A zigzag pattern in micromagnetics

Radu Ignat* Roger Moser†

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Abstract

We study a simplified model for the micromagnetic energy functional in a specific asymptotic regime. The analysis includes a construction of domain walls with an internal zigzag pattern and a lower bound for the energy of a domain wall. Under certain conditions, the two results yield matching upper and lower estimates for the asymptotic energy. The combination of these then gives a Γ-convergence result.

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1 Introduction

Ferromagnetic materials display a variety of different microstructures. Among the most common phenomena are domain walls, i.e., layers of rapid changes between domains of almost constant magnetization. The internal structure of the domain walls is sometimes fairly simple (e.g., for a so-called Bloch wall), but sometimes it has a rich structure, typically at a scale different from its thickness. An example of such behavior is the cross-tie wall studied by several authors [29, 30, 1, 14]. In this paper, we study a simple model for the free energy of a ferromagnetic sample that gives rise to another type of domain walls with internal microstructure. In this case, what we see is a zigzag pattern.

1.1 Micromagnetics

Our starting point is the theory of micromagnetics. Suppose that $\Sigma \subset \mathbb{R}^3$ represents the shape of a ferromagnetic sample. Its magnetization is represented by a vector field $m : \Sigma \to \mathbb{R}^3$. Below the Curie point, m has a constant length, and after a renormalization, we may assume that $|m| = 1$ in Σ. Sometimes it is convenient to think of m as a map into the unit sphere S^2 rather than a vector field. In the absence of an external magnetic field, the free energy of m is of the form

$$\mathcal{E}^{3\mathrm{D}}(m) = d^2 \int_\Sigma |\nabla m|^2 \, dx + \int_\Sigma a(m) \, dx + \int_{\mathbb{R}^3} |H|^2 \, dx.$$

*Laboratoire de Mathématiques, Université Paris-Sud 11, bât. 425, 91405 Orsay, France (e-mail: Radu.Ignat@math.u-psud.fr)

†Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom (e-mail: r.moser@bath.ac.uk)

The first term on the right hand side is called the exchange energy and models quantum mechanic spin interaction. The parameter d is a material constant, called the exchange length. The function $a : S^2 \to [0, \infty)$ is fixed and models crystalline anisotropy, and the vector field $H : \mathbb{R}^3 \to \mathbb{R}^3$ represents the stray field induced by m. The latter is determined by the static Maxwell equations

$$\begin{cases} \nabla \times H = 0 & \text{in } \mathbb{R}^3, \\ \nabla \cdot (H + m\chi_\Sigma) = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where χ_Σ is the characteristic function of Σ and $m\chi_\Sigma$ denotes the extension of m by 0 outside of Σ. This gives rise to the formula

$$H = \nabla(-\Delta)^{-1}\nabla \cdot (m\chi_\Sigma),$$

and thus the third energy term, the so-called magnetostatic energy, is

$$\begin{aligned} \int_{\mathbb{R}^3} |H|^2 \, dx &= \|\nabla \cdot (m\chi_\Sigma)\|^2_{\dot{H}^{-1}(\mathbb{R}^3)} \\ &= \sup \left\{ \left(\int_\Sigma m \cdot \nabla v \, dx \right)^2 : v \in C_0^1(\mathbb{R}^3) \text{ with } \|\nabla v\|_{L^2(\mathbb{R}^3)} \leq 1 \right\}. \end{aligned}$$

The various patterns observed in experiments are typically explained by the competition between the three energy terms. The exchange energy favors constant magnetizations, the anisotropy energy favors specific directions of m, while the magnetostatic energy favors divergence free vector fields. The last condition is the most subtle, as it involves not only the bulk charge $\nabla \cdot m$ in Σ, but also a surface charge on the boundary of Σ if m is not tangent to $\partial\Sigma$. This means that a simultaneous minimization of all three energy contributions is typically impossible.

Depending on the relative sizes of the material constants involved and the geometry of the sample, the theory gives rise to a variety of phenomena—which can also be observed experimentally. Mathematically, the corresponding patterns are usually obtained as limits or solutions of limiting problems in specific asymptotic regimes. There is a rich literature on the subject, especially in the context of thin films. The corresponding papers are too numerous to be listed here, so we refer to some survey papers [15, 26].

1.2 A simplified model

We first reduce the complexity of the problem by passing from a 3-dimensional domain $\Sigma \subset \mathbb{R}^3$ to a 2-dimensional one $\omega \subset \mathbb{R}^2$. We study maps $m : \omega \to S^2$, which can also be interpreted as unit vector fields on a cylinder $\omega \times \mathbb{R}$ that are constant in the third direction. (This represents a considerable simplification and some of our results will not carry over directly to the three-dimensional case. But the construction below can still give some insight into possible structures in a 3D model.) We consider an anisotropy of the form $a(m) = Qm_2^2$ for a constant Q. We neglect the surface charges of the magnetostatic energy on the boundary of ω, since we are interested in the structure of m in the interior of ω. We work in the space

$$H^1(\omega; S^2) = \left\{ m \in H^1(\omega; \mathbb{R}^3) : |m| = 1 \text{ a.e. in } \omega \right\}.$$

For $m \in H^1(\omega; S^2)$, we write

$$\nabla \cdot m = \frac{\partial m_1}{\partial x_1} + \frac{\partial m_2}{\partial x_2}.$$

If $\dot{H}^{-1}(\omega)$ denotes the dual space of $H_0^1(\omega)$ with the norm

$$\|v\|_{\dot{H}^{-1}(\omega)} = \sup \left\{ \int_\omega vu \, dx : u \in H_0^1(\omega) \text{ with } \|\nabla u\|_{L^2(\omega)} \leq 1 \right\},$$

then a natural 2-dimensional counterpart to the energy $\mathcal{E}^{\mathrm{3D}}$ is

$$\mathcal{E}^{\mathrm{2D}}(m) = \int_\omega (d^2 |\nabla m|^2 + Q m_2^2) \, dx + \|\nabla \cdot m\|_{\dot{H}^{-1}(\omega)}^2.$$

We study an asymptotic regime characterized by certain relations between the constants d, Q, and the length scale of the 2-dimensional domain ω, measured in terms of $\ell = \operatorname{diam} \omega$. Before we give the details, it is convenient to renormalize ω to unit size. We set $\Omega = \omega/\ell$ and $\tilde{m}(x) = m(\ell x)$. Furthermore, we set $\epsilon = d/(\ell\sqrt{Q})$ and $\eta = 2d\sqrt{Q}/\ell$. Then

$$\mathcal{E}^{\mathrm{2D}}(m) = 2\ell d\sqrt{Q} \left(\frac{1}{2} \int_\Omega \left(\epsilon |\nabla \tilde{m}|^2 + \frac{\tilde{m}_2^2}{\epsilon} \right) dx + \frac{1}{\eta} \|\nabla \cdot \tilde{m}\|_{\dot{H}^{-1}(\Omega)}^2 \right).$$

The asymptotic regime that we study corresponds to the conditions that $\epsilon \to 0^+$, while η is of the order ϵ^s for some number $s \in (1, 2)$. From now on, we drop the tilde and write m instead of $\tilde{m}$. Moreover, we renormalize the energy. Then we obtain the functional that we study in the sequel:

$$E_\epsilon(m) = \frac{1}{2} \int_\Omega \left(\epsilon |\nabla m|^2 + \frac{1}{\epsilon} m_2^2 \right) dx + \frac{1}{\epsilon^s} \|\nabla \cdot m\|_{\dot{H}^{-1}(\Omega)}^2$$

for $m \in H^1(\Omega; S^2)$.

1.3 Limiting energy

Suppose that we have a family of maps $m_\epsilon \in H^1(\Omega; S^2)$ with

$$\limsup_{\epsilon \to 0^+} E_\epsilon(m_\epsilon) < \infty. \tag{1}$$

What can we say about the asymptotic behavior of m_ϵ and the energy $E_\epsilon(m_\epsilon)$ as $\epsilon \to 0^+$?

It is natural to study a question of this type in the framework of Γ-convergence. To this end, we first need to fix a topology on the space of admissible magnetizations. The topology of $L^1(\Omega, \mathbb{R}^3)$ is often used in such a context, but it turns out that E_ϵ is not coercive enough to deduce compactness from (1) in this space (cf. Proposition 5.1 below). Another possibility is the weak* topology in $L^\infty(\Omega, \mathbb{R}^3)$. Clearly the limit m (as $\epsilon \to 0^+$) must have a vanishing second component m_2 and a vanishing distributional divergence $\nabla \cdot m = 0$ in Ω. However, we obtain more information about the limit if we first apply a nonlinear transformation to m. In order to do so, we use spherical coordinates (φ, ϑ) so that

$$m = (\cos\varphi \cos\vartheta, \sin\varphi, \cos\varphi \sin\vartheta).$$

The quantity that we need to study is

$$\psi = \sin\vartheta - \vartheta\cos\vartheta,$$

at least if we work in the hemisphere where $|\vartheta| \leq \frac{\pi}{2}$. We will show that as long as ϑ remains sufficiently small, the functional

$$E_0(\psi) = \sup\left\{\int_\Omega \frac{\partial v}{\partial x_1}\psi\,dx : v \in C_0^1(\Omega) \text{ with } \sup_\Omega |v| \leq 1\right\}$$

can be identified as the limiting energy. For a sufficiently regular ψ, this is of course

$$E_0(\psi) = \int_\Omega \left|\frac{\partial\psi}{\partial x_1}\right| dx.$$

The lack of a penalization of $\frac{\partial\psi}{\partial x_2}$ means that we can have very rough limiting configurations. On the other hand, almost every restriction to a horizontal line $\Omega \cap (\mathbb{R} \times \{x_2\})$ will be a function of bounded variation. There can be jumps, but these jumps contribute to the energy proportionally to the jump height. It is convenient to imagine here that the magnetization depends only on x_1, and then we can think of a jump as a domain wall. It is worth noting that in general, the wall energy given by E_0 is *not* achieved by a 1-dimensional transition between the two states on either side of the wall. Instead, in order to obtain the optimal limiting energy given by E_0, a transition with an additional zigzag structure is required.

1.4 Related models

The phenomenon studied in this paper depends crucially on the interaction between the anisotropy and the magnetostatic energy (but involving also the exchange energy). In particular, the spatial orientation of the anisotropy (relative to the expected domain walls in the corresponding 3-dimensional configuration) is important. If m_2^2 is replaced by m_3^2, then the limiting behavior is described in terms of Bloch walls, which are 1-dimensional transition layers between two mesoscopic directions of m within S^2, as shown by Ignat and Merlet [21].

A related problem has been studied by Moser [27]. In a 3-dimensional model with a different anisotropy, it is shown that similar zigzag walls are to be expected (unsurprisingly, as this is exactly a situation for which the phenomenon is described in the physics literature [18, Chapter 3.6]). An upper bound is given for the limiting wall energy through a zigzag construction similar to what we explain later (see Section 3). A preliminary lower bound is also given, but there is so far no Γ-convergence result, as the two estimates do not match.

If we ignore the magnetostatic energy in our model, then we obtain an energy similar to the Ginzburg-Landau functionals studied by Bethuel, Brezis, and Hélein [7] and many other authors, including André and Shafrir [4], Sandier [31], Hang and Lin [17] in the context of S^2-valued maps. On the other hand, since the penalization of the magnetostatic energy is very strong, it is perhaps more appropriate to compare our model with a theory involving the constraint $\nabla \cdot m = 0$. If Ω is simply connected, then under such a condition, there exists a function u such that $m = (\nabla^\perp u, m_3)$. The energy is then

$$E_\epsilon(m) = \frac{1}{2}\int_\Omega \left(\epsilon|\nabla^2 u|^2 + \epsilon|\nabla m_3|^2 + \frac{1}{\epsilon}\left(\frac{\partial u}{\partial x_1}\right)^2\right) dx.$$

This has some similarity to the functional

$$\mathrm{AG}_\epsilon(u) = \frac{1}{2}\int_\Omega \left(\epsilon|\nabla^2 u|^2 + \frac{1}{\epsilon}(1-|\nabla u|^2)^2\right)\,dx$$

introduced by Aviles and Giga [5] and also studied by others [6, 25, 3, 13, 12, 9, 28]. A variant of the problem with applications to micromagnetics has been considered by Jabin, Otto, and Perthame [24] and by De Lellis and Otto [12].

In contrast to the problem studied in this paper, the optimal transition profiles between two phases are 1-dimensional for the Aviles-Giga problem. This is indeed the case for most problems involving phase transitions where the limiting energy is explicitly known. In some cases, it is not difficult to see that it will not be sufficient to study 1-dimensional transitions. For certain classes of such problems, a Γ-limit has been described in terms of other variational problems by Fonseca and Popovici [16] and Conti, Fonseca, and Leoni [10]. But we are aware of only one other situation where the Γ-limit is explicitly known for a problem involving similar microstructures: the problem leading to cross-tie walls in thin ferromagnetic films [29, 30, 1]. The cross-tie wall consists in a mixture of vortices and Néel walls (1-dimensional transition layers similar to Bloch walls, but taking values only in S^1). Remarkably, the function $\sin\theta - \theta\cos\theta$ plays an important role in that context as well, although this may be a mere coincidence. We also mention some other works related to patterns in thin-film micromagnetics that involve Néel walls and (interior or boundary) vortices (see Ignat-Otto [22, 23], Ignat-Knüpfer [20]).

2 Main results

2.1 The periodic case

For simplicity, we first focus on the periodic situation

$$\Omega = (-1,1)\times\mathbb{R}/\mathbb{Z}.$$

For a fixed transition angle $\theta \in (0,\pi/2)$, we set the mesoscopic directions

$$m^\pm = (\cos\theta, 0, \pm\sin\theta) \in S^2$$

and we consider magnetizations (periodic in the tangential direction x_2 to the wall) with the desired transition imposed at the boundary:

$$M = M(\theta) := \left\{ m \in H^1(\Omega, S^2) \;:\; m(\pm1,\cdot) = m^\pm \quad \text{in } H^{1/2}(\mathbb{R}/\mathbb{Z}) \right\}.$$

Set

$$F(\theta) = \sin\theta - \theta\cos\theta.$$

The associated 2D stray field $h(m)$ is assumed to be x_2-periodic. Then the stray field energy per unit length in x_2-direction is given by:

$$\begin{aligned}\int_\Omega |h(m)|^2\,dx &= \|\nabla\cdot m\|^2_{\dot H^{-1}_{\mathrm{per}}(\Omega)} &(2)\\ &= \sup\left\{\left(\int_\Omega u\nabla\cdot m\,dx\right)^2 : u \in H^1_{\mathrm{per}}(\Omega) \text{ with } \|\nabla u\|_{L^2(\Omega)} \le 1\right\},\end{aligned}$$

where

$$H^1_{\mathrm{per}}(\Omega) = \left\{ u \in H^1(\Omega) : u(\pm 1, \cdot) = 0 \text{ in } H^{1/2}(\mathbb{R}/\mathbb{Z}) \right\}.$$

Here, we will always use the periodic stray field energy (2) as the last term in the energy E_ε:

$$E_\varepsilon(m) = \frac{\varepsilon}{2} \int_\Omega |\nabla m|^2 \, dx + \frac{1}{2\varepsilon} \int_\Omega m_2^2 \, dx + \frac{1}{\varepsilon^s} \int_\Omega |h(m)|^2 \, dx,$$

for $s \in (1, 2)$. We state the following asymptotic minimal value of E_ε on the set $M(\theta)$ for small transition angles θ:

Theorem 2.1. *There exists an angle $\theta_0 \in (0, \frac{\pi}{2})$ such that the following holds: for every $\theta \in (0, \theta_0]$,*

$$\min_{m_\varepsilon \in M(\theta)} E_\varepsilon(m_\varepsilon) = 2F(\theta) + o(1) \quad \text{as} \quad \varepsilon \to 0.$$

The idea of the proof is to match an upper bound coming from the zigzag wall construction with a lower bound based on generalized entropies. Let us

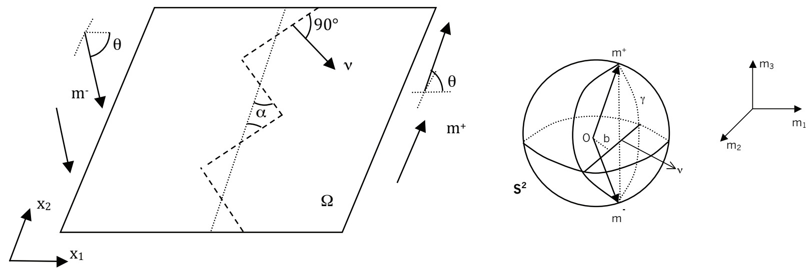

Figure 1: The zigzag pattern

explain the heuristics of deducing the limit energy in Theorem 2.1 (as an upper bound). Let $\alpha \in [0, \frac{\pi}{2})$ and consider in $\mathbb{R}^3$ the plane containing the two points $m^\pm \in S^2$ so that $\nu = (\cos\alpha, -\sin\alpha, 0)$ is the normal vector to the plane. The construction will involve a transition path from m^- to m^+ along the curve on S^2 within this plane (see Figure 1). More precisely, we define

$$b = \cos\theta \cos\alpha \quad \text{and} \quad \sigma = \arcsin \frac{\sin\theta}{\sqrt{1 - b^2}};$$

the smallest arc connecting $m^\pm$ on the circle of radius $\sqrt{1-b^2}$ whose plane is perpendicular to ν is given by

$$\gamma(t) = b\nu + \sqrt{1 - b^2}(\sin\alpha \cos t, \cos\alpha \cos t, \sin t) \tag{3}$$

for $-\sigma \le t \le \sigma$. For a transition along $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, the expected energy per unit wall length is

$$K(\alpha) = \int_{-\sigma}^{\sigma} \gamma_2(t) |\dot\gamma(t)| \, dt.$$

In order to keep the magnetostatic energy small, we will have to use this transition across pieces of a zigzag wall that are tilted with respect to $\{0\} \times (0, 1)$ by the angle α. This increases the length of the wall by the factor $\frac{1}{\cos\alpha}$, and in the limit we expect the energy density

$$g(\alpha) = \frac{K(\alpha)}{\cos\alpha}. \tag{4}$$

One can check that g is a decreasing function (see Proposition 6.1 in Appendix) and conclude that

$$\inf_{0\leq\alpha<\frac{\pi}{2}} g(\alpha) = \lim_{\alpha\to\frac{\pi}{2}^-} g(\alpha) = 2F(\theta). \tag{5}$$

We observe that the energy cost of a transition of small angle θ is cubic, so that it is asymptotically cheaper than the quadratic energy cost of a Bloch wall transition of the same angle.

We explain the precise construction that leads to the above wall energy in Section 3. We thereby obtain an upper bound for the limiting energy (this construction is done for arbitrary angles $\theta \in (0, \frac{\pi}{2}]$). We show in Section 4 that the upper bound is optimal at least when θ is small. To this end, we use an "entropy method" introduced by Jin and Kohn [25], Aviles and Giga [6], DeSimone, Kohn, Müller, and Otto [13] and used in a context similar to this problem by Ignat and Merlet [21].

2.2 $\Gamma-$convergence for small transition angles

We now concentrate on families of uniformly bounded energy configurations $\{m_k = (m_{k,1}, m_{k,2}, m_{k,3}) \in H^1(\Omega; S^2)\}$ in a smooth, bounded, simply-connected domain $\Omega \subset \mathbb{R}^2$, i.e.,

$$\limsup_{k\to\infty} E_{\epsilon_k}(m_k) < \infty, \tag{6}$$

with $\varepsilon_k \to 0$ as $k \to \infty$. The aim is to establish the structure of limiting configurations of such families and to determine their limit energy, according to the $\Gamma-$convergence method. The first issue is to find out the appropriate topology for the desired $\Gamma-$convergence result. Obviously, (6) entails $m_{k,2} \to 0$ strongly in $L^2(\Omega)$. However, as we will see in Proposition 5.1, families $\{m_k\}$ satisfying (6) are in general *not* relatively compact in the strong L^1 topology and the limiting configurations m are not necessarily taking values into S^2 (in general, one only has $|m| \leq 1$ a.e. in Ω). Therefore, one alternative would be to choose the weak* $L^\infty-$topology for $\{(m_{k,1}, m_{k,3})\}$. Rather than studying the limiting behavior of $(m_{k,1}, m_{k,3})$, we focus on the quantity

$$\psi_k = f(m_k), \tag{7}$$

where $f : S^2 \to \mathbb{R}$ is the function defined by

$$f(m) = \begin{cases} F(\arctan(m_3/m_1)) & \text{if } m_1 > 0, \\ 2 + F(\arctan(m_3/m_1)) & \text{if } m_1 < 0 \text{ and } m_3 \geq 0, \\ -2 + F(\arctan(m_3/m_1)) & \text{if } m_1 < 0 \text{ and } m_3 < 0, \end{cases} \tag{8}$$

extended continuously where $m_1 = 0$ and $m_2 \neq \pm 1$ (here, $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$). This function has a discontinuity along the semicircle $\{m \in S^2 :$

$m_3 = 0,\, m_1 \le 0\}$, and from a geometric point of view, it would be more appropriate to regard f as a function from S^2 into $\mathbb{R}/4\mathbb{Z}$. Since we work mostly in a hemisphere below, we keep $\mathbb{R}$ as the target anyway. The discontinuities at the poles $\pm\mathbf{e}_2$, of course, are unavoidable. Since $|\psi_k| \le 2$ a.e. in Ω, we choose the weak* L^∞-topology for $\{\psi_k\}$ as appropriate for the $\Gamma-$convergence result. We define the limiting functional $E_0 : L^\infty(\Omega) \to [0,\infty]$ by

$$E_0(\psi) = \int_\Omega \left|\frac{\partial \psi}{\partial x_1}\right| := \sup\left\{ \int_\Omega \frac{\partial v}{\partial x_1}\psi\, dx : v \in C_0^1(\Omega) \text{ with } \sup_\Omega |v| \le 1\right\},$$

for every $\psi \in L^\infty(\Omega)$, i.e., $E_0(\psi)$ is the total variation of ψ in the x_1-direction.

We prove the following $\Gamma-$convergence result for small transition angles:

Theorem 2.2. *There exists an angle $\theta_0 \in (0, \frac{\pi}{2})$ such that the following holds true.*

1) (Compactness and Lower bound) Let $\{\varepsilon_k\} \subset (0,\infty)$ with $\varepsilon_k \to 0$ as $k \to \infty$ and let $\{m_k\} \subset H^1(\Omega; S^2)$ with (6). Consider the sequence $\{\psi_k\}$ associated to $\{m_k\}$ via (7). Then discarding a subsequence,

$$\psi_k \overset{*}{\rightharpoonup} \psi \text{ in } L^\infty(\Omega) \text{ and } m_{k,2} \to 0 \text{ in } L^2(\Omega). \tag{9}$$

If $|\psi_k| \le F(\theta_0)$ a.e. in Ω and for every positive integer k, then

$$E_0(\psi) \le \liminf_{k\to\infty} E_{\epsilon_k}(m_k).$$

2) (Upper bound) For every $\psi \in L^\infty(\Omega)$ with $|\psi| \le F(\theta_0)$ a.e. in Ω, there exist sequences $\{\varepsilon_k\} \subset (0,\infty)$ with $\varepsilon_k \to 0$ and $\{m_k\} \subset H^1(\Omega; S^2)$ such that (9) holds and

$$E_0(\psi) = \lim_{k\to\infty} E_{\epsilon_k}(m_k).$$

The proof of this result is presented in Section 5.

3 Upper bound in Theorem 2.1: The zigzag wall

Let $\theta \in (0, \frac{\pi}{2}]$ be an arbitrary angle. For the mesoscopic directions $m^\pm = (\cos\theta, 0, \pm\sin\theta) \in S^2$, we show that the energy $2F(\theta)$ can be achieved by a zigzag transition layer in the limit $\varepsilon \to 0$. To this end, we first reparametrize the curve γ defined in (3) as follows. In the sequel, we will always use the notation introduced in subsection 2.1.

Fix $\delta > 0$. (This number will determine the length scale of the zigzag layer Ω_δ in our construction.) We define

$$\xi_\delta(t) = \int_0^t \frac{|\dot\gamma(s)|}{\sqrt{(\gamma_2(s))^2 + \delta^2}}\, ds, \quad -\sigma \le t \le \sigma.$$

Let $T_\delta = \xi_\delta(\sigma)$ and note that $-T_\delta = \xi_\delta(-\sigma)$ by symmetry. The function ξ_δ is strictly increasing, and therefore we have an inverse $\zeta_\delta = \xi_\delta^{-1} : [-T_\delta, T_\delta] \to [-\sigma, \sigma]$. We compute

$$\dot\zeta_\delta = \frac{\sqrt{(\gamma_2 \circ \zeta_\delta)^2 + \delta^2}}{|\dot\gamma \circ \zeta_\delta|}.$$

Extend ζ_δ to $\mathbb{R}$ by $\zeta_\delta(s) = \pm\sigma$ for $T_\delta < \pm s$. Then the curve $c^\delta = \gamma \circ \zeta_\delta$ satisfies

$$|\dot{c}^\delta(s)| = \sqrt{(c_2^\delta(s))^2 + \delta^2}, \quad -T_\delta < s < T_\delta,$$

which means

$$\frac{1}{2}\int_{-\infty}^{\infty} \left(|\dot{c}^\delta|^2 + (c_2^\delta)^2\right) ds \to f(\alpha)$$

as $\delta \to 0$.

We consider the layer $\Omega_\delta = (-1,1) \times (0,\delta)$ and $\nu = (\cos\alpha, -\sin\alpha, 0)$. In Ω_δ, the vertical limit wall $\{0\} \times (0,\delta)$ is tilted by the angle α so that the transition between the directions $m^\pm$ corresponds to a Bloch wall transition in the direction ν (see Figure 2). This layer of scale δ is to be reflected with respect to the horizontal axis and then, the new layer of thickness 2δ is to be repeated in a periodic way in the x_2-direction in order to get the global zigzag pattern. Therefore, for $x \in \Omega_\delta$, we define a 1-dimensional transition layer in the normal direction ν:

$$\tilde{m}^{\epsilon\delta}(x) = c^\delta\left(\frac{x \cdot \nu}{\varepsilon}\right).$$

The transition path from m^- to m^+ follows the curve γ in S^2 within the plane orthogonal to ν (as explained in subsection 2.1), so that

$$\nabla \cdot \tilde{m}^{\epsilon\delta} = 0 \text{ in } \Omega_\delta.$$

Moreover, we compute

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} \frac{1}{2\delta}\int_{\Omega_\delta} \left(\epsilon|\nabla\tilde{m}^{\epsilon\delta}|^2 + \frac{1}{\epsilon}(\tilde{m}_2^{\epsilon\delta})^2\right) dx = g(\alpha).$$

Notice that $\tilde{m}^{\epsilon\delta}$ is locally constant away from the set $B_{\epsilon\delta} := \{x \in \Omega_\delta : |x\cdot\nu| \leq \varepsilon T_\delta\}$ (where the transition between m^- and m^+ takes place); more precisely,

$$\tilde{m}^{\epsilon\delta} \equiv m^- \quad \text{on the left side of } \Omega_\delta \setminus B_{\epsilon\delta}$$

and

$$\tilde{m}^{\epsilon\delta} \equiv m^+ \quad \text{on the right side of } \Omega_\delta \setminus B_{\epsilon\delta}.$$

In order to extend this layer periodically in the x_2-direction (which will eventually yield a zigzag pattern), we need to replace $\tilde{m}^{\epsilon\delta}$ by a new vector field $m^{\epsilon\delta} \in H^1(\Omega_\delta; S^2)$ with $m_2^{\epsilon\delta} = 0$ on $\partial\Omega_\delta$ (see Figure 2). This is to avoid discontinuities on $\partial\Omega_\delta \cap (\{x_2 = 0\} \cup \{x_2 = \delta\})$. To this end, we set $L_\delta = (\tan\alpha + \frac{1}{\cos\alpha})T_\delta$ and define

$$A_{\epsilon\delta} = (-L_\delta\epsilon, L_\delta\epsilon) \times (0, T_\delta\epsilon) \cup (\delta\tan\alpha - L_\delta\epsilon, \delta\tan\alpha + L_\delta\epsilon) \times (\delta - T_\delta\epsilon, \delta).$$

Modifying $\tilde{m}^{\epsilon\delta}$ in $A_{\epsilon\delta}$, we can construct vector fields $m^{\epsilon\delta} \in H^1(\Omega_\delta; S^2)$ such that $m^{\epsilon\delta} = \tilde{m}^{\epsilon\delta}$ in $\Omega_\delta \setminus A_{\epsilon\delta}$,

$$m_2^{\epsilon\delta}(x_1, 0) = m_2^{\epsilon\delta}(x_1, \delta) = 0 \quad \text{for every } x_1 \in (-1,1),$$

and so that

$$|\nabla m^{\epsilon\delta}| \leq \frac{C_1}{T_\delta\epsilon}$$

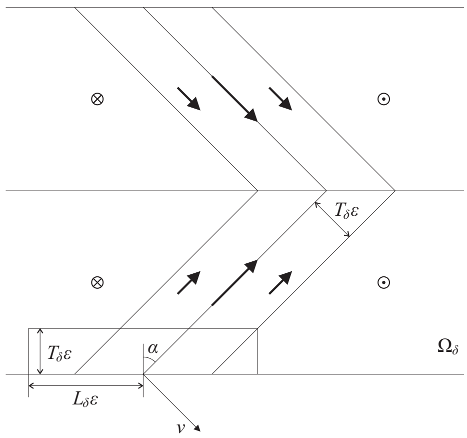

Figure 2: The microstructure of the zigzag layer for $\theta = \pi/2$. The arrows stand for the projection of the magnetization on the horizontal plane.

for a constant C_1 that depends only on θ and α. Hence we still have

$$\lim_{\delta\to0}\lim_{\epsilon\to0}\frac{1}{2\delta}\int_{\Omega_\delta}\left(\epsilon|\nabla m^{\epsilon\delta}|^2+\frac{1}{\epsilon}(m_2^{\epsilon\delta})^2\right)dx = g(\alpha)$$

and also

$$\int_{\Omega_\delta}|\nabla\cdot m^{\epsilon\delta}|^p\,dx \le C_2(T_\delta\epsilon)^{2-p}$$

for every $p\in[1,\infty)$, where C_2 is another constant depending only on θ and α.

Now we reflect $m^{\epsilon\delta}$ with respect to the horizontal axis: We extend $m^{\epsilon\delta}$ to $(-1,1)\times(-\delta,\delta)$ by

$$\begin{aligned} m_1^{\epsilon\delta}(x_1,-x_2) &= m_1^{\epsilon\delta}(x_1,x_2),\\ m_2^{\epsilon\delta}(x_1,-x_2) &= -m_2^{\epsilon\delta}(x_1,x_2),\\ m_3^{\epsilon\delta}(x_1,-x_2) &= m_3^{\epsilon\delta}(x_1,x_2). \end{aligned}$$

(Since $m_2^{\epsilon\delta}=0$ on $\partial\Omega_\delta$, no discontinuities are induced in the reflected domain.) Finally, we extend it to $(-1,1)\times\mathbb{R}$ periodically in x_2. The resulting vector field satisfies

$$\lim_{\delta\to0}\lim_{\epsilon\to0}\int_{\Omega}\left(\frac{\varepsilon}{2}|\nabla m^{\epsilon\delta}|^2+\frac{1}{2\varepsilon}(m_2^{\epsilon\delta})^2\right)dx = g(\alpha).$$

Furthermore, for any $p\in[1,\frac{4}{s+2})$, the Sobolev embedding theorem implies the existence of a universal constant C_3 such that

$$\epsilon^{-s}\|\nabla\cdot m^{\epsilon\delta}\|^2_{\dot H^{-1}(\Omega)} \le C_3\epsilon^{-s}\|\nabla\cdot m^{\epsilon\delta}\|^2_{L^p(\Omega)} \to 0 \qquad (10)$$

as $\epsilon\to0$ for any $\delta>0$. Notice that the assumption $s<2$ is essential here so that a p exists in this interval. Hence

$$\lim_{\delta\to0}\lim_{\epsilon\to0}E_\epsilon(m^{\epsilon\delta}) = g(\alpha).$$

This construction works for any $\alpha<\frac{\pi}{2}$. If we have a given sequence $\epsilon_k\to0^+$, by (5), we can apply a diagonal sequence argument for some angles $\alpha_k\to\frac{\pi}{2}^-$ in order to find a sequence of vector fields $m_k\in H^1(\Omega;S^2)$ such that

$$\lim_{k\to\infty}E_{\epsilon_k}(m_k) = 2F(\theta).$$

We highlight the fact that this result holds for arbitrary angles $\theta\in(0,\frac{\pi}{2}]$.

4 Lower bound in Theorem 2.1

4.1 Entropies

In order to obtain the above lower bound we introduce (as in [21]) a class of maps Φ for which $\int \nabla \cdot \{\Phi(m)\}\, dx$ is controlled by the energy. This idea comes from the concept of entropies (borrowed from the scalar conservation laws) and was introduced by Jin and Kohn [25], Aviles and Giga [6], DeSimone, Kohn, Müller, and Otto [13]. More precisely, we systematically study the particular class of Lipschitz continuous maps $\Phi = (\Phi_1, \Phi_2) \in \mathrm{Lip}(S^2, \mathbb{R}^2)$ and $\alpha \in \mathrm{Lip}(S^2)$ such that for every smooth $m \in C^\infty(\Omega, S^2)$, there holds

$$\nabla \cdot \{\Phi(m)\} + \alpha(m)\nabla \cdot m \le \frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}m_2^2 \qquad \text{a.e. in } \Omega, \tag{11}$$

where $\varepsilon > 0$ is a small parameter. The condition (11) yields some necessary pointwise bounds for an admissible triplet $(\Phi = (\Phi_1, \Phi_2), \alpha)$.

Lemma 4.1. *Let $\varepsilon > 0$ and $(\Phi = (\Phi_1, \Phi_2), \alpha) \in \mathrm{Lip}(S^2, \mathbb{R}^2) \times \mathrm{Lip}(S^2)$ satisfying (11). For every $\tau \in [-\pi, \pi)$, we set*

$$\nu_\tau = (-\sin\tau, \cos\tau, 0) \in S^2 \quad \text{and} \quad \Psi_\tau = -\sin\tau\, \Phi_1 + \cos\tau\, \Phi_2 \in \mathrm{Lip}(S^2).$$

Then for almost every point $m \in S^2$, we have

$$|D\Psi_\tau(m) + \alpha(m)\Pi_m \nu_\tau| \le |m_2|, \tag{12}$$

where $D\Psi_\tau(m) \in T_m S^2$ is the gradient of Ψ_τ at m and Π_m denotes the orthogonal projection onto $T_m S^2$.

Proof. Let $\mathbf{e}_1 = (1,0,0)$ and $\mathbf{e}_2 = (0,1,0)$. We define the following operator L: for a.e. $\tilde m \in S^2$ (that is a Lebesgue point of $D\Phi$), $L(\tilde m) : (T_{\tilde m} S^2)^2 \to \mathbb{R}$ is the linear functional such that for every $v = (v_1, v_2) \in (T_{\tilde m} S^2)^2$,

$$\begin{aligned} &L(\tilde m)(v) = L_1(\tilde m)(v_1) + L_2(\tilde m)(v_2), \\ \text{with} \quad &L_k(\tilde m)(v_k) := \Big(D\Phi_k(\tilde m) + \alpha(\tilde m)\Pi_{\tilde m}\mathbf{e}_k; v_k\Big), \quad k = 1, 2, \end{aligned}$$

where $(\cdot;\cdot)$ denotes the scalar product in the Euclidean space $\mathbb{R}^3$. Then for every smooth map $m \in C^\infty(\Omega, S^2)$, inequality (11) means that

$$\begin{aligned} L(m)(\partial_{x_1} m, \partial_{x_2} m) &= \nabla \cdot \{\Phi(m)\} + \alpha(m)\nabla \cdot m \\ &\le \frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}m_2^2 \text{ for a.e. } x \in \Omega. \end{aligned} \tag{13}$$

Now let $\tilde x \in \Omega$ be fixed and $\tilde m \in S^2$ be a Lebesgue point of $D\Phi$. For every nonzero vector $\tilde v \in T_{\tilde m} S^2 \setminus \{0\}$ such that $|\tilde v| = |\tilde m_2|/\varepsilon$, we choose a smooth map m such that $m(\tilde x) = \tilde m$ and $(\partial_1 m, \partial_2 m)(\tilde x) := (-\sin\tau\, \tilde v, \cos\tau\, \tilde v)$. Applying (13) at $\tilde x$, we obtain

$$\Big(D\Psi_\tau(\tilde m) + \alpha(\tilde m)\Pi_{\tilde m}\nu_\tau; \frac{\tilde v}{|\tilde v|}\Big) = \frac{1}{|\tilde v|} L(\tilde m)(-\sin\tau \tilde v, \cos\tau \tilde v) \le |\tilde m_2|.$$

Since $\tilde m$ is an arbitrary point in a dense set of S^2, the conclusion follows. ☐

4.2 Adapted triplet (Φ_1, Φ_2, α)

Inequality (11) is useful if Φ takes the appropriate values on the circle on S^2 given by $\{m_2 = 0\}$. More precisely, we introduce the following concept:

Definition 4.1. *For $\theta \in (0, \pi/2]$, recall that*

$$F(\theta) = \sin\theta - \theta\cos\theta \quad \textit{and} \quad m^{\pm} = (\cos\theta, 0, \pm\sin\theta) \in S^2. \tag{14}$$

We will say that a triplet $(\Phi = (\Phi_1, \Phi_2), \alpha) \in \mathrm{Lip}(S^2, \mathbb{R}^2) \times \mathrm{Lip}(S^2)$ is adapted to the jump (m^-, m^+) if

$$\Phi_1(m^+) - \Phi_1(m^-) \ = 2F(\theta) \tag{15}$$

and there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, inequality (11) holds for every map $m \in C^\infty(\Omega, S^2)$.

We prove an existence result for walls of small transition angles $\theta \in [0, \theta_0]$, where θ_0 is determined in the proof of Proposition 4.1 (see Claim 1):

Proposition 4.1. *There exist an angle $\theta_0 \in (0, \frac{\pi}{2})$ and a Lipschitz triplet $(\Phi = (\Phi_1, \Phi_2), \alpha)$ that is adapted to the jump $m^\pm$ for every $\theta \in [0, \theta_0]$.*

For the biggest jump $\pm\mathbf{e}_3$, we prove a nonexistence result. This result suggests that the zigzag pattern may not be optimal for large angles.

Proposition 4.2. *There is no smooth triplet $(\Phi = (\Phi_1, \Phi_2), \alpha)$ adapted to the jump $m^\pm$ for $\theta = \pi/2$.*

In the proofs of Propositions 4.1 and 4.2, we will use spherical coordinates on S^2. Suppose $m \in H^1(\Omega, S^2)$ can be written in the form

$$m = (\cos\varphi\cos\vartheta, \sin\varphi, \cos\varphi\sin\vartheta) \tag{16}$$

for two functions $\varphi \in H^1(\Omega, [-\pi/2, \pi/2])$ and $\vartheta \in H^1(\Omega, \mathbb{R})$, where the range of ϑ is fixed by imposing the condition $\vartheta(x_0) \in (-\pi, \pi]$ for some Lebesgue point $x_0 \in \Omega$ of m. Then we compute

$$|\nabla m|^2 = |\nabla\varphi|^2 + \cos^2\varphi|\nabla\vartheta|^2 \quad \text{in} \quad L^1(\Omega)$$

and

$$\nabla \cdot m = -\sin\varphi\cos\vartheta\frac{\partial\varphi}{\partial x_1} - \cos\varphi\sin\vartheta\frac{\partial\vartheta}{\partial x_1} + \cos\varphi\frac{\partial\varphi}{\partial x_2} \quad \text{in} \quad L^2(\Omega).$$

Remark 1. *i) In general, a vector field $m \in H^1(\Omega, S^2)$ cannot be written in the form (16) with $\varphi, \vartheta \in H^1(\Omega, \mathbb{R})$. The standard example is the vortex type configuration in the unit disk $\Omega := B^2 \subset \mathbb{R}^2$:*

$$(m_1, m_3)(x) = \sin(\frac{\pi}{2}|x|)\frac{x}{|x|} \quad \textit{and} \quad m_2(x) = \cos(\frac{\pi}{2}|x|) \quad \textit{for } x \in B^2.$$

Indeed, $m \in H^1(B^2, S^2)$ and the 2D vector field (m_1, m_3) has a topological degree 1 at the boundary ∂B^2, which forbids the existence of a lifting $\vartheta \in H^1(\Omega, \mathbb{R})$ such that $\frac{(m_1, m_3)}{|(m_1, m_3)|} = (\cos\vartheta, \sin\vartheta)$ a.e. in B^2. (In fact, in this case, one can *find a lifting $\vartheta \in BV(B^2, \mathbb{R})$ with a jump set concentrated on a radius of B^2, see e.g. [11, 19], while $\varphi \in H^1(\Omega, [0, \pi/2])$ is given by $\varphi(x) = \frac{\pi}{2}(1 - |x|)$, $x \in B^2$.)*

ii) However, a vector field $m \in H^1(\Omega, S^2)$ can be written in the form (16) *with $\vartheta \in H^1(\Omega)$ and $\varphi \in H^1(\Omega, (-\pi/2, \pi/2))$ if $\operatorname{ess\,sup} |m_2| < 1$. Indeed, if we denote $v = (m_1, m_3)$, then $(|v|, m_2) \in H^1(\Omega, S^1)$ has a lifting $\varphi \in H^1(\Omega)$, i.e., $(|v|, m_2) = (\cos\varphi, \sin\varphi)$ (see [8]). Moreover, $\cos\varphi = |v| \geq 0$, therefore the range of φ satisfies $\operatorname{Im}\varphi \subset [-\pi/2, \pi/2] + 2\pi\mathbb{Z}$. Since $\varphi \in H^1(\Omega) \subset \mathrm{VMO}(\Omega)$, we deduce that $\operatorname{Im}\varphi$ is connected (here, Ω is supposed to be simply-connected); thus, up to an additive constant, $\varphi \in H^1(\Omega, (-\pi/2, \pi/2))$ where we used that $\operatorname{ess\,sup} |\sin\varphi| < 1$. Then $\operatorname{ess\,inf} |v| > 0$ so that $\frac{v}{|v|} \in H^1(\Omega, S^1)$ has a lifting $\vartheta \in H^1(\Omega, \mathbb{R})$ and* (16) *holds. Obviously, up to an additive constant in $2\pi\mathbb{Z}$, we can always assume that $\operatorname{Im}\vartheta \cap [-\pi, \pi] \neq \emptyset$. The representation* (16) *is unique if one imposes $\vartheta(x_0) \in (-\pi, \pi]$ for some Lebesgue point $x_0 \in \Omega$ of m.*

4.3 Existence of an adapted triplet for small angles

Proof of Proposition 4.1. We divide the proof in several steps:

Step 1. An "almost" adapted triplet $(\Phi = (\Phi_1, \Phi_2), \alpha)$. There is no general recipe for finding an adapted triplet $(\Phi = (\Phi_1, \Phi_2), \alpha)$ for a transition angle θ. However, Lemma 4.1 gives some useful constraints when trying to construct a triplet adapted to a certain range of angles θ. In particular, Φ_1 and α are determined on the circle $S^2 \cap \{m_2 = 0\}$: in the spherical coordinates (ϕ, θ), we use the anzatz that $\Phi_1(0, \cdot)$ is an odd function in $\theta \in [-\pi, \pi]$. Then (12) (for $\tau = 0$ and $\tau = \frac{\pi}{2}$) and (15) lead to

$$\Phi_1(0, \theta) = F(\theta), \quad \alpha(0, \theta) = \theta \quad \text{and} \quad \frac{\partial \Phi_2}{\partial \theta}(0, \theta) = 0.$$

Motivated by these facts, we consider the following triplet $(\Phi = (\Phi_1, \Phi_2), \alpha) : \mathbb{R}^2 \to \mathbb{R}^3$ with all components written in the spherical coordinates (16):

$$\Phi_1(\phi, \theta) = F(\theta)\cos^3\phi + G(\theta)\sin^2\phi\cos\phi,$$

$$\Phi_2(\phi, \theta) = -\theta\sin\phi\cos^2\phi,$$

and

$$\alpha(\phi, \theta) = \theta\cos^2\phi,$$

where $G : \mathbb{R} \to \mathbb{R}$ is defined by

$$G(\theta) = \frac{3}{2}\sin\theta - \theta\cos\theta, \quad \theta \in \mathbb{R}.$$

Let $m \in C^\infty(\Omega, S^2)$ be a smooth vector field that can be written in the form (16) for two smooth functions $\varphi, \vartheta \in C^\infty(\Omega, \mathbb{R})$ (this representation is unique when we impose the condition $\vartheta(x_0) \in (-\pi, \pi]$ for some fixed point $x_0 \in \Omega$). We compute

$$\nabla \cdot [\Phi(m)] + \alpha(m)\nabla \cdot m = -G(\vartheta)\sin^3\varphi\frac{\partial\varphi}{\partial x_1} + G'(\vartheta)\sin^2\varphi\cos\varphi\frac{\partial\vartheta}{\partial x_1}$$
$$+ 2\vartheta\sin^2\varphi\cos\varphi\frac{\partial\varphi}{\partial x_2} - \sin\varphi\cos^2\varphi\frac{\partial\vartheta}{\partial x_2}.$$

In particular, if we define $\ell : \mathbb{R}^2 \to \mathbb{R}$ with

$$\ell(\phi,\theta) = (G(\theta))^2 \sin^4\phi + (G'(\theta))^2 \sin^2\phi + 4\theta^2 \sin^2\phi \cos^2\phi + \cos^2\phi$$

for every $(\phi,\theta) \in \mathbb{R}^2$, then we obtain

$$0 \leq \ell(\phi,\theta) \leq 1 \quad \text{for every } \phi \in \mathbb{R}, |\theta| \leq \tilde{\theta}_0, \tag{17}$$

where $\tilde{\theta}_0$ is defined in Step 2, and

$$|\nabla \cdot [\Phi(m)] + \alpha(m)\nabla \cdot m| \leq \sqrt{\ell(\varphi,\vartheta)}|m_2||\nabla m|.$$

It follows that the triplet $(\Phi = (\Phi_1,\Phi_2),\alpha)$ is "almost" adapted for angles $\theta \in [0,\tilde{\theta}_0]$ in the sense that (15) holds for $\theta \in [0,\pi/2]$, but (11) holds *only* for vector fields m satisfying (16) for smooth functions φ,ϑ with the constraint that $|\vartheta| \leq \tilde{\theta}_0$ in Ω. (The inequality (11) is indeed satisfied, since by Young's inequality,

$$|m_2||\nabla m| \leq \frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}m_2^2 \quad \text{in } \Omega$$

for every $\varepsilon > 0$.)

Step 2. Estimate of $\tilde{\theta}_0$. The transition angle $\tilde{\theta}_0 \in (0,\pi/2]$ is the largest angle where (17) holds for every $\phi \in [-\pi/2,\pi/2]$ and $|\theta| \leq \tilde{\theta}_0$. We want to determine this angle $\tilde{\theta}_0$. For a fixed θ, setting $t := \sin^2\phi \in [0,1]$, the function ℓ can be seen as a polynomial function of second degree in t, i.e.,

$$\ell(\phi,\theta) = a(\theta)t^2 + b(\theta)t + 1,$$

where $a(\theta) = (G(\theta))^2 - 4\theta^2$ and $b(\theta) = (G'(\theta))^2 + 4\theta^2 - 1$. First we show that $a(\theta) \leq 0$ for $\theta \in [-\pi/2,\pi/2]$. Indeed, setting

$$\tilde{a}(\theta) = G(\theta) - 2\theta = \frac{3}{2}\sin\theta - \theta(\cos\theta + 2),$$

we compute that $\tilde{a}''(\theta) = \frac{1}{2}\sin\theta + \theta\cos\theta \geq 0$ for $\theta \in [0,\pi/2]$. Thus, $\tilde{a}$ is convex on $[0,\pi/2]$; since $\tilde{a}(0) = 0$ and $\tilde{a}(\pi/2) \leq 0$, we conclude that $\tilde{a} \leq 0$ in $[0,\pi/2]$, which implies that a has the same property on $[0,\pi/2]$. Since the function a is even, we conclude that $a \leq 0$ in $[-\pi/2,\pi/2]$. Observe now that $\ell(\phi,\theta) = 1$ if $t = 0$. In order that $\ell(\phi,\theta) \leq 1$ for every $t \in [0,1]$, one should impose that $b(\theta) \leq 0$ for every $|\theta| \leq \tilde{\theta}_0$. (We see that $b(0) = -3/4$ so that $\tilde{\theta}_0 > 0$.) The optimal $\tilde{\theta}_0 \in [0,\pi/2]$ is given by the condition $b(\tilde{\theta}_0) = 0$, i.e.,

$$(G'(\tilde{\theta}_0))^2 + 4\tilde{\theta}_0^2 = 1, \quad \text{i.e.,} \quad \tilde{\theta}_0 = 0.3948752981179... \tag{18}$$

Step 3. An adapted triplet (Ψ,β) for any transition angle $\theta \in [0,\theta_0]$. Motivated by *Step 1*, we now truncate the triplet (Φ,α) constructed above at a level $\theta_0 < \tilde{\theta}_0$, where $\theta_0 \in (0,\pi/2)$ will be given later (see Claim 1). Consider the map $\tilde{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2$ and the function $\tilde{\alpha} : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$\tilde{\Phi}_1(\phi,\theta) = \tilde{F}(\cos\theta)\cos^3\phi + \tilde{G}(\theta)\sin^2\phi\cos\phi,$$

$$\tilde{\Phi}_2(\phi,\theta) = \tilde{F}'(\cos\theta)\sin\phi\cos^2\phi$$

and

$$\tilde{\alpha}(\phi,\theta) = -\tilde{F}'(\cos\theta)\cos^2\phi,$$

where $\tilde{F} : [-1,1] \to \mathbb{R}$ is the C^1 function defined by

$$\tilde{F}(t) = \begin{cases} \frac{q_0}{2}(t+1)^2 & \text{if} \quad t \in [-1,0], \\ -\frac{q_0+\theta_0}{2\cos\theta_0}t^2 + q_0 t + \frac{q_0}{2} & \text{if} \quad t \in [0,1] \end{cases}$$

with

$$q_0 = \frac{2\sin\theta_0 - \theta_0\cos\theta_0}{1+\cos\theta_0}$$

and $\tilde{G} : \mathbb{R} \to \mathbb{R}$ is the $2\pi-$periodic, even Lipschitz function defined by

$$\tilde{G}(\theta) = G(\theta_0)\frac{\theta-\pi}{\theta_0-\pi}, \quad \theta \in [0,\pi]$$

(see Figure 3). Observe that for $\theta_0 > 0$ small, then $q_0 > 0$ is small so that $\|\tilde{F}\|_{L^\infty}$, $\|\tilde{F}'\|_{L^\infty}$ and $\|\tilde{F}''\|_{L^\infty}$ are small together with $\|\tilde{G}\|_{L^\infty}$ and $\|\tilde{G}'\|_{L^\infty}$.

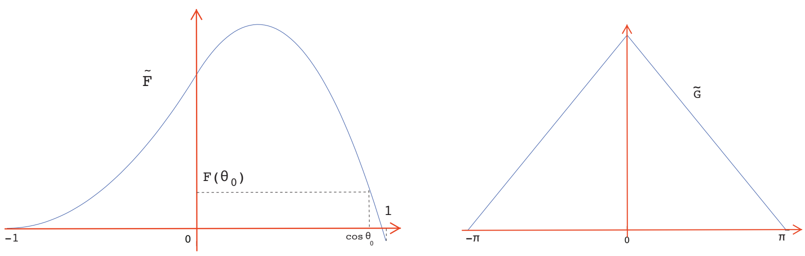

Figure 3: The functions $\tilde{F}$ and $\tilde{G}$.

We define $(\Psi,\beta) : (-\frac{\pi}{2},\frac{\pi}{2}) \times (-\pi,\pi) \to \mathbb{R}^3$ as follows: for every $\phi \in (-\frac{\pi}{2},\frac{\pi}{2})$,

$$\Psi(\phi,\theta) = \begin{cases} -\tilde{\Phi}(\phi,\theta) & \text{if } -\pi < \theta < -\theta_0, \\ \Phi(\phi,\theta) & \text{if } -\theta_0 \le \theta \le \theta_0, \\ \tilde{\Phi}(\phi,\theta) & \text{if } \pi > \theta > \theta_0, \end{cases}$$

and

$$\beta(\phi,\theta) = \begin{cases} -\tilde{\alpha}(\phi,\theta) & \text{if } -\pi < \theta < -\theta_0, \\ \alpha(\phi,\theta) & \text{if } -\theta_0 \le \theta \le \theta_0, \\ \tilde{\alpha}(\phi,\theta) & \text{if } \pi > \theta > \theta_0. \end{cases}$$

Then we extend this triplet to $(\Psi,\beta) : \mathbb{R}^2 \to \mathbb{R}^3$ that is $\pi-$periodic in ϕ and $2\pi-$periodic in θ. Observe that (Ψ,β) is a Lipschitz triplet on $\mathbb{R}^2$ and satisfies (15) for every mesoscopic wall $m^\pm = (\cos\theta, 0, \pm\sin\theta)$ with $\theta \in [0,\theta_0]$. Therefore, it makes sense to see the triplet as defined on S^2, i.e., $(\Psi,\beta) : S^2 \to \mathbb{R}^3$. The aim is to show that (Ψ,β) is an adapted triplet for angles wall $\theta \in [0,\theta_0]$.

Step 4. Proof of (11) *for* (Ψ,β). Let $m \in C^\infty(\Omega, S^2)$ and we will prove that (11) holds for (Ψ,β) for a.e. $x_0 \in \Omega$.

Case 1: $|m_2(x_0)| < 1$. There exists a closed ball $B \subset \Omega$ centered at x_0 such that $|m_2(x)| < 1$ for every $x \in B$. As explained in Remark 1 (ii), m can be written in the spherical coordinates (16) for some smooth $\varphi \in C^\infty(B, [-\pi/2, \pi/2])$ and $\vartheta \in C^\infty(B, \mathbb{R})$ with the range of ϑ determined by $\vartheta(x_0) \in (-\pi, \pi]$ and this representation is unique. Then we compute, as in Step 1, that a.e. in B:

$$\frac{1}{\sin\varphi}\Big(\nabla\cdot[\tilde\Phi(m)] + \tilde\alpha(m)\nabla\cdot m\Big)$$
$$= \Big(-3\tilde F(\cos\vartheta)\cos^2\varphi + \tilde G(\vartheta)(2\cos^2\varphi - \sin^2\varphi) + \tilde F'(\cos\vartheta)\cos^2\varphi\cos\vartheta\Big)\frac{\partial\varphi}{\partial x_1}$$
$$-2\tilde F'(\cos\vartheta)\sin\varphi\cos\varphi\frac{\partial\varphi}{\partial x_2} + \tilde G'(\theta)\sin\varphi\cos\varphi\frac{\partial\vartheta}{\partial x_1} - \tilde F''(\cos\vartheta)\cos^2\varphi\sin\vartheta\frac{\partial\vartheta}{\partial x_2}.$$

Defining $\tilde\ell : \mathbb{R}^2 \to \mathbb{R}$ by

$$\tilde\ell(\phi,\theta) = \Big(-3\tilde F(\cos\theta)\cos^2\phi + \tilde G(\theta)(2\cos^2\phi - \sin^2\phi) + \tilde F'(\cos\theta)\cos^2\phi\cos\theta\Big)^2$$
$$+4\left(\tilde F'(\cos\theta)\sin\phi\cos\phi\right)^2 + \left(\tilde G'(\theta)\sin\phi\right)^2 + \left(\tilde F''(\cos\theta)\cos\phi\sin\theta\right)^2,$$

we find

$$\Big|\nabla\cdot[\tilde\Phi(m)] + \tilde\alpha(m)\nabla\cdot m\Big| \le \sqrt{\tilde\ell(\varphi,\vartheta)}|m_2||\nabla m| \quad \text{a.e. in } \ B.$$

Claim 1. *$0 \le \tilde\ell(\phi,\theta) \le 1$ for every $\phi,\theta \in \mathbb{R}$ if $\theta_0 > 0$ is small.*

The proof of Claim 1 is a straightforward consequence of the definition of functions $\tilde F$ and $\tilde G$. Our angle θ_0 is the maximal angle $\theta_0 \in (0, \tilde\theta_0]$ (where $\tilde\theta_0$ is given at Step 2) that satisfies the constraint $\tilde\ell(\phi,\theta) \le 1$ for every $\phi,\theta \in \mathbb{R}$.

We conclude that inequality (11) is indeed satisfied for $(\tilde\Phi, \tilde\alpha)$ and m in B. Together with Step 1, since $\theta_0 < \tilde\theta_0$, we conclude that (11) holds for the triplet (Ψ, β) and m in B (in particular, at x_0).

Case 2: $|m_2(x_0)| = 1$, i.e., $m(x_0)$ is one of the poles $P_\pm = (0, \pm1, 0)$. Notice that $\Psi(P_\pm) = 0$ and $\beta(P_\pm) = 0$. We may assume that both sides of (11) are well-defined at x_0, because the chain rule applies almost everywhere [2, Corollary 3.2]. If $\nabla m(x_0) = 0$, then (11) is trivially satisfied at x_0. Otherwise, $\nabla m(x_0) \neq 0$. By the implicit function theorem, the set $\{x_0 \in \Omega : m(x_0) \in \{P_\pm\}, \nabla m(x_0) \neq 0\}$ is a countable union of curves, in particular of vanishing $\mathcal{L}^2-$measure. Therefore, we conclude that (11) holds for a.e. $x_0 \in \Omega$. □

As consequence, we prove Theorem 2.1:

Proof of Theorem 2.1. Let $\theta \in (0, \theta_0]$. First, by Schoen-Uhlenbeck's density result and the continuity of E_ε on H^1, it is enough to prove the theorem for smooth vector fields $m_\varepsilon \in M(\theta)$. By Proposition 4.1, we choose a triplet (Φ, α) adapted to the jump $m^\pm$. Integrating (11) on Ω, one gets

$$\int_\Omega \nabla\cdot\{\Phi(m_\varepsilon)\}\,dx + \int_\Omega \alpha(m_\varepsilon)\nabla\cdot m_\varepsilon\,dx \le E_\varepsilon(m_\varepsilon).$$

Since m_ε is periodic in x_2, integration by parts yields

$$\int_\Omega \nabla \cdot \{\Phi(m_\varepsilon)\}\, dx = \Phi_1(m^+) - \Phi_1(m^-) = 2F(\theta),$$

while by duality, we deduce

$$\begin{aligned}\left|\int_\Omega \alpha(m_\varepsilon)\nabla \cdot m_\varepsilon\, dx\right| &\le \|\nabla \cdot m_\varepsilon\|_{\dot{H}^{-1}_{\mathrm{per}}(\Omega)} \|\nabla \alpha\|_{L^\infty} \|\nabla m_\varepsilon\|_{L^2(\Omega)} \\ &\le o(1)E_\varepsilon(m_\varepsilon)\end{aligned}$$

as $\varepsilon \to 0$ (the assumption $s > 1$ is essential here). Therefore, $\min_{M(\theta)} E_\varepsilon \ge 2F(\theta) + o(1)$ as $\varepsilon \to 0$. In Section 3, we saw that the reverse inequality also holds, so that the conclusion is now straightforward. $\square$

4.4 Non-existence of smooth adapted triplet for the maximal jump

Proof of Proposition 4.2. Assume for contradiction that there exists a triplet $(\Phi = (\Phi_1, \Phi_2), \alpha) \in \mathrm{Lip}(S^2, \mathbb{R}^3)$ adapted to the wall $\pm\mathbf{e}_3$ and of class C^2 away from the poles $\pm\mathbf{e}_3$. Fix $\tau \in (-\pi, \pi]$. As in Lemma 4.1, we define

$$\nu_\tau = (-\sin\tau, \cos\tau, 0)$$

and

$$\Psi_\tau = -\sin\tau\Phi_1 + \cos\tau\Phi_2.$$

Furthermore, consider the semicircle

$$C_\tau = \left\{m \in S^2 : m \cdot \nu_\tau = 0,\, m \cdot \nu_\tau^\perp < 0\right\},$$

where $\nu_\tau^\perp = -(\cos\tau, \sin\tau, 0)$. By Lemma 4.1, we have

$$|D\Psi_\tau(m) + \alpha(m)\Pi_m\nu_\tau|^2 \le m_2^2, \quad m \in S^2. \tag{19}$$

We choose a new set of spherical coordinates $(s,t) \in [-\pi/2, \pi/2] \times [-\pi, \pi]$ such that

$$m = (\cos s\cos t, \cos s\sin t, \sin s) \in S^2$$

and we identify $\Phi(m) := \Phi(s,t)$ and $\alpha(m) := \alpha(s,t)$. Then (19) becomes

$$\begin{aligned}&\left(\frac{\partial\Psi_\tau}{\partial s}(m) + (\sin\tau\sin s\cos t - \cos\tau\sin s\sin t)\alpha(m)\right)^2 \\ &\qquad + \left(\frac{\frac{\partial\Psi_\tau}{\partial t}(m)}{\cos s} + (\sin\tau\sin t + \cos\tau\cos t)\alpha(m)\right)^2 \le \cos^2 s\sin^2 t.\end{aligned} \tag{20}$$

On C_τ, this means

$$\left(\frac{\partial\Psi_\tau}{\partial s}(m)\right)^2 + \left(\frac{\frac{\partial\Psi_\tau}{\partial t}(m)}{\cos s} + \alpha(m)\right)^2 \le \cos^2 s\sin^2\tau \quad \text{on } C_\tau. \tag{21}$$

As a consequence, note that if $\tau = 0$, then $\Psi_\tau \equiv \Phi_2$ is constant on C_τ. In particular, Φ_2 takes the same value at the poles $\pm\mathbf{e}_3$, i.e.,

$$\Phi_2(\mathbf{e}_3) = \Phi_2(-\mathbf{e}_3).$$

Combined with our assumption $\Phi_1(\mathbf{e}_3) - \Phi_1(-\mathbf{e}_3) = 2$, we deduce $\Psi_\tau(\pi/2, \tau) - \Psi_\tau(-\pi/2, \tau) = -2\sin\tau$ for every $\tau \in (-\pi, \pi]$. Combined with (21), we deduce

$$2|\sin\tau| \leq \int_{-\pi/2}^{\pi/2} \left|\frac{\partial \Psi_\tau}{\partial s}\right| ds \leq |\sin\tau| \int_{-\pi/2}^{\pi/2} \cos s \, ds = 2|\sin\tau|.$$

It follows that

$$\frac{\partial \Psi_\tau}{\partial s}(s, \tau) = -\sin\tau\cos s \quad \text{ on } C_\tau \tag{22}$$

and

$$\frac{\partial \Psi_\tau}{\partial t} = -\alpha(s, \tau)\cos s \quad \text{ on } C_\tau. \tag{23}$$

(The sign in (22) is determined by the fact that $\Psi_\tau(\pi/2, \tau) < \Psi_\tau(-\pi/2, \tau)$ whenever $\sin\tau > 0$.) Moreover, on C_τ, we have equality in (20). Hence for every $s \in (-\pi/2, \pi/2)$, the function

$$t \mapsto \left(\frac{\partial \Psi_\tau}{\partial s}(s, t) + (\sin\tau \sin s \cos t - \cos\tau \sin s \sin t)\alpha(s, t)\right)^2 - \cos^2 s \sin^2 t$$

has a maximum ($= 0$) at $t = \tau$. Differentiating in t at $t = \tau$, we obtain

$$\left(\frac{\partial^2 \Psi_\tau}{\partial s \partial t}(s, \tau) - \alpha(s, \tau)\sin s\right) \frac{\partial \Psi_\tau}{\partial s}(s, \tau) - \sin\tau\cos\tau\cos^2 s = 0.$$

Combined with (22), we obtain

$$\frac{\partial^2 \Psi_\tau}{\partial s \partial t}(s, \tau) = \alpha(s, \tau)\sin s - \cos\tau\cos s, \quad s \in (-\pi/2, \pi/2), \tau \in (-\pi, \pi) \setminus \{0\}.$$

Differentiating (23), we also find

$$\frac{\partial^2 \Psi_\tau}{\partial s \partial t}(s, \tau) = \alpha(s, \tau)\sin s - \frac{\partial \alpha}{\partial s}(s, \tau)\cos s.$$

(The hypothesis $\Psi \in C^2$ and $\alpha \in C^1$ is needed in the above two identities.) Therefore,

$$\frac{\partial \alpha}{\partial s}(s, \tau) = \cos\tau, \quad s \in (-\pi/2, \pi/2), \tau \in [-\pi, \pi].$$

Integrating in s, the continuity of α in S^2 yields

$$\alpha(s, \tau) = s\cos\tau + c(\tau) \quad s \in [-\pi/2, \pi/2], \tau \in [-\pi, \pi],$$

for some function $c = c(\tau)$ depending only on τ. The contradiction arises when we evaluate α at the poles $\pm\mathbf{e}_3$:

$$\alpha(\mathbf{e}_3) - \alpha(-\mathbf{e}_3) = \alpha(\pi/2, \tau) - \alpha(-\pi/2, \tau) = \pi\cos\tau$$

which is absurd since the above LHS cannot depend on τ. $\square$

5 Proof of $\Gamma-$convergence result in Theorem 2.2

We start by proving compactness and lower bound for our energy in the context of an arbitrary domain Ω:

Proof of Theorem 2.2 1). It is straightforward to check (9). Suppose now that $|\psi_k| \le F(\theta_0)$ a.e. in Ω. Again, by Schoen-Uhlenbeck's density result, due to the continuity of E_{ε_k} on H^1, we can assume that $m_k \in C^1(\Omega, S^2)$. Let $(\Psi = (\Psi_1, \Psi_2), \beta)$ be the triplet constructed in the proof of Proposition 4.1. By the definition of Ψ, there exists a constant C such that $|F(\theta) - \Psi_1(m)| \le C|m_2|$ for every point $m = (\cos\phi\cos\theta, \sin\phi, \cos\phi\sin\theta)$ with $|\theta| \le \theta_0$ and $\phi \in [-\pi/2, \pi/2]$. By (9), the condition $|\psi_k| \le F(\theta_0)$ in Ω then yields $\lim_k \Psi_1(m_k) = \lim_k \psi_k = \psi$ weakly* in $L^\infty(\Omega)$. Let $v \in C_0^1(\Omega)$. By (11), integration by parts yields:

$$\left|\int_\Omega \Big(\nabla v \cdot \Psi(m_k) - v\beta(m_k)\nabla \cdot m_k\Big)\, dx\right| \le \sup_\Omega |v| E_{\epsilon_k}(m_k). \tag{24}$$

By definition of Ψ_2 and (9), we deduce that

$$\int_\Omega \frac{\partial v}{\partial x_2}\Psi_2(m_k)\, dx \to 0 \quad \text{as } k \to \infty.$$

Moreover, since

$$\limsup_{k\to\infty} \epsilon_k \|v\beta(m_k)\|^2_{H_0^1(\Omega)} < \infty \quad \text{and} \quad \limsup_{k\to\infty} \varepsilon_k^{-s}\|\nabla \cdot m_k\|^2_{\dot{H}^{-1}_{\text{per}}(\Omega)} < \infty,$$

by duality, it follows that

$$\int_\Omega v\beta(m_k)\nabla \cdot m_k\, dx \to 0$$

as well. Combining with (24), it follows that

$$\int_\Omega \frac{\partial v}{\partial x_1}\psi\, dx = \lim_{k\to\infty}\int_\Omega \frac{\partial v}{\partial x_1}\Psi_1(m_k)\, dx \le \sup_\Omega |v| \liminf_{k\to\infty} E_{\epsilon_k}(m_k);$$

thus, $E_0(\psi) \le \liminf_{k\to\infty} E_{\epsilon_k}(m_k)$. ◻

Let us now prove the recovery sequence step for the $\Gamma-$convergence:

Proof of Theorem 2.2 2). For simplicity, we set $\Omega = (-1,1)^2$ (all the following arguments adapt to a general smooth bounded simply-connected domain Ω). Suppose that $\psi \in L^\infty(\Omega)$ with $|\psi| \le F(\theta_0)$ almost everywhere and $E_0(\psi) < \infty$. Then it follows that the distributional derivative of ψ with respect to x_1 is represented by a Radon measure $\frac{\partial\psi}{\partial x_1}$ on Ω, and

$$E_0(\psi) = \left|\frac{\partial \psi}{\partial x_1}\right|(\Omega).$$

We want to construct a sequence

$$m_k = (\cos\varphi_k\cos\vartheta_k, \sin\varphi_k, \cos\varphi_k\sin\vartheta_k), \quad k \in \mathbb{N},$$

such that $|\vartheta_k| \le \theta_0$ in Ω and

$$F(\vartheta_k) = \psi_k \overset{*}{\rightharpoonup} \psi \quad \text{in } L^\infty(\Omega) \text{ as } k \to \infty$$

and a corresponding sequence $\epsilon_k \to 0$ such that

$$\limsup_{k\to\infty} E_{\epsilon_k}(m_k) \le E_0(\psi).$$

Step 1. Approximating ψ by step functions $\{\tilde{\psi}^\ell\}_{\ell\in\mathbb{N}}$. Fix $\ell \in \mathbb{N}$. We divide Ω in squares of length $2^{-\ell}$, i.e.,

$$Q_{ij}^\ell = (s_i, s_{i+1}) \times (s_j, s_{j+1}),$$

where $s_i = 2^{-\ell} i$ for $i = -2^\ell, \dots, 2^\ell - 1$. Consider the mean values

$$a_{ij}^\ell = \fint_{s_j}^{s_{j+1}} \psi(s_{i+1}, x_2)\, dx_2 \in [-F(\theta_0), F(\theta_0)], \quad i, j = -2^\ell, \dots, 2^\ell - 1.$$

Define the rectangles $P_{ij}^\ell \subset Q_{ij}^\ell$ by

$$P_{ij}^\ell = (s_i, s_{i+1}) \times \left(s_j, s_j + 2^{-\ell-1}(1 + \frac{a_{ij}^\ell}{F(\theta_0)}) \right)$$

and let χ_{ij}^ℓ be the characteristic function of P_{ij}^ℓ. Let $\tilde{\psi}^\ell : \Omega \to \{\pm F(\theta_0)\}$ be the following step function:

$$\tilde{\psi}^\ell = F(\theta_0)\bigg(-1 + 2 \sum_{i,j=-2^\ell}^{2^\ell - 1} \chi_{ij}^\ell \bigg).$$

The choice of P_{ij}^ℓ was made so that

$$\fint_{Q_{ij}^\ell} \tilde{\psi}^\ell \, dx = a_{ij}^\ell.$$

We claim that $\{\tilde{\psi}^\ell\}$ converges weakly* to ψ in $L^\infty(\Omega)$. This can be seen as follows. Note first that

$$\begin{aligned}\left| \int_{Q_{ij}^\ell} \psi\, dx - 2^{-2\ell} a_{ij}^\ell \right| &= 2^{-2\ell} \left| \fint_{s_j}^{s_{j+1}} \fint_{s_i}^{s_{i+1}} \big(\psi(x_1, x_2) - \psi(s_{i+1}, x_2)\big)\, dx_1 dx_2 \right| \\ &\le 2^{-\ell} \left| \frac{\partial \psi}{\partial x_1} \right| (Q_{ij}^\ell).\end{aligned}$$

Thus for any $v \in C^1(\overline{\Omega})$, we have

$$\begin{aligned}\left| \int_\Omega \left(\psi - \tilde{\psi}^\ell\right) v\, dx \right| &\le \left| \sum_{i,j=-2^\ell}^{2^\ell-1} \left(\int_{Q_{ij}^\ell} \psi\, dx - 2^{-2\ell} a_{ij}^\ell \right) v(s_i, s_j) \right| \\ &\quad + 16F(\theta_0) 2^{-\ell} \|\nabla v\|_{L^\infty(\Omega)} \\ &\le 2^{-\ell} \bigg(E_0(\psi) + 16F(\theta_0) \bigg) \|v\|_{C^1(\Omega)}.\end{aligned}$$

Since the sequence $\{\tilde{\psi}^\ell\}_{\ell\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, there exists a subsequence which converges weakly*. But by the above estimates, the limit of any such subsequence must be ψ. Hence we have weak* convergence to ψ of the entire sequence.

Step 2. Recovery sequence for each step function $\tilde{\psi}^\ell$. For a fixed ℓ, we now want to construct a sequence

$$m_k^\ell = (\cos\varphi_k^\ell \cos\vartheta_k^\ell, \sin\varphi_k^\ell, \cos\varphi_k^\ell \sin\vartheta_k^\ell), \quad k \in \mathbb{N}, \tag{25}$$

such that

$$F(\vartheta_k^\ell) \overset{*}{\rightharpoonup} \tilde{\psi}^\ell \quad \text{as } k \to \infty$$

weakly* in $L^\infty(\Omega)$. The construction, if carried out in detail, is technically quite complicated, but not difficult in principle, and the underlying ideas have been discussed in the previous sections. We therefore give a description of the construction rather than the full technical details. We expect that this will be more illuminating to the reader.

Zigzag construction for vertical jumps. Fix $\delta > 0$. Consider the jump set of $\tilde{\psi}^\ell$ and note that it consists of horizontal and vertical line segments. Consider first a vertical piece, say $\{s_i\} \times (s_j + r, s_j + q)$ for some fixed indices i and j and $r < q$ with $r, q \in (0, 2^{-\ell})$. Suppose that $\tilde{\psi}^\ell = -F(\theta_0)$ in $(s_{i-1}, s_i) \times (s_j + r, s_j + q)$ and $\tilde{\psi}^\ell = F(\theta_0)$ in $(s_i, s_{i+1}) \times (s_j + r, s_j + q)$, say. Then there exists a constant $c > 0$, such that the construction from Section 3 for a transition between the mesoscopic directions $(\cos\theta_0, 0, -\sin\theta_0)$ and $(\cos\theta_0, 0, \sin\theta_0)$ yields a family of maps $\hat{m}_\epsilon$ with

$$\begin{aligned}
\hat{m}_\epsilon &= (\cos\theta_0, 0, -\sin\theta_0) \quad \text{in } (s_{i-1}, s_i - c\epsilon) \times (s_j + r, s_j + q), \\
\hat{m}_\epsilon &= (\cos\theta_0, 0, \sin\theta_0) \quad \text{in } (s_i + c\epsilon, s_{i+1}) \times (s_j + r, s_j + q),
\end{aligned}$$

and

$$\limsup_{\epsilon\to 0} \int_{s_{i-1}}^{s_{i+1}} \int_{s_j+r}^{s_j+q} \left(\frac{\epsilon}{2} |\nabla \hat{m}_\epsilon|^2 + \frac{\hat{m}_{\epsilon,2}^2}{2\epsilon} \right) dx_1\, dx_2 \le (2F(\theta_0) + \delta)(q - r).$$

The divergence of $\hat{m}_\epsilon$ satisfies an estimate similar to (10). Moreover, $\arctan \frac{\hat{m}_{\epsilon,3}}{\hat{m}_{\epsilon,1}} \in [-\theta_0, \theta_0]$.

Bloch wall for horizontal jumps. If we have a horizontal piece of the jump set, say $(s_i, s_{i+1}) \times \{s_j + q\}$ with $\tilde{\psi}^\ell = -F(\theta_0)$ in $(s_i, s_{i+1}) \times (s_j, s_j + q)$ and $\tilde{\psi}^\ell = F(\theta_0)$ in $(s_i, s_{i+1}) \times (s_j + q, s_{j+1})$, then we use a Bloch wall instead of the zigzag wall. That is, we choose a function $v \in C^\infty(\mathbb{R}, [-\theta_0, \theta_0])$ with $v \equiv -\theta_0$ in $(-\infty, -1]$ and $v \equiv \theta_0$ in $[1, \infty)$, and we set

$$\check{m}_\epsilon(x_1, x_2) = \left(\cos v\left(\frac{x_2 - s_j - q}{c\epsilon} \right), 0, \sin v\left(\frac{x_2 - s_j - q}{c\epsilon} \right) \right),$$

for some $c > 0$. Then

$$\limsup_{\epsilon\to 0} \int_{s_i}^{s_{i+1}} \int_{s_j}^{s_{j+1}} \left(\frac{\epsilon}{2} |\nabla \check{m}_\epsilon|^2 + \frac{\check{m}_{\epsilon,2}^2}{2\epsilon} \right) dx_1\, dx_2 \lesssim \frac{1}{c}.$$

If the constant $c = c(\delta)$ is chosen sufficiently large, then we have

$$\limsup_{\epsilon \to 0} \int_{s_i}^{s_{i+1}} \int_{s_j}^{s_{j+1}} \left(\frac{\epsilon}{2} |\nabla \check{m}_\epsilon|^2 + \frac{\check{m}_{\epsilon,2}^2}{2\epsilon} \right) dx_1\, dx_2 \leq \delta.$$

Moreover, a vector field of this form is divergence free.

Final construction. Now we construct a family of unit vector fields $\tilde{m}_\epsilon : \Omega \to S^2$ that behaves like $\hat{m}_\epsilon$ near the vertical jump set and like $\check{m}_\epsilon$ near the horizontal jumps. This requires a modification at the corners. The situation here is not essentially different, however, from the internal corners of the zigzag wall. Thus we can use the same arguments as in section 3 again and we obtain an estimate of the form

$$\int_\Omega |\nabla \cdot \tilde{m}_\epsilon|^p \, dx \leq C_1 \epsilon^{2-p}$$

for a constant $C_1 > 0$ that is independent of ϵ. Thus the contribution of the magnetostatic energy will be negligible in the limit $\epsilon \to 0$. We then obtain another constant C_2, independent of δ, such that

$$\limsup_{\epsilon \to 0} E_\epsilon(\tilde{m}_\epsilon) \leq E_0(\tilde{\psi}^\ell) + C_2 \delta.$$

Letting $\delta \to 0$, we can now construct a sequence $\{m_k^\ell\}_{k \in \mathbb{N}}$ of the form (25) with

$$F(\vartheta_k^\ell) \overset{*}{\rightharpoonup} \tilde{\psi}^\ell \quad \text{as } k \to \infty$$

weakly* in $L^\infty(\Omega)$, and a corresponding sequence $\epsilon_k \to 0$ with

$$\begin{aligned} \lim_{k \to \infty} E_{\epsilon_k}(m_k^\ell) \leq E_0(\tilde{\psi}^\ell) &= 2^{-\ell} \sum_{i=-2^\ell}^{2^\ell - 2} \sum_{j=-2^\ell}^{2^\ell - 1} |a_{i+1,j}^\ell - a_{ij}^\ell| \\ &\leq \sum_{i=-2^\ell+1}^{2^\ell - 1} \int_{-1}^{1} |\psi(s_{i+1}, x_2) - \psi(s_i, x_2)| \, dx_2 \leq E_0(\psi). \end{aligned}$$

We are working in a bounded subset of $L^\infty(\Omega)$, where the weak*-topology is metrizable. Therefore, we can construct another diagonal sequence with the desired properties. □

Remark 2. *The construction does not depend on the assumption that $|\psi| \leq F(\theta_0)$ and can also be done in the more general context of $|\psi| \leq 1$ almost everywhere. This will yield a sequence, however, that is not compatible with the results we proved for the lower bound (see Section 4). This is why we presented above the more restrictive condition.*

Let us end this section by showing why the loss of compactness in strong L^1−topology does occur in our model:

Proposition 5.1. *There exist sequences $\{\varepsilon_k\} \subset (0, \infty)$ with $\varepsilon_k \to 0$ and $\{m_k\} \subset H^1(\Omega; S^2)$ such that*

$$\lim_{k \to \infty} E_{\epsilon_k}(m_k) = 0$$

and $\{m_k\}$ is not relatively compact in $L^1(\Omega)$.

Proof. The idea is to construct sequences of magnetizations m_k having $2^{k+1}-1$ Bloch wall transitions between the poles $\pm\mathbf{e}_3$, each transition concentrating on horizontal segments so that their energy is very small. As before, we restrict to the case $\Omega = (-1,1)^2$. For each $k \in \mathbb{N}$, we set $s_j = 2^{-k}j$ for $j = -2^k + 1, \ldots, 2^k - 1$. On each horizontal segment $(-1,1) \times \{s_j\}$ we place a mesoscopic transition between the directions $\pm\mathbf{e}_3$. At the microscopic level, this transition is replaced by a smooth Bloch wall. More precisely, we choose an odd function $v \in C^\infty(\mathbb{R}, [-\frac{\pi}{2}, \frac{\pi}{2}])$ with $v \equiv -\frac{\pi}{2}$ in $(-\infty, -1]$ and $v \equiv \frac{\pi}{2}$ in $[1, \infty)$. Then we set for every odd $j \in \{-2^k+1, \ldots, 2^k-1\}$, $x_1 \in (-1,1)$, $x_2 \in (s_j - 2^{-2k}, s_j + 2^{-2k})$:

$$m_k = m_k(x_2) = \left(\cos\left((-1)^{\frac{j-1}{2}} v\left(\frac{x_2 - s_j}{c\epsilon}\right)\right), 0, \sin\left((-1)^{\frac{j-1}{2}} v\left(\frac{x_2 - s_j}{c\epsilon}\right)\right)\right),$$

for some $c \geq 1$, $\varepsilon > 0$ so that $2^{-2k} \geq c\varepsilon$. (One completes the definition of m_k in the remaining parts of Ω by the obvious constant $\pm\mathbf{e}_3$ so that m_k is continuous.) Then the vector field m_k is divergence free, $m_{2,k} = 0$ and

$$E_\varepsilon(m_k) = \int_\Omega \frac{\epsilon}{2} |\nabla m_k|^2 \, dx_1 \, dx_2 \lesssim \frac{2^k}{c}.$$

If the constant $c = c(k)$ is chosen sufficiently large and $\varepsilon := \varepsilon_k = 2^{-2k}/c$, then we have

$$E_{\varepsilon_k}(m_k) \leq \frac{1}{k}.$$

A standard computation shows that $m_k \rightharpoonup (0,0,0)$ weakly in $L^2(\Omega)$ so that $\{m_k\}$ cannot be relatively compact in $L^1(\Omega, S^2)$.

$\square$

6 Appendix

Let us prove that the energy density g defined in (4) achieves the minimum as $\alpha \to \frac{\pi}{2}^-$:

Proposition 6.1. *The function $g : [0, \frac{\pi}{2}) \to \mathbb{R}$ defined in* (4) *is decreasing and*

$$\inf_{0 \leq \alpha < \frac{\pi}{2}} g(\alpha) = 2F(\theta).$$

Proof. Recall that the expected energy per unit wall length is given by

$$\begin{aligned} K(\alpha) &= \int_{-\sigma}^{\sigma} \gamma_2(t) |\dot\gamma(t)| \, dt = \int_{-\sigma}^{\sigma} \left(\sqrt{1-b^2}\cos\alpha\cos t - b\sin\alpha\right)\sqrt{1-b^2}\, dt \\ &= 2\sqrt{1-b^2}\Big(\cos\alpha\sin\theta - b\sigma\sin\alpha\Big) \\ &= 2\cos\alpha\sqrt{1-\cos^2\theta\cos^2\alpha}\left(\sin\theta - \cos\theta\sin\alpha\arcsin\frac{\sin\theta}{\sqrt{1-\cos^2\theta\cos^2\alpha}}\right). \end{aligned}$$

We have $g(\alpha) = \frac{K(\alpha)}{\cos\alpha}$. That is,

$$g(\alpha) \;=\; 2\sqrt{1-\cos^2\theta\cos^2\alpha}\left(\sin\theta \,-\, \cos\theta\sin\alpha\arcsin\frac{\sin\theta}{\sqrt{1-\cos^2\theta\cos^2\alpha}}\right).$$

First we prove that this function is decreasing in $\alpha \in [0, \frac{\pi}{2})$. To see this, set

$$y := y(\alpha) = \frac{\sin\theta}{\sqrt{1 - \cos^2\theta\cos^2\alpha}}$$

which is a decreasing function in $\alpha \in [0, \frac{\pi}{2})$. Then

$$g(\alpha) = 2\sin^2\theta\tilde{g}(y) \quad \text{ with } \tilde{g}(y) = \frac{1}{y^2}\left(y - \sqrt{1-y^2}\arcsin y\right).$$

We have $0 < \sin\theta \le y \le 1$ and

$$\tilde{g}'(y) = -\frac{2}{y^2} + \frac{2-y^2}{y^3\sqrt{1-y^2}}\arcsin y.$$

We show that $\tilde{g}' \ge 0$ in $(\sin\theta, 1)$; indeed, we have $2 - y^2 \ge 2\sqrt{1-y^2}$ for $|y| \le 1$, therefore,

$$\tilde{g}'(y) \ge -\frac{2}{y^2} + \frac{2}{y^3}\arcsin y = \frac{2}{y^3}(\arcsin y - y) \ge 0.$$

Thus, we find that $\tilde{g}$ is increasing and g is a decreasing function. We conclude that

$$\inf_{0\le\alpha<\frac{\pi}{2}} g(\alpha) = \lim_{\alpha\to\frac{\pi}{2}^-} g(\alpha) = 2(\sin\theta - \theta\cos\theta).$$

□

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