$$

Therefore, in order that F is a bi-Lipschitz homeomorphism, we show that F is injective on $Dom(F)$. Assume by contradiction that there exist two points $(s, t) \neq (\tilde{s}, \tilde{t})$ in $Dom(F)$ so that $F(s, t) = F(\tilde{s}, \tilde{t})$. Then $t, \tilde{t} \neq 0$ and $\frac{\tilde{t}}{t} = \frac{\cos\Theta(s)}{\cos\Theta(\tilde{s})}$. It follows that

$$|s-\tilde{s}| = \left|t\left(\sin\Theta(s) - \frac{\tilde{t}}{t}\sin\Theta(\tilde{s})\right)\right| = \left|\frac{t}{\cos\Theta(\tilde{s})}\sin\bigl(\Theta(s)-\Theta(\tilde{s})\bigr)\right|$$
$$\leqslant \frac{2}{3\|\frac{d}{ds}\Theta\|_{L^\infty}}\bigl|\Theta(s)-\Theta(\tilde{s})\bigr| \leqslant \frac{2}{3}|s-\tilde{s}|$$

which would mean that $s=\tilde{s}$ and then, $t=\tilde{t}$ which is a contradiction. (In the above inequalities, we used again (16).) Denoting by $G=(G_1,G_2):\Omega\to Dom(F)$ the inverse of F, we have that

$$\nabla G\bigl(F(s,t)\bigr) = \frac{1}{\cos\Theta(s) - t\frac{d}{ds}\Theta(s)}\begin{pmatrix} \cos\Theta(s) & \sin\Theta(s) \\ t\sin\Theta(s)\frac{d}{ds}\Theta(s) & 1-t\cos\Theta(s)\frac{d}{ds}\Theta(s)\end{pmatrix}. \tag{17}$$

We now define $m:\Omega\to S^1$ by

$$m\bigl(F(s,t)\bigr) = m(s,0) \quad \text{for every } (s,t)\in Dom(F).$$

Obviously, m is a Lipschitz function in Ω with values in S^1. Since the open segment $(0,1)\times\{0\}\subset\Omega$ and Θ is not C^1 in $(0,1)$, then m is not a C^1 map in Ω. Finally, we check that $\nabla\cdot m = 0$ in Ω. Indeed, if $(x_1,x_2)=F(s,t)\in\Omega$, then $m(x)=m(G_1(x),0)=e^{i\Theta(G_1(x))}$ and

$$\nabla\cdot m(x) = \frac{\partial}{\partial x_1}\bigl[\cos\Theta\bigl(G_1(x)\bigr)\bigr] + \frac{\partial}{\partial x_2}\bigl[\sin\Theta\bigl(G_1(x)\bigr)\bigr]$$
$$= \left(-\sin\Theta(s)\frac{\partial}{\partial x_1}G_1\bigl(F(s,t)\bigr) + \cos\Theta(s)\frac{\partial}{\partial x_2}G_1\bigl(F(s,t)\bigr)\right)\frac{d}{ds}\Theta(s) \overset{(17)}{=} 0. \qquad \square$$

4.2. H^1 vector fields (1) that are not globally Lipschitz

The case of nonconvex domains Ω. We consider the kink domain

$$\Omega = \bigl\{re^{i\theta}:\ r\in(0,1),\ |\theta|<r\bigr\}$$

and the boundary vortex configuration in the origin: $m(x)=\frac{x^\perp}{|x|}$ for $x\in\Omega$. Then $m\in H^1(\Omega,S^1)$ and m is not globally Lipschitz in Ω (but only locally Lipschitz).

The case of convex domains Ω. There exist a convex domain Ω and a vector field $m\in H^1_{div}(\Omega,S^1)$, such that m is not globally Lipschitz in Ω. Indeed, we use the construction in the proof of Proposition 1. Let $\Theta:(0,1/10)\to(-\infty,0)$ be given by $\Theta(s)=-\sqrt{s}$ for $s\in(0,1/10)$. On the "space" axis $(0,1)\times\{0\}$, we define m by

$$m(s,0) = \bigl(\cos\Theta(s),\sin\Theta(s)\bigr) = e^{i\Theta(s)}\in S^1 \quad \text{for every } s\in(0,1/10).$$

The flow of characteristics is defined by

$$F: Dom(F) = \bigl\{(s,t):\ s\in(0,1/10),\ 0<t<\sqrt{s}/10\bigr\}\to\mathbf{R}^2$$

as

$$F(s,t) = (s,0) + tm(s,0)^\perp = s + ite^{i\Theta(s)}\in\mathbf{C} \quad \text{for every } (s,t)\in Dom(F).$$

Then F is locally a C^1-diffeomorphism since

$$\det \nabla F(s,t) = \cos\Theta(s) - t\frac{d}{ds}\Theta(s) \geqslant \cos\Theta(1/10) > 0 \quad \text{in } Dom(F).$$

We show that no characteristics intersect inside $Dom(F)$ (i.e., F is injective in $Dom(F)$): Assume by contradiction that there exist two points $(s,t) \neq (\tilde{s},\tilde{t})$ in $Dom(F)$ so that $F(s,t) = F(\tilde{s},\tilde{t})$. Then $t,\tilde{t} \neq 0$ and $\frac{\tilde{t}}{t} = \frac{\cos\Theta(s)}{\cos\Theta(\tilde{s})}$ which would imply

$$|s-\tilde{s}| = \left| t\left(\sin\Theta(s) - \frac{\tilde{t}}{t}\sin\Theta(\tilde{s})\right)\right| = \left|\frac{t}{\cos\sqrt{\tilde{s}}}\sin(\sqrt{s}-\sqrt{\tilde{s}})\right| \leqslant \frac{\sqrt{s}}{5}\frac{|s-\tilde{s}|}{\sqrt{s}+\sqrt{\tilde{s}}} \leqslant \frac{1}{5}|s-\tilde{s}|$$

and would mean that $s = \tilde{s}$ and then $t = \tilde{t}$ which is a contradiction. Here, we used that $0 < t < \frac{\sqrt{s}}{10}$ and $\cos\Theta(s) \geqslant 1/2$ if $s \in (0, 1/10)$. Let Ω be any convex domain included in $F(Dom(F))$ such that $\bar{\Omega}$ contains the segment $[0, 1/10] \times \{0\}$ and $G : \Omega \to F^{-1}(\Omega)$ be the inverse of F. Then (17) holds and we set

$$m\big(F(s,t)\big) = m(s,0) \quad \text{for every } (s,t) \in G(\Omega).$$

We compute that

$$\nabla m\big(F(s,t)\big) = \left(\frac{dm}{ds}(s,0)\ 0_{\mathbf{R}^2}\right) \cdot \nabla G\big(F(s,t)\big)$$

so that (17) implies

$$|\nabla m|\big(F(s,t)\big) = \frac{1}{\det\nabla F(s,t)}\left|\frac{dm}{ds}(s)\right| = \frac{1}{\cos(\sqrt{s}) + \frac{t}{2\sqrt{s}}}\left|\frac{d\Theta}{ds}(s)\right|.$$

Then

$$\begin{aligned}\int_\Omega \left|\nabla m(x)\right|^2 dx &= \int_{G(\Omega)} |\nabla m|^2\big(F(s,t)\big)\det\nabla F(s,t)\,ds\,dt \\ &\leqslant \int_0^{1/10}\left|\frac{d\Theta}{ds}(s)\right| ds \int_0^{\sqrt{s}/10} \frac{1}{\cos(\sqrt{s}) - t\frac{d\Theta}{ds}(s)}\left|\frac{d\Theta}{ds}(s)\right| dt \\ &\leqslant \int_0^{1/10} \frac{ds}{2\sqrt{s}}\Big(\ln\big(\cos(\sqrt{s}) + 1/20\big) - \ln\big(\cos(\sqrt{s})\big)\Big) < \infty.\end{aligned}$$

Therefore, $m \in H^1_{div}(\Omega, S^1)$ and m is not globally Lipschitz in Ω since $\frac{d\Theta}{ds}$ blows up at $s = 0$. $\square$

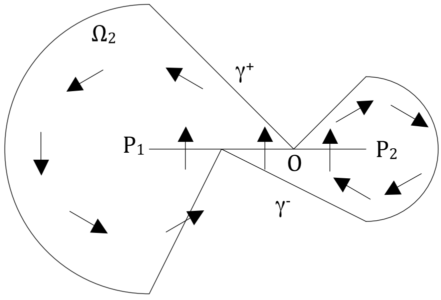

Fig. 3. Two vortex-point singularities of different orientation.

4.3. *Vector fields (1) with arbitrary many vortex-point singularities*

The geometry of the domain Ω determines the number of vortex-point singularities of a vector field (1).

Remark 6. (i) If Ω is a convex domain and $m \in W^{1/p,p}_{div}(\Omega, S^1)$ (with $p \in [1,2]$), then m has either no (interior) vortex-point singularities, or one interior vortex singularity in Ω. It is a consequence of the final step in the proof of Theorem 1 (see [25] for more details).

(ii) If $\Omega \neq \mathbf{R}^2$ is a smooth simply connected domain and additionally we impose the boundary condition $m \cdot n = 0$ on $\partial\Omega$, then either Ω is a disk and m has a vortex singularity placed in the center of the disk, or Ω is a strip and m is constant (see [25]).

Proof of Proposition 2. First of all, we construct domains Ω_n and $W^{1,q}$ vector fields (1) defined on Ω_n with n vortex-point singularities for every $n \geqslant 1$. For $n = 1$, we choose $\Omega_1 := B(0,2)$ to be the ball centered at the origin of radius 2 and $m_1(x) = \frac{x^\perp}{|x|}$ in Ω_1. Then $m_1 \in W^{1,q}_{div}(\Omega_1, S^1)$ and $\|\nabla m_1\|^q_{L^q(\Omega_1)} = 2\pi \frac{2^{2-q}}{2-q}$, for any $q \in [1,2)$. For $n = 2$, the construction is the following: Set $f, g : \mathbf{R} \to \mathbf{R}$ with $f(t) = |t|$ and

$$g(t) = \begin{cases} 2(t+1) & \text{for } t \leqslant -1, \\ -\frac{1}{2}(t+1) & \text{for } t \geqslant -1 \end{cases}$$

and define the curves

$$\gamma^+ = \big\{\big(x_1, f(x_1)\big)\colon x_1 \in [-2,1]\big\} \quad \text{and} \quad \gamma^- = \big\{\big(x_1, g(x_1)\big)\colon x_1 \in [-2,1]\big\}.$$

Fixing the vortex points $P_1 := (-2,0)$ and $P_2 = (1,0)$, we define the domain

$$\begin{aligned} \Omega_2 := {} & \big(B(P_1,2) \cap \{x_1 \leqslant -2\}\big) \cup \big\{(x_1,x_2)\colon x_1 \in [-2,1],\ g(x_1) < x_2 < f(x_1)\big\} \\ & \cup \big(B(P_2,1) \cap \{x_1 \geqslant 1\}\big) \end{aligned}$$

(see Fig. 3). We define $m_2 : \Omega_2 \to S^1$ as follows:

$$m_2(x) = \begin{cases} \frac{(x-P_1)^\perp}{|x-P_1|} & \text{in } \{x \in \Omega_2\colon x_1 \leqslant -1 \text{ or } (x_1 \in (-1,0) \text{ and } x_2 > 0)\}, \\ -\frac{(x-P_2)^\perp}{|x-P_2|} & \text{in } \{x \in \Omega_2\colon x_1 \geqslant 0 \text{ or } (x_1 \in (-1,0) \text{ and } x_2 < 0)\}. \end{cases}$$

Then $m_2 \in W^{1,q}_{div}(\Omega, S^1)$ has two vortex-point singularities P_1 and P_2 of degree 1, but with different orientation. One easily checks that $\|\nabla m_2\|^q_{L^q(\Omega_1)} \lesssim \frac{1+2^{2-q}}{2-q}$, $q \in [1,2)$.

For arbitrary n, the above construction is to be repeated by an inductive argument: for every positive integer n, we construct a nonconvex Lipschitz simply-connected domain Ω_n and a vector field $m_n \in W^{1,q}_{div}(\Omega_n, S^1)$ having n vortex singularities and such that there exists a vortex point $P_n \in \Omega_n$ where $(P_n + \mathbf{R}_+ \times \mathbf{R}) \cap \Omega_n$ is a half disk of radius 2^{-n+2} and $m_n(x) = (-1)^{n-1}\frac{(x-P_n)^\perp}{|x-P_n|}$ for every $x \in (P_n + \mathbf{R}_+ \times \mathbf{R}) \cap \Omega_n$ and

$$\|\nabla m_n\|^q_{L^q(\Omega_1)} \lesssim \frac{2^{2-q} + \sum_{k=0}^{n-2} 2^{-k(2-q)}}{2-q} \leqslant C(q), \quad q \in [1,2),$$

where the constant $C(q) > 0$ is independent of n.

Letting $n \to \infty$, one gets

$$\Omega = \{x \in \mathbf{R}^2\colon\ \exists n(x) \geqslant 1,\ \forall n \geqslant n(x),\ x \in \Omega_n\} \tag{18}$$

which is a nonconvex piecewise Lipschitz simply-connected domain. We define $m : \Omega \to S^1$ as follows: for every $x \in \Omega$, set $m(x) = m_n(x)$, for $n \geqslant n(x)$ (given in (18)). Then $m \in W^{1,q}_{div}(\Omega, S^1)$ for every $q \in [1,2)$ and has infinitely many vortex-point singularities $\{P_1, P_2, \ldots\}$. (Here, the sequence of points $\{P_k\}$ accumulates on some point $P \in \partial\Omega$, therefore $\partial\Omega \setminus \{P\}$ is a Lipschitz curve, but not $\partial\Omega$.) □

Remark 7. In the above construction, the vortex singularities have alternative orientations. However, one can construct domains where the vortex singularities have the same orientation. Here is the example of two vortex configuration in $P_1 = (-3,0)$ and $P_2 = (1,0)$: $\omega = \omega_1 \cup \omega_2 \cup \omega_3$ with ω_1 be a union of a square and a rectangle

$$\omega_1 := \big((-5,-1) \times (-2,2)\big) \cup \big((-5,-3) \times (-4,-2)\big)$$

and $\omega_2 := ((0,2) \times (-1,1)) \cup ((1,2) \times (-4,-1))$ and $\omega_3 = (-3,1) \times (-4,-3)$. Then choose

$$m(x) = \begin{cases} \frac{(x-P_1)^\perp}{|x-P_1|} & \text{in } \omega_1, \\ \frac{(x-P_2)^\perp}{|x-P_2|} & \text{in } \omega_2, \\ (1,0) & \text{in } \omega_3. \end{cases}$$

Then $m \in W^{1,q}_{div}(\omega, S^1)$ for every $q \in [1,2)$ (see Fig. 4).

Let us explain why in general a domain Ω satisfying the properties in Proposition 2 (i.e. admitting configurations with infinitely many vortex-point singularities) is not Lipschitz.

Proposition 7. *If Ω is a Lipschitz domain and $m \in W^{1/p,p}_{div}(\Omega, S^1)$ (with $p \in [1,2]$), then m has only a finite number of interior vortex singularities.*

Proof. Assume by contradiction that m has infinitely many interior vortex singularities $\{P_1, P_2, \ldots\}$. Obviously, discarding a subsequence, we can assume that the points P_k converge to a point $P \in \bar{\Omega}$.

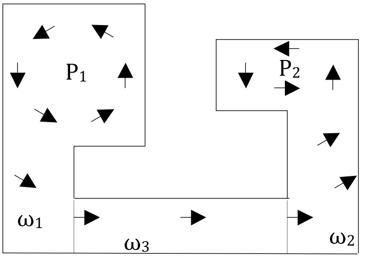

Fig. 4. Two vortex-point singularities with the same orientation.

Claim. *There exist two cones $\mathcal{C}_{k_1} \subset \Omega$ and $\mathcal{C}_{k_2} \subset \Omega$ centered in P_{k_1} and P_{k_2} (of some positive height and positive angle) such that $\mathcal{C}_{k_1} \cap \mathcal{C}_{k_2}$ has nonempty interior.*

Proof of Claim. The case of a limit point P belonging to the interior of Ω is obvious. Let us suppose that $P \in \partial\Omega$. Since $\partial\Omega$ is Lipschitz, we may assume that there exists a ball B centered at P such that $\partial\Omega \cap B$ is the Lipschitz graph $\{(x_1, \gamma(x_1)): x_1 \in (-\delta, \delta)\}$ with $\gamma(0) = \delta$, $P = (0, \gamma(0))$ and

$$U = \{(x_1, x_2): 0 \leqslant x_2 < \gamma(x_1),\ x_1 \in (-\delta, \delta)\} \subset \Omega \cap B.$$

Therefore, for every $P_k = (x_{1,k}, x_{2,k}) \in U$, the vertical segment S_k between $(x_{1,k}, 0)$ and P_k belongs to U. Since γ is a Lipschitz function, there exists an angle

$$\alpha = \alpha(\gamma) := \pi/2 - \arctan\left(\left\| \frac{d\gamma}{dx_1} \right\|_{L^\infty(-\delta,\delta)}\right) > 0$$

such that the cone $\mathcal{C}_k$ centered at P_k of angle α and having S_k as height is included in Ω. Due to the fact that P_k converges to P, it follows that there exists $k_0 \geqslant 1$ such that two cones $\mathcal{C}_{k_1}$ and $\mathcal{C}_{k_2}$ have nonempty interior intersection for any $k_1, k_2 \geqslant k_0$. This finishes the proof of Claim.

The contradiction comes from the structure proved in Theorem 1 since $m(x) = \pm\frac{(x-P_{k_j})^\perp}{|x-P_{k_j}|}$ in $\mathcal{C}_{k_1} \cap \mathcal{C}_{k_2}$ for both $j = 1, 2$ which is absurd. $\square$

Remark 8. For a Lipschitz domain Ω, we say that a point $P \in \partial\Omega$ is a *boundary vortex* singularity of m if there exists a cone $\mathcal{C} \subset \bar{\Omega}$ (of some height > 0 and angle $\beta > 0$) centered at P such that $m(x) = \pm\frac{(x-P)^\perp}{|x-P|}$ for $x \in \mathcal{C} \setminus \{P\}$. By Jabin–Otto–Perthame strategy, a boundary vortex point P of m is always assigned to a maximal cone $\mathcal{C}$, in the sense that for every point $x \in \Omega \setminus \mathcal{C}$ outside the maximal cone, the characteristic of m passing through x stays outside the cone $\mathcal{C}$. For $q < 2$, one can construct $W^{1,q}$ vector fields (1) in a Lipschitz simply connected domain with infinitely many boundary vortex-point singularities (that accumulate on the boundary).

5. Non-density results. Proof of Propositions 4 and 5

Proof of Proposition 5. Assume by contradiction that such a sequence exists. By dominated convergence theorem, it would mean in particular that $m_n \to m$ in $L^1(B^2)$, i.e.,

$$\int_0^1 \int_{\partial B_r} |m_n - m| \, d\mathcal{H}^1 \, dr \to 0 \quad \text{as } n \to \infty.$$

Up to a subsequence, there exists $0 < r < 1$ such that

$$\int_{\partial B_r} |m_n - m| \, d\mathcal{H}^1 \to 0 \quad \text{as } n \to \infty. \tag{19}$$

The key point of the proof consists in a dynamical system argument related to the topology of the flow of $m_n^\perp$: Using the technique in [13,24,21], we will consider the autonomous system

$$\dot{X} = m_n^\perp(X). \tag{20}$$

First of all, (20) has no critical point and no cycle (i.e., no closed loop): Since $|m_n^\perp| = 1$ in B^2 and $m_n^\perp$ is smooth, the degree of $m_n^\perp$ on a closed curve in B^2 is zero and therefore, an orbit of (20) cannot be closed in B^2. Now set X_n be the orbit of (20) passing by 0 (see Fig. 5), i.e.,

$$\begin{cases} \dot{X}_n(t) = m_n^\perp(X_n(t)), \\ X_n(0) = 0. \end{cases}$$

Then either the orbit X_n reaches the boundary ∂B^2 in finite time, or the limit points of X_n (see [10, Chapter 16]) belong to the boundary ∂B^2: Suppose that this is not the case, i.e., there is a limit point inside the ball B^2. Since (20) has no critical point, Poincaré–Bendixson's theorem (see [10, Theorem 2.1]) implies that the limit set of X_n should contain a periodic orbit which is a contradiction with the nonexistence of cycles for (20). Hence, the orbit X_n separates the ball B_r into a right side G_n (where m_n is the inner normal vector to ∂G_n) and a left side $B_r \setminus G_n$ (see Fig. 5). We define

$$\chi_n = \begin{cases} \frac{1}{2} & \text{in } G_n, \\ -\frac{1}{2} & \text{in } B_r \setminus G_n, \\ 0 & \text{in } B^2 \setminus B_r. \end{cases}$$

Then $\chi_n \in BV(B^2)$ with

$$\begin{aligned} D\chi_n &= m_n \mathcal{H}^1 \llcorner \big(\{X_n\} \cap B_r\big) + \chi_n^- m^\perp \mathcal{H}^1 \llcorner \partial B_r \\ &= m_n |D\chi_n| \llcorner B_r + \chi_n^- m^\perp \mathcal{H}^1 \llcorner \partial B_r \quad \text{in } \mathcal{D}'\big(B^2\big), \end{aligned} \tag{21}$$

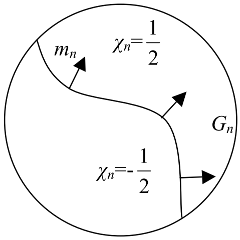

Fig. 5. The orbit X_n of the vector field $m_n^\perp$ passing by 0 in the ball B^2.

where χ_n^- is the interior trace of χ_n on ∂B_r with respect to the normal outer vector $\frac{x}{|x|} = -m^\perp(x)$. Notice that $|\chi_n^-| = \frac{1}{2}$ on ∂B_r. Moreover, we have that

$$\int_{B_r} |D\chi_n| = \mathcal{H}^1\big(\{X_n \in B_r\}\big) \geqslant 2r \quad \text{and} \quad \int_{B^2} |D\chi_n| = \int_{B_r} |D\chi_n| + \pi r. \tag{22}$$

Integration by parts leads to

$$\begin{aligned}
2r &\overset{(22)}{\leqslant} \int_{B_r} |D\chi_n| \overset{(21)}{=} \int_{B_r} m_n \cdot D\chi_n \\
&= -\int_{\partial B_r} m_n \cdot m^\perp \chi_n^- \, d\mathcal{H}^1 - \int_{B_r} \nabla \cdot m_n \chi_n \, dx \\
&\overset{(19)}{=} o(1) - \int_{B_r} \nabla \cdot m_n \chi_n \, dx \quad \text{as } n \to \infty.
\end{aligned} \tag{23}$$

If (a) holds, then the contradiction follows immediately from (23): since $|\chi_n| = 1/2$ in B_r, then $\int_{B_r} \nabla \cdot m_n \chi_n \, dx = o(1)$ which would mean that $o(1) \geqslant r > 0$ that is absurd. If (b) holds, we prove first that $\{\chi_n\}$ is uniformly bounded in $BV(B^2)$. Indeed, since $\{\nabla \cdot m_n\}$ is bounded in $H^{-s}(B^2)$ (here $s \in [0, 1/2)$), by Gagliardo–Nirenberg's inequality (see Proposition 8 in Appendix A) we have

$$\begin{aligned}
\left| \int_{B_r} \nabla \cdot m_n \chi_n \, dx \right| &= \left| \int_{B^2} \nabla \cdot m_n \chi_n \, dx \right| \leqslant \|\nabla \cdot m_n\|_{H^{-s}(B^2)} \|\chi_n\|_{H^s(B^2)} \\
&\overset{\text{Proposition 8}}{\leqslant} C\|\nabla \cdot m_n\|_{H^{-s}(B^2)} \big(\|\chi_n\|_{L^\infty(B^2)}^{1/2} \|\chi_n\|_{BV(B^2)}^{1/2} + \|\chi_n\|_{L^\infty(B^2)}\big) \\
&\overset{(22)}{\leqslant} C\bigg(1 + \bigg(\int_{B_r} |D\chi_n|\bigg)^{1/2}\bigg).
\end{aligned}$$

By (23), we deduce that $\{\chi_n\}$ is uniformly bounded in $BV(B_r)$ and also in $BV(B^2)$ due to (22). Therefore, $\{\chi_n\}$ is relatively compact in $H^s(B^2)$ (since $s \in [0, 1/2)$, see Proposition 8 in Appendix A) and we conclude by assumption (b) that

$$\int_{B_r} \nabla \cdot m_n \chi_n \, dx = \int_{B^2} \nabla \cdot m_n \chi_n \, dx = o(1)$$

which is a contradiction with (23). □

Proof of Proposition 4. Obviously, the statement of Proposition 4 is a direct consequence of Proposition 5(a). However, there is an easier proof in this particular case. More precisely, we assume by contradiction that there exists a sequence of divergence-free vector fields $m_n \in C^\infty(\Omega, S^1)$ satisfying $m_n \to m$ a.e. in B^2. In particular, dominated convergence theorem leads to $m_n \to m$ in $L^3(B^2)$. First of all, Poincaré's lemma yields the existence of smooth stream functions $\varphi_n \in C^\infty(\Omega)$ such that $m_n = \nabla^\perp \varphi_n$ in B^2. Since $|m_n| = 1$, φ_n is a 1-Lipschitz function. Observe that $m = \nabla^\perp \varphi$ with $\varphi(x) = |x|$ in B^2. Subtracting eventually a constant, we can assume that $\int_{B^2} \varphi_n \, dx = \int_{B^2} \varphi \, dx$ so that $\varphi_n \to \varphi$ uniformly in B^2: indeed, for every $x \in B^2$, one has

$$\begin{aligned}
\big|\varphi_n(x) - \varphi(x)\big| &= \bigg| \fint_{B^2} \big[(\varphi_n - \varphi)(x) - (\varphi_n - \varphi)(y)\big] \, dy \bigg| \\
&\leqslant \fint_{B^2} \int_0^1 |x-y| \, \big|\nabla(\varphi_n - \varphi)\big|\big(x + t(y-x)\big) \, dy dt \\
&\leqslant C \int_0^1 \bigg(\int_{B^2} \big|\nabla(\varphi_n - \varphi)\big|^3 \big(x + t(y-x)\big) \, dy \bigg)^{1/3} dt \\
&\leqslant C \|m_n - m\|_{L^3(B^2)} \int_0^1 \frac{dt}{t^{2/3}} \to 0 \quad \text{uniformly in } x, \quad \text{as } n \to \infty.
\end{aligned}$$

Since 0 is a strict minimum of φ, it would imply that φ_n has a minimum inside B^2, i.e., $\nabla \varphi_n$ vanishes somewhere inside B^2 which is a contradiction with the fact that $|\nabla \varphi_n| = 1$ in B^2. □

6. Dense subsets: Proof of Theorems 2 and 3

We start by proving Theorem 2:

Proof of Theorem 2. By Proposition 7, we know that m has only a finite number of (interior) vortex singularities in Ω (call this set $\mathcal{A}$).

Case 1. There are no interior vortex-point singularities, i.e., $\mathcal{A} = \emptyset$. By Theorem 1 we know that m is locally Lipschitz in Ω. For $d > 0$ small, we choose a smooth simply connected subdomain Ω^d of Ω that is close to Ω in the sense that $\operatorname{dist}(x, \partial\Omega) \in (d, 3d/2)$ for every $x \in \partial\Omega^d$. Then, by Theorem 1, we have that m is globally Lipschitz in Ω^d. For each characteristic of m passing through a point $x \in \Omega^d$ we call *extremal points* in $\bar{\Omega}^d$ the two intersection points $P(x)$ and $Q(x)$ of the characteristic with $\partial\Omega^d$. By the Jabin–Otto–Perthame procedure (as recalled in the proof of Theorem 1), any two characteristics of m passing through two points $x, y \in \Omega^d$ intersect outside Ω^d at a distance larger than d with respect to their extremal points. As a consequence, $|m(x) - m(y)| \leqslant \frac{1}{d}|x - y|$ for every $x, y \in \Omega^d$, i.e., m is globally Lipschitz in Ω^d with the Lipschitz constant $\leqslant 1/d$ (see [25]).

A standard geometry argument shows the existence of a (at most) countable set of segments $\gamma_j : [0, 1] \to \bar{\Omega}^d$, $j \in J \subset \mathbb{N}$ such that

(1) every two (open) segments $\gamma_j((0, 1))$ and $\gamma_k((0, 1))$ are disjoint, $j \neq k$;

(2) the characteristics of m passing through $\gamma_j(t)$ for every $t \in [0, 1]$ and $j \in J$ cover the whole domain $\bar{\Omega}^d$, i.e., $\bar{\Omega}^d = \bigcup_{j\in J, t\in[0,1]}[P(t)Q(t)]$ where $P(t) \in \partial\Omega^d$ and $Q(t) \in \partial\Omega^d$ are the extremal points in $\bar{\Omega}^d$ of the characteristic passing through the point $\gamma_j(t)$.

(3) for every $t \in (0, 1)$ and $j \in J$, the characteristic $[P(t)Q(t)]$ passing through $\gamma_j(t)$ intersects the set of segments $\cup_{k\in J}\{\gamma_k([0, 1])\}$ only at $\{\gamma_j(t)\}$.

One could have two end points P_j and P_k of segments γ_j and γ_k that coincide, or the characteristic passing through P_j may intersect γ_k at the end point P_k. (There exist smooth vector fields m with (1) on C^2 simply connected domain such that any set of curves $\{\gamma_j\}$ satisfying (1), (2), (3) are necessary infinitely countable, i.e., J is not finite; one could think of a local boundary given by the graph $\{(x_1, \gamma(x_1))\}$ with $\gamma : x_1 \mapsto x_1^4 \sin\frac{1}{x_1}$ around 0 and m be a small perturbation of the constant vector field $\mathbf{e}_2$.)

We consider the lifting Θ (unique up to a constant) of m in Ω^d, so $\Theta \in W^{1,\infty}(\Omega^d)$. The smoothing procedure is the following: for each segment $\gamma_j([0, 1])$, we approximate the lifting $\Theta|_{\gamma_j}$ in $H^1(\{\gamma_j\})$ (or any $W^{1,q}$ with $q < \infty$) by a sequence of C^∞ liftings $\Theta_n|_{\gamma_j}$ such that $\Theta_n|_{\partial\gamma_j} = \Theta|_{\partial\gamma_j}$ and the Lipschitz constant of $e^{i\Theta_n}$ is less than $1/(d - 1/n)$. Then we set $m_n := e^{i\Theta_n}|_{\gamma_j}$ on the segment γ_j; after that, m_n is uniquely smoothly extended along the characteristics starting from any point $\gamma_j(t)$ due to the initial value $m_n(\gamma_j(t))$ since these characteristics (passing through points $\gamma_j(t)$) could intersect only outside Ω^d at a distance $\geqslant d - \frac{1}{n}$ with respect to their extremal points in Ω^d. The new vector fields m_n satisfy (1) in Ω^d and approximate m in $H^1(\Omega^d)$; they are smooth on the domain covered by the characteristics passing through $\gamma_j((0, 1))$. However, globally in Ω^d, they could be only Lipschitz (and not smooth) at the end points $\gamma_j(\{0, 1\})$. Let us call these corresponding characteristics as "bad" characteristics of m_n. In order to smooth everywhere m_n, we will proceed as follows: we will restrict to a subdomain $\tilde{\Omega}^d$ of Ω^d such that $0 < \operatorname{dist}(x, \partial\Omega^d) < d/2$ for every $x \in \partial\tilde{\Omega}^d$. Then any "bad" characteristic S has the following property: $S \cap \tilde{\Omega}^d$ splits $\tilde{\Omega}^d$ into two open subdomains ω_n and $\tilde{\omega}_n$ so that m_n is locally smooth around S in ω_n respectively in $\tilde{\omega}_n$. Then one considers a small segment $\tilde{S} = [A\tilde{A}]$ orthogonal to S such that the middle point $M := (A + \tilde{A})/2$ of $\tilde{S}$ belongs to the "bad" characteristic S and m_n is smooth on $(AM) \subset \omega_n$ respectively $(\tilde{A}M) \subset \tilde{\omega}_n$. Considering the lifting Θ_n of m_n on $\tilde{S}$, one repeats the same smoothing argument as above, but asking that the new vector field $\tilde{m}_n$ coincides with m_n in a neighborhood of $\partial\tilde{S}$. Therefore, we conclude that m can be regularized in $H^1(\tilde{\Omega}^d)$ by smooth vector fields with (1) defined in $\tilde{\Omega}^d$. Letting $d \to 0$, the above argument applies (slightly changed when "bad" characteristics appear as in Fig. 3) and the conclusion follows in Case 1 (the approximation holding in $H^1_{loc}(\Omega)$).

Case 2. There are interior vortex-point singularities, i.e., $\mathcal{A} \neq \emptyset$. We recall that $\mathcal{A}$ is finite. As before, for $d > 0$ small, we choose a smooth simply connected subdomain Ω^d of Ω that is close to Ω in the sense that $\operatorname{dist}(x, \partial\Omega) \in (d, 3d/2)$ for every $x \in \partial\Omega^d$ and $\mathcal{A} \subset \Omega^d$ and $\operatorname{dist}(\mathcal{A}, \partial\Omega^d) \geqslant d$. Therefore, all the characteristics passing through Ω^d intersect either in a vortex point of $\mathcal{A}$, or at a distance larger than d outside Ω^d. We decompose the domain in a partition $\Omega^d := \Omega_1(m) \cup \Omega_2(m)$ where

$$\Omega_1(m) = \left\{ x \in \Omega^d\text{: there exist } r > 0,\ P \in \mathcal{A},\ m(z) = \pm\frac{(z-P)^\perp}{|z-P|} \text{ for all } z \in B(x,r) \subset \Omega^d \right\}.$$

Then $\Omega_1(m)$ is an open subset of Ω^d, $\bar{\Omega}_1(m)$ contains all the (interior) vortex singularities of m (i.e. $\mathcal{A} \subset \Omega_1(m)$) and $m \in C^\infty(\Omega_1(m) \setminus \mathcal{A})$.

To construct the desired m_n in Ω^d, the idea is the following: Set first $m_n := m$ on $\Omega_1(m)$ so that m_n has the same (interior) vortex-point singularities $\mathcal{A}$ as m and $m_n \in C^\infty(\Omega_1(m) \setminus \mathcal{A})$. It remains to smooth m on $\Omega_2(m)$; since $\mathcal{A}$ is not included in $\Omega_2(m)$ (in fact, $\mathcal{A}$ is placed at distance larger than d outside $\partial\Omega_2(m)$), it means that any two characteristics of m passing through $\Omega_2(m)$ intersect at a distance larger than d outside $\Omega_2(m)$, i.e., m is globally Lipschitz on $\Omega_2(m)$ with a Lipschitz constant less than $1/d$. We will construct m_n on $\Omega_2(m)$ as at Case 1: we find a countable set of segments $\gamma_j : [0,1] \to \bar{\Omega}_2(m)$, $j \in J \subset \mathbb{N}$ such that

(1) every two (open) lines $\gamma_j((0,1))$ and $\gamma_k((0,1))$ are disjoint, $j \neq k$;

(2) the characteristics of m passing through $\gamma_j(t)$ for every $t \in [0,1]$ and $j \in J$ cover the whole domain $\bar{\Omega}_2(m)$, i.e., $\bar{\Omega}_2(m) = \bigcup_{j\in J, t\in[0,1]} P(t)Q(t)$ where $P(t) \in \partial\Omega_2(m)$ and $Q(t) \in \partial\Omega_2(m)$ are the extremal points of the characteristic passing through $\gamma(t)$ in $\Omega_2(m)$.

(3) for every $t \in (0,1)$ and $j \in J$, the characteristic $P(t)Q(t)$ passing through $\gamma_j(t)$ intersects the set of curves $\bigcup_{k\in J}\{\gamma_k\}$ only at $\{\gamma_j(t)\}$.

It could happen that some lines γ_j have their end points in $\partial\Omega_1(m)$ (i.e., $\gamma_j(\{0,1\}) \subset \partial\Omega_1(m)$). Since $\Omega_1(m)$ is open, it means that one can extend a little bit γ_j by the line $\gamma_j \cup \tilde{\gamma}_j$ in the interior of $\Omega_1(m)$ (i.e., $\tilde{\gamma}_j \subset \Omega_1(m)$); for such lines, the smoothing process of Case 1 is to be repeated by adding the constraint that the approximation m_n coincides with m on $\tilde{\gamma}_j$. On the other lines γ_j (not intersecting $\partial\Omega_1(m)$), the smoothing process in Case 1 is to be repeated. As before, m_n is not smooth around "bad" characteristics. Eventually by considering a subdomain $\tilde{\Omega}^d \subset \Omega^d$, one can also smooth m_n around the "bad" characteristics. Therefore, by letting $d \to 0$, the above argument applies (with a slight change by approximating $\mathcal{A}$ by a set $\mathcal{A}_n$ of vortices of m_n when "bad" characteristics appear as in Fig. 3) and the conclusion follows also in Case 2 (the approximation holding in $W^{1,q}_{loc}(\Omega)$ for any $q < 2$). $\square$

Proof of Theorem 3. Let $\mathcal{A} = \{P_k\colon k \in K\}$ be the set of interior vortex-point singularities of m with K be a finite set. For each interior vortex point P_k of m, we can find a cone $\mathcal{C}_k \subset \Omega$ (of center A_k and some small angle) such that $P_k \in \operatorname{int}(\mathcal{C}_k)$ and $\Omega \setminus \mathcal{C}_k$ is a Lipschitz simply-connected domain (see Fig. 6). Set $d_k = \operatorname{dist}(P_k, \partial\mathcal{C}_k) > 0$ and $\tilde{\Omega} := \Omega \setminus \bigcup_{k\in K} \mathcal{C}_k$. The smoothing process is the following: in a first step, we smooth m inside each cone $\mathcal{C}_k$ by nonvanishing divergence vector fields of unit-length, and in a second step we smooth m by divergence-free vector fields inside $\tilde{\Omega}$.

Step 1. Smoothing inside the cones. We will use the strategy in [21]. For simplicity of the writing, we suppose that $P_k = O(0,0)$ is the origin, $m(x) = \frac{x^\perp}{|x|}$ in $\mathcal{C}_k$ and for every small ε, denoting

$$\lambda = \lambda(\varepsilon) = \frac{1}{|\ln\varepsilon|^2},$$

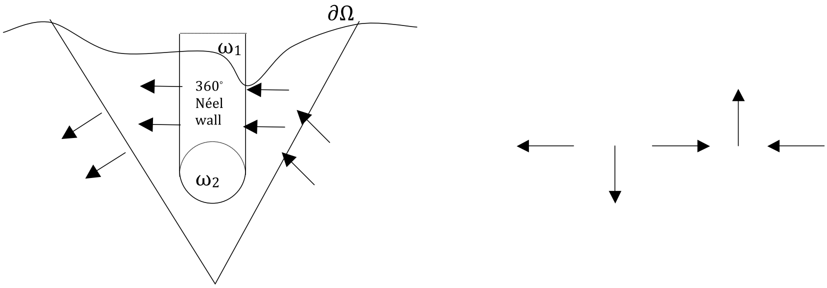

Fig. 6. Smooth vortex approximation inside a cone using a 360°-Néel wall transition (described in the right picture).

we will assume that the cone $\mathcal{C}_k$ contains the sets

$$\omega_1 = \omega_1(\varepsilon) = \left\{x \in \mathbf{R}^2\colon\ \lambda \leqslant |x| := \sqrt{x_1^2 + x_2^2} < 1,\ x_2 \geqslant 0,\ |x_1| \leqslant \lambda\right\} \quad \text{and}$$
$$\omega_2 = \omega_2(\varepsilon) = \left\{x \in \mathbf{R}^2\colon\ |x| < \lambda\right\}$$

and $\partial\Omega \cap \{x_2 > 0,\ |x_1| \leqslant \lambda\}$ is contained in $\bar{\omega}_1$ (see Fig. 6). We construct a family $\{m_\varepsilon \in H^1(\mathcal{C}_k, S^1)\}$ such that $m_\varepsilon \to m$ a.e. in $\mathcal{C}_k$ and $(\nabla \cdot m_\varepsilon)\mathbf{1}_{\mathcal{C}_k} \to 0$ in $\dot{H}^{-1/2}(\mathbf{R}^2)$ as $\varepsilon \to 0$.

Definition of $\boldsymbol{m_\varepsilon}$. We will denote the phase of m_ε by Θ_ε, i.e. $m_\varepsilon = e^{i\Theta_\varepsilon}$. In the region $\mathcal{C}_k \setminus (\omega_1 \cup \omega_2)$, m_ε will coincide with the vortex vector field m (in particular, m_ε is a smooth vector field (1) in $\mathcal{C}_k \setminus (\omega_1 \cup \omega_2)$). More precisely, in polar coordinates, the phase is given by

$$\Theta_\varepsilon(r,\theta) = \theta + \pi/2 \quad \text{in } \mathcal{C}_k \setminus (\omega_1 \cup \omega_2).$$

In the region ω_1, m_ε will turn clockwise as a transition layer of degree -1 (known in micromagnetic jargon as 360°-Néel wall of initial angle 0, see [21]) and in the region ω_2 (standing for the core of the vortex), we apply some linear cut-off in the radius for the phase of m_ε.

Transition layer of degree -1. Let

$$\delta = \delta(\varepsilon) = \varepsilon|\ln\varepsilon|^3.$$

In ω_1, we first denote by $(\tilde{u}_\delta, \tilde{v}_\delta) = e^{i\tilde{\varphi}_\delta} : \mathbf{R} \to S^1$, the following approximation of the 360°-Néel wall of initial angle 0 (magnetization turning clockwise, i.e., of topological degree -1):

$$\tilde{u}_\delta(t) = \begin{cases} 1 - \frac{2}{|\ln\delta|}\ln\frac{1}{\sqrt{t^2+\delta^2}} & \text{if } |t| \leqslant \sqrt{\frac{1}{4} - \delta^2}, \\ \cos\tilde{\theta}_\delta(|t|) & \text{if } \sqrt{\frac{1}{4} - \delta^2} \leqslant |t| \leqslant 1, \\ 1 & \text{if } |t| \geqslant 1, \end{cases} \quad \text{and}$$
$$\tilde{v}_\delta(t) = \begin{cases} -\sqrt{1 - \tilde{u}_\delta^2(t)} & \text{if } t < 0, \\ \sqrt{1 - \tilde{u}_\delta^2(t)} & \text{if } t > 0, \end{cases} \tag{24}$$

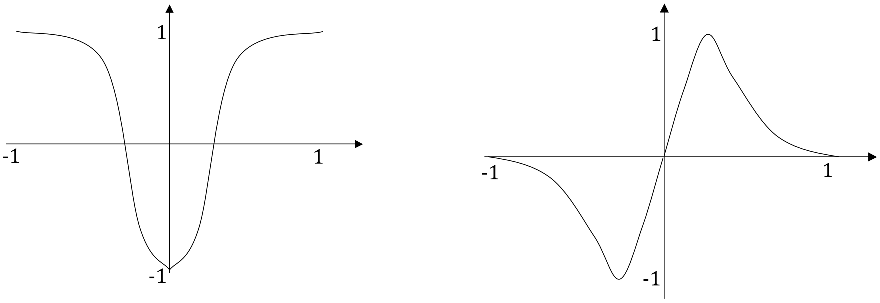

Fig. 7. The components $\tilde{u}_\delta$ and $\tilde{v}_\delta$ of a 360° Néel wall transition of degree −1.

where $\tilde{\theta}_\delta : [\sqrt{1/4-\delta^2}, 1] \to [0, \frac{\pi}{2}]$ is defined by

$$\tilde{\theta}_\delta := \text{linear function with } \tilde{\theta}_\delta\big(\sqrt{1/4-\delta^2}\big) = \arccos\frac{|\ln 4\delta|}{|\ln\delta|} \quad \text{and} \quad \tilde{\theta}_\delta(1) = 0 \tag{25}$$

(see Fig. 7). In view of (24), we may assume that $\tilde{\varphi}_\delta(0) = -\pi$. We then have $\tilde{\varphi}_\delta(-\infty) = 0$ and $\tilde{\varphi}_\delta(+\infty) = -2\pi$ and since $\tilde{\varphi}_\delta + \pi$ is an odd function, we also get

$$\tilde{\varphi}_\delta(-t) + \tilde{\varphi}_\delta(t) = -2\pi \quad \text{for every } t \in \mathbf{R}. \tag{26}$$

We rescale the transition layer $(\tilde{u}_\delta, \tilde{v}_\delta)$ so that it is contained in ω_1: In polar coordinates, for each arc of radius $r \in [\lambda, 1)$ fixed in ω_1 and where the angle θ varies inside the interval $\theta \in (\frac{\pi}{2} \pm \arcsin\frac{\lambda}{r})$, we define the rescaled transition layer $(u_\varepsilon, v_\varepsilon)$ with phase φ_ε by

$$(u_\varepsilon, v_\varepsilon)(\theta) \;=\; e^{i\varphi_\varepsilon(\theta)} := (\tilde{u}_\delta, \tilde{v}_\delta)\left(\frac{\theta - \frac{\pi}{2}}{\arcsin\lambda}\right),$$

or equivalently, the rescaled phase is given by

$$\varphi_\varepsilon(\theta) = \tilde{\varphi}_\delta\left(\frac{\theta - \frac{\pi}{2}}{\arcsin\lambda}\right). \tag{27}$$

The profile m_ε is defined in terms of its components in radial direction $\vec{r}$ and angular direction $\vec{\theta}$ on arcs with fixed radius in ω_1:

$$m_\varepsilon(r,\theta) = e^{i\Theta_\varepsilon(r,\theta)} = u_\varepsilon(\theta)\vec{\theta} - v_\varepsilon(\theta)\vec{r} \quad \text{in } \omega_1.$$

Notice that in ω_1 (as well as in $\mathcal{C}_k \setminus (\omega_1 \cup \omega_2)$), the profile m_ε (together with its phase Θ_ε) are invariant in r. Also, we have the following relation between φ_ε and Θ_ε:

$$\Theta_\varepsilon(r,\theta) = \theta + \varphi_\varepsilon(\theta) + \frac{\pi}{2} \quad \text{in } \omega_1 \tag{28}$$

and the phase Θ_ε is continuous in $\mathcal{C}_k \setminus \omega_2$. Finally, in the core region ω_2, we define the profile $m_\varepsilon(r,\theta) = e^{i\Theta_\varepsilon(r,\theta)}$ in polar coordinates by

$$\Theta_\varepsilon(r,\theta) = \frac{r}{\lambda}\Theta_\varepsilon(\lambda,\theta) \quad \text{in } \omega_2, \tag{29}$$

where we recall by (28) that $\Theta_\varepsilon(\lambda,\theta) = \theta + \frac{\pi}{2} + \tilde{\varphi}_\delta(\frac{\theta-\frac{\pi}{2}}{\arcsin\lambda})$ for every $\theta \in (0, 2\pi)$.

Energy estimates. We refer to [21] for the following estimates:

$$\int_{\mathcal{C}_k} |\nabla m_\varepsilon|^2\,dx = o\left(\frac{1}{\varepsilon|\ln\varepsilon|}\right) \quad \text{and} \quad \left\|(\nabla\cdot m_\varepsilon)\mathbf{1}_{\mathcal{C}_k}\right\|^2_{\dot{H}^{-1/2}(\mathbf{R}^2)} = O\left(\frac{1}{|\ln\varepsilon|}\right). \tag{30}$$

Notice that m_ε is continuous, but not smooth in $\mathcal{C}_k$. Obviously, one can approximate m_ε by $\tilde{m}_{\varepsilon,\eta} \in C^\infty(\mathcal{C}_k, S^1)$ in $H^1(\mathcal{C}_k)$ as $\eta \to 0$, so that by a diagonal argument, m is approximated by smooth vector fields $m_n \in C^\infty(\mathcal{C}_k, S^1)$ in L^1 such that $(\nabla\cdot m_n)\mathbf{1}_{\mathcal{C}_k} \to 0$ in $\dot{H}^{-1/2}(\mathbf{R}^2)$.

Step 2. Smoothing outside the cones. Observe that m doesn't have any interior vortex-point singularities in $\tilde{\Omega}$ that is Lipschitz simply connected. By the proof of Theorem 2 our vector field m can be approximated in $W^{1,q}_{loc}(\tilde{\Omega})$ for $q < \infty$ (in particular, a.e. in $\tilde{\Omega}$) by smooth vector fields $m_n \in C^\infty(\tilde{\Omega}, S^1)$ of vanishing divergence. Moreover, $m_n = m$ in $\mathcal{C}_k$ around $\partial\mathcal{C}_k \cap \partial\tilde{\Omega}$ and the characteristics of m passing through $\tilde{\Omega}$ around $\mathcal{C}_k$ intersect at a distance larger than d_k outside $\tilde{\Omega}$ so that the proof of Theorem 2 enables m_n to be chosen C^∞ around the boundary of $\partial\tilde{\Omega} \cap \partial\mathcal{C}_k$.

Therefore, we have constructed smooth vector fields $m_n \in C^\infty(\Omega, S^1)$ converging to m a.e. that are divergence-free in $\tilde{\Omega}$. It remains to show that $(\nabla\cdot m_n)\mathbf{1}_\Omega \to 0$ in $\dot{H}^{-1/2}(\Omega)$. Observe that $(\nabla\cdot m_n)\mathbf{1}_\Omega = (\nabla\cdot m_n)\mathbf{1}_{\bigcup_{k\in K}\mathcal{C}_k} \in L^2(\mathbf{R}^2)$ (since $m_n \in H^1(\mathcal{C}_k)$ for each $k \in K$). Therefore, by (30), it follows

$$\left\|(\nabla\cdot m_n)\mathbf{1}_\Omega\right\|_{\dot{H}^{-1/2}(\mathbf{R}^2)} \to 0.$$

which concludes our statement. □

Remark 9. Let $\Omega \subset \mathbf{R}^2$ be a Lipschitz bounded simply-connected domain. If $m : \Omega \to S^1$ satisfies (1), m is locally Lipschitz except at the vortex points $\{P_1, \ldots, P_k\}$ and m belongs to H^1 in a neighborhood of the boundary $\partial\Omega$, then m is a zero-energy state of the line-energy Ginzburg–Landau E_ε (defined at Remark 1). Let us sketch the proof of this statement. For that, we choose some disjoint disks $\mathcal{D}_j \subset \Omega$ of center P_j and radius $R_j > 0$ for $j = 1, \ldots, k$. By Theorem 1 we know that $m(x) = \alpha_j(x - P_j)^\perp/|x - P_j|$ with $\alpha_j \in \{\pm 1\}$ for every $x \in \mathcal{D}_j \setminus \{P_j\}$ and $1 \leqslant j \leqslant k$ and m is locally Lipschitz outside the disks $\bigcup_j \mathcal{D}_j$.

Step 1. First of all, note that the approximating family $m_\varepsilon \in H^1(\Omega, \mathbf{R}^2)$ with $m_\varepsilon \to m$ in $L^1(\Omega)$ and $E_\varepsilon(m_\varepsilon) \to 0$ as $\varepsilon \to 0$ cannot be chosen with values into S^1 (as in Theorems 2 and 3). Indeed, if $|m_\varepsilon| = 1$ in Ω and $m_\varepsilon \to m$ in $L^1(\Omega)$, then Rivière and Serfaty [32] proved that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(m_\varepsilon) \geqslant 2\sum_{j=1}^k \mathcal{H}^1(\partial\mathcal{D}_j) > 0.$$

Step 2. Therefore, the approximating family m_ε should vanish somewhere inside each disk $\mathcal{D}_j$. The family m_ε is chosen in the following: for $\varepsilon > 0$ sufficiently small, we set $m_\varepsilon = m$ in $\Omega \setminus \bigcup_j \mathcal{D}_{j,\varepsilon}$ where $\mathcal{D}_{j,\varepsilon}$ is the disk of center P_j and radius ε and $m_\varepsilon(x) = \alpha_j(x - P_j)^\perp/\varepsilon$ for

$x \in \mathcal{D}_{j,\varepsilon}$ and $1 \leqslant j \leqslant k$. Observe that m_ε are divergence-free vector fields in Ω and the constraint $|m_\varepsilon| = 1$ fails inside the disks $\mathcal{D}_{j,\varepsilon}$. Obviously $m_\varepsilon \to m$ in $L^1(\Omega)$. To conclude, we need to prove that $E_\varepsilon(m_\varepsilon) \to 0$ as $\varepsilon \to 0$. For that, let $\mathcal{N}$ be a neighborhood of $\partial\Omega$ such that $m \in H^1(\mathcal{N})$. Then one computes:

$$E_\varepsilon(m_\varepsilon) = \varepsilon \int_{\mathcal{N}} |\nabla m|^2 + \varepsilon \int_{\Omega \setminus (\mathcal{N} \cup \bigcup_j \mathcal{D}_j)} |\nabla m|^2 + \sum_{j=1}^k \left(\varepsilon \int_{\mathcal{D}_j} |\nabla m_\varepsilon|^2 + \frac{1}{\varepsilon} \int_{\mathcal{D}_{j,\varepsilon}} \left(1 - |m_\varepsilon|^2\right)^2 \right)$$

$$\leqslant \varepsilon \int_{\mathcal{N}} |\nabla m|^2 + \varepsilon |\Omega| \sup_{x \in \Omega \setminus (\mathcal{N} \cup \bigcup_j \mathcal{D}_j)} \left|\nabla m(x)\right|^2 + 2\pi\varepsilon \sum_j \left(3 + \log \frac{R_j}{\varepsilon}\right) \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus, m is a zero-energy state of $\{E_\varepsilon\}_{\varepsilon \downarrow 0}$ in Ω. This argument also shows that without the H^1-regularity assumption of m around the boundary $\partial\Omega$, m is still a zero-energy state in any subdomain $\tilde{\Omega} \subset\subset \Omega$ of the energy $\tilde{E}_\varepsilon$ corresponding to E_ε on $\tilde{\Omega}$.

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Appendix A

For $s \in (0, 1)$, $p > 1$ and $\Omega \subset \mathbf{R}^d$ a Lipschitz domain, the Sobolev space $W^{s,p}(\Omega)$ is a Banach space defined as

$$W^{s,p}(\Omega) = \left\{ f \in L^p(\Omega)\colon \|f\|^p_{\dot{W}^{s,p}(\Omega)} := \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}}\, dx\, dy < \infty \right\}$$

and endowed by the norm $\|f\|^p_{W^{s,p}(\Omega)} := \|f\|^p_{L^p(\Omega)} + \|f\|^p_{\dot{W}^{s,p}(\Omega)}$ for every $f \in W^{s,p}(\Omega)$. (Convention: $H^s(\Omega) := W^{s,2}(\Omega)$.) It is known that for $s \in (0, \frac{1}{p}]$,

$$W^{s,p}(\Omega) = W^{s,p}_0(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{s,p}(\Omega)}$$

where $\mathcal{D}(\Omega) = C^\infty_c(\Omega)$ (see [18]). For $p' = p/(p-1)$ the conjugate of p and $s \in (0, \frac{1}{p}]$, the dual space $W^{-s,p'}(\Omega)$ of $W^{s,p}(\Omega)$ with respect $\langle \cdot, \cdot \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}$ is defined as

$$W^{-s,p'}(\Omega) = \left\{ u \in \mathcal{D}'(\Omega)\colon \|u\|_{W^{-s,p'}(\Omega)} := \sup_{\substack{f \in \mathcal{D}(\Omega) \\ \|f\|_{W^{s,p}(\Omega)} \leqslant 1}} \langle u, f \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} < \infty \right\}.$$

We introduce a closed subspace of $W^{1/p,p}(\Omega)$:

$$W^{1/p,p}_{00}(\Omega) = \left\{ f \in W^{1/p,p}(\Omega)\colon |f|^p_{0,p} = \|f\|^p_{\dot{W}^{1/p,p}(\Omega)} + \int_\Omega \frac{|f(x)|^p}{\rho(x)}\, dx < \infty \right\}$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. In fact, $W^{1/p,p}_{00}(\Omega)$ can be seen as the closure of $C^\infty_c(\Omega)$ in $W^{1/p,p}(\mathbf{R}^d)$. Recall that the map $u \mapsto \zeta u$ is continuous from $W^{1/p,p}(\Omega)$ to $W^{1/p,p}_{00}(\Omega)$ for any $\zeta \in C^\infty_c(\Omega)$. We denote by $\mathbf{W}^{-1/p,p'}(\Omega)$ the dual of $(W^{1/p,p}_{00}(\Omega), |\cdot|_{0,p})$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$. So, $W^{-1/p,p'}(\Omega)$ is a closed subspace of $\mathbf{W}^{-1/p,p'}(\Omega)$. Recall that $u \mapsto \nabla u$ is continuous as map between the spaces $W^{1/p,p}(\Omega)$ to $\mathbf{W}^{-1/p',p}(\Omega)$ which is the dual of $W^{1/p',p'}_{00}(\Omega)$ (see details in [18]).

The space $BV(\Omega)$ is the set of functions $f \in L^1(\Omega)$ such that the derivative of f (in the sense of the distributions) is a finite Radon measure, i.e.,

$$\int_\Omega |\nabla f| = \sup\left\{ \int_\Omega f \nabla\cdot\zeta\, dx\colon\ \zeta \in C^1_c\big(\Omega, \mathbf{R}^d\big),\ \big|\zeta(x)\big| \leqslant 1,\ \forall x \in \Omega \right\} < \infty,$$

where $|\cdot|$ is the Euclidean norm in $\mathbf{R}^d$. We denote the BV-norm as follows:

$$\|f\|_{BV(\Omega)} := \|f\|_{L^1(\Omega)} + \int_\Omega |\nabla f|.$$

One has that $W^{1,1}(\Omega) = \overline{C^\infty(\Omega)}^{BV(\Omega)}$. It is known that the zero extension operator $T : BV(\Omega) \to BV(\mathbf{R}^d)$ defined as $Tf(x) = f(x)$ for $x \in \Omega$ and $Tf(x) = 0$ for $x \in \mathbf{R}^d \setminus \Omega$ is a linear continuous operator and we denote $\|T\|_{BV(\Omega)}$ the norm of operator T (more details in [2]).

Proposition 8 *(Gagliardo–Nirenberg's inequality). Let $\Omega \subset \mathbf{R}^d$ be a bounded Lipschitz domain, $s \in (0,1)$ and $p > 1$ with $sp < 1$. Then the embedding $BV \cap L^\infty(\Omega) \subset W^{s,p}(\Omega)$ is compact and one has*

$$\|f\|^p_{\dot W^{s,p}(\Omega)} \leqslant C\|T\|_{BV(\Omega)} \frac{\operatorname{diam}(\Omega)^{1-sp}}{1-sp} \|f\|^{p-1}_{L^\infty(\Omega)} \int_\Omega |\nabla f| \quad \text{for every } f \in BV \cap L^\infty(\Omega),$$

where $C > 0$ is some positive constant (depending on p and d).

Proof. If $f \in C^\infty(\Omega)$, one has

$$\begin{aligned}
\int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}}\, dx\, dy &= \int_\Omega\int_\Omega \frac{|Tf(x) - Tf(y)|^p}{|x-y|^{d+sp}}\, dx\, dy \\
&\leqslant 2^{p-1}\|f\|^{p-1}_{L^\infty(\Omega)} \int_{|h|\leqslant \operatorname{diam}(\Omega)} \int_{\mathbf{R}^n} \frac{|Tf(x+h) - Tf(x)|}{|h|^{d+sp}}\, dh\, dx \\
&\leqslant 2^{p-1}\|f\|^{p-1}_{L^\infty(\Omega)} \int_{|h|\leqslant \operatorname{diam}(\Omega)} \frac{dh}{|h|^{d+sp-1}} \int_{\mathbf{R}^n}\int_0^1 \big|\nabla(Tf)(x+th)\big|\, dt\, dx \\
&\leqslant C\|T\|_{BV(\Omega)} \frac{\operatorname{diam}(\Omega)^{1-sp}}{1-sp} \|f\|^{p-1}_{L^\infty(\Omega)} \int_\Omega |\nabla f|.
\end{aligned}$$

The inequality holds true for general $f \in BV \cap L^\infty(\Omega)$ due to the density of smooth functions in BV endowed with the topology induced by the strict convergence, i.e., there exists $f_n \in C^\infty \cap BV(\Omega)$ such that $|f_n| \leqslant \|f\|_{L^\infty}$ and $f_n \to f$ a.e. in Ω and $\|f_n\|_{BV(\Omega)} \to \|f\|_{BV(\Omega)}$ as $n \to \infty$ (see Remark 3.22 in [2]). Obviously, $\|f\|_{L^p(\Omega)} \leqslant C\|f\|_{L^\infty(\Omega)}$. Therefore, one obtains the embedding $BV \cap L^\infty(\Omega) \subset W^{s,p}(\Omega)$. Moreover, this embedding is compact since for some $s' \in (s, \frac{1}{p})$, one has $BV \cap L^\infty(\Omega) \subset W^{s',p}(\Omega)$ and $W^{s',p}(\Omega) \subset W^{s,p}(\Omega)$ is compact (see e.g. [18]). □

Proposition 9. *For every open set $\Omega \subset \mathbf{R}^d$, $d \geqslant 1$, one has $W^{1,1}(\Omega, S^1) \setminus \bigcup_{p>1} W^{1/p,p}(\Omega, S^1) \neq \emptyset$.*

Proof. The idea is the following (as explained in [29]): it is known that $BV \cap L^\infty \not\subset W^{1/p,p}$ in any dimension and any $p > 1$ and the counter-example is given by any jump function, e.g., $\varphi_0 = \mathbf{1}_{(0,1)}$ in $\Omega = (-1, 1)$. Moreover, if one regularizes φ_0 by $\varphi_n = \varphi_0$ in $\Omega \setminus (0, 1/n)$ and $\varphi_n(x) = nx$ for $x \in (0, 1/n)$, then we have

$$\int_\Omega |\varphi_n'| = 1, \qquad \|\varphi_n\|_{L^\infty} = 1 \quad \text{and} \quad \|\varphi_n\|_{W^{1/p,p}} \to \infty.$$

Based on this idea, one can construct a function $\varphi \in W^{1,1}(\Omega) \setminus \bigcup_{p>1} W^{1/p,p}(\Omega)$ with $\Omega = (-1, 1)$ and set $m : \Omega \to S^1$ with $m = e^{i\varphi}$. Then m satisfies the desired properties. In higher dimensions, the same example (depending on a single variable) holds (see [29]). □

Proposition 10. *Let $s \in (0, 1)$, $p > 1$ and $\Omega \subset \mathbf{R}^d$ be a bounded open set.*

(a) (*Stability by composition in $W^{s,p}$.*) *Let $h_n, h \in W^{s,p}(\Omega, \mathbf{R}^k)$ ($k = 1, 2$) and $\Psi : \mathbf{R}^k \to \mathbf{R}$ be a Lipschitz function. If $h_n \to h$ in $W^{s,p}$ then $\Psi(h_n) \to \Psi(h)$ in $W^{s,p}$.*
(b) (*Stability by complex multiplication in $W^{s,p}$.*) *Let $h, m_n, m \in W^{s,p}(\Omega, S^1)$ and $m_n \to m$ in $W^{s,p}$ then $hm_n \to hm$ in $W^{s,p}$ (the last terms are to be interpreted as product of complex numbers).*

Proof. (a) Let

$$R(h, x, y) := \frac{h(x) - h(y)}{|x - y|^{(d+sp)/p}}, \quad x \neq y \in \Omega.$$

Observe that $h_n \to h$ in $W^{s,p}$ yields $h_n \to h$ in $L^p(\Omega)$ and $R(h_n, \cdot,\cdot) \to R(h, \cdot,\cdot)$ in $L^p(\Omega \times \Omega)$. Discarding eventually a subsequence, we may assume that $h_n \to h$ a.e. in Ω and $|R(h_n, \cdot,\cdot)| \leqslant T$ for a.e. $x, y \in \Omega$ for some $T \in L^p(\Omega \times \Omega, \mathbf{R}_+)$. Therefore, $|R(\Psi(h_n), \cdot,\cdot)| \leqslant \|\Psi\|_{Lip} T$ and $R(\Psi(h_n), \cdot,\cdot) \to R(\Psi(h), \cdot,\cdot)$ a.e. in $\Omega \times \Omega$. By dominated convergence theorem, we get that $R(\Psi(h_n), \cdot,\cdot) \to R(\Psi(h), \cdot,\cdot)$ in $L^p(\Omega \times \Omega)$. Since $|\Psi(h_n) - \Psi(h)| \leqslant \|\Psi\|_{Lip}|h_n - h|$ a.e. in $\Omega \times \Omega$, we conclude that $\Psi(h_n) \to \Psi(h)$ in $W^{s,p}$. (Since the limit is unique, the above argument holds for the whole sequence n in the metric space $W^{s,p}$.)

(b) Discarding eventually a subsequence, we may assume as before that $m_n \to m$ a.e. in Ω. Let $v_n := m_n \bar{m}$ where $\bar{m}$ is the complex conjugate of m. Then $v_n \to 1$ in $W^{s,p}$. Indeed, we first have that $|v_n| = 1$ and $v_n \to 1$ a.e. in Ω. Then $R(v_n, \cdot,\cdot) \to 0$ a.e. in $\Omega \times \Omega$ and $|R(v_n, \cdot,\cdot)| \leqslant$

$|R(m_n,\cdot,\cdot)| + |R(m,\cdot,\cdot)| \in L^p(\Omega \times \Omega)$. The dominated convergence theorem yields $v_n \to 1$ in $W^{s,p}$. Finally, we conclude

$$\|hm_n - hm\|^p_{\dot W^{s,p}} = \big\|hm(v_n - 1)\big\|^p_{\dot W^{s,p}} \leqslant 2^{p-1}\|v_n\|^p_{\dot W^{s,p}} + 2^{p-1}\int_\Omega\int_\Omega \big|v_n(y) - 1\big|^p \big|R(mh, x, y)\big|^p\,dx\,dy \to 0 \quad \text{as } n \to \infty$$

by dominated convergence theorem (since $mh \in W^{s,p}$). $\square$

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