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ON AN OPEN PROBLEM ABOUT HOW TO RECOGNIZE CONSTANT FUNCTIONS

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Abstract. We find necessary and sufficient conditions for the function ω in order that any measurable function $f : \Omega \to \mathbb{R}$ which satisfies

$$\int_\Omega \int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|^N} < +\infty, \tag{1}$$

is constant (a.e. in Ω). We also study what regularity on f should be assumed so that for any function ω which is continuous, $\omega(0) = 0$ and $\omega(t) > 0$ for every $t > 0$, if (1) holds, then f is a constant.

1. Introduction

In this paper we investigate an open question posed by Brezis in [2]. Its motivation came from the following result (see [2]):

Theorem 1.1. *Let Ω be a domain (i.e. a connected open set) in $\mathbb{R}^N$. If $f : \Omega \to \mathbb{R}$ is a measurable function which satisfies*

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|} \frac{dx\,dy}{|x-y|^N} < +\infty,$$

then f is a constant (a.e. in Ω). More generally, if $p \geq 1$ and

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \frac{dx\,dy}{|x-y|^N} < +\infty,$$

then the same conclusion holds.

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We denote

$$\mathcal{W} = \{\omega \in C(\mathbb{R}_+, \mathbb{R}_+) \,|\, \omega(0) = 0,\, \omega(t) > 0,\, \forall t > 0\}\,.$$

The following problem now arises:

Problem 1. *Find a necessary and sufficient condition for $\omega \in \mathcal{W}$ so that any measurable function $f : \Omega \to \mathbb{R}$ which satisfies*

$$\int_\Omega \int_\Omega \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|^N} < +\infty, \tag{2}$$

is constant (a.e. in Ω).

Observe that the restriction $\omega \in \mathcal{W}$ is natural. Indeed, the continuity of ω is needed to make the left hand side of (2) well-defined. Also, $\omega(0) = 0$ (since for any constant function f, (2) should hold) and $\omega(t) > 0, \forall t > 0$ (if $\omega(t) = 0$ for some $t > 0$, take $N = 1$ and $f(x) = tx$). Henceforth it is assumed that $\omega \in \mathcal{W}$.

Three theorems are established concerning Problem 1. Theorem 1.2 gives a necessary condition and Theorems 1.3 and 1.4 provide sufficient conditions. The question whether the necessary condition in Theorem 1.2 is also sufficient remains open.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $\omega \in \mathcal{W}$ be such that any measurable function $f : \Omega \to \mathbb{R}$ that satisfies (2) is constant (a.e. in Ω). Then $\int_1^{+\infty} \frac{\omega(t)}{t^2}\, dt = +\infty$.*

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N$ be a domain, $f : \Omega \to \mathbb{R}$ be a measurable function and $\omega \in \mathcal{W}$ such that $\liminf_{t\to+\infty} \frac{\omega(t)}{t} > 0$. If (2) holds, then f is constant (a.e. in Ω).*

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^N$ be a domain, $f : \Omega \to \mathbb{R}$ be a measurable function and $\omega \in \mathcal{W}$. Define $\phi : (0, +\infty) \mapsto (0, +\infty)$, $\phi(t) = t^{-1}\omega(t)$ for all $t > 0$. Assume that ω is a non-decreasing function such that*

$$\int_1^{+\infty} \frac{\omega(t)}{t^2}\, dt = +\infty \text{ and } \sup_{0<s\le t} \frac{\phi(t)}{\phi(s)} < +\infty.$$

If (2) holds, then f is constant (a.e. in Ω).

Open question 1. *Is the condition $\int_1^{+\infty} \frac{\omega(t)}{t^2}\, dt = +\infty$ sufficient for Problem 1 (of course, under the assumption $\omega \in \mathcal{W}$)?*

In the second part of the paper, we investigate the following problem:

Problem 2. *What regularity on f should be assumed so that for any $\omega \in \mathcal{W}$, (2) implies f is a constant?*

The motivation is clear: if we do not want any restriction on $\omega \in \mathcal{W}$, an additional condition on f should be imposed in order that (2) yields f to be a constant. We establish the following results for Problem 2. Theorem 1.5 establishes that the condition $f \in W^{1,1}_{loc}(\Omega)$ guarantees that Problem 2 has a positive answer. The other two theorems deal with the question raised by Brezis in [2]: *Is the continuity (or even the $C^{0,\alpha}_{loc}$ regularity) of f sufficient for Problem 2?* The answer is negative in general. In the end, we state another open question (related to the previous one).

Theorem 1.5. *Let Ω be a domain in $\mathbb{R}^N$ and $f \in W^{1,1}_{loc}(\Omega)$. For any $\omega \in \mathcal{W}$, if (2) holds, then f is constant a.e in Ω.*

Theorem 1.6. *Let Ω be the unit cube in $\mathbb{R}^N$ i.e. $\Omega = (0,1)^N$. For every $0 < \alpha < 1$, there is a nonconstant α-Hölder continuous function $f : [0,1]^N \mapsto \mathbb{R}$ of bounded variation which satisfies (2), for every bounded function $\omega \in \mathcal{W}$.*

Theorem 1.7. *Let $\Omega = (0,1)^N$. For every $0 < \alpha < 1$, there is a nonconstant α-Hölder continuous function $f : [0,1]^N \mapsto \mathbb{R}$ of bounded variation which satisfies*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^\theta}{|x-y|^\theta} \frac{dx\,dy}{|x-y|^N} < +\infty, \quad \forall \theta \in (0,1).$$

Open question 2. *Let $\omega \in \mathcal{W}$ be such that $\int_1^{+\infty} \frac{\omega(t)}{t^2}\,dt = +\infty$. Suppose f is continuous (or even $C^{0,\alpha}_{loc}$ for some $0 < \alpha < 1$) and satisfies (2). Is f constant?*

In this paper, we also present some remarkable properties concerning a generalized Cantor set and Cantor function, results that we use in the proofs of the last theorems.

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2. Necessary condition for Problem 1

In this section we prove Theorem 1.2 i.e., the condition

$$\int_1^{+\infty} \frac{\omega(t)}{t^2}\,dt = +\infty$$

is necessary for Problem 1. Firstly, we present a preliminary result. It states that the above condition is needed in order to prevent f from being a step function.

Lemma 2.1. *Let $\Omega = (-1,1) \times (0,1)^{N-1}$ and $\omega \in \mathcal{W}$. Let f be the characteristic function of the unit cube i.e. $f = \chi_{(0,1)^N}$. Then (2) holds if and only if $\int_1^\infty \frac{\omega(t)}{t^2}\,dt < +\infty$.*

PROOF. We denote $x = (x_1, x_2, \ldots, x_N) = (x_1, x') \in \mathbb{R}^N$ and

$$I = \int\limits_\Omega \int\limits_\Omega \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|^N}.$$

After a change of variable $t = x_1 - y_1$ we get $I = 2(I_1 + I_2)$ where

$$I_1 = \int\limits_{(0,1)^{N-1}} \int\limits_{(0,1)^{N-1}} dx'\,dy' \int_0^1 \omega\bigg(\frac{1}{\sqrt{|x'-y'|^2 + t^2}}\bigg) \frac{t}{(|x'-y'|^2+t^2)^{\frac{N}{2}}}\,dt$$

$$I_2 = \int\limits_{(0,1)^{N-1}} \int\limits_{(0,1)^{N-1}} dx'\,dy' \int_1^2 \omega\bigg(\frac{1}{\sqrt{|x'-y'|^2 + t^2}}\bigg) \frac{2-t}{(|x'-y'|^2+t^2)^{\frac{N}{2}}}\,dt.$$

We remark that $|I_2| \le ||\omega||_{L^\infty[0,1]}$ and

$$I_1 = 2^{N-1} \underbrace{\int_0^1 \ldots \int_0^1}_{\text{N times}} \omega\left(\frac{1}{|x|}\right) \frac{x_1 \prod\limits_{i=2}^N (1-x_i)}{|x|^N}\,dx.$$

If $N = 1$, then $I_1 = \int_0^1 \omega\left(\frac{1}{x}\right)\,dx = \int_1^\infty \frac{\omega(z)}{z^2}\,dz$. If $N \ge 2$, after the change of variable $z = \frac{1}{\sqrt{x_1^2 + |x'|^2}}$ for each x', we get $I_1 = 2^{N-1}(I_3 + I_4)$ where

$$I_3 = \int_{\frac{1}{\sqrt{N}}}^1 \omega(z) z^{N-3} \int\limits_{(0,1)^{N-1}} \prod_{i=2}^N (1-x_i) \cdot \chi_{\left(\frac{1}{\sqrt{|x'|^2+1}}, \frac{1}{|x'|}\right)}(z)\,dx'\,dz$$

$$I_4 = \int_1^\infty \omega(z) z^{N-3} \int\limits_{\substack{|x'| \le \frac{1}{z} \\ x' \in [0,1]^{N-1}}} \prod_{i=2}^N (1-x_i)\,dx'\,dz.$$

Note that $|I_3| \le ||\omega||_{L^\infty[0,1]}$. Therefore it is sufficient to show that $I_4 < +\infty$ if and only if $\int_1^\infty \frac{\omega(t)}{t^2} dt < +\infty$. For $0 < t < 1$, define

$$T_N(t) = \int\limits_{\substack{x \in [0,1]^N \\ |x| \le t}} \prod_{i=1}^N (1-x_i)\,dx.$$

Then

$$\int_{[0,\frac{t}{\sqrt{N}}]^N} \prod_{i=1}^{N}(1-x_i)\,dx \le T_N(t) \le \int_{[0,t]^N} \prod_{i=1}^{N}(1-x_i)\,dx;$$

so there is a constant $c_N = (\frac{1}{2\sqrt{N}})^N$ such that

$$c_N t^N \le T_N(t) \le t^N \text{ for all } t \in (0,1).$$

This yields $I_4 \approx \int_1^\infty \frac{\omega(z)}{z^2}\,dz$. □

Proof of Theorem 1.2. Assume the contrary i.e. $\int_1^{+\infty} \frac{\omega(t)}{t^2}\,dt < +\infty$. Since Ω is bounded, $\Omega \subset (-r,r)^N$ for some $r > 0$. For the simplicity, we suppose that $0 \in \Omega$. Take now the characteristic function $f = \chi_{(0,r)\times(-r,r)^{N-1}}$. By Lemma 2.1,

$$\int_{(-r,r)^N} \int_{(-r,r)^N} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|^N} < +\infty.$$

Therefore (2) holds which contradicts the hypothesis that f is not constant on Ω. □

3. Sufficient conditions for Problem 1

In this section, the proofs of Theorem 1.3 and Theorem 1.4 are presented. We call *mollifiers* in $\mathbb{R}^N$, any family $(\rho_\varepsilon)_{\varepsilon>0}$ of functions in $L^1_{loc}(0,\infty)$ satisfying the following properties

$$\begin{cases} \rho_\varepsilon \ge 0 \text{ a.e. in } (0,+\infty), \\ \displaystyle\int_0^\infty \rho_\varepsilon(t)\,t^{N-1}\,dt = 1 \quad \forall \varepsilon > 0, \\ \displaystyle\lim_{\varepsilon\to 0} \int_\delta^\infty \rho_\varepsilon(t)\,t^{N-1}\,dt = 0 \quad \forall \delta > 0. \end{cases}$$

Recall the following result of Brezis (for the proof see e.g. [6] Proposition 1 and Lemma 4):

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain, (ρ_ε) be mollifiers in $\mathbb{R}^N$, $f \in L^1_{loc}(\Omega)$ and $\omega \in \mathcal{W}$ be a convex function. If*

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \rho_\varepsilon(|x-y|)\,dx\,dy = 0$$

then f is constant (a.e. in Ω).

Proof of Theorem 1.3. Firstly, since $\omega \in \mathcal{W}$ we can construct a convex function $\tilde{\omega} \in \mathcal{W}$ such that $\tilde{\omega}(t) \le \omega(t), \forall t \in [0,1]$ and $\tilde{\omega}(t) = at + b, \forall t \ge 1$ for some $a, b > 0$. The hypothesis $\liminf_{t\to\infty} \frac{\omega(t)}{t} > 0$ implies the existence of a constant $c > 0$ such that $\omega(t) \ge c\,\tilde{\omega}(t), \forall t \ge 0$. Therefore

$$\int_\Omega \int_\Omega \tilde{\omega}\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|^N} < +\infty.$$

Consider the mollifiers in $\mathbb{R}^N$

$$\rho_\varepsilon(t) = \begin{cases} \frac{\varepsilon}{t^{N-\varepsilon}} & \text{if } 0 < t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}. \tag{3}$$

By the dominated convergence theorem,

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \tilde{\omega}\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \rho_\varepsilon(|x-y|)\,dx\,dy = 0.$$

If $f \in L^1_{loc}(\Omega)$, we conclude by Theorem 3.1. In the general case of a measurable function f, we consider

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \le n \\ n & \text{if } f(x) \ge n \\ -n & \text{if } f(x) \le -n \end{cases}.$$

So $f_n \in L^1_{loc}(\Omega)$, $f_n \to f$ a.e. in Ω and

$$|f_n(x) - f_n(y)| \le |f(x) - f(y)| \quad \forall x, y \in \Omega.$$

Since $\tilde{\omega}$ is increasing, we get for all $n \ge 1$,

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \tilde{\omega}\left(\frac{|f_n(x) - f_n(y)|}{|x-y|}\right) \rho_\varepsilon(|x-y|)\,dx\,dy = 0.$$

This yields $f_n \equiv c_n$ et $c_n \to f$ a.e. in Ω. Thus f is constant. $\square$

Proof of Theorem 1.4. Since ω is non-decreasing, using the same argument as in the proof of Theorem 1.3, it is sufficient to show that the conclusion holds for $f \in L^\infty_{loc}(\Omega)$. Firstly, assume that the function ϕ is non-increasing on $(0, +\infty)$. Take an arbitrary ball $\bar{B} \subset \Omega$. For simplicity, we suppose that $|f| \le \frac{1}{2}$ a.e. in B. By these assumptions we get

$$\int_B \int_B \frac{|f(x) - f(y)|}{|x-y|} \phi\left(\frac{1}{|x-y|}\right) \frac{dx\,dy}{|x-y|^N} < +\infty.$$

For each $\varepsilon > 0$, set

$$0 < c_\varepsilon := \int_0^1 \phi\left(\frac{1}{t}\right)\frac{\varepsilon}{t^{1-\varepsilon}}\,dt \le \phi(1).$$

Consider the functions

$$\rho_\varepsilon(t) = \begin{cases} \frac{1}{c_\varepsilon}\phi\left(\frac{1}{t}\right)\frac{\varepsilon}{t^{N-\varepsilon}} & \text{if } 0 < t < 1 \\ 0 & \text{if } t \ge 1 \end{cases} \qquad \forall \varepsilon > 0.$$

Using the hypothesis that $\int_0^1 \phi\left(\frac{1}{t}\right)\frac{dt}{t} = +\infty$, we see that (ρ_ε) are mollifiers in $\mathbb{R}^N$. We also notice that $\lim_{\varepsilon\to 0}\frac{\varepsilon}{c_\varepsilon} = 0$. By dominated convergence theorem we obtain

$$\lim_{\varepsilon\to 0}\int_B\int_B \frac{|f(x)-f(y)|}{|x-y|}\rho_\varepsilon(|x-y|)\,dx\,dy = 0.$$

Hence Theorem 3.1 implies f is constant (a.e. in B) and since Ω is connected, we conclude that f is constant (a.e. in Ω). We now consider the general case when $c := \sup_{0<s\le t}\frac{\phi(t)}{\phi(s)} < +\infty$. Set $\phi(0) = \frac{\phi(1)}{c}$ and define

$$\tilde{\phi} : [0,+\infty) \mapsto (0,+\infty),\ \tilde{\phi}(t) = \min_{s\in[0,t]}\phi(s) \quad \forall t \ge 0.$$

So $\tilde{\phi}$ is continuous and non-increasing on $[0,+\infty)$ and $\tilde{\phi}(t) \le \phi(t), \forall t > 0$. From here,

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\tilde{\phi}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{dx\,dy}{|x-y|^N} < +\infty.$$

We also have that $\phi(t) \le c^2\,\tilde{\phi}(t), \forall t \ge 1$ and thus $\int_0^1 \tilde{\phi}\left(\frac{1}{t}\right)\frac{dt}{t} = +\infty$. By the previous case, f is constant (a.e. in Ω). □

4. The case of $W^{1,1}_{loc}$ functions

In this section, we show that for $f \in W^{1,1}_{loc}(\Omega)$ (in particular for Lipschitz functions), the answer to Problem 2 is positive. We will present two different approaches for solving this case.

Proof of Theorem 1.5. Let $x_0 \in \Omega$. Take $r > 0$ such that $\tilde{B} = B(x_0, 2r) \subset \Omega$ and denote $B = B(x_0, r)$. Then $f \in W^{1,1}(B)$ i.e. $f \in L^1(B)$ and $\nabla f \in \left(L^1(B)\right)^N$. So it makes sense to speak of $f(x)$ and $\nabla f(x)$ for a.e. $x \in B$. Let $\sigma \in S^{N-1}$. By Fubini's theorem we find that for a.e. $x \in B$ there is a small $t_x > 0$ such that

$I_x = \{x + t\sigma \,|\, t \in (-t_x, t_x)\} \subset B$ and $f \in W^{1,1}(I_x)$ i.e., f is absolutely continuous on I_x. Therefore for every $\sigma \in S^{N-1}$,

$$\lim_{t\to 0} \frac{f(x+t\sigma) - f(x)}{t} = \nabla f(x) \cdot \sigma \quad \text{for a.e.} \quad x \in B. \tag{4}$$

Write

$$\int_{\tilde{B}} \int_{\tilde{B}} \omega\Big(\frac{|f(x) - f(y)|}{|x-y|}\Big) \frac{dx\, dy}{|x-y|^N} \geq \int_B dx \int_{S^{N-1}} d\sigma \int_0^r \omega\Big(\frac{|f(x+t\sigma) - f(x)|}{t}\Big) \frac{dt}{t}$$

and by (2) deduce that for a.e. $x \in B$ and for a.e. $\sigma \in S^{N-1}$,

$$\int_0^r \omega\left(\frac{|f(x+t\sigma) - f(x)|}{t}\right) \frac{dt}{t} < +\infty.$$

Using $\int_0^r \frac{dt}{t} = \infty$, we get

$$\liminf_{t\to 0} \omega\left(\frac{|f(x+t\sigma) - f(x)|}{t}\right) = 0.$$

ω being continuous, by (4) one can find N linear independent directions $(\sigma_i)_{1\leq i\leq N}$ such that $\omega\left(|\nabla f(x) \cdot \sigma_i|\right) = 0$ for a.e. $x \in B$ and for every $i \in \{1, ..., N\}$. This implies $\nabla f = 0$ a.e. in B. By the Poincaré-Wirtinger inequality, we have that

$$\left\| f - \frac{1}{|B|} \int_B f \right\|_{L^1(B)} \leq C \left\| \nabla f \right\|_{L^1(B)} = 0$$

i.e. f is constant (a.e. in B). Since x_0 was arbitrarly chosen and Ω is connected, we conclude that f is constant (a.e. in Ω). $\square$

Remark. One could prove this result using another method, as follows. Define $\tilde{\omega} : [0, +\infty) \mapsto [0, 1]$, $\tilde{\omega}(t) = \min(\omega(t), 1)$ for every $t \geq 0$. Take an arbitrary ball $\bar{B} \subset \Omega$. Then

$$\int_B \int_B \tilde{\omega}\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\, dy}{|x-y|^N} < +\infty.$$

Consider the mollifiers (3) in $\mathbb{R}^N$. By the dominated convergence theorem, we obtain

$$\lim_{\varepsilon\to 0} \int_B \int_B \tilde{\omega}\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \rho_\varepsilon\left(|x-y|\right)\, dx\, dy = 0.$$

On the other hand, one can show that for a bounded continuous function $\tilde{\omega}$ on $[0, +\infty)$ and $f \in W^{1,1}(B)$,

$$\lim_{\varepsilon\to 0} \int_B \int_B \tilde{\omega}\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \rho_\varepsilon\left(|x-y|\right)\, dx\, dy = \int_B \int_{S^{N-1}} \tilde{\omega}\left(|\nabla f(x) \cdot \sigma|\right)\, dx\, d\sigma$$

(see e.g. [6] Lemma 5). As before, this yields $\nabla f = 0$ a.e. in B for every ball $\bar{B} \subset \Omega$; since $f \in W^{1,1}_{loc}(\Omega)$ and Ω is connected, f is constant (a.e. in Ω).

5. Some generalized Cantor sets and Cantor functions

Let $0 < \beta < 1$. We recall the definition of some general Cantor sets, called here *β-Cantor sets*, all homeomorphic to the standard one and which can be obtained by deleting a sequence of pairwise disjoint open intervals from the interior of the segment $I_0^{(0)} = [0,1]$, as follows (see [5]). Firstly, remove the centered open interval from $I_0^{(0)}$ which has length $\beta = \beta \cdot \left| I_0^{(0)} \right|$ i.e., delete the interval $J_0^{(1)} = \left(\frac{1-\beta}{2}, \frac{1+\beta}{2} \right)$ and leave two segments $I_0^{(1)} = \left[0, \frac{1-\beta}{2} \right]$ and $I_1^{(1)} = \left[\frac{1+\beta}{2}, 1 \right]$. The second step consists in deleting the open subinterval of length $\beta \cdot \left| I_0^{(1)} \right| = \beta \cdot \left| I_1^{(1)} \right| = \beta \frac{1-\beta}{2}$ from the center of each of the segments $I_0^{(1)}$ and $I_1^{(1)}$, namely $J_0^{(2)} = \left(\frac{(1-\beta)^2}{4}, \frac{1-\beta^2}{4} \right)$ and $J_1^{(2)} = 1 - J_0^{(2)}$; thus, there remains 2^2 segments, denoted $I_0^{(2)}, I_1^{(2)}, I_2^{(2)}$ and $I_3^{(2)}$. We iterate this procedure; at the $(n+1)$ step, remove the centered open subinterval $J_k^{(n+1)}$ of length $\beta \cdot \left| I_k^{(n)} \right|$ from each segment $I_k^{(n)} = [a_k^{(n)}, b_k^{(n)}]$ and leave the two segments

$$I_{2k}^{(n+1)} = [a_{2k}^{(n+1)}, b_{2k}^{(n+1)}] \text{ and } I_{2k+1}^{(n+1)} = [a_{2k+1}^{(n+1)}, b_{2k+1}^{(n+1)}] \text{ for } k = 0, 1, \ldots, 2^n - 1.$$

The limit set is the *β-Cantor set*, denoted by C_β. It is a compact set, containing an uncountable infinity of points; it has Lebesgue measure zero and it is nowhere dense (i.e. it has no interior). We will give the specific form of C_β. In order to do that, let us consider σ_n and δ_n the length of the removed interval $J_k^{(n)}$ and respectively, of the remaining segment $I_k^{(n)}$ at the n step. A simple computation yields

$$\delta_n = \left(\frac{1-\beta}{2}\right)^n, \ \sigma_n = \beta\delta_{n-1} \quad \forall n \geq 1 \text{ (here } \delta_0 = 1).$$

Set $\varepsilon_n = \delta_n + \sigma_n$. Then one can deduce (see [5]) that

$$C_\beta = \left\{ \sum_{k=1}^{\infty} \alpha_k \varepsilon_k \mid \alpha_k \in \{0,1\}, k = 0, 1, \ldots \right\}.$$

In fact, the binary decomposition

$$j = \alpha_n + 2\alpha_{n-1} + \cdots + 2^{n-1}\alpha_1 = (\alpha_1 \ldots \alpha_n)_2$$

gives $a_j^{(n)} = \sum_{k=1}^{n} \alpha_k \varepsilon_k$ and $b_j^{(n)} = a_j^{(n)} + \sum_{k \geq n+1} \varepsilon_k$.

We define now the *β-Cantor function*, denoted here by f_β (see [3]). Set $f_\beta(0) = 0$ and $f_\beta(1) = 1$. So f_β is specified at the endpoints of $I_0^{(0)}$. Define $f_\beta(x) = \frac{1}{2}$ if $x \in cl J_0^{(1)}$. Thus $f_\beta(x)$ is the average of the values of f_β at the endpoints of $I_0^{(0)}$ when x belongs to the removed interval $J_0^{(1)}$ and f_β is specified at the endpoints of $I_0^{(1)}$ and $I_1^{(1)}$. At the $n+1$ step, define $f_\beta \equiv \frac{f_\beta(b_k^{(n)}) - f_\beta(a_k^{(n)})}{2}$ on the closure of each $J_k^{(n+1)}$, the removed interval from $I_k^{(n)} = [a_k^{(n)}, b_k^{(n)}]$. By that, f_β is defined in every endpoint of $I_{2k}^{(n+1)}$ and $I_{2k+1}^{(n+1)}$ for $k = 0, 1, \ldots, 2^n - 1$; then we can iterate the process.

Suppose f_β is not yet defined at x. At each n step, x is in the interior of exactly one of the 2^n retained segments, say $[a_n, b_n]$ of length δ_n. Moreover, $b_n = a_n + \delta_n$, $f_\beta(b_n) = f_\beta(a_n) + 2^{-n}$, $a_n \leq a_{n+1} < b_{n+1} \leq b_n$ and $f_\beta(a_n) \leq f_\beta(a_{n+1}) < f_\beta(b_{n+1}) \leq f_\beta(b_n)$; then $f_\beta(x)$ is defined by

$$\lim_{n\to\infty} f_\beta(a_n) = f_\beta(x) = \lim_{n\to\infty} f_\beta(b_n).$$

Furthermore, f_β is a continuous, nondecreasing map of $[0,1]$ onto $[0,1]$ (so f_β is a function of bounded variation on $[0,1]$) and $f'_\beta(x)=0$ for a.e. $x\in[0,1]$. One can easily check that on the β-Cantor set we have

$$f_\beta\left(\sum_{k=1}^{\infty}\alpha_k\varepsilon_k\right)=\sum_{k=1}^{\infty}\alpha_k 2^{-k}.$$

We now show that each β-Cantor function is Hölder continuous with Hölder exponent equal to the Hausdorff dimension of C_β i.e. $H_\beta=\frac{1}{1-\log_2(1-\beta)}$ (see also [4]).

Theorem 5.1. *The β-Cantor function is α-Hölder if and only if $0<\alpha\le H_\beta$.*

Proof. Since C_β is nowhere dense and f_β is continuous, it is sufficient to prove that for every $\alpha\le H_\beta$, there exists $l_\alpha>0$ such that

$$|f_\beta(x)-f_\beta(y)|\le l_\alpha|x-y|^\alpha\quad\forall x,y\in[0,1]\backslash C_\beta. \tag{5}$$

Take $x<y, x,y\in[0,1]\backslash C_\beta$ i.e. x and y are in the interior of two removed intervals in the construction of C_β, say (b,a) and $(\tilde b,\tilde a)$. Write $a=\sum_{k=1}^{n}\alpha_k\varepsilon_k, \alpha_k\in\{0,1\}$, $\alpha_n=1$ and $\tilde a=\sum_{j=1}^{m}\gamma_j\varepsilon_j, \gamma_j\in\{0,1\}, \gamma_m=1$. Then $b=a-\sigma_n$, $\tilde b=\tilde a-\sigma_m$. If the two removed intervals coincide, then $f_\beta(x)=f_\beta(y)$ and (5) is obvious. Otherwise, $a<\tilde b$. Take $s\ge 1$ such that $\alpha_j=\gamma_j$ for $j=1,\dots,s-1$ and $\alpha_s\ne\gamma_s$ (we may consider $\alpha_j=0,\forall j>n$). Thus $\gamma_s=1$, $\alpha_s=0$ and $s\le m$.

If $s<n$, we get

$$\begin{aligned}
f_\beta(y)-f_\beta(x) &= \sum_{j=1}^{m}\gamma_j 2^{-j}-\sum_{k=1}^{n}\alpha_k 2^{-k}\\
&= 2^{-n}+\sum_{j=s+1}^{m}\gamma_j 2^{-j}+\sum_{k=s+1}^{n}(1-\alpha_k)2^{-k},\\
y-x\ge\tilde b-a &= \sum_{j=1}^{m}\gamma_j\varepsilon_j-\sigma_m-\sum_{k=1}^{n}\alpha_k\varepsilon_k\\
&\ge \delta_n+\sum_{j=s+1}^{m}\gamma_j\delta_j+\sum_{k=s+1}^{n}(1-\alpha_k)\delta_k
\end{aligned}$$

(here we used $\varepsilon_s = \sigma_s + \delta_s = \sigma_s + \varepsilon_{s+1} + \cdots + \varepsilon_n + \delta_n$). Otherwise, $s > n$ (since $s \neq n$) and we have

$$f_\beta(y) - f_\beta(x) = \sum_{j=s}^{m} \gamma_j 2^{-j},$$
$$y - x \geq \tilde{b} - a = \sum_{j=s}^{m} \gamma_j \varepsilon_j - \sigma_m \geq \sum_{j=s}^{m} \gamma_j \delta_j.$$

So in both cases, we can write

$$f_\beta(y) - f_\beta(x) = \sum_{j=1}^{M} h_j 2^{-j} \text{ and } y - x \geq \sum_{j=1}^{M} h_j \delta_j$$

where $M \geq 1, h_j \in \{0,1,2\}, j = 1, \ldots, M$. We distinguish three cases:
Case 1: $0 < \alpha < H_\beta$. Set $\varepsilon = H_\beta - \alpha > 0$. By Hölder's inequality, we get

$$\sum_{j=1}^{M} h_j 2^{-j} = \sum_{j=1}^{M} h_j^\alpha \delta_j^\alpha h_j^{1-\alpha} \delta_j^\varepsilon \leq \left(\sum_{j=1}^{M} h_j \delta_j \right)^\alpha \left(\sum_{j=1}^{M} h_j \delta_j^{\frac{\varepsilon}{1-\alpha}} \right)^{1-\alpha}.$$

Since $h_j \in \{0,1,2\}$, we deduce

$$\sum_{j=1}^{M} h_j \delta_j^{\frac{\varepsilon}{1-\alpha}} \leq 2 \sum_{j\geq 1} \left(\delta_1^{\frac{\varepsilon}{1-\alpha}} \right)^j =: l_\alpha^{\frac{1}{1-\alpha}} < +\infty.$$

So $|f(x) - f(y)| \leq l_\alpha |x - y|^\alpha$.
Case 2: $\alpha = H_\beta$ i.e. $\delta_j^\alpha = 2^{-j}, \forall j \geq 0$. Take the smallest $j_0 \geq 1$ such that $h_{j_0} \neq 0$. Then

$$\frac{\sum\limits_{j=j_0}^{M} h_j \delta_j^\alpha}{\left(\sum\limits_{j=j_0}^{M} h_j \delta_j \right)^\alpha} \leq \frac{2 \sum\limits_{j \geq j_0} \delta_j^\alpha}{\delta_{j_0}^\alpha} = 2 \sum_{j \geq 0} 2^{-j} = 4.$$

Thus, (5) is satisfied.
Case 3: $\alpha > H_\beta$. Take $x = \varepsilon_n$ and $y = \delta_{n-1} = \sum\limits_{k \geq n} \varepsilon_k$. Then

$$\frac{f(y) - f(x)}{|y - x|^\alpha} = \frac{2^{-n}}{|\delta_{n-1} - \varepsilon_n|^\alpha} = \frac{2^{-n}}{\delta_n^\alpha} \to \infty \text{ if } n \to \infty.$$

So, in this case, f_β is not an α-Hölder continuous function. $\square$

6. Some counter-examples

In this section, we present some counter-examples for Problem 2 in the case of regularity $C^{0,\alpha}$. We will assume that Ω is the unit cube in $\mathbb{R}^N$ i.e. $\Omega = (0,1)^N$.

Theorem 6.1. *For every $\alpha \in (0,1)$, there is a nonconstant α-Hölder function $f : [0,1]^N \mapsto \mathbb{R}$ of bounded variation which satisfies* (2)*, for all $\omega \in \mathcal{W}$ with the property that $\omega(t) \le \frac{1}{t}, \forall t > 0$.*

Proof. : Let $\alpha \in (0,1)$. Consider the unique $\beta \in (0,1)$ such that $\alpha = H_\beta$.
Case 1: $N = 1$. Let f be the β-Cantor function. Take an arbitrary $\omega \in \mathcal{W}$ such that $\omega(t) \le \frac{1}{t}, \forall t > 0$. Denote by $\mathcal{J}$ the (countable) set of all removed intervals in the construction of the β-Cantor set i.e.

$$\mathcal{J} = \left\{ J_k^{(n+1)} : n \ge 0, k = 0, 1, \dots, 2^n - 1 \right\}.$$

We have

$$\begin{aligned} I &= \int_0^1 \int_0^1 \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\, dy}{|x-y|} \\ &= \sum_{J \in \mathcal{J}} \sum_{\tilde{J} \in \mathcal{J}} \int_J \int_{\tilde{J}} \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\, dy}{|x-y|} \\ &= 2 \sum_{\substack{J, \tilde{J} \in \mathcal{J} \\ J < \tilde{J}}} \int_J \int_{\tilde{J}} \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\, dy}{|x-y|} \end{aligned}$$

(we denote $J = (b,a) < \tilde{J} = (\tilde{b}, \tilde{a})$ if $a < \tilde{b}$). We want to prove that $I < +\infty$. Take two removed intervals $J = (b,a)$ and $\tilde{J} = (\tilde{b}, \tilde{a})$ such that $J < \tilde{J}$. Write $a = \sum_{k=1}^{n} \alpha_k \varepsilon_k$, $\alpha_k \in \{0,1\}$, $\alpha_n = 1$ and $\tilde{a} = \sum_{j=1}^{m} \gamma_j \varepsilon_j$, $\gamma_j \in \{0,1\}$, $\gamma_m = 1$; here $b = a - \sigma_n$, $\tilde{b} = \tilde{a} - \sigma_m$. Take $r = f|_{\tilde{J}} - f|_J = \sum_{j=1}^{m} \gamma_j 2^{-j} - \sum_{k=1}^{n} \alpha_k 2^{-k} > 0$. *We use these notations in the rest of the paper.* Since $\omega(t) \le \frac{1}{t}, \forall t > 0$ we get

$$\int_J \int_{\tilde{J}} \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \frac{dx\, dy}{|x-y|} \le \int_J \int_{\tilde{J}} \frac{dx\, dy}{r} = \frac{|J| \cdot |\tilde{J}|}{r} = \frac{\sigma_n \sigma_m}{r}.$$

The aim is to estimate

$$S = \sum_{\substack{J < \tilde{J} \\ J, \tilde{J} \in \mathcal{J}}} \frac{|J| \cdot |\tilde{J}|}{f|_{\tilde{J}} - f|_J}.$$

Firstly, consider the interval $J=(b,a)$ fix. Let $\tilde{J}=(\tilde{b},\tilde{a})$ be a variable removed interval (in the construction of C_β) such that $\tilde{J}>J$ (i.e. $a<\tilde{a}$). Each time, we consider the first s step (in the construction of C_β) when J and $\tilde{J}$ do not belong anymore to the same remaining interval; that means the biggest $1\le s\le n$ such that $\alpha_j=\gamma_j$ for $j=1,\dots,s-1$ (if $\alpha_1\ne\gamma_1$ then $s=1$). Notice that $s\le m$, $\gamma_s=1$ and $\alpha_s=\gamma_s \iff s=n$.

If $s<m$ i.e. $dist(J,\tilde{J})\ge\delta_m$ then

$$r=f|_{\tilde{J}}-f|_J=\sum_{j=1}^m\gamma_j2^{-j}-\sum_{k=1}^n\alpha_k2^{-k}\ge\sum_{j=s+1}^m\gamma_j2^{-j}.$$

If we sum up over these $\tilde{J}$, we get:

$$\begin{aligned}\sum_{\substack{\tilde{J}\in\mathcal{J},J<\tilde{J}\\ dist(J,\tilde{J})\ge\delta_m}}\frac{|\tilde{J}|}{f|_{\tilde{J}}-f|_J}&=\sum_{s=1}^n\sum_{m\ge s+1}\sum_{\substack{\gamma_j\in\{0,1\},\gamma_s=\gamma_m=1\\ s+1\le j\le m-1}}\frac{\sigma_m}{r}\\ &\le\sum_{s=1}^n\sum_{m\ge s+1}\sigma_m\sum_{\substack{\gamma_j\in\{0,1\},\gamma_m=1\\ s+1\le j\le m-1}}\frac{1}{\sum\limits_{j=s+1}^m\gamma_j2^{-j}}\\ &\le\sum_{s=1}^n\sum_{m\ge s+1}\sigma_m2^m\sum_{j=1}^{2^{m-s}-1}\frac{1}{j}\\ &\le\sum_{s=1}^n\sum_{m\ge s+1}\sigma_m2^m(m-s)\\ &\le nL\end{aligned}$$

where $L=\sum\limits_{m\ge1}\sigma_m2^mm=\frac{\beta}{\delta_1}\sum\limits_{m\ge1}(1-\beta)^mm<+\infty$.

Otherwise, $s=m$ i.e. $dist(J,\tilde{J})<\delta_m$. Thus $s<n$ and

$$r=f|_{\tilde{J}}-f|_J=2^{-s}-\sum_{k=s+1}^n\alpha_k2^{-k}=\sum_{k=s+1}^{n-1}(1-\alpha_k)2^{-k}+2^{-n}.$$

We get

$$\sum_{\substack{\tilde{J}\in\mathcal{J},J<\tilde{J}\\ dist(J,\tilde{J})<\delta_m}}\frac{|\tilde{J}|}{f|_{\tilde{J}}-f|_J}=\sum_{s=1}^{n-1}\frac{\sigma_s}{\sum\limits_{k=s+1}^{n-1}(1-\alpha_k)2^{-k}+2^{-n}}.$$

Finally, if we let J be variable in $\mathcal{J}$, we deduce

$$\begin{aligned}
S &\le \sum_{n\ge 1} \sum_{\substack{\alpha_k\in\{0,1\}\\ 1\le k\le n-1}} \sigma_n \left(nL + \sum_{s=1}^{n-1} \frac{\sigma_s}{\sum\limits_{k=s+1}^{n-1}(1-\alpha_k)2^{-k} + 2^{-n}} \right) \\
&= \sum_{n\ge 1} n\sigma_n 2^{n-1} L + \sum_{n\ge 1} \sigma_n 2^n \sum_{s=1}^{n-1} \sigma_s \sum_{\substack{\tilde\alpha_k\in\{0,1\}\\ 1\le k\le n-1}} \frac{1}{1+\sum\limits_{k=1}^{n-s-1} \tilde\alpha_k 2^k} \\
&\le L^2 + \sum_{n\ge 1} \sigma_n \cdot 2^n \sum_{s=1}^{n-1} \sigma_s 2^s (n-s) \\
&\le 2L^2.
\end{aligned}$$

Case 2: $N \ge 2$. We denote $x = (x_1, x') = (x_1, x_2, \dots, x_N) \in [0,1]^N$. Take $f(x) = f_\beta(x_1), \forall x \in [0,1]^N$. So $f \in C^{0,\alpha} \cap BV(\Omega)$. Choose any $\omega \in \mathcal{W}$ with the property that $\omega(t) \le \frac{1}{t}$ for all $t > 0$. Firstly, remark that

$$\begin{aligned}
I &= \int\limits_{(0,1)^N} \int\limits_{(0,1)^N} \omega\left(\frac{|f(x)-f(y)|}{|x-y|} \right) \frac{dx\,dy}{|x-y|^N} \\
&= 2^{N-1} \int_0^1 \int_0^1 \int\limits_{(0,1)^{N-1}} \omega\left(\frac{|f_\beta(x_1)-f_\beta(y_1)|}{\sqrt{|x'|^2+(x_1-y_1)^2}} \right) \frac{\prod\limits_{i=2}^{N}(1-x_i)\,dx_1\,dy_1\,dx'}{(|x'|^2+(x_1-y_1)^2)^{\frac{N}{2}}} \\
&\le 2^N \sum_{\substack{J,\tilde J\in\mathcal{J}\\ J<\tilde J}} \int\limits_J \int\limits_{\tilde J} \int\limits_{(0,1)^{N-1}} \omega\left(\frac{|f_\beta(x_1)-f_\beta(y_1)|}{\sqrt{|x'|^2+(x_1-y_1)^2}} \right) \frac{dx_1\,dy_1\,dx'}{(|x'|^2+(x_1-y_1)^2)^{\frac{N}{2}}} \\
&\le 2^N |S^{N-2}| \sum_{\substack{J,\tilde J\in\mathcal{J}\\ J<\tilde J}} \frac{1}{f_\beta|_{\tilde J} - f_\beta|_J} \int\limits_J \int\limits_{\tilde J} dx_1\,dy_1 \int_0^{N-1} \frac{t^{N-2}}{(t^2+(x_1-y_1)^2)^{\frac{N-1}{2}}}\,dt.
\end{aligned}$$

On the other hand, we have

$$\int_0^{N-1} \frac{t^{N-2}\,dt}{(t^2+(x_1-y_1)^2)^{\frac{N-1}{2}}} \le 2\int_0^{N-1} \frac{dt}{y_1-x_1+t} \le 2\left(\ln N + \ln\frac{1}{y_1-x_1} \right)$$

for every $0 \le x_1 < y_1 \le 1$. Therefore there is a constant $c = c(N) > 0$ such that

$$I \le c(N)\Big(\sum_{\substack{J,\tilde{J}\in\mathcal{J} \\ J<\tilde{J}}} \frac{|J|\cdot|\tilde{J}|}{f_\beta|_{\tilde{J}} - f_\beta|_J} + \sum_{\substack{J,\tilde{J}\in\mathcal{J} \\ J<\tilde{J}}} \frac{|J|\cdot|\tilde{J}|}{f_\beta|_{\tilde{J}} - f_\beta|_J} \ln \frac{1}{dist(J,\tilde{J})} \Big).$$

We have already proved that the first sum converges; it remains to show that the second one is convergent, too. As before, fix $J = (b, a)$ and let $\tilde{J} = (\tilde{b}, \tilde{a})$ be such that $J < \tilde{J}$; write $a = \sum_{k=1}^{n} \alpha_k \varepsilon_k$, $b = a - \sigma_n$ and $\tilde{a} = \sum_{j=1}^{m} \gamma_j \varepsilon_j$, $\tilde{b} = \tilde{a} - \sigma_m$. Set $r = f_\beta|_{\tilde{J}} - f_\beta|_J$. We have that $dist(J,\tilde{J}) = \tilde{b} - a$. Using the same argument as in the case $N = 1$, we get

$$\begin{aligned}
\sum_{\substack{\tilde{J}\in\mathcal{J}, J<\tilde{J} \\ dist(J,\tilde{J})\ge\delta_m}} \frac{|\tilde{J}|}{f_\beta|_{\tilde{J}} - f_\beta|_J} \ln \frac{1}{dist(J,\tilde{J})} &\le \sum_{s=1}^{n} \sum_{m\ge s+1} \sum_{\substack{\gamma_j\in\{0,1\},\gamma_m=1 \\ s+1\le j\le m-1}} \frac{\sigma_m}{r} \ln \frac{1}{\delta_m} \\
&\le \sum_{s=1}^{n} \sum_{m\ge s+1} m\sigma_m 2^m \sum_{j=1}^{2^{m-s}-1} \frac{1}{j} \ln \frac{1}{\delta_1} \\
&\le n\tilde{L} \ln \frac{1}{\delta_1}
\end{aligned}$$

where $\tilde{L} = \sum_{m\ge1} \sigma_m 2^m m^2 < +\infty$. Since $dist(J,\tilde{J}) \ge min(\delta_n, \delta_m)$, it results

$$\sum_{\substack{\tilde{J}\in\mathcal{J}, J<\tilde{J} \\ dist(J,\tilde{J})<\delta_m}} \frac{|\tilde{J}|}{f_\beta|_{\tilde{J}} - f_\beta|_J} \ln \frac{1}{dist(J,\tilde{J})} \le \sum_{s=1}^{n-1} \frac{\sigma_s}{\sum_{k=s+1}^{n-1} (1-\alpha_k)2^{-k} + 2^{-n}} \ln \frac{1}{\delta_n}.$$

Similarly, allowing J to be variable in $\mathcal{J}$ we conclude that:

$$\sum_{\substack{J,\tilde{J}\in\mathcal{J} \\ J<\tilde{J}}} \frac{|J|\cdot|\tilde{J}|}{f_\beta|_{\tilde{J}} - f_\beta|_J} \ln \frac{1}{dist(J,\tilde{J})} \le 2L\,\tilde{L} \ln \frac{1}{\delta_1}.$$

□

We now prove Theorem 1.7:

Proof of Theorem 1.7. Let $\alpha \in (0, 1)$. Take $\beta \in (0, 1)$ such that $\alpha = H_\beta$.

Case 1: $N = 1$. Let f be the β-Cantor function. Choose an arbitrary $\theta \in (0,1)$ and set $\omega(t) = t^\theta, \forall t \geq 0$. Like in the previous proof, we want to show that

$$\sum_{\substack{J,\tilde{J}\in\mathcal{J} \\ J<\tilde{J}}} \int_J \int_{\tilde{J}} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{dx\,dy}{|x-y|} < +\infty.$$

As before, consider the interval $J = (b,a)$ fix. Let $\tilde{J} = (\tilde{b},\tilde{a})$ be a variable removed interval such that $a < \tilde{a}$. Each time, we consider the first s step (in the construction of C_β) when J and $\tilde{J}$ do not belong anymore to the same remaining interval. Let us denote $p = \frac{1}{\delta_1} > 2$ and we use the same notations $r = f|_{\tilde{J}} - f|_J$, $b = a - \sigma_n$, $\tilde{b} = \tilde{a} - \sigma_m$, $a = \sum\limits_{k=1}^{n} \alpha_k \varepsilon_k$, $\alpha_k \in \{0,1\}$, $\alpha_n = 1$ and $\tilde{a} = \sum\limits_{j=1}^{m} \gamma_j \varepsilon_j$, $\gamma_j \in \{0,1\}$, $\gamma_m = 1$.

If $dist(J,\tilde{J}) \geq \delta_m$ i.e. $s < m$, we distinguish two cases:
i) $dist(J,\tilde{J}) \geq \delta_n$ i.e. $s < n$. Here we have $\tilde{b} - a \geq \sigma_s$ and $r \leq 2^{-s+1}$. We write:

$$\begin{aligned} E(J,\tilde{J}) &= \int_J \int_{\tilde{J}} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{dx\,dy}{|x-y|} \\ &= \int_0^1 \int_0^1 \frac{\omega(r)\sigma_n\sigma_m\,dt\,dz}{(\tilde{b}-a+t\sigma_n+z\sigma_m)^{1+\theta}} \leq \frac{r^\theta \sigma_n \sigma_m}{(\tilde{b}-a)^{1+\theta}}. \end{aligned}$$

If we sum up over these $\tilde{J}$, we get:

$$\begin{aligned} \sum_{\substack{\tilde{J}\in\mathcal{J},J<\tilde{J} \\ dist(J,\tilde{J})\geq max\{\delta_m,\delta_n\}}} E(J,\tilde{J}) &\leq \sigma_n \sum_{s=1}^{n-1} \sum_{m\geq s+1} \sum_{\substack{\gamma_j\in\{0,1\} \\ s+1\leq j\leq m-1}} \frac{\sigma_m}{\sigma_s}\frac{1}{(2^{s-1}\sigma_s)^\theta} \\ &\leq \sigma_n \sum_{s=1}^{n-1} \frac{1}{(2^{s-1}\sigma_s)^\theta} \sum_{m\geq s+1}\left(\frac{2}{p}\right)^{m-s} \\ &\leq c\sigma_n \sum_{s=0}^{n-2}\left(\frac{p}{2}\right)^{s\theta} L_1 \\ &\leq c\sigma_n L_1 \left(\frac{p}{2}\right)^{\theta(n-1)} \end{aligned}$$

where for $q > 0$ we denote $L_q = \sum\limits_{m\geq 0}\left(\frac{2}{p}\right)^{mq} < +\infty$ and $c = c(\beta,\theta)$ is a constant that depends only on β and θ.

ii) $dist(J,\tilde J)<\delta_n$ i.e. $s=n$. In this case,

$$E(J,\tilde J)\le \int_0^1 \frac{r^\theta \sigma_n\sigma_m\,dt}{(\tilde b-a+t\sigma_n)^{1+\theta}}.$$

We have $\tilde b-a=\sum\limits_{j=n+1}^m \gamma_j\varepsilon_j-\sigma_m\ge \sum\limits_{j=n+1}^m \gamma_j\delta_j$ and $r=\sum\limits_{j=n+1}^m \gamma_j 2^{-j}$. From here, we obtain

$$\sum_{\substack{\tilde J\in\mathcal J,\tilde J>J\\ \delta_m\le dist(J,\tilde J)<\delta_n}} E(J,\tilde J)\le c\,L_\theta\,L_{1-\theta}\,\sigma_n\left(\frac{p}{2}\right)^{n\theta}$$

where $c=c(\beta,\theta)$ is a constant that depends only on β and θ. If we let J be variable in $\mathcal J$, we deduce

$$\begin{aligned}\sum_{\substack{J,\tilde J\in\mathcal J,J<\tilde J\\ dist(J,\tilde J)\ge\delta_m}} E(J,\tilde J)&\le c(\beta,\theta)\sum_{n\ge1}\sum_{\substack{\alpha_k\in\{0,1\}\\ 1\le k\le n-1}}\sigma_n\left(\frac{p}{2}\right)^{n\theta}\\ &\le c(\beta,\theta)\sum_{n\ge1}\left(\frac{2}{p}\right)^{n(1-\theta)}\\ &<+\infty.\end{aligned}$$

Otherwise, $dist(J,\tilde J)<\delta_m$ i.e. $s=m$. Thus $m<n$,

$$\tilde b-a=\delta_m-\sum_{k=m+1}^n \alpha_k\varepsilon_k\ge\sum_{k=m+1}^{n-1}(1-\alpha_k)\delta_k+\delta_n$$

$$r=\sum_{k=m+1}^{n-1}(1-\alpha_k)2^{-k}+2^{-n}\ \text{and}\ E(J,\tilde J)\le\int_0^1\frac{r^\theta\sigma_n\sigma_m\,dz}{(\tilde b-a+z\sigma_m)^{1+\theta}}.$$

We get

$$\sum_{\substack{\tilde J\in\mathcal J,J<\tilde J\\ dist(J,\tilde J)<\delta_m}} E(J,\tilde J)\le\sigma_n\sum_{m=1}^{n-1}\int_0^1\frac{\sigma_m\Big(\sum\limits_{k=m+1}^{n-1}(1-\alpha_k)2^{-k}+2^{-n}\Big)^\theta dz}{\Big(\sum\limits_{k=m+1}^{n-1}(1-\alpha_k)\delta_k+\delta_n+z\sigma_m\Big)^{1+\theta}}.$$

Finally, if we let J be variable in $\mathcal J$, we find

$$\sum_{\substack{J,\tilde J\in\mathcal J,J<\tilde J\\ dist(J,\tilde J)<\delta_m}} E(J,\tilde J)\le c(\beta,\theta)L_\theta M_{1-\theta}$$

where $M_{1-\theta} = \sum_{n\geq 1} n \left(\frac{2}{p}\right)^{n(1-\theta)} < +\infty$.

Case 2: $N \geq 2$. Let $f(x) = f_\beta(x_1), \forall x \in [0,1]^N$. As before, take $\theta \in (0,1)$ and set $\omega(t) = t^\theta, \forall t \geq 0$. Write

$$\begin{aligned}
I &= \int_{(0,1)^N} \int_{(0,1)^N} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \frac{dx\,dy}{|x-y|^N} \\
&\leq 2^N \sum_{\substack{J,\tilde{J}\in\mathcal{J} \\ J<\tilde{J}}} \int_J \int_{\tilde{J}} \int_{(0,1)^{N-1}} \omega\left(\frac{|f_\beta(x_1)-f_\beta(y_1)|}{\sqrt{|x'|^2+(x_1-y_1)^2}}\right) \frac{dx_1\,dy_1\,dx'}{(|x'|^2+(x_1-y_1)^2)^{\frac{N}{2}}} \\
&\leq 2^N |S^{N-2}| \sum_{\substack{J,\tilde{J}\in\mathcal{J} \\ J<\tilde{J}}} \int_J \int_{\tilde{J}} \omega(r)\,dx_1\,dy_1 \int_0^{N-1} \frac{t^{N-2}}{(t^2+(x_1-y_1)^2)^{\frac{N+\theta}{2}}}\,dt
\end{aligned}$$

(here we denote $r = f_\beta|_{\tilde{J}} - f_\beta|_J$). On the other hand, we have

$$\int_0^{N-1} \frac{t^{N-2}\,dt}{(t^2+(x_1-y_1)^2)^{\frac{N+\theta}{2}}} \leq 4\int_0^{N-1} \frac{dt}{(y_1-x_1+t)^{2+\theta}} \leq \frac{4}{(y_1-x_1)^{1+\theta}}$$

for every $0 \leq x_1 < y_1 \leq 1$. Therefore there is a constant $c = c(N) > 0$ such that

$$I \leq c(N) \sum_{\substack{J,\tilde{J}\in\mathcal{J} \\ J<\tilde{J}}} \int_J \int_{\tilde{J}} \omega\left(\frac{|f_\beta(x_1)-f_\beta(y_1)|}{|x_1-y_1|}\right) \frac{dx_1 dy_1}{|x_1-y_1|}.$$

By Case 1, the conclusion follows. □

Theorem 1.6 is a consequence of the previous two "counter-examples"; indeed, for some $0 < \theta < 1$ a bounded function ω satisfies $\omega(t) \leq ||\omega||_{L^\infty} \cdot \left(\frac{1}{t} + t^\theta\right)$ for every $t > 0$.

References

[1] R.A. Adams, *Sobolev spaces*, Academic Press, New York, 1975 (Pure and Applied Mathematics, 65)

[2] H. Brezis, *How to recognize constant functions. A connection with Sobolev spaces* (Russian), Uspekhi Mat. Nauk **57**(2002), 59-74. English version to appear in Russian Math. Surveys, volume in honor of M. Vishik

[3] R.B. Darst, *Some Cantor sets and Cantor functions*, Math. Mag. **45** (1972), 2-7

[4] M. Giaquinta, G. Modica and J. Soucek, *Cartesian Currents in the Calculus of Variations*, vol. I, Springer, 1998

[5] J.H. Halton, *Random sequences in generalized Cantor sets*, J. Sci. Comput.**6** (1991), no.4, 415-423

[6] A. Ponce, *A new approach to Sobolev spaces and connections to* Γ*-convergence*, Calc. Var. Partial Differential Equations, to appear

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