A compactness result in thin-film micromagnetics and the optimality of the Néel wall

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Abstract

In this paper, we study a model for the magnetization in thin ferromagnetic films. It comes as a variational problem for S^1-valued maps m' (the magnetization) of two variables x':

$$E_\varepsilon(m') = \varepsilon \int |\nabla' \cdot m'|^2 \, dx' + \frac{1}{2} \int \left| |\nabla'|^{-1/2} \nabla' \cdot m' \right|^2 dx'.$$

We are interested in the behavior of minimizers as $\varepsilon \to 0$. They are expected to be S^1-valued maps m' of vanishing distributional divergence $\nabla' \cdot m' = 0$, so that appropriate boundary conditions enforce line discontinuities. For finite $\varepsilon > 0$, these line discontinuities are approximated by smooth transition layers, the so-called Néel walls. Néel walls have a line energy density of the order $\frac{1}{|\ln \varepsilon|}$. One of the main results is to show that the boundedness of $\{|\ln \varepsilon| \, E_\varepsilon(m'_\varepsilon)\}$ implies the compactness of $\{m'_\varepsilon\}_{\varepsilon \downarrow 0}$, so that indeed limits m' will be S^1-valued and weakly divergence-free. Moreover, we show the optimality of the 1-d Néel wall under 2-d perturbations as $\varepsilon \downarrow 0$.

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1 Introduction

In this paper we analyze a 2-d approximation of the micromagnetic energy of a thin-film in the absence of external field and crystalline anisotropy. Following [5, 7], the setting is determined by our goal to prove the optimality of Néel walls under 2-d variations. Let $\Omega' = (-1, 1) \times \mathbb{R}$ be a 2-d sheet (the cross section of the thin ferromagnetic sample) (see Figure 1). The admissible magnetizations are *smooth* 2-d *unit-length* vector fields

$$m' = (m_1, m_2) : \mathbb{R}^2 \to S^1$$

that macroscopically connect two magnetizations which form an angle (see Figure 2), i.e.,

$$m'(x') = \begin{pmatrix} m_{1,\infty} \\ \pm\sqrt{1 - m_{1,\infty}{}^2} \end{pmatrix} \text{ for } \pm x_1 \geq 1, \, x_2 \in \mathbb{R}, \tag{1}$$

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where $m_{1,\infty} \in [0,1)$ is some fixed number and we use the shorthand notation $x' = (x_1, x_2)$. Here and in the sequel, the prime always indicates an in-plane quantity. Next to the magnetization $m' : \mathbb{R}^2 \to S^1$, the stray field $h = (h_1, h_2, h_3) : \mathbb{R}^3 \to \mathbb{R}^3$ is of importance. It is related to the magnetization via the following variational formulation:

$$\int_{\mathbb{R}^3} h \cdot \nabla\zeta \, dx = \int_{\mathbb{R}^2} \zeta \, \nabla' \cdot m' \, dx', \quad \forall \zeta \in C_c^\infty(\mathbb{R}^3), \tag{2}$$

where we write $x = (x', x_3) \in \mathbb{R}^3$ and $\nabla' \cdot m'$ for the in-plane divergence of m'. Classically, this is,

$$\begin{cases} \nabla \cdot h = 0 & \text{in } \ \mathbb{R}^3 \setminus (\mathbb{R}^2 \times \{0\}), \\ [h_3] = -\nabla' \cdot m' & \text{on } \ \mathbb{R}^2 \times \{0\}, \end{cases} \tag{3}$$

where $[h_3]$ denotes the jump of the out-of-plane component of h across the plane $\mathbb{R}^2 \times \{0\}$.

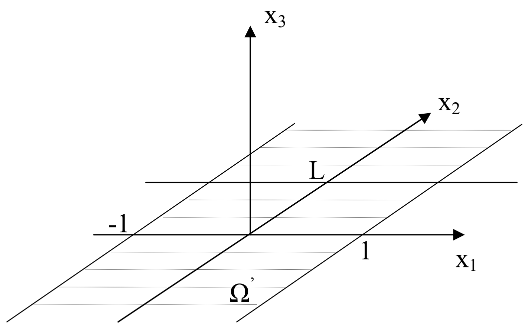

Figure 1: The infinite domain Ω'

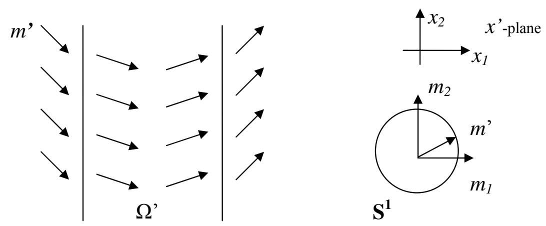

Figure 2: The admissible magnetization m'

The micromagnetic model states that the experimentally observed ground state for the magnetization m' and for the stray field is (local) minimizer of the micromagnetic energy. In order to assign the energy density for this configuration we assume that

$$m' \text{ and } h \text{ are } L\text{-periodic in the infinite } x_2\text{-direction}, \tag{4}$$

where L is an arbitrary positive number. In this paper we focus on the following non-dimensionalized energy functional:

$$E_\varepsilon(m', h) = \varepsilon \int_{\mathbb{R}\times[0,L)} |\nabla' \cdot m'|^2 \, dx' + \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} |h|^2 \, dx \tag{5}$$

where $\varepsilon > 0$ is a small parameter. In fact, ε is a non-dimensional quantity formed from three length scales: a material length scale, the film thickness and the film width (see [5, 7]). The first term in (5) comes from the exchange energy (in fact, it is smaller than the usual exchange energy term represented by the Dirichlet integral of m') and the energy of the stray field is also called the magnetostatic energy. Notice that the stray field h can be minimized out. Given m', it is characterized by (2) respectively (3) and $\nabla \times h = 0$. We so recover the static part of the Maxwell equations. In particular, $h = -\nabla u$, where the potential u is characterized by

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^3 \setminus (\mathbb{R}^2 \times \{0\}), \\ \left[\frac{\partial u}{\partial x_3}\right] = \nabla' \cdot m' & \text{on } \mathbb{R}^2 \times \{0\}. \end{cases} \tag{6}$$

Rewriting this in Fourier space, we see that the stray field energy is given by the homogeneous $H^{-1/2}$-norm of $\nabla' \cdot m'$:

$$\min_{h \text{ with } (2)} \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} |h|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}\times[0,L)} \left| |\nabla'|^{-1/2} \nabla' \cdot m' \right|^2 dx'.$$

In view of (6) and the electrostatic analogy, one thinks of $\nabla' \cdot m'$ as a "magnetic charge density". Hence, the energy forces magnetizations with small charge density $\nabla' \cdot m'$, a principle which is called "pole avoidance". Now we shall informally explain how the principle of pole avoidance leads to the formation of walls (i.e., transition layers). For this discussion, we neglect the first term in (5). For simplicity, we assume that the mesoscopic transition angle imposed by (1) on the boundary $\partial\Omega'$ is $180°$, i.e., $m' \cdot \nu' = 0$ on $\partial\Omega'$. The boundary effects in the tangential direction are excluded by our choice of Ω' which is infinite in x_2-direction. The magnetostatic energy will try to enforce the divergence-free condition for m', i.e., $\nabla' \cdot m' = 0$ in Ω'. Therefore, we arrive at

$$|m'| = 1 \text{ and } \nabla' \cdot m' = 0 \text{ in } \Omega'. \tag{7}$$

We notice that the conditions in (7) are too rigid for smooth magnetization m'. This can be seen

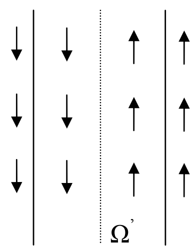

Figure 3: Landau state in Ω'

by writing $m' = \nabla'^{\perp}\psi$ with the help of a "stream function" ψ. Then (7) implies that ψ is a solution of the Dirichlet problem for the eikonal equation:

$$|\nabla'\psi| = 1 \text{ in } \Omega'. \tag{8}$$

Using the method of characteristics, it follows that there is no smooth solution of the equation (8) such that m' satisfies the boundary conditions (1). On the other hand, there are many continuous solutions that satisfy (8) away from a set of vanishing Lebesgue measure. One of them is the "viscosity solution" given by the distance function

$$\psi(x') = \mathrm{dist}\,(x', \partial\Omega')$$

that corresponds to the so-called Landau state for the magnetization m' (see Figure 3). Hence, the divergence-free equation in (7) has to be interpreted in the distribution sense and the boundary conditions (1) are expected to induce line-singularities for solutions m'. These ridges ("ridges" from the point of view of ψ) are an idealization of walls in thin-film elements at the mesoscopic level. At the microscopic level, they are replaced by smooth transition layers where the magnetization varies very quickly on a small length scale, which we will address below. A final remark is that the normal component of m' does not jump across these discontinuity lines (because of (7)) and therefore, the normal of the mesoscopic wall is determined by the angle between the mesoscopic levels in the adjacent domains.

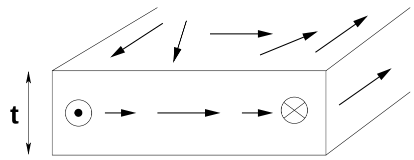

Figure 4: Néel wall in a 3-d cylinder

We now have a closer look at the transition layer itself, which is called Néel wall in the micromagnetics jargon (see Figure 4). As usual, one first consider 1-d transition layers, i.e.,

$$m' = (m_1(x_1), m_2(x_1)). \tag{9}$$

Notice that the continuous transition layers are necessarily not charge-free

$$\nabla' \cdot m' = \frac{dm_1}{dx_1} \neq 0.$$

Hence there is a competition between the first and the second term in (5). The prototype is the 180° Néel wall which corresponds to the boundary conditions (1) for $m_{1,\infty} = 0$, i.e.,

$$m'(x_1) = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \text{ for } \pm x_1 \geq 1. \tag{10}$$

Let us now discuss about the scaling of the energy of the prototypical Néel wall. For magnetizations (9), the specific energy (5) reduces to

$$E_\varepsilon^{1d}(m') = \varepsilon \int_{\mathbb{R}} \left|\frac{dm_1}{dx_1}\right|^2 dx_1 + \frac{1}{2}\int_{\mathbb{R}} \left|\left|\frac{d}{dx_1}\right|^{1/2} m_1\right|^2 dx_1. \tag{11}$$

We define the Néel wall as the 1-d minimizer of (11) under the boundary constraint (10). The Néel wall is a two length scale object: a small core ($|x_1| \lesssim w_{core}$) with fast varying rotation and a logarithmically decaying tail ($w_{core} \lesssim |x_1| \lesssim 1$). The finiteness of Ω' in x_1-direction in our setting

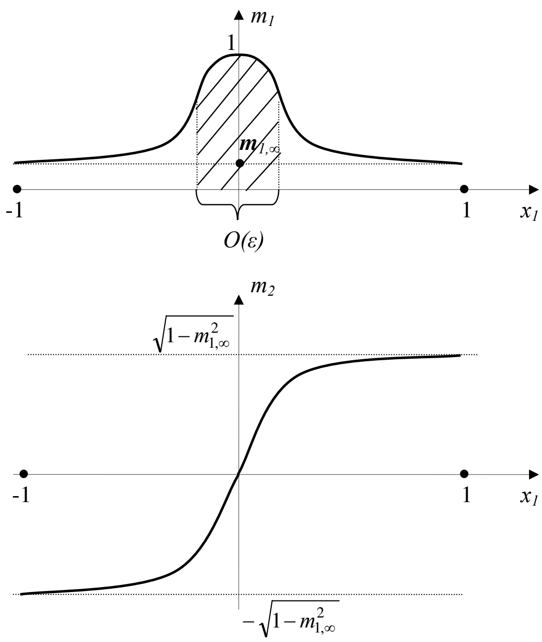

Figure 5: Qualitative behavior of the Néel wall

serves as the confining mechanism for the Néel wall tail. This two-scale structure permits the Néel wall to decrease the specific energy by a logarithmic factor. The prediction of the logarithmic decay was formally proved by Riedel and Seeger [13]; a detailed mathematical discussion of their results was carried out by Garcia-Cervera [8]. Finally, Melcher rigorously established in [11, 12] the exact logarithmic scaling for the 180° Néel wall tails: The minimizer m_1 with $m_1(0) = 1$ is symmetric around 0 ($w_{core} \sim \varepsilon$) and satisfies

$$m_1(x_1) \sim \frac{\ln \frac{1}{|x_1|}}{|\ln \varepsilon|} \quad \text{for } \varepsilon \lesssim |x_1| \lesssim 1$$

(see Figure 5). Moreover, the leading order term of the minimal energy level is

$$\min_{(9),(10)} E_\varepsilon^{1d}(m') \approx \frac{\pi}{2|\ln \varepsilon|} \quad \text{for } \varepsilon \ll 1.$$

The stability of 180° Néel walls under arbitrary 2-d modulation was proved by DeSimone, Knüpfer and Otto in [5]:

$$\min_{\substack{m',h \\ m'=m'(x_1,x_2) \text{ with } (10)}} E_\varepsilon(m',h) \approx \min_{\substack{m',h \\ m'=m'(x_1) \text{ with } (10)}} E_\varepsilon(m',h) \approx \frac{\pi L}{2|\ln \varepsilon|} \quad \text{for } \varepsilon \ll 1.$$

This means that asymptotically, the minimal energy E_ε is assumed by a straight wall. More precisely, the variations of the optimal 1-d transition layer in x_2-direction will not decrease the leading order term in the energy.

Our first result is a qualitative property of the optimal 1-d transition layers: We prove that asymptotically, the minimal energy can be assumed *only* by the straight walls. This property holds for general boundary conditions (1). It is based on a compactness result for magnetizations $\{m'_\varepsilon\}_{\varepsilon\downarrow 0}$ with energies E_ε close to the minimal energy level: Any accumulation limit m' has the singularities concentrated on a vertical line (see Figure 6).

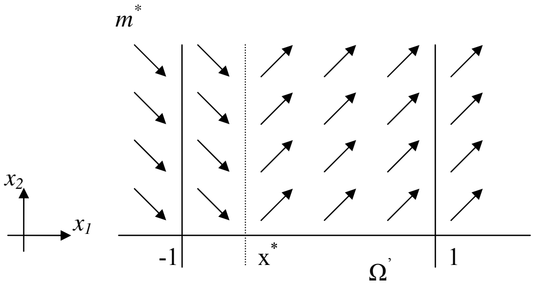

Figure 6: Straight wall

Theorem 1 *Let $m_{1,\infty} \in [0,1)$ and $L > 0$ be given. For any $\delta > 0$ there exists $\varepsilon_0 > 0$ with the following property: Given $m' : \mathbb{R}^2 \to S^1$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that*

$$\begin{aligned} & m' \text{ and } h \text{ are } L\text{-periodic in } x_2, \text{ i.e., (4) holds,} \\ & m' \text{ satisfies the boundary conditions (1),} \\ & m' \text{ and } h' \text{ are related by (2),} \\ & |\ln\varepsilon| E_\varepsilon(m',h) \le L\frac{\pi}{2}(1-m_{1,\infty})^2 + \varepsilon_0, \quad \text{for some } 0 < \varepsilon \le \varepsilon_0, \end{aligned} \tag{12}$$

then we have

$$\int_{\mathbb{R}\times[0,L)} |m' - m^*| \, dx' \le \delta, \tag{13}$$

where m^* is a straight wall given by

$$m^*(x_1,x_2) = \begin{pmatrix} m_{1,\infty} \\ \pm\sqrt{1-{m_{1,\infty}}^2} \end{pmatrix} \text{ for } \pm x_1 > \pm x_1^*, \tag{14}$$

for some $x_1^* \in [-1,1]$.

Remark 1 The estimate (13) also holds in L^p for any $\delta_p > 0$ and $1 \le p < \infty$.

Let us first discuss the compactness result for the case of 1-d magnetizations. We are interested in the asymptotics as $\varepsilon \to 0$ of families of 1-d magnetizations in the more general context of an energy regime $O(\frac{1}{|\ln\varepsilon|})$. We show that such a sequence of magnetizations is relatively compact in L^1_{loc} and the accumulation points in L^1_{loc} concentrate on a finite number of walls (see Figure 7). As a direct consequence, we obtain the optimality of the straight walls over 1-d perturbations in the asymptotic regime of the minimal energy.

Theorem 2 *Let $m_{1,\infty} \in [0,1)$. Consider a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \downarrow 0$. For $k \in \mathbb{N}$, let $m'_k = (m_{1,k}, m_{2,k}) : \mathbb{R} \to S^1$ such that (1) holds and*

$$\limsup_{k\to\infty} |\ln \varepsilon_k| \left(\varepsilon_k \int_{\mathbb{R}} \left|\frac{dm_{1,k}}{dx_1}\right|^2 dx_1 + \frac{1}{2}\int_{\mathbb{R}} \left| \left|\frac{d}{dx_1}\right|^{1/2} m_{1,k} \right|^2 dx_1 \right) < \infty. \tag{15}$$

Then $\{m'_k\}_{k\uparrow\infty}$ is relatively compact in $L^1_{loc}(\mathbb{R})$. Moreover, any accumulation point $m' : \mathbb{R} \to S^1$ of the sequence $\{m'_k\}_{k\uparrow\infty}$ in L^1_{loc} is of bounded variation and can be written as

$$m' = \sum_{n=1}^{2N} \begin{pmatrix} m_{1,\infty} \\ (-1)^n \sqrt{1 - {m_{1,\infty}}^2} \end{pmatrix} 1_{(b_{n-1}, b_n)},$$

where $-\infty = b_0 < b_1 < \cdots < b_{2N-1} < b_{2N} = +\infty$ and $b_n \in [-1,1]$ for $n = 1, \ldots, 2N-1$.

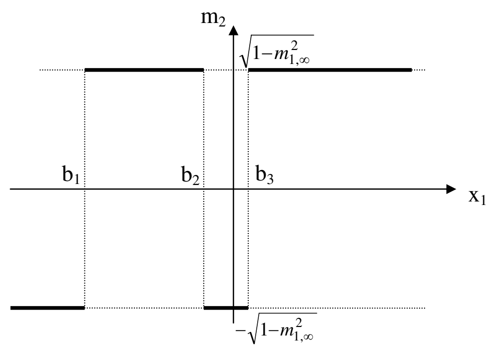

Figure 7: The m_2 component of a limit with three walls

One may ask whether the above sequences of 1-d magnetizations are relatively compact in BV since their limit has bounded variation. The answer is negative in general. For that, we construct a family of 1-d magnetizations with the energy level in the regime $O(\frac{1}{|\ln \varepsilon|})$ such that the total variations of $\{m_{1,k}\}$ blow-up as $k \to \infty$:

Theorem 3 *There exists a sequence $\{m'_k : \mathbb{R} \to S^1\}_{k\in\mathbb{N}}$ with the properties:*

$$\begin{aligned} &(1) \text{ holds for some } m_{1,\infty} \in [0,1), \\ &\lim_{k\to\infty} \int_{\mathbb{R}} \left|\frac{dm_{1,k}}{dx_1}\right| dx_1 = \infty, \\ &(15) \text{ holds for some } \{\varepsilon_k\}_{k\uparrow\infty} \text{ with } \varepsilon_k \to 0. \end{aligned}$$

Now we investigate the asymptotics as $\varepsilon \to 0$ of families of 2-d magnetizations when the energy $E_\varepsilon(m'_\varepsilon, h_\varepsilon)$ is placed in the regime $O(\frac{1}{|\ln \varepsilon|})$. One of the issues we discuss here is the question of the L^1_{loc}-compactness of the magnetizations $\{m'_\varepsilon\}_{\varepsilon\downarrow 0}$ in the above energy regime, i.e., whether the topological constraint $|m'_\varepsilon| = 1$ passes to the limit. The difficulty arises from the fact that in general the sequence of divergences $\{\nabla' \cdot m'_\varepsilon\}$ is not uniformly bounded in L^1 (a counter-example is given in Theorem 3). This was one of the particularities used in the entropy methods for proving compactness results for the Modica-Mortola type problems; we refer to the studies of Jin and Kohn [10], Ambrosio, De Lellis and Mantegazza [3], DeSimone, Kohn, Müller and Otto [6], Alouges, Rivière and Serfaty [2], Rivière and Serfaty [14], Jabin, Otto and Perthame [9]. For our model, the idea is to use a duality argument in the spirit of [5] based on an ε-perturbation of a logarithmically failing Gagliardo-Nirenberg inequality (see Section 2). Since the compactness result is a local issue, we state it in the context of the unit ball $B_1 \subset \mathbb{R}^3$ with no imposed boundary conditions:

Theorem 4 *Consider a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \downarrow 0$. For $k \in \mathbb{N}$, let $m'_k : B'_1 \to S^1$ and $h_k : B_1 \to \mathbb{R}^3$ be related by*

$$\int_{B_1} h_k \cdot \nabla\zeta \, dx = \int_{B'_1} \zeta \, \nabla' \cdot m'_k \, dx', \, \forall \zeta \in C_c^\infty(B_1). \tag{16}$$

Suppose that

$$\limsup_{k\to\infty} |\ln \varepsilon_k| \left(\varepsilon_k \int_{B'_1} |\nabla' \cdot m'_k|^2 \, dx' + \int_{B_1} |h_k|^2 \, dx \right) < \infty. \tag{17}$$

Then $\{m'_k\}_{k\uparrow\infty}$ is relatively compact in $L^1(B'_1)$ and any accumulation point $m' : B'_1 \to \mathbb{R}^2$ satisfies

$$|m'| = 1 \text{ a.e. in } B'_1 \quad \text{and} \quad \nabla' \cdot m' = 0 \text{ distributionally in } B'_1. \tag{18}$$

We now focus on the behavior of the finite-energy states m'. As in (8), we formally have by (18) that $m' = \nabla'^{\perp}\psi$ where ψ satisfies the eikonal equation $|\nabla'\psi| = 1$. We discuss the case of *zero-energy states*, i.e., m' is an accumulation point of sequences $\{m'_\varepsilon\}_{\varepsilon\downarrow 0}$ such that the limit in (17) vanishes for some stray fields $\{h_\varepsilon\}$ (in the absence of any boundary conditions). We show that every zero-energy state m' is locally Lipschitz continuous. The main tool is the principle of characteristics for the eikonal equation. The difference with respect to the zero-energy states for the Ginzburg-Landau models treated in [9] consists in the avoidance of vortices. Our result can be stated as follows:

Theorem 5 *Consider a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \downarrow 0$. For $k \in \mathbb{N}$, let $m'_k : B'_1 \to S^1$ and $h_k : B_1 \to \mathbb{R}^3$ be related by (16). Suppose that*

$$\lim_{k\to\infty} |\ln \varepsilon_k| \left(\varepsilon_k \int_{B'_1} |\nabla' \cdot m'_k|^2 \, dx' + \int_{B_1} |h_k|^2 \, dx \right) = 0. \tag{19}$$

Then any accumulation point $m' : B'_1 \to \mathbb{R}^2$ of $\{m'_k\}_{k\uparrow\infty}$ in $L^1(B'_1)$ satisfies

a) m' is locally Lipschitz in B'_1;

b) m' satisfies the principle of characteristics related to (18), i.e., for any $x'_0 \in B'_1$ we have that

$$m'(x'_0 + tm'(x'_0)^\perp) = m'(x'_0) \text{ for any } t \in \mathbb{R} \text{ with } x'_0 + tm'(x'_0)^\perp \in B'_1$$

(see Figure 8).

Remark 2 In general, a function m' satisfying *a)* and *b)* in Theorem 5 is not globally Lipschitz in B'_1; an example is given by

$$m'(x') = \left(\frac{x' - P}{|x' - P|} \right)^\perp \quad \text{for any } x' \in B'_1,$$

for some $P \in \partial B'_1$ (P plays the role of a vortex on the boundary).

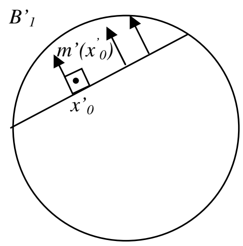

Figure 8: Principle of characteristics

The outline of the paper is as follows. In Section 2, we give some fundamental estimates based on a duality argument and a logarithmically failing interpolation inequality. In Section 3, we prove Theorem 4. In Section 4, we focus on the zero-energy states: We establish a list of lemmas that lead to Theorem 5. In Section 5, we show the optimality of the straight walls in Theorem 1 as an application of Theorems 4 and 5. In Section 6 we discuss the behavior of 1-d magnetizations by proving Theorems 2 and 3.

2 Some fundamental localized estimates

We present some inequalities in the spirit of [5] that are to be used in the next sections. Obviously, it is important to draw information from the fact that $\sigma := \nabla' \cdot m'$ is controlled and that $|m'| = 1$. Following [5], we will do this by working with characteristic functions $\chi \in \{-\frac{1}{2}, \frac{1}{2}\}$ which have the property that the outer normal to the set $\{\chi = -\frac{1}{2}\}$ is given by m' (see Figure 9). In the language

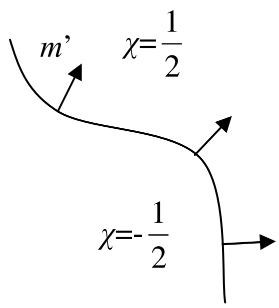

Figure 9: The characteristic function χ

of $BV-$functions, this means

$$D'\chi = m' \, |D'\chi| \tag{20}$$

as an identity between measures. The purpose of (20) is that an integration by parts yields for any localizing function η:

$$\int \eta^2 \chi \sigma \, dx' = -\int \eta^2 |D'\chi| - \int \chi \nabla' \eta^2 \cdot m' \, dx'. \tag{21}$$

The merit of (21) is that it gives a control of the length of the interface $\partial\{\chi = -\frac{1}{2}\}$ by σ (and boundary data), see Step 2 in the proof of Theorem 4. Let's be more precise: The energy E_ε gives us a control over σ in form of

$$\varepsilon \int \sigma^2 \, dx' \tag{22}$$

and

$$\int |h|^2 \, dx = \frac{1}{2} \int \left| |\nabla'|^{-1/2} \sigma \right|^2 dx' = \frac{1}{2} \int \frac{1}{|\xi'|} |\mathcal{F}(\sigma)|^2 \, d\xi', \tag{23}$$

where $\mathcal{F}(\sigma)$ denotes the Fourier transform of σ. The question which immediately arises is whether the control (23) of σ is sufficient to estimate $\int \eta^2 \chi \sigma \, dx'$. We simplify the question even further: Are (23), $\int |D'\chi|$ and $\sup |\chi|$ enough to estimate the expression $\int \chi \sigma \, dx'$? By duality, this question can be rephrased as follows: Can we control

$$\int \left| |\nabla'|^{1/2} \chi \right|^2 dx' = \int |\xi'| |\mathcal{F}(\chi)|^2 \, d\xi' \tag{24}$$

by $\int |D'\chi|$ and $\sup |\chi|$? The answer is no: For $1-$d characteristic function

$$\chi = \begin{cases} \frac{1}{2} & \text{for} \quad x_1 < 0, \\ -\frac{1}{2} & \text{for} \quad x_1 > 0, \end{cases}$$

we have $|\mathcal{F}(\chi)|(\xi_1) = \frac{1}{\sqrt{2\pi}|\xi_1|}$ so that $\int |\xi_1| |\mathcal{F}(\chi)|^2 \, d\xi_1$ diverges logarithmically. However, the divergence is only logarithmic, which is also borne out by the fact that

$$\int \left| |\nabla'|^{1/2} \chi \right|^2 dx' \quad \text{and} \quad \sup |\chi| \int |D'\chi| \tag{25}$$

have the same scaling. In fact, it follows from the analysis in [5] that

$$\int_{\{1 \le |\xi'| \le 1/\varepsilon\}} |\xi'| |\mathcal{F}(\chi)|^2 \, d\xi' \le \frac{2}{\pi} (|\ln \varepsilon| + C) \sup |\chi| \int |D'\chi| \tag{26}$$

for some universal constant C. (We will reprove a variant of (26) below). Hence the Gagliardo-Nirenberg estimate in (25) holds when one cuts off the very small and very large length scales - but the constant blows up logarithmically in the ratio of large and small cut-off length. Hence the control (22) is necessary to deal with the small length scales. It has to be combined with the estimate

$$\int_{\{|\xi'| > 1/\varepsilon\}} |\mathcal{F}(\chi)|^2 \, d\xi' \le C \varepsilon \sup |\chi| \int |D'\chi|. \tag{27}$$

Finally, provided that

$$\operatorname{supp} \chi \subset B_1',$$

the large length scales are easily controlled

$$\int_{\{|\xi'| < 1\}} |\mathcal{F}(\chi)|^2 \, d\xi' \le C (\sup |\chi|)^2. \tag{28}$$

The combination of (23), (26), (27) and (28) yields

$$
\begin{aligned}
\left|\int \chi\sigma\, dx'\right| &\le \left(\frac{4}{\pi}(|\ln\varepsilon| + C)\,\sup|\chi|\,\int |D'\chi| \int |h|^2\,dx\right)^{1/2} \\
&\quad + C\left(\varepsilon\sup|\chi|\int |D'\chi|\int \sigma^2\,dx'\right)^{1/2} \\
&\quad + C\sup|\chi|\left(\int |h|^2\,dx\right)^{1/2} \\
&\le \left(\frac{4}{\pi}|\ln\varepsilon|\,\sup|\chi|\,\int |D'\chi| \int |h|^2\,dx\right)^{1/2} \\
&\quad + C\left(\varepsilon\int \sigma^2\,dx' + \int |h|^2\,dx\right)^{1/2} \qquad (29)\\
&\quad \times \left(\sup|\chi| + \int |D'\chi|\right).
\end{aligned}
$$

In fact, we need the following estimate of the localized expression $\int \eta^2\chi\sigma\,dx'$:

Proposition 1 *Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be related by*

$$
\int_{\mathbb{R}^3} h\cdot\nabla\zeta\,dx = \int_{\mathbb{R}^2} \sigma\zeta\,dx', \quad \forall\zeta \in C_c^\infty(B_1), \tag{30}
$$

where $x' = (x_1, x_2) \in \mathbb{R}^2$ and $x = (x', x_3) \in \mathbb{R}^3$. Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function of locally bounded variation and $\eta \in C_c^\infty(\mathbb{R}^3)$ be such that

$$
\operatorname{supp}\eta \subset B_1 \subset \mathbb{R}^3. \tag{31}
$$

Then there exists a universal constant $C > 0$ such that for all $\varepsilon \in (0, 1]$,

$$
\left|\int_{\mathbb{R}^2} \eta^2\chi\sigma\,dx'\right| \le \left(\frac{4}{\pi}|\ln\varepsilon|\,\sup_{\mathbb{R}^2}|\chi|\,\int_{\mathbb{R}^2}\eta^2|D'\chi|\int_{\mathbb{R}^3}\eta^2|h|^2\,dx\right)^{1/2} \tag{32}
$$

$$
\begin{aligned}
&+ C\left(\varepsilon\int_{B_1'}\sigma^2\,dx' + \int_{B_1}|h|^2\,dx\right)^{1/2} \\
&\times\left(\sup_{\mathbb{R}^3}|\eta| + \sup_{\mathbb{R}^3}|\nabla\eta|\right)\left(\sup_{\mathbb{R}^3}|\eta|\,\sup_{\mathbb{R}^2}|\chi| + \int_{\mathbb{R}^2}|\eta||D'\chi|\right),
\end{aligned} \tag{33}
$$

where D' denotes the in-plane derivatives (∂_1, ∂_2).

As a direct consequence, we have:

Corollary 1 *Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be related by (30). Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function of bounded variation such that*

$$
\operatorname{supp}\chi \subset B_1'.
$$

Then there exists a universal constant $C > 0$ such that for all $\varepsilon \in (0, 1]$,

$$\left| \int_{\mathbb{R}^2} \chi\sigma \, dx' \right| \leq \left(\frac{4}{\pi} |\ln \varepsilon| \sup_{\mathbb{R}^2} |\chi| \int_{B_1'} |D'\chi| \int_{B_1} |h|^2 \, dx \right)^{1/2} \tag{34}$$

$$+ \frac{C}{d} \left(\varepsilon \int_{B_1'} \sigma^2 \, dx' + \int_{B_1} |h|^2 \, dx \right)^{1/2} \times \left(\sup_{\mathbb{R}^2} |\chi| + \int_{B_1'} |D'\chi| \right), \tag{35}$$

where $d = \operatorname{dist}(\operatorname{supp} \chi, \partial B_1') > 0$.

Proof of Proposition 1. Let C denotes a generic universal constant. Our heuristic derivation of estimate (29) above was based on the decomposition in Fourier space, following [5]. However, Fourier space methods do not seem to be appropriate for our localized version. We thus need to find a real space counterpart of the Fourier space representation (24) and of the small scale cut-off $\{|\xi'| \leq \frac{1}{\varepsilon}\}$ in (26). To circumvent (24), we work with the harmonic extension $\bar{\zeta} : \mathbb{R}^3 \to \mathbb{R}$ of a function $\zeta : \mathbb{R}^2 \to \mathbb{R}$, i.e.,

$$\begin{cases} \Delta \bar{\zeta} = 0 & \text{in} \quad \mathbb{R}^3 \setminus (\mathbb{R}^2 \times \{0\}), \\ \bar{\zeta}(\cdot, 0) = \zeta & \text{on} \quad \mathbb{R}^2. \end{cases} \tag{36}$$

Notice that equation (36) can be solved explicitly. The Fourier transform of (36) in the horizontal variables yields an ODE in x_3 with ξ' as a parameter. This ODE is solved by

$$\mathcal{F}(\bar{\zeta})(\xi', x_3) = \mathcal{F}(\zeta)(\xi') e^{-|\xi'||x_3|}.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^2} |\xi'| \, |\mathcal{F}(\zeta)|^2 \, d\xi' &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \left(|\xi'|^2 \, |\mathcal{F}(\bar{\zeta})|^2 + |\partial_3 \mathcal{F}(\bar{\zeta})|^2 \right) d\xi' \, dx_3 \\ &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \bar{\zeta}|^2 \, dx. \end{aligned} \tag{37}$$

To avoid the Fourier based decomposition into a small length scale part and the remainder, we introduce a convolution $\zeta_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ of a function $\zeta : \mathbb{R}^2 \to \mathbb{R}$ with a universal kernel ρ_ε of the form

$$\rho_\varepsilon(x') = \frac{1}{\varepsilon^2} \rho_1(\frac{x'}{\varepsilon}) \quad \text{where} \quad \rho_1 \in C_c^\infty(B_1') \text{ is radial, } \rho_1 \geq 0, \int_{B_1'} \rho_1(x') \, dx' = 1.$$

Indeed, the convolution allows for a decomposition of ζ into the small scale part $\zeta - \zeta_\varepsilon$ and the remainder ζ_ε.

We prove the estimate for $\chi \in W^{1,1}_{loc} \cap L^\infty(\mathbb{R}^2)$. In the general case of a function $\chi \in BV_{loc} \cap L^\infty(\mathbb{R}^2)$, it will follow by a density argument, using a sequence $\{\chi_\delta\} \subset W^{1,1}_{loc} \cap L^\infty(\mathbb{R}^2)$ such that $\chi_\delta \to \chi$ a.e. in B_1', $\sup_{\mathbb{R}^2} |\chi_\delta| \leq \sup_{\mathbb{R}^2} |\chi|$ and $\int_{B_1'} |\nabla' \chi_\delta| \, dx' \to \int_{B_1'} |D'\chi|$ (hence, $|D'\chi_\delta| \overset{w^*}{\rightharpoonup} |D'\chi|$ weakly* as measures in B_1').

We rewrite the left-hand side of (32) of our estimate as follows:

$$\int_{\mathbb{R}^2} \eta^2 \chi\sigma \, dx' = \int_{\mathbb{R}^2} \eta\sigma \big(\eta\chi - (\eta\chi)_\varepsilon \big) \, dx' + \int_{\mathbb{R}^2} \eta\sigma \overline{(\eta\chi)_\varepsilon} \, dx'$$

and by (30) (where $\operatorname{supp} \eta\overline{(\eta\chi)_\varepsilon} \subset B_1$),

$$\begin{aligned}\int_{\mathbb{R}^2} \eta\sigma\overline{(\eta\chi)_\varepsilon}\, dx' &= \int_{\mathbb{R}^3} h \cdot \nabla\big(\eta\overline{(\eta\chi)_\varepsilon}\big)\, dx \\ &= \int_{\mathbb{R}^3} \overline{(\eta\chi)_\varepsilon} h \cdot \nabla\eta\, dx + \int_{\mathbb{R}^3} \eta h \cdot \nabla\overline{(\eta\chi)_\varepsilon}\, dx.\end{aligned}$$

Hence, we obtain the estimate

$$\begin{aligned}\Big| \int_{\mathbb{R}^2} \eta^2\chi\sigma\, dx' \Big| &\le \Big(\int_{\mathbb{R}^2} \eta^2\sigma^2\, dx' \Big)^{1/2} \Big(\int_{\mathbb{R}^2} \big|\eta\chi - (\eta\chi)_\varepsilon\big|^2\, dx' \Big)^{1/2} \\ &\quad + \sup_{\mathbb{R}^3} |\overline{(\eta\chi)_\varepsilon}| \int_{\mathbb{R}^3} |h|\,|\nabla\eta|\, dx + \Big(\int_{\mathbb{R}^3} \eta^2|h|^2\, dx \Big)^{1/2} \Big(\int_{\mathbb{R}^3} \big|\nabla\overline{(\eta\chi)_\varepsilon}\big|^2\, dx \Big)^{1/2} \\ &\overset{(31)}{\le} \sup_{\mathbb{R}^3} |\eta| \Big(\int_{B_1'} \sigma^2\, dx' \Big)^{1/2} \Big(\int_{\mathbb{R}^2} \big|\eta\chi - (\eta\chi)_\varepsilon\big|^2\, dx' \Big)^{1/2} \qquad (38) \\ &\quad + C \sup_{\mathbb{R}^3} |\nabla\eta| \sup_{\mathbb{R}^3} |\overline{(\eta\chi)_\varepsilon}| \Big(\int_{B_1} |h|^2\, dx \Big)^{1/2} \qquad (39) \\ &\quad + \Big(\int_{\mathbb{R}^3} \eta^2|h|^2\, dx \Big)^{1/2} \Big(\int_{\mathbb{R}^3} \big|\nabla\overline{(\eta\chi)_\varepsilon}\big|^2\, dx \Big)^{1/2}. \qquad (40)\end{aligned}$$

As we shall see, only the term (40) contributes to the leading order term (32). We first address (38) and (39). For (39), we observe that by the maximum principle,

$$\sup_{\mathbb{R}^3} |\overline{(\eta\chi)_\varepsilon}| \le \sup_{\mathbb{R}^2} |(\eta\chi)_\varepsilon| \le \sup_{\mathbb{R}^2} |\eta\chi| \le \sup_{\mathbb{R}^3} |\eta| \sup_{\mathbb{R}^2} |\chi|,$$

so that (39) can indeed be absorbed into (33). For (38), we have

$$\begin{aligned}\int_{\mathbb{R}^2} \big|\eta\chi - (\eta\chi)_\varepsilon\big|^2\, dx' &\le (\sup_{\mathbb{R}^2} |(\eta\chi)_\varepsilon| + \sup_{\mathbb{R}^2} |\eta\chi|) \int_{\mathbb{R}^2} \big|\eta\chi - (\eta\chi)_\varepsilon\big|\, dx' \\ &\le 2\varepsilon \sup_{\mathbb{R}^2} |\eta\chi| \int_{\mathbb{R}^2} \big|\nabla'(\eta\chi)\big|\, dx' \\ &\le 2\varepsilon \sup_{\mathbb{R}^2} |\eta\chi| \int_{\mathbb{R}^2} \big(|\eta||\nabla'\chi| + |\chi||\nabla'\eta|\big)\, dx' \\ &\overset{(31)}{\le} C\varepsilon \sup_{\mathbb{R}^3} |\eta| \sup_{\mathbb{R}^2} |\chi| \\ &\qquad \times \Big(\int_{\mathbb{R}^2} |\eta||\nabla'\chi|\, dx' + \sup_{\mathbb{R}^3} |\nabla\eta| \sup_{\mathbb{R}^2} |\chi| \Big).\end{aligned}$$

Hence, (38) can be absorbed into (33).

We now turn to (40). In order to have the desired inequality, it is sufficient to prove that

$$\begin{aligned}\int_{\mathbb{R}^3} \big|\nabla\overline{(\eta\chi)_\varepsilon}\big|^2\, dx &\le \frac{4}{\pi} |\ln\varepsilon| \sup_{\mathbb{R}^2} |\chi| \int_{\mathbb{R}^2} \eta^2 |\nabla'\chi|\, dx' \qquad (41) \\ &\quad + C \Big(\sup_{\mathbb{R}^3} |\eta| \sup_{\mathbb{R}^2} |\chi| + \sup_{\mathbb{R}^3} |\nabla\eta| \sup_{\mathbb{R}^2} |\chi| + \int_{\mathbb{R}^2} |\eta||\nabla'\chi|\, dx' \Big)^2. \qquad (42)\end{aligned}$$

We appeal to the following identity, which was already used in [5],

$$\int_{\mathbb{R}^3} |\nabla\bar{\phi}|^2\, dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|z'|^3} \int_{\mathbb{R}^2} |\phi(x'+z') - \phi(x')|^2\, dx'\, dz', \qquad (43)$$

which we apply to $\phi = (\eta\chi)_\varepsilon$. Actually, (43) is easy to establish. First of all, by homogeneity and isotropy, it results that for every $\xi' \in \mathbb{R}^2$,

$$
\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|z'|^3}|1-e^{i\xi'\cdot z'}|^2\,dz' &= \frac{|\xi'|}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|\tilde z'|^3}|1-e^{i\tilde z_1}|^2\,d\tilde z' \\
&= \frac{2|\xi'|}{\pi}\int_0^{2\pi}\left(\int_0^\infty \frac{1}{r^2}\sin^2\big(\frac{r|\cos\theta|}{2}\big)\,dr\right)d\theta \\
&= \frac{|\xi'|}{\pi}\int_0^{2\pi}\int_0^\infty \frac{|\cos\theta|}{s^2}\sin^2 s\,ds\,d\theta \\
&= \frac{|\xi'|}{\pi}\int_0^{2\pi}|\cos\theta|\,d\theta\int_0^\infty\frac{\sin^2 s}{s^2}\,ds = 2|\xi'|. \qquad (44)
\end{aligned}
$$

(Here, we used that $\int_0^\infty \frac{\sin^2 s}{s^2}\,ds = \int_0^\infty \frac{\sin 2s}{s}\,ds = \frac{\pi}{2}$, see e.g. [1], 5.2.25.) By (37), we have

$$
\begin{aligned}
\int_{\mathbb{R}^3}|\nabla\bar\phi|^2\,dx &\overset{(37)}{=} 2\int_{\mathbb{R}^2}|\xi'|\,|\mathcal{F}(\phi)(\xi')|^2\,d\xi' \\
&\overset{(44)}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|1-e^{i\xi'\cdot z'}|^2|\mathcal{F}(\phi)(\xi')|^2\,d\xi'\,dz' \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|\phi(x'+z')-\phi(x')|^2\,dx'\,dz',
\end{aligned}
$$

i.e., (43) holds. We split the $z'-$integral on the right-hand side of (43) into three different regions:

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx'\,dz' \\
&\quad= \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B_1'}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx'\,dz' \qquad (45)\\
&\qquad+ \frac{1}{2\pi}\int_{B_\varepsilon'}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx'\,dz' \qquad (46)\\
&\qquad+ \frac{1}{2\pi}\int_{B_1'\setminus B_\varepsilon'}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx'\,dz'. \qquad (47)
\end{aligned}
$$

As we shall see, only the intermediate term (47) contributes to the leading order term (41).

We first address (45) and (46). We start with the term (45) corresponding to the long wave length (i.e., $|z'|\geq 1$). Since

$$
\begin{aligned}
\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx' &\leq 2\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon|^2\,dx' \\
&\leq 2\int_{\mathbb{R}^2}|\eta\chi|^2\,dx' \overset{(31)}{\leq} C\sup_{\mathbb{R}^3}|\eta|^2\sup_{\mathbb{R}^2}|\chi|^2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B_1'}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)_\varepsilon(x'+z')-(\eta\chi)_\varepsilon(x')|^2\,dx'\,dz' \\
&\qquad\leq C\sup_{\mathbb{R}^3}|\eta|^2\sup_{\mathbb{R}^2}|\chi|^2\int_{\mathbb{R}^2\setminus B_1'}\frac{1}{|z'|^3}\,dz' \leq C\sup_{\mathbb{R}^3}|\eta|^2\sup_{\mathbb{R}^2}|\chi|^2,
\end{aligned}
\qquad (48)
$$

i.e., (45) is absorbed by (42). We now tackle the short wave length term (46). We have

$$\begin{aligned}\int_{\mathbb{R}^2} |(\eta\chi)_\varepsilon(x'+z') - (\eta\chi)_\varepsilon(x')|^2\,dx' &\le |z'|^2 \int_{\mathbb{R}^2} |\nabla'(\eta\chi)_\varepsilon|^2\,dx' \\ &\le |z'|^2 \sup_{\mathbb{R}^2} |\nabla'(\eta\chi)_\varepsilon| \int_{\mathbb{R}^2} |\nabla'(\eta\chi)_\varepsilon|\,dx' \\ &\le \frac{C}{\varepsilon}|z'|^2 \sup_{\mathbb{R}^2}|\eta\chi| \int_{\mathbb{R}^2} |\nabla'(\eta\chi)|\,dx'\end{aligned}$$

and thus,

$$\begin{aligned}&\frac{1}{2\pi}\int_{B'_\varepsilon} \frac{1}{|z'|^3} \int_{\mathbb{R}^2} |(\eta\chi)_\varepsilon(x'+z') - (\eta\chi)_\varepsilon(x')|^2\,dx'\,dz' \\ &\quad\overset{(31)}{\le} C \sup_{\mathbb{R}^3}|\eta| \sup_{\mathbb{R}^2}|\chi| \Big(\sup_{\mathbb{R}^3}|\nabla\eta| \sup_{\mathbb{R}^2}|\chi| + \int_{\mathbb{R}^2} |\eta||\nabla'\chi|\,dx' \Big) \frac{1}{\varepsilon}\int_{B'_\varepsilon} \frac{dz'}{|z'|},\end{aligned} \tag{49}$$

i.e., (46) can also be absorbed by (42).

We finally address the medium wave length term (47). We start by observing that

$$\int_{\mathbb{R}^2} |(\eta\chi)_\varepsilon(x'+z') - (\eta\chi)_\varepsilon(x')|^2\,dx' \le \int_{\mathbb{R}^2} |(\eta\chi)(x'+z') - (\eta\chi)(x')|^2\,dx'.$$

We consider the integrand, which we shall rewrite in form of

$$\begin{aligned}|(\eta\chi)(x'+z') - (\eta\chi)(x')|^2 = (\chi(x'+z') - \chi(x')) \int_0^1 \eta^2(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt \\ + \mathit{remainder}.\end{aligned}$$

To do so, we proceed as follows

$$\begin{aligned}(\eta\chi)(x'+z') - (\eta\chi)(x') &= \int_0^1 \nabla'(\eta\chi)(x'+tz')\cdot z'\,dt \\ &= \int_0^1 \eta(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt + \int_0^1 \chi(x'+tz')\nabla'\eta(x'+tz')\cdot z'\,dt,\end{aligned}$$

and thus,

$$\begin{aligned}|(\eta\chi)(x'+z') - (\eta\chi)(x')|^2 = (\chi(x'+z') - \chi(x')) &\int_0^1 \eta^2(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt \\ &+ \chi(x'+z')\int_0^1 (\eta(x'+z') - \eta(x'+tz'))\eta(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt \\ &- \chi(x')\int_0^1 (\eta(x') - \eta(x'+tz'))\eta(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt \\ &+ \int_0^1 \eta(x'+tz')\nabla'\chi(x'+tz')\cdot z'\,dt \int_0^1 \chi(x'+tz')\nabla'\eta(x'+tz')\cdot z'\,dt \\ &+ \Big(\int_0^1 \chi(x'+tz')\nabla'\eta(x'+tz')\cdot z'\,dt\Big)^2.\end{aligned}$$

This yields the estimate

$$
\begin{aligned}
|(\eta\chi)(x'+z')-(\eta\chi)(x')|^2 \le 2\sup_{\mathbb{R}^2}|\chi| &\int_0^1 \eta^2(x'+tz')|\nabla'\chi(x'+tz')\cdot z'|\,dt \\
&+3|z'|^2\sup_{\mathbb{R}^2}|\chi|\sup_{\mathbb{R}^3}|\nabla\eta|\int_0^1 |\eta(x'+tz')||\nabla'\chi(x'+tz')|\,dt \\
&+|z'|^2\sup_{\mathbb{R}^2}|\chi|^2\sup_{\mathbb{R}^3}|\nabla\eta|\int_0^1 |\nabla'\eta(x'+tz')|\,dt.
\end{aligned}
$$

Integration in x' gives

$$
\begin{aligned}
&\int_{\mathbb{R}^2}|(\eta\chi)(x'+z')-(\eta\chi)(x')|^2\,dx' \\
&\quad\le 2\sup_{\mathbb{R}^2}|\chi|\int_{\mathbb{R}^2}\eta^2|\nabla'\chi\cdot z'|\,dx' + 3|z'|^2\sup_{\mathbb{R}^2}|\chi|\sup_{\mathbb{R}^3}|\nabla\eta|\int_{\mathbb{R}^2}|\eta||\nabla'\chi|\,dx' \\
&\qquad + |z'|^2\sup_{\mathbb{R}^2}|\chi|^2\sup_{\mathbb{R}^3}|\nabla\eta|\int_{\mathbb{R}^2}|\nabla'\eta|\,dx' \\
&\quad\le 2\sup_{\mathbb{R}^2}|\chi|\int_{\mathbb{R}^2}\eta^2|\nabla'\chi\cdot z'|\,dx' \\
&\qquad + C|z'|^2\sup_{\mathbb{R}^3}|\nabla\eta|\sup_{\mathbb{R}^2}|\chi|\left(\sup_{\mathbb{R}^3}|\nabla\eta|\sup_{\mathbb{R}^2}|\chi| + \int_{\mathbb{R}^2}|\eta||\nabla'\chi|\,dx'\right).
\end{aligned}
$$

Integration in z' yields

$$
\begin{aligned}
&\int_{B_1'\setminus B_\varepsilon'}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)(x'+z')-(\eta\chi)(x')|^2\,dx'\,dz' \\
&\quad\le 2\sup_{\mathbb{R}^2}|\chi|\int_{\mathbb{R}^2}\eta^2(x')\int_{B_1'\setminus B_\varepsilon'}\frac{1}{|z'|^3}|\nabla'\chi(x')\cdot z'|\,dz'\,dx' \\
&\qquad + C\left(\sup_{\mathbb{R}^3}|\eta|\sup_{\mathbb{R}^2}|\chi| + \sup_{\mathbb{R}^3}|\nabla\eta|\sup_{\mathbb{R}^2}|\chi| + \int_{\mathbb{R}^2}|\eta||\nabla'\chi|\,dx'\right)^2\int_{B_1'\setminus B_\varepsilon'}\frac{dz'}{|z'|}.
\end{aligned}
\tag{50}
$$

Notice that for any $v'\in\mathbb{R}^2$,

$$
\begin{aligned}
\int_{B_1'\setminus B_\varepsilon'}\frac{1}{|z'|^3}|v'\cdot z'|\,dz' &\int_0^{2\pi}\int_\varepsilon^1\frac{1}{r^3}\left|v'\cdot\begin{pmatrix} r\cos\theta \\ r\sin\theta\end{pmatrix}\right| r\,dr\,d\theta \\
&= |v'|\int_0^{2\pi}|\cos\theta|\,d\theta\int_\varepsilon^1\frac{1}{r}\,dr = 4|\ln\varepsilon|\,|v'|.
\end{aligned}
$$

Hence (50) turns into

$$
\begin{aligned}
\frac{1}{2\pi}\int_{B_1'\setminus B_\varepsilon'}\frac{1}{|z'|^3}\int_{\mathbb{R}^2}|(\eta\chi)(x'+z')-(\eta\chi)(x')|^2\,dx'\,dz' &\le \frac{4}{\pi}|\ln\varepsilon|\sup_{\mathbb{R}^2}|\chi|\int_{\mathbb{R}^2}\eta^2|\nabla'\chi|\,dx' \\
&+ C\left(\sup_{\mathbb{R}^3}|\eta|\sup_{\mathbb{R}^2}|\chi| + \sup_{\mathbb{R}^3}|\nabla\eta|\sup_{\mathbb{R}^2}|\chi| + \int_{\mathbb{R}^2}|\eta||\nabla'\chi|\,dx'\right)^2.
\end{aligned}
\tag{51}
$$

Combining identity (43) with the estimates (48), (49) and (51), we conclude that (41) holds. □

Proof of Corollary 1. It directly follows from Proposition 1 by choosing $\eta\in C_c^\infty(B_1)$ such that

$$
\eta = 1 \text{ on } \operatorname{supp}\chi,\ |\eta|\le 1 \text{ and } |\nabla\eta|\le\frac{C}{d} \text{ in } B_1.
$$

□

By rescaling length in Proposition 1 from unity to some $R > 0$, we obtain:

Corollary 2 *Let $R > 0$ and $x_0 = (x_0', 0) \in \mathbb{R}^2 \times \{0\}$. Consider $h : \mathbb{R}^3 \to \mathbb{R}^3$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be related by*

$$\int_{\mathbb{R}^3} h \cdot \nabla\zeta \, dx = \int_{\mathbb{R}^2} \sigma\zeta \, dx', \quad \forall \zeta \in C_c^\infty(B(x_0, R)).$$

Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function of locally bounded variation and $\eta \in C_c^\infty(\mathbb{R}^3)$ be such that

$$\operatorname{supp} \eta \subset B(x_0, R) \subset \mathbb{R}^3. \tag{52}$$

Then there exists a universal constant $C > 0$ such that for all $\varepsilon \in (0, R]$,

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \eta^2 \chi\sigma \, dx' \right| \le{}& \left(\frac{4}{\pi} |\ln \varepsilon| \sup_{\mathbb{R}^2} |\chi| \int_{\mathbb{R}^2} \eta^2 |D'\chi| \int_{\mathbb{R}^3} \eta^2 |h|^2 \, dx \right)^{1/2} \\ &+ C(1 + |\ln R|^{1/2}) \left(\varepsilon \int_{B'(x_0', R)} \sigma^2 \, dx' + \int_{B(x_0,R)} |h|^2 \, dx \right)^{1/2} \\ &\times \left(\sup_{\mathbb{R}^3} |\eta| + R \sup_{\mathbb{R}^3} |\nabla\eta| \right) \left(R^{1/2} \sup_{\mathbb{R}^3} |\eta| \sup_{\mathbb{R}^2} |\chi| + R^{-1/2} \int_{\mathbb{R}^2} |\eta||D'\chi| \right). \end{aligned} \tag{53}$$

Proof. The change of variables $x = R\hat{x} + x_0$ (and $\varepsilon = R\hat{\varepsilon}$) preserves (30) and turns (52) into $\operatorname{supp} \eta \subset \hat{B}_1$, so that we may apply Proposition 1. It yields in the original variables:

$$\begin{aligned} \left| R^{-2} \int_{\mathbb{R}^2} \eta^2 \chi\sigma \, dx' \right| \le{}& \left(\frac{4}{\pi} |\ln \frac{\varepsilon}{R}| \sup_{\mathbb{R}^2} |\chi| \, R^{-1} \int_{\mathbb{R}^2} \eta^2 |D'\chi| \, R^{-3} \int_{\mathbb{R}^3} \eta^2 |h|^2 \, dx \right)^{1/2} \\ &+ C \left(\frac{\varepsilon}{R} R^{-2} \int_{B'(x_0', R)} \sigma^2 \, dx' + R^{-3} \int_{B(x_0,R)} |h|^2 \, dx \right)^{1/2} \\ &\times (\sup_{\mathbb{R}^3} |\eta| + R \sup_{\mathbb{R}^3} |\nabla\eta|) \left(\sup_{\mathbb{R}^3} |\eta| \sup_{\mathbb{R}^2} |\chi| + R^{-1} \int_{\mathbb{R}^2} |\eta||D'\chi| \right), \end{aligned}$$

that is,

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \eta^2 \chi\sigma \, dx' \right| \le{}& \left(\frac{4}{\pi} (|\ln \varepsilon| + |\ln R|) \sup_{\mathbb{R}^2} |\chi| \int_{\mathbb{R}^2} \eta^2 |D'\chi| \int_{\mathbb{R}^3} \eta^2 |h|^2 \, dx \right)^{1/2} \\ &+ C \left(\varepsilon \int_{B'(x_0', R)} \sigma^2 \, dx' + \int_{B(x_0,R)} |h|^2 \, dx \right)^{1/2} \\ &\times (\sup_{\mathbb{R}^3} |\eta| + R \sup_{\mathbb{R}^3} |\nabla\eta|) \left(R^{1/2} \sup_{\mathbb{R}^3} |\eta| \sup_{\mathbb{R}^2} |\chi| + R^{-1/2} \int_{\mathbb{R}^2} |\eta||D'\chi| \right). \end{aligned}$$

The conclusion is now straightforward. □

If one drops the test function η and localizes the function χ in Corollary 2, the following result comes out:

Corollary 3 *Let $R > 0$ and $x_0 = (x_0', 0) \in \mathbb{R}^2 \times \{0\}$. Consider $h : \mathbb{R}^3 \to \mathbb{R}^3$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be related by*

$$\int_{\mathbb{R}^3} h \cdot \nabla\zeta \, dx = \int_{\mathbb{R}^2} \sigma\zeta \, dx', \quad \forall \zeta \in C_c^\infty(B(x_0, 2R)).$$

Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function of bounded variation such that

$$\operatorname{supp}\chi \subseteq \bar{B}'(x_0', R) \subset \mathbb{R}^2.$$

Then there exists a constant $C(R) > 0$ only depending on R such that for all $\varepsilon \in (0, 2R]$,

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \chi\sigma \, dx'\right| \leq & \left(\frac{4}{\pi}|\ln\varepsilon| \sup_{\mathbb{R}^2}|\chi| \int_{\mathbb{R}^2} |D'\chi| \int_{B(x_0,2R)} |h|^2 \, dx\right)^{1/2} \\ & + C(R)\left(\varepsilon \int_{B'(x_0',2R)} \sigma^2 \, dx' + \int_{B(x_0,2R)} |h|^2 \, dx\right)^{1/2} \\ & \times \left(\sup_{\mathbb{R}^2}|\chi| + \int_{\mathbb{R}^2} |D'\chi|\right).\end{aligned}$$

Proof. Let $\eta \in C_c^\infty(B(x_0, 2R))$ be such that

$$\eta = 1 \text{ in } B'(x_0', R) \times \{0\},\ |\eta| \leq 1 \text{ and } |\nabla\eta| \leq \frac{C}{R} \text{ in } B(x_0, 2R), \tag{54}$$

where $C > 0$ is some generic constant. We apply Corollary 2:

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \eta^2\chi\sigma dx'\right| \overset{(54)}{\leq} & \left(\frac{4}{\pi}|\ln\varepsilon| \sup_{\mathbb{R}^2}|\chi| \int_{\mathbb{R}^2} |D'\chi| \int_{B(x_0,2R)} |h|^2 \, dx\right)^{1/2} \\ & + C(1+|\ln R|^{1/2})\left(\varepsilon \int_{B'(x_0',2R)} \sigma^2 \, dx' + \int_{B(x_0,2R)} |h|^2 \, dx\right)^{1/2} \\ & \times \left(R^{1/2}\sup_{\mathbb{R}^2}|\chi| + R^{-1/2}\int_{\mathbb{R}^2} |D'\chi|\right),\end{aligned}$$

and the conclusion is straightforward. □

A periodic version of Proposition 1 is the following:

Corollary 4 *Let $L > 0$ be a positive number. Consider $h : \mathbb{R}^3 \to \mathbb{R}^3$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be related by*

$$\int_{\mathbb{R}^3} h \cdot \nabla\zeta \, dx = \int_{\mathbb{R}^2} \sigma\zeta \, dx', \quad \forall\zeta \in C_c^\infty(\mathbb{R}^3).$$

Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function of bounded variation in $\mathbb{R} \times [0, L)$ and $\eta \in C^\infty(\mathbb{R}^3)$ be such that

$$\operatorname{supp}\eta \subset (-2, 2) \times \mathbb{R} \times (-1, 1). \tag{55}$$

Assume that the functions

$$h, \sigma, \chi \text{ and } \eta \text{ are } L-\text{periodic in } x_2. \tag{56}$$

Then there exists a constant $C(L) > 0$ only depending on L such that for all $\varepsilon \in (0, L]$,

$$\begin{aligned}\left|\int_{\mathbb{R}\times[0,L)} \eta^2\chi\sigma \, dx'\right| \leq & \left(\frac{4}{\pi}|\ln\varepsilon| \sup_{\mathbb{R}^2}|\chi| \int_{\mathbb{R}\times[0,L)} \eta^2|D'\chi| \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2|h|^2 \, dx\right)^{1/2} \\ & + C(L)\left(\varepsilon \int_{\mathbb{R}\times[0,L)} \sigma^2 \, dx' + \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} |h|^2 \, dx\right)^{1/2} \\ & \times (\sup_{\mathbb{R}^3}|\eta| + \sup_{\mathbb{R}^3}|\nabla\eta|)\left(\sup_{\mathbb{R}^3}|\eta| \sup_{\mathbb{R}^2}|\chi| + \int_{\mathbb{R}\times[0,L)} |\eta||D'\chi|\right).\end{aligned} \tag{57}$$

Proof. Select a universal $\zeta \in C_c^\infty(\mathbb{R})$ such that

$$\operatorname{supp} \zeta \subset (-1,1),\ |\zeta| \leq 1,\ \sum_{k\in\mathbb{Z}} \zeta^2(x_2+k) = 1,\ \forall x_2 \in \mathbb{R} \tag{58}$$

and set

$$\tilde{\eta}(x_1,x_2,x_3) = \zeta(\frac{x_2}{L})\eta(x_1,x_2,x_3),\ \forall (x_1,x_2,x_3) \in \mathbb{R}^3. \tag{59}$$

In view of (55) and (58) we have that

$$\operatorname{supp} \tilde{\eta} \subset B_R$$

for some radius

$$\tilde{L} \leq R \leq 2\tilde{L}, \tag{60}$$

where $\tilde{L} = \max\{2, L\}$. Hence, we may apply (53) to σ, h, χ and $\tilde{\eta}$. Notice that because of (56) and (58),

$$\begin{aligned}
\int_{\mathbb{R}^2} \tilde{\eta}^2 \chi\sigma\, dx' &= \int_{\mathbb{R}\times[0,L)} \eta^2\chi\sigma\, dx',\\
\int_{\mathbb{R}^3} \tilde{\eta}^2 |h|^2\, dx &= \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2 |h|^2\, dx,\\
\int_{\mathbb{R}^2} \tilde{\eta}^2 |D'\chi| &= \int_{\mathbb{R}\times[0,L)} \eta^2 |D'\chi|,\\
\int_{\mathbb{R}^2} |\tilde{\eta}||D'\chi| &\leq C \int_{\mathbb{R}\times[0,L)} |\eta||D'\chi|.
\end{aligned}$$

Furthermore, we have because of (56) and (60),

$$\begin{aligned}
\int_{B'_R} \sigma^2\, dx' &\leq C\frac{\tilde{L}}{L} \int_{\mathbb{R}\times[0,L)} \sigma^2\, dx',\\
\int_{B_R} |h|^2\, dx &\leq C\frac{\tilde{L}}{L} \int_{\mathbb{R}\times[0,L)\times\mathbb{R}} |h|^2\, dx.
\end{aligned}$$

Finally, it follows from (58) and (59),

$$\begin{aligned}
\sup_{\mathbb{R}^3} |\tilde{\eta}| &\leq \sup_{\mathbb{R}^3} |\eta|\\
\sup_{\mathbb{R}^3} |\nabla\tilde{\eta}| &\leq \frac{C}{L} \sup_{\mathbb{R}^3} |\eta| + \sup_{\mathbb{R}^3} |\nabla\eta|.
\end{aligned}$$

Hence, (53) yields (57). □

Remark 3 The conclusion of Corollary 4 holds true for a more general support of η than (55) (for example, $(-a,a)\times\mathbb{R}\times(a,a)$ for every $a > 0$). The choice of the interval $(-2,2)$ in (55) (as support in x_1 variable) is needed in the proof of Theorem 1 due to the choice of the boundary data (1).

3 Compactness of the Néel wall

This section is devoted to the proof of the compactness result for magnetizations in the energy regime $O(\frac{1}{|\ln \varepsilon|})$:

Proof of Theorem 4. We proceed in several steps:
Step 1. *Some preliminaries.* Since $|m'_k| = 1$ in B'_1, it results that the sequence $\{\|m'_k\|_{L^\infty(B'_1)}\}_{k\in\mathbb{N}}$ is bounded and therefore, there exists $m' \in L^\infty(B'_1, \mathbb{R}^2)$ such that up to a subsequence,

$$m'_k \overset{w^*}{\rightharpoonup} m' \quad \text{weakly}^* \text{ in } L^\infty. \tag{61}$$

In particular,

$$|m'| \leq 1 \text{ a.e. in } B'_1. \tag{62}$$

In order to have the strong convergence in some L^p with $1 \leq p < \infty$, we need to show that $|m'| = 1$ a.e. in B'_1. Indeed, that will imply $\|m'_k\|_{L^2(B'_1)} \to \|m'\|_{L^2(B'_1)}$ and by the weak convergence in L^2, it will lead to the strong convergence in L^2 and then, in any other L^p, $1 \leq p < \infty$.

We introduce the finite positive measures $\{e_k\}_{k\in\mathbb{N}} \subset \mathcal{M}(B_1)$ as

$$\int_{\mathbb{R}^3} \zeta\, de_k = \frac{2}{\pi}|\ln \varepsilon_k| \left(\varepsilon_k \int_{\mathbb{R}^2} \zeta |\nabla' \cdot m'_k|^2 \, dx' + \int_{\mathbb{R}^3} \zeta |h_k|^2 \, dx \right), \quad \forall \zeta \in C_c(B_1). \tag{63}$$

Then by (17), the family of positive measures $\{e_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{M}(B_1)$ and hence, there exists a positive measure $e \in \mathcal{M}(B_1)$ such that

$$e_k \overset{w^*}{\rightharpoonup} e \quad \text{weakly}^* \text{ in } \mathcal{M}(B_1) \tag{64}$$

for a subsequence.
Step 2. *Some topology.* Let $x'_0 \in B'_1$. Using the technique in [5], we will identify the "characteristic" of m' passing at x'_0. Recall that the admissible magnetizations m'_k are assumed to be smooth and let us consider the autonomous equation

$$\dot{X} = {m'_k}^\perp(X). \tag{65}$$

First of all, (65) has no critical point and no cycle (i.e., closed orbit): Since $|{m'_k}^\perp| = 1$ in B'_1 and ${m'_k}^\perp$ is smooth, the degree of ${m'_k}^\perp$ on a closed curve in B'_1 is zero and therefore, an orbit of (65) cannot be closed. Now set X_k be the orbit of (65) passing by x'_0 (see Figure 10), i.e.,

$$\begin{cases} \dot{X}_k(t) = {m'_k}^\perp(X_k(t)), \\ X_k(0) = x'_0. \end{cases}$$

Then either the orbit X_k reaches the boundary $\partial B'_1$ in finite time, or the limit points of X_k (see [4], Chapter 16) belong to the boundary $\partial B'_1$: Suppose that this is not the case, i.e., there is a limit point inside the ball B'_1. Since (65) has no critical point, Poincaré-Bendixson's theorem (see [4], Theorem 2.1) implies that the limit set of X_k should contain a periodic orbit which is a contradiction with the nonexistence of cycles for (65). Hence, the orbit X_k separates the ball B'_1 into a right side G'_k (where m'_k is the inner normal vector to $\partial G'_k$) and a left side $B'_1 \setminus G'_k$. We define

$$\chi_k = \begin{cases} \frac{1}{2} & \text{in} \quad G'_k, \\ -\frac{1}{2} & \text{in} \quad B'_1 \setminus G'_k. \end{cases} \tag{66}$$

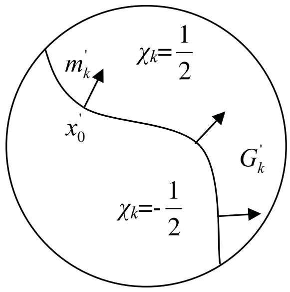

Figure 10: The orbit X_k of the vector field ${m'_k}^\perp$ passing by x'_0 in the ball B'_1

Then $\chi_k \in BV_{loc}(B'_1)$ with

$$D'\chi_k = m'_k\,\mathcal{H}^1 \llcorner X_k = m'_k\,|D'\chi_k|. \tag{67}$$

Moreover, in the ball $B'(x'_0, 1-|x'_0|) \subset B'_1$ we have that for every $r \in (0, 1-|x'_0|)$,

$$\int_{B'(x'_0,r)} |D'\chi_k| = \mathcal{H}^1(\{X_k \in B'(x'_0,r)\}) \geq 2r. \tag{68}$$

Step 3. *The sequence $\{\chi_k\}_{k\in\mathbb{N}}$ is uniformly locally bounded in $BV(B'_1)$ and any accumulation point χ of $\{\chi_k\}_{k\uparrow\infty}$ in $L^1_{loc}(B'_1)$ belongs to $BV_{loc}(B'_1, \{-\frac{1}{2}, \frac{1}{2}\})$.* (The jump set of χ is concentrated on the characteristic of m' passing at x'_0, for almost all $x'_0 \in B'_1$.) It is enough to prove that $\{\chi_k\}_{k\in\mathbb{N}}$ is bounded in $BV(B'(z'_0,r))$ for any $z'_0 \in B'_1$ such that $B'(z'_0, 2r) \subset B'_1$. We apply Corollary 3 in the ball $B(z_0, 2r)$ for the restriction of $\chi_k\big|_{B'(z'_0,r)}$ where $z_0 = (z'_0, 0) \in B_1$:

$$\begin{aligned}
\Big|\int_{B'(z'_0,r)} \chi_k \nabla'\cdot m'_k\,dx'\Big| &\overset{(66)}{\leq} \Big(\frac{2}{\pi}|\ln\varepsilon_k| \int_{\mathbb{R}^2} \big|D'\big(\chi_k\big|_{B'(z'_0,r)}\big)\big| \int_{B_1} |h_k|^2\,dx\Big)^{1/2} \\
&\qquad + C(r)\big(\frac{1}{|\ln\varepsilon_k|} e_k(B_1)\big)^{1/2}\Big(1 + \int_{\mathbb{R}^2} \big|D'\big(\chi_k\big|_{B'(z'_0,r)}\big)\big|\Big) \\
&\leq \frac{1}{2}\int_{\mathbb{R}^2} \big|D'\big(\chi_k\big|_{B'(z'_0,r)}\big)\big| + \frac{1}{2} e_k(B_1) \\
&\qquad + C(r)\big(\frac{1}{|\ln\varepsilon_k|} e_k(B_1)\big)^{1/2}\Big(1 + \int_{\mathbb{R}^2} \big|D'\big(\chi_k\big|_{B'(z'_0,r)}\big)\big|\Big) \\
&\leq \Big(\frac{1}{2} + C(r)\big(\frac{1}{|\ln\varepsilon_k|} e_k(B_1)\big)^{1/2}\Big) \int_{\mathbb{R}^2} \big|D'\big(\chi_k\big|_{B'(z'_0,r)}\big)\big| \\
&\qquad + \frac{1}{2} e_k(B_1) + C(r)\big(\frac{1}{|\ln\varepsilon_k|} e_k(B_1)\big)^{1/2},
\end{aligned}$$

where $C(r) > 0$ denotes a generic constant only depending on r and we used the Cauchy-Schwarz inequality. Since

$$\int_{\mathbb{R}^2} \big|D'\big(\chi_k\big|_{B'(z'_0,r)}\big)\big| \overset{(66)}{\leq} \int_{B'(z'_0,r)} |D'\chi_k| + \pi r,$$

we deduce that

$$\begin{aligned}
\Big|\int_{B'(z'_0,r)} \chi_k \nabla'\cdot m'_k\,dx'\Big| \leq &\Big(\frac{1}{2} + C(r)\big(\frac{1}{|\ln\varepsilon_k|} e_k(B_1)\big)^{1/2}\Big) \int_{B'(z'_0,r)} |D'\chi_k| \\
&+ \frac{1}{2} e_k(B_1) + C(r)\Big(1 + \big(\frac{1}{|\ln\varepsilon_k|} e_k(B_1)\big)^{1/2}\Big).
\end{aligned} \tag{69}$$

By (67), the integration by parts leads to

$$\int_{B'(z_0',r)} \chi_k \,\nabla'\cdot m_k'\, dx' = \int_{\partial B'(z_0',r)} \chi_k^- m_k' \cdot \nu\, d\mathcal{H}^1 - \int_{B'(z_0',r)} |D'\chi_k|,$$

where χ_k^- denotes the interior trace of $\chi_k\big|_{\partial B'(z_0',r)}$ and ν is the unit outer normal vector on $\partial B'(z_0',r)$. In combination with (69), this yields

$$\begin{aligned}
&\Big(\frac{1}{2} - C(r)\big(\frac{1}{|\ln\varepsilon_k|}e_k(B_1)\big)^{1/2}\Big)\int_{B'(z_0',r)} |D'\chi_k| \\
&\quad\le \int_{\partial B'(z_0',r)} \chi_k^- m_k'\cdot\nu\, d\mathcal{H}^1 + \frac{1}{2}e_k(B_1) + C(r)\Big(1 + \big(\frac{1}{|\ln\varepsilon_k|}e_k(B_1)\big)^{1/2}\Big) \\
&\quad\le \frac{1}{2}e_k(B_1) + C(r)\Big(1 + \big(\frac{1}{|\ln\varepsilon_k|}e_k(B_1)\big)^{1/2}\Big).
\end{aligned}$$

By assumption, $\varepsilon_k \to 0$ and the sequence $\{e_k(B_1)\}$ is bounded, this estimate implies the boundedness of $\Big\{\int_{B'(z_0',r)} |D'\chi_k|\Big\}$. Since z_0' was arbitrarily chosen, we conclude that $\{\chi_k\}$ is uniformly locally bounded in $BV(B_1')$. Thus, there exists a function $\chi \in BV_{loc}(B_1', \{-\frac{1}{2}, \frac{1}{2}\})$ such that up to a subsequence,

$$\chi_k \to \chi \quad \text{in} \quad L^1(B_1'). \tag{70}$$

Step 4. *We prove that*

$$-\int_{\mathbb{R}^2} \chi m'\cdot\nabla'\eta^2\, dx' + \frac{1}{4\delta}\int_{\mathbb{R}^3} \eta^2\, de \ge 2(1-\delta)\int_0^\infty \eta^2\, dr, \tag{71}$$

for any small $\delta > 0$ and any $\eta \in C_c^\infty(B(x_0,R))$ such that η^2 is a decreasing function of $r = |x - x_0|$ only, where $R \in (0, 1-|x_0'|]$. For such an η, (68) implies that

$$\int_{\mathbb{R}^2} \eta^2 |D'\chi_k| = \int_0^\infty \Big(-\frac{d}{dr}\eta^2\Big)\int_{B'(x_0',r)} |D'\chi_k|\, dr \ge 2\int_0^\infty \Big(-\frac{d}{dr}\eta^2\Big) r\, dr = 2\int_0^\infty \eta^2\, dr. \tag{72}$$

Using (67), the integration by parts leads to

$$-\int_{\mathbb{R}^2} \chi_k m_k'\cdot\nabla'\eta^2\, dx' = \int_{\mathbb{R}^2} \eta^2\chi_k\,\nabla'\cdot m_k'\, dx' + \int_{\mathbb{R}^2} \eta^2 |D'\chi_k|. \tag{73}$$

We apply Corollary 2:

$$\begin{aligned}
\Big|\int_{\mathbb{R}^2} \eta^2\chi_k\nabla'\cdot m_k'\, dx'\Big| &\overset{(66)}{\le} \Big(\frac{2}{\pi}|\ln\varepsilon_k|\int_{\mathbb{R}^2} \eta^2|D'\chi_k|\int_{B_1} \eta^2|h_k|^2\, dx\Big)^{1/2} \\
&\qquad + C(R,\eta)\big(\frac{1}{|\ln\varepsilon_k|}e_k(B_1)\big)^{1/2}\Big(1 + \int_{\mathbb{R}^2} |\eta||D'\chi_k|\Big) \\
&\le \delta\int_{\mathbb{R}^2} \eta^2|D'\chi_k| + \frac{1}{4\delta}\int_{\mathbb{R}^3} \eta^2\, de_k \\
&\qquad + C(R,\eta)\big(\frac{1}{|\ln\varepsilon_k|}e_k(B_1)\big)^{1/2}\Big(1 + \int_{\mathbb{R}^2} |\eta||D'\chi_k|\Big),
\end{aligned}$$

where $C(R,\eta) > 0$ denotes a generic constant only depending on R and η and we used Young's inequality for some small $\delta > 0$. Combining with (72) and (73), this yields

$$\begin{aligned}
&-\int_{\mathbb{R}^2} \chi_k m_k' \cdot \nabla' \eta^2 \, dx' + \frac{1}{4\delta} \int_{\mathbb{R}^3} \eta^2 \, de_k \\
&\quad \geq (1-\delta) \int_{\mathbb{R}^2} \eta^2 |D'\chi_k| - C(R,\eta) \big(\frac{1}{|\ln \varepsilon_k|} e_k(B_1) \big)^{1/2} \Big(1 + \int_{\mathbb{R}^2} |\eta||D'\chi_k| \Big) \\
&\quad \geq 2(1-\delta) \int_0^\infty \eta^2 \, dr - C(R,\eta) \big(\frac{1}{|\ln \varepsilon_k|} e_k(B_1) \big)^{1/2} \Big(1 + \int_{\mathbb{R}^2} |\eta||D'\chi_k| \Big).
\end{aligned} \tag{74}$$

We may pass to the limit $k \to \infty$ in order to conclude with (71). Indeed, by (61) and (70), it follows that

$$\int_{\mathbb{R}^2} \chi_k m_k' \cdot \nabla' \eta^2 \, dx' \to \int_{\mathbb{R}^2} \chi m' \cdot \nabla' \eta^2 \, dx'.$$

Because of (64), we have

$$\frac{1}{4\delta} \int_{\mathbb{R}^3} \eta^2 \, de_k \to \frac{1}{4\delta} \int_{\mathbb{R}^3} \eta^2 \, de.$$

Because of assumption (17), Step 3 and $\varepsilon_k \to 0$, the last term in (74) vanishes as $k \to \infty$.

Step 5. *We show that* $|m'| = 1$ *a.e. in* B_1'. Let $x_0' \in B_1'$ be a Lebesgue point of m' and of vanishing $\mathcal{H}^1$-density of e, i.e.,

$$\lim_{r \to 0} \frac{1}{R^2} \int_{B'(x_0',R)} |m'(x') - m'(x_0')| \, dx' = 0 \quad \text{and} \quad \limsup_{R \to 0} \frac{e(B(x_0,R))}{R} = 0 \tag{75}$$

where $x_0 = (x_0', 0)$. (By Lebesgue decomposition theorem and Vitali covering lemma, almost every point in B_1' has the above properties). By (62) and (75), it follows that $|m'(x_0')| \leq 1$. In order to have that $|m'(x_0')| = 1$, we show that $|m'(x_0')| \geq 1$. For that, we now specify in (71) that η^2 is of the form:

$$\eta^2(r) = \eta_R^2(r) := \frac{1}{R} \eta_1^2 (\frac{r}{R}), \quad \forall r \in (0,R),$$

where $\eta_1^2 \in C_c^\infty(B(x_0,1))$ is a decreasing function of $r = |x - x_0|$ only. Since $\eta_R^2 \leq \frac{C}{R}$ and $|\nabla' \eta_R^2| \leq \frac{C}{R^2}$, by (75), we have that

$$\lim_{R \to 0} \int_{\mathbb{R}^3} \eta_R^2 \, de = 0,$$

$$\lim_{R \to 0} \left| \int_{\mathbb{R}^2} \chi m' \cdot \nabla' \eta_R^2 \, dx' - \int_{\mathbb{R}^2} \chi m'(x_0') \cdot \nabla' \eta_R^2 \, dx' \right| = 0.$$

Hence, we obtain from (71),

$$-\limsup_{R \to 0} \int_{\mathbb{R}^2} \chi m'(x_0') \cdot \nabla' \eta_R^2 \, dx' \geq 2(1-\delta) \lim_{R \to 0} \int_0^\infty \eta_R^2 \, dr = 2(1-\delta) \int_0^\infty \eta_1^2 \, dr.$$

Since $\delta > 0$ was arbitrary, this leads to

$$-\limsup_{R \to 0} \int_{\mathbb{R}^2} \chi m'(x_0') \cdot \nabla' \eta_R^2 \, dx' \geq 2 \int_0^\infty \eta_1^2 \, dr. \tag{76}$$

On the other hand, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\chi m'(x_0')\cdot\nabla'\eta_R^2\,dx'\right| &\le |m'(x_0')|\int_{\mathbb{R}^2}\left|\frac{\partial}{\partial x_1}\eta_R^2\right|dx'\\ &= |m'(x_0')|\int_{\mathbb{R}^2}\left|\frac{\partial}{\partial x_1}\eta_1^2\right|dx'\\ &= |m'(x_0')|\int_0^\infty\Big(-\frac{d}{dr}\eta_1^2\Big)\,r\,dr\int_0^{2\pi}|\cos\theta|\,d\theta\\ &= 2|m'(x_0')|\int_0^\infty\eta_1^2\,dr.\end{aligned}$$

Thus, we obtain from (76) that

$$2|m'(x_0')|\int_0^\infty\eta_1^2\,dr\ge 2\int_0^\infty\eta_1^2\,dr,$$

i.e., $|m'(x_0')|\ge 1$.

Step 6. *End of proof.* Let now m' be an accumulation point of the sequence $\{m_k'\}_{k\uparrow\infty}$ in $L^1_{loc}(B_1')$. Since $|m_k'|=1$, we deduce that $|m'|=1$ a.e. in B_1'. By (17), we have that $\int_{B_1}|h_k|^2\,dx\to 0$ as $k\to\infty$ and therefore, (16) yields that

$$\lim_{k\to\infty}\int_{B_1'}\zeta\,\nabla'\cdot m_k'\,dx'=0,\forall\zeta\in C_c^\infty(B_1').$$

Thus, $\nabla'\cdot m'=0$ distributionally in B_1'. □

4 Zero-energy states

Recall that a zero-energy state is an accumulation point $m':B_1'\to S^1$ of a sequence of magnetizations $\{m_k':B_1'\to S^1\}_{k\uparrow\infty}$ in $L^1_{loc}(B_1')$ such that (19) holds for a sequence $\varepsilon_k\to 0$ and some stray fields $\{h_k:B_1\to\mathbb{R}^3\}_{k\uparrow\infty}$ related to $\{m_k'\}$ by (16). In order to prove Theorem 5, we proceed in several steps. A key ingredient to Theorem 5 is the following additional property of limits m':

Lemma 1 *Next to (18), any accumulation point $m':B_1'\to\mathbb{R}^2$ of $\{m_k'\}_{k\uparrow\infty}$ in $L^1(B_1')$ has the following property: For all $x_0'\in B_1'$ there exists $\chi:B_1'\to\{-\frac12,\frac12\}$ such that*

$$\nabla'\cdot(\chi m')=|D'\chi|\quad\textit{distributionally in}\ \ B_1',\tag{77}$$

$$\int_{B'(x_0',r)}|D'\chi|\ge 2r,\quad\textit{for all}\ \ 0<r<1-|x_0'|.\tag{78}$$

Proof of Lemma 1. Let $x_0'\in B_1'$ be given. Let $\{\chi_k\}$ be defined in B_1' as in Step 2 the proof of Theorem 4 (see (66)). By Step 3 in the proof of Theorem 4, we know that the sequence

$$\left\{\int_{B_r'}|D'\chi_k|\right\}_{k\uparrow\infty}\text{ is bounded for all }0<r<1.\tag{79}$$

Hence, after passage to a subsequence, we may assume that there exists $\chi : B_1' \to \{-\frac{1}{2}, \frac{1}{2}\}$ of locally bounded variation such that

$$\chi_k \to \chi \text{ in } L^1(B_1'). \tag{80}$$

It remains to argue that χ satisfies (77) and (78). For a given $\zeta \in C_c^\infty(B_1')$, we shall establish the following four statements:

$$-\int_{B_1'} \chi_k \nabla'\zeta \cdot m_k' \, dx' - \int_{B_1'} \zeta |D'\chi_k| \to 0, \tag{81}$$

$$-\int_{B_1'} \chi \nabla'\zeta \cdot m' \, dx' - \int_{B_1'} \zeta |D'\chi| \geq 0 \quad \text{if } \zeta \geq 0, \tag{82}$$

$$-\int_{B_1'} \chi \nabla'\zeta \cdot m' \, dx' - \int_{B_1'} \zeta |D'\chi| \leq 0 \quad \text{if } \zeta \geq 0, \tag{83}$$

$$\int_{B_1'} \zeta |D'\chi_k| \to \int_{B_1'} \zeta |D'\chi| \quad \text{if } \zeta \geq 0. \tag{84}$$

In order to establish (81), we will use again the identity (67) based on the construction of χ_k, i.e., $m_k' \cdot D'\chi_k = |D'\chi_k|$; namely,

$$\begin{aligned}
-\int_{B_1'} \nabla'\zeta \cdot m_k'\chi_k \, dx' - \int_{B_1'} \zeta |D'\chi_k| &= \int_{B_1'} \zeta\chi_k \, \nabla'\cdot m_k' \, dx' + \int_{B_1'} \zeta m_k' \cdot D'\chi_k - \int_{B_1'} \zeta |D'\chi_k| \\
&= \int_{B_1'} \zeta\chi_k \, \nabla'\cdot m_k' \, dx'.
\end{aligned} \tag{85}$$

The second ingredient is Corollary 1, applied for the function $\zeta\chi_k$ in the ball B_1' et $d = \operatorname{dist}(\operatorname{supp}\zeta, \partial B_1')$. Because of $\sup|\chi_k| = \frac{1}{2}$, we obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} (\zeta\chi_k) \nabla' \cdot m_k' \, dx' \right| \leq & \left(\frac{2}{\pi} |\ln \varepsilon_k| \sup|\zeta| \int_{B_1'} |D'(\zeta\chi_k)| \int_{B_1} |h_k|^2 \, dx \right)^{1/2} \\
& + \frac{C}{d} \left(\varepsilon_k \int_{B_1'} |\nabla' \cdot m_k'|^2 \, dx' + \int_{B_1} |h_k|^2 \, dx \right)^{1/2} \\
& \times \left(\sup|\zeta| + \int_{B_1'} |D'(\zeta\chi_k)| \right).
\end{aligned}$$

Since $|D'(\zeta\chi_k)| \leq \dfrac{1}{2}|\nabla'\zeta| + |\zeta||D'\chi_k|$, by (79) we deduce that the sequence $\left\{ \int_{B_1'} |D'(\zeta\chi_k)| \right\}$ is bounded and by (19), it follows that

$$\int_{B_1'} \zeta\chi_k \, \nabla'\cdot m_k' \, dx' \to 0 \text{ as } k \to \infty. \tag{86}$$

Now (81) follows from (85) and (86). Statement (82) follows easily from (81). Indeed, because of (80) and $m_k' \to m'$ in $L^1(B_1')$, we have

$$\int_{B_1'} \chi_k \nabla'\zeta \cdot m_k' \, dx' \to \int_{B_1'} \chi \nabla'\zeta \cdot m' \, dx'; \tag{87}$$

on the other hand, the lower semicontinuity of $|D'\chi_k|$ under (80) implies

$$\int_{B_1'} \zeta |D'\chi| \le \liminf_{k\to\infty} \int_{B_1'} \zeta |D'\chi_k| \quad \text{if } \zeta \ge 0 \text{ in } B_1'.$$

Statement (83) is a general fact which follows from (18). Indeed, let $\{m_\delta'\}_{\delta\downarrow 0}$ denote the mollification of m' by convolution. For any $r<1$ and sufficiently small δ, we then have in a classical sense:

$$\nabla'\cdot m_\delta' = 0 \text{ and } |m_\delta'|^2 \le 1 \text{ in } B_r'. \tag{88}$$

Therefore,

$$\int_{B_1'} \chi\nabla'\zeta\cdot m_\delta'\,dx' \overset{(88)}{=} \int_{B_1'} \chi\nabla'\cdot(\zeta m_\delta')\,dx' = -\int_{B_1'} \zeta m_\delta'\cdot D'\chi \overset{(88)}{\le} \int_{B_1'} \zeta|D'\chi| \quad \text{if } \zeta\ge 0.$$

Statement (84) is a straightforward consequence of the previous ones:

$$\lim_{k\to\infty}\int_{B_1'} \zeta|D'\chi_k| \overset{(81)}{=} -\lim_{k\to\infty}\int_{B_1'} \chi_k\nabla'\zeta\cdot m_k'\,dx' \overset{(87)}{=} -\int_{B_1'} \chi\nabla'\zeta\cdot m'\,dx' \overset{(82),(83)}{=} \int_{B_1'} \zeta|D'\chi|,$$

if $\zeta \ge 0$.

We now argue that (77) and (78) are true. We start with (77). From (82) and (83), we already know that

$$-\int_{B_1'} \chi\nabla'\zeta\cdot m'\,dx' = \int_{B_1'} \zeta|D'\chi| \quad \text{for all } \zeta\in C_c^\infty(B_1') \text{ with } \zeta\ge 0. \tag{89}$$

Since any $\zeta\in C_c^\infty(B_1')$ can be approximated both in $H^1(B_1')$ and $C_c(B_1')$ by ζ_δ's of the form

$$\zeta_\delta = \zeta_\delta^+ - \zeta_\delta^- \text{ with the positive and negative parts } \zeta_\delta^+,\zeta_\delta^-\in C_c^\infty(B_1'), \tag{90}$$

(89) implies (77). An approximation of the form (90) can be constructed as follows

$$\zeta_\delta = \phi_\delta(\zeta),$$

where $\{\phi_\delta\}_{\delta\downarrow 0}\subset C^\infty(\mathbb{R})$ is an approximation of the identity with the following properties:

$$\phi_\delta(t) = 0 \text{ for } |t|\le\delta,\ \frac{d\phi_\delta}{dt}(t)\to 1 \text{ for } t\ne 0,\ \left|\frac{d\phi_\delta}{dt}(t)\right| \le 1 \text{ for all } t.$$

We now address (78). Let $0<r<1-|x_0'|$ be given. We will derive (78) from the corresponding property (68) of χ_k and (84). Let $\{\eta_\delta\}_{\delta\downarrow 0}\subset C_c^\infty(B_1')$ be an approximation of the characteristic function $1_{B'(x_0',r)}$ in the following sense

$$\eta_\delta(x') = 0 \text{ for } x'\notin B'(x_0',r),\ \eta_\delta(x') = 1 \text{ for } x'\in B'(x_0',r-\delta),\ 0\le\eta_\delta(x')\le 1 \text{ for } x'\in B_1'.$$

We have

$$\int_{B'(x_0',r)} |D'\chi| \ge \int_{B_1'} \eta_\delta|D'\chi| \overset{(84)}{=} \lim_{k\to\infty}\int_{B_1'} \eta_\delta|D'\chi_k| \ge \liminf_{k\to\infty}\int_{B'(x_0',r-\delta)} |D'\chi_k| \overset{(68)}{\ge} 2(r-\delta),$$

and we conclude with (78) by passing $\delta\to 0$. $\square$

The next lemma establishes that the χ's from Lemma 1 are minimal (perimeter minimizing). It is a well-known general fact that sets whose normal can be extended to a divergence-free unit-length vector field are minimal.

Lemma 2 *Let $\chi : B_1' \to \{-\frac{1}{2}, \frac{1}{2}\}$ have the property* (77) *for some $m' : B_1' \to S^1$ with*

$$\nabla' \cdot m' = 0 \quad \textit{distributionally in } B_1'.$$

Then χ is minimal in B_1' in the sense that for any function $\tilde{\chi} : B_1' \to \{-\frac{1}{2}, \frac{1}{2}\}$ with $\mathrm{supp}(\tilde{\chi}-\chi) \subset\subset B_1'$*, we have*

$$|D'\chi|(B_1') \leq |D'\tilde{\chi}|(B_1').$$

Proof of Lemma 2 . Let $0 < r < 1$ be such that $\mathrm{supp}(\tilde{\chi} - \chi) \subset B_r'$. Select an $\zeta \in C_c^\infty(B_1')$ with $\zeta = 1$ in B_r' and $\zeta \geq 0$ in B_1'. Then we have

$$\begin{aligned} |D'\chi|(B_r') - |D'\tilde{\chi}|(B_r') &= \int_{B_1'} \zeta |D'\chi| - \int_{B_1'} \zeta |D'\tilde{\chi}| \\ &\overset{(77)}{=} - \int_{B_1'} \chi \nabla' \zeta \cdot m' \, dx' - \int_{B_1'} \zeta |D'\tilde{\chi}| \\ &= - \int_{B_1'} \tilde{\chi} \nabla' \zeta \cdot m' \, dx' - \int_{B_1'} \zeta |D'\tilde{\chi}|. \end{aligned}$$

The argument used to establish the inequality (83) in the proof of Lemma 1 also yields this lemma (with χ replaced by $\tilde{\chi}$). $\square$

For convenience of the reader, the following lemma gives an elementary proof for the fact that minimal sets in two dimensions are locally half-spaces.

Lemma 3 *Let $\chi : B_1' \to \{-\frac{1}{2}, \frac{1}{2}\}$ satisfy*

$$\chi \textit{ is minimal in } B_1', \tag{91}$$

$$\int_{B_r'} |D'\chi| \geq 2r \textit{ for all } r \in (0,1). \tag{92}$$

Then χ is the characteristic function of a centered half-space in $B_{1-\frac{\pi}{4}}'$ (see Figure 11), i.e., there exists $\nu \in S^1$ such that

$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \textit{for } x' \cdot \nu > 0 \\ -\frac{1}{2} & \textit{else} \end{array} \right\} \mathcal{L}^2\textit{-a.e. in } B_{1-\frac{\pi}{4}}'.$$

Proof of Lemma 3. We start by arguing that

$$|D'\chi|(B_1') \leq \pi. \tag{93}$$

Let $0 < r < 1$ be arbitrary. We compare χ to $\tilde{\chi}_+$, $\tilde{\chi}_-$ given by

$$\tilde{\chi}_+ = \left\{ \begin{array}{cc} \frac{1}{2} & \text{in } B_r' \\ \chi & \text{else} \end{array} \right\}, \quad \tilde{\chi}_- = \left\{ \begin{array}{cc} -\frac{1}{2} & \text{in } B_r' \\ \chi & \text{else} \end{array} \right\}.$$

By assumption (91), we obtain that

$$\begin{aligned} |D'\chi|(B_r') \leq \min \Big\{ & |D'\tilde{\chi}_-|(B_r') + \int_{\partial B_r'} |\chi^- - \tilde{\chi}_-^-| \, d\mathcal{H}^1, \\ & |D'\tilde{\chi}_+|(B_r') + \int_{\partial B_r'} |\chi^- - \tilde{\chi}_+^-| \, d\mathcal{H}^1 \Big\}, \end{aligned}$$

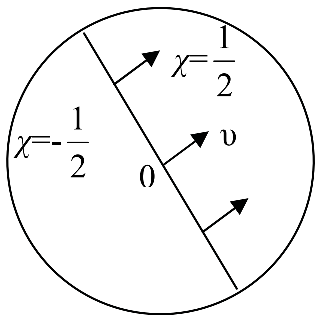

Figure 11: The characteristic χ in the ball $B'_{1-\frac{\pi}{4}}$

where χ^-, $\tilde\chi_-^-$ and $\tilde\chi_+^-$ denote the interior traces of $\chi\big|_{\partial B'_r}$, $\tilde\chi_-\big|_{\partial B'_r}$ and $\tilde\chi_+\big|_{\partial B'_r}$ respectively. In view of the form of $\tilde\chi_-$, $\tilde\chi_+$, this turns into

$$\begin{aligned}|D'\chi|(B'_r) &\leq \min\Big\{\int_{\partial B'_r}|\chi^-+\frac{1}{2}|\,d\mathcal{H}^1, \int_{\partial B'_r}|\chi^- -\frac{1}{2}|\,d\mathcal{H}^1\Big\}\\ &= \min\Big\{\pi r+\int_{\partial B'_r}\chi^-\,d\mathcal{H}^1, \pi r-\int_{\partial B'_r}\chi^-\,d\mathcal{H}^1\Big\}\\ &\leq \pi r.\end{aligned}$$

¿From this, we deduce (93) by monotone convergence under $r\uparrow 1$.

We now argue that there exists an $r\in[1-\frac{\pi}{4},1)$ such that

$$\int_{\partial B'_r}|D_\theta\chi^-|\in\{0,2\} \tag{94}$$

where $\int_{\partial B'_r}|D_\theta\chi^-|$ denotes the total variation of the trace χ^- on $\partial B'_r$. Indeed, we have

$$\begin{aligned}\mathcal{L}^1\Big(\Big\{r\in(0,1)\,:\,\int_{\partial B'_r}|D_\theta\chi^-|\geq 4\Big\}\Big) &\leq \frac{1}{4}\int_0^1\big(\int_{\partial B'_r}|D_\theta\chi^-|\big)\,dr\\ &\leq \frac{1}{4}|D'\chi|(B'_1)\overset{(93)}{\leq}\frac{\pi}{4}.\end{aligned}$$

Hence, there exists $1-\frac{\pi}{4}\leq r<1$ such that

$$\int_{\partial B'_r}|D_\theta\chi^-|<4. \tag{95}$$

But since $\chi^-\in\{-\frac{1}{2},\frac{1}{2}\}$, we have that $\int_{\partial B'_r}|D_\theta\chi^-|\in\{0,2,4,\dots\}$, so that (95) entails (94).

We now argue that there exists $\nu\in S^1$ such that

$$\chi^-=\left\{\begin{array}{cc}\frac{1}{2} & \text{for } x'\cdot\nu>0\\ -\frac{1}{2} & \text{else}\end{array}\right\}\ \mathcal{H}^1\text{-a.e. on } \partial B'_r, \tag{96}$$

where r is as in (94). Indeed, because of (94), there exist $\nu\in S^1$ and $\alpha\in\mathbb{R}$ such that

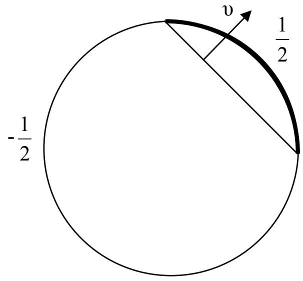

Figure 12: The trace χ^- on $\partial B'_r$

$$\chi^- = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > \alpha \\ -\frac{1}{2} & \text{else} \end{array} \right\} \mathcal{H}^1\text{-a.e. on } \partial B'_r \tag{97}$$

(see Figure 12). We compare χ with $\tilde{\chi}$ given by

$$\tilde{\chi} = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > \alpha \text{ and } x' \in B'_r, \\ -\frac{1}{2} & \text{for } x' \cdot \nu \le \alpha \text{ and } x' \in B'_r, \\ \chi & \text{else.} \end{array} \right\}$$

Because of (97), the traces of $\chi\big|_{\partial B'_r}$ and $\tilde{\chi}\big|_{\partial B'_r}$ coincide. Hence we obtain by the assumption (91),

$$|D'\chi|(B'_r) \le |D'\tilde{\chi}|(B'_r). \tag{98}$$

Because of assumption (92) this yields

$$2r \le \mathcal{H}^1(\{x' \cdot \nu = \alpha\} \cap B'_r),$$

which enforces $\alpha = 0$ so that (97) turns into (96). We finally argue that

$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > 0 \\ -\frac{1}{2} & \text{else} \end{array} \right\} \mathcal{L}^2\text{-a.e. in } B'_r, \tag{99}$$

where ν is as in (96). Indeed, (96) implies that

$$\int_{B'_r} \nu \cdot D'\chi = \int_{\partial B'_r} \nu \cdot \frac{x'}{r} \chi^- \, d\mathcal{H}^1 = 2r,$$

whereas (98) yields

$$|D'\chi|(B'_r) \le \mathcal{H}^1(\{x' \cdot \nu = 0\} \cap B'_r) \le 2r.$$

Hence we necessarily have

$$D'\chi = \nu |D'\chi| \quad |D'\chi|\text{-a.e. in } B'_r.$$

Since $\chi \in \{-\frac{1}{2}, \frac{1}{2}\}$, this implies that

$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > \alpha \\ -\frac{1}{2} & \text{else} \end{array} \right\} \mathcal{L}^2\text{-a.e. in } B'_r,$$

for some $\alpha \in \mathbb{R}$. Since its trace χ^- is given by (96), χ must indeed be of form (99). $\square$

The next lemma establishes that the characteristic functions from Lemma 1 are locally ordered.

Lemma 4 *Let $m' : B_1' \to \mathbb{R}^2$ satisfy* (18). *Let $\chi_0 : B_1' \to \{-\frac{1}{2}, \frac{1}{2}\}$ have the properties:*

- *χ_0 is the characteristic function of a centered half-space, i.e., there exists $\nu_0 \in S^1$ such that*

$$\chi_0 = \left\{ \begin{array}{cc} \frac{1}{2} & \textit{for } x' \cdot \nu_0 > 0 \\ -\frac{1}{2} & \textit{else} \end{array} \right\} \quad \textit{in } B_1';$$

- *χ_0 satisfies* (77).

Let $\chi : B_1' \to \{-\frac{1}{2}, \frac{1}{2}\}$ have the properties:

- *χ is the characteristic function of a half-space, i.e., there exist $\nu \in S^1$ and $\alpha \in \mathbb{R}$ such that*

$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \textit{for } x' \cdot \nu > \alpha \\ -\frac{1}{2} & \textit{else} \end{array} \right\} \quad \textit{in } B_1';$$

- *χ satisfies* (77).

Then $\chi \le \chi_0$ in $B'_{1-\frac{\pi}{4}}$ or $\chi \ge \chi_0$ in $B'_{1-\frac{\pi}{4}}$.

Proof of Lemma 4. We distinguish three cases.

Case 1: $\mathcal{H}^0(\{x' \cdot \nu_0 = 0\} \cap \{x' \cdot \nu = \alpha\}) \le 1$ and $\alpha \le 0$. In this case, we consider $\tilde{\chi}$ given by

$$\tilde{\chi} = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu_0 > 0 \text{ and } x' \cdot \nu > \alpha \\ -\frac{1}{2} & \text{else} \end{array} \right\} \quad \text{in } B_1'$$

(see Figure 13). We argue that

$$\nabla' \cdot (\tilde{\chi} m') = |D'\tilde{\chi}| \text{ distributionally in } B_1', \tag{100}$$

$$\int_{B_r'} |D'\tilde{\chi}| \ge 2r \text{ for all } r \in (0,1). \tag{101}$$

Indeed, (100) holds distributionally in

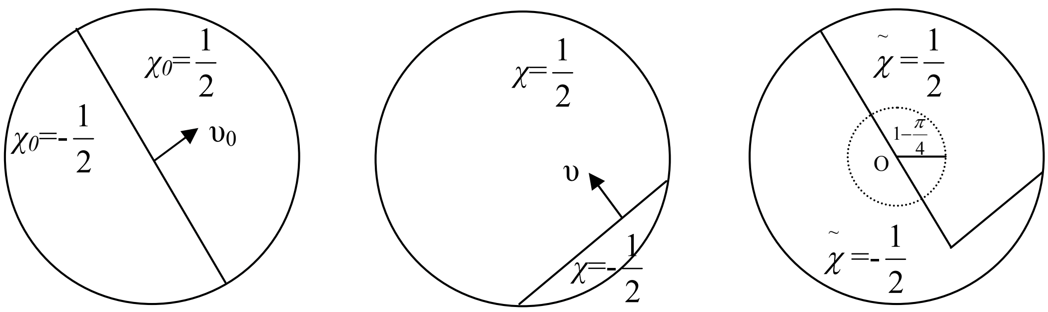

Figure 13: The characteristics χ_0, χ and $\tilde{\chi}$ in the ball B_1'

- $B_1' \cap \{x' \cdot \nu_0 > 0\}$, since there $\tilde{\chi} = \chi$, so that (100) follows from the property (77) of χ;

- $B'_1 \cap \{x' \cdot \nu_0 < 0\}$, since there $\tilde{\chi} = -\frac{1}{2}$, so that (100) follows from (18);
- $B'_1 \cap \{x' \cdot \nu > \alpha\}$, since there $\tilde{\chi} = \chi_0$, so that (100) follows from the property (77) of χ_0;
- $B'_1 \cap \{x' \cdot \nu < \alpha\}$, since there $\tilde{\chi} = -\frac{1}{2}$, so that (100) follows from (18).

Hence, (100) holds distributionally in $B'_1 \setminus (\{x' \cdot \nu_0 = 0\} \cap \{x' \cdot \nu = \alpha\})$. By assumption, $\{x' \cdot \nu_0 = 0\} \cap \{x' \cdot \nu = \alpha\}$ consists of at most a single point. But (100) is oblivious to sets of vanishing $\mathcal{H}^1$-measure. This establishes (100). (101) follows from the fact that $0 \in \partial(\{x' \cdot \nu_0 > 0\} \cap \{x' \cdot \nu > \alpha\})$, which is a consequence of our assumption $\alpha \leq 0$. According to Lemma 2, (18) and (100) imply that $\tilde{\chi}$ is minimal in B'_1. According to Lemma 3, this and (101) imply that $\tilde{\chi}$ is the characteristic function of a centered half-space in $B'_{1-\frac{\pi}{4}}$. Hence $\{x' \cdot \nu_0 > 0\} \cap \{x' \cdot \nu > \alpha\}$ is a centered half-space in $B'_{1-\frac{\pi}{4}}$. In view of $\alpha \leq 0$, this yields

$$\{x' \cdot \nu_0 > 0\} \cap \{x' \cdot \nu > \alpha\} \cap B'_{1-\frac{\pi}{4}} \{x' \cdot \nu_0 > 0\} \cap B'_{1-\frac{\pi}{4}},$$

that is

$$x' \cdot \nu > \alpha \text{ in } \{x' \cdot \nu_0 > 0\} \cap B'_{1-\frac{\pi}{4}},$$

whence $\chi \geq \chi_0$ in $B'_{1-\frac{\pi}{4}}$.

Case 2: $\mathcal{H}^0(\{x' \cdot \nu_0 = 0\} \cap \{x' \cdot \nu = \alpha\}) \leq 1$ and $\alpha \geq 0$. In this case, we consider $\tilde{\chi}$ given by

$$\tilde{\chi} = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu_0 > 0 \text{ or } x' \cdot \nu > \alpha \\ -\frac{1}{2} & \text{else} \end{array} \right\} \text{ in } B'_1$$

and we argue as before to arrive at $\chi \leq \chi_0$ in $B'_{1-\frac{\pi}{4}}$.

Case 3: $\mathcal{H}^0(\{x' \cdot \nu_0 = 0\} \cap \{x' \cdot \nu = \alpha\}) > 1$. In this case, we necessarily have

$$\alpha = 0 \quad \text{and} \quad (\nu = \nu_0 \text{ or } \nu = -\nu_0).$$

In the case of $\nu = \nu_0$, we have $\chi = \chi_0$. The case of $\nu = -\nu_0$ cannot occur since then

$$\chi_0 + \chi = 0 \;\; \mathcal{L}^2\text{-a.e. in } B'_1$$

so that (77) could yield

$$|D'\chi_0| + |D'\chi| = \nabla' \cdot (\chi_0 m') + \nabla' \cdot (\chi m') = 0,$$

in particular, $D'\chi_0 = 0$ which is a contradiction. $\square$

The next lemma establishes Lipschitz continuity of m' locally in B'_1. Because of translation and scaling invariance, it suffices to prove the following:

Lemma 5 *Let m' be as in Lemma 1. Let 0 and $y' \in B'_1$ be Lebesgue points of m'. Then*

$$|m'(y') - m'(0)| \leq \frac{2\sqrt{2}}{(1-\frac{\pi}{4})^2}|y'| \quad \textit{for all } y' \in B'_{\frac{1}{2}(1-\frac{\pi}{4})^2}.$$

Proof of Lemma 5. Let χ_0 and χ denote the characteristic functions associated to 0 and y' respectively, according to Lemma 1. According to Lemmas 2 and 3, there exist ν_0, $\nu \in S^1$ such that

$$\chi_0 = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu_0 > 0 \\ -\frac{1}{2} & \text{else} \end{array} \right\} \text{ in } B'_{1-\frac{\pi}{4}}, \tag{102}$$

$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } (x' - y') \cdot \nu > 0 \\ -\frac{1}{2} & \text{else} \end{array} \right\} \text{ in } B'(y', (1 - \frac{\pi}{4})(1 - |y'|)). \tag{103}$$

Since

$$|y'| \leq \frac{1}{2}(1 - \frac{\pi}{4})^2 \leq \frac{\frac{1}{2}(1 - \frac{\pi}{4})}{2 - \frac{\pi}{4}},$$

we have

$$B'\Big(y', (1 - \frac{\pi}{4})(1 - |y'|)\Big) \supset B'\Big(0, (1 - \frac{\pi}{4})(1 - |y'|) - |y'|\Big) \supset B'_{\frac{1}{2}(1-\frac{\pi}{4})},$$

so that both (102) and (103) hold in $B'_{\frac{1}{2}(1-\frac{\pi}{4})}$. Thus an application of Lemma 4 yields

$$\chi \leq \chi_0 \text{ in } B'_{\frac{1}{2}(1-\frac{\pi}{4})^2} \quad \text{or} \quad \chi \geq \chi_0 \text{ in } B'_{\frac{1}{2}(1-\frac{\pi}{4})^2}.$$

W.l.o.g. we consider only the first alternative, that is,

$$\{x' \cdot \nu_0 \leq 0\} \cap B'_{\frac{1}{2}(1-\frac{\pi}{4})^2} \subset \{(x' - y') \cdot \nu \leq 0\}.$$

Thus, $\nu \cdot \nu_0 > 0$. We introduce the abbreviations

$$\delta := \frac{y' \cdot \nu}{r}, \ r := \frac{1}{2}(1 - \frac{\pi}{4})^2.$$

By elementary geometry (see Figure 14), this implies

$$|\nu - \nu_0|^2 \leq 2\delta^2. \tag{104}$$

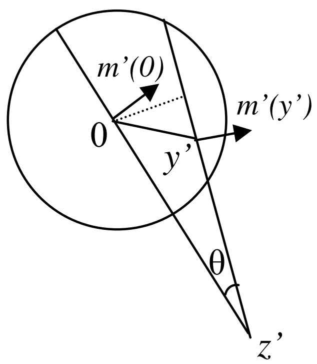

Figure 14: Geometry of characteristics

Indeed, if $\nu = \nu_0$, then (104) is obvious. Otherwise, $\nu \neq \nu_0$ and therefore, the point of intersection z' of the two lines respectively orthogonal to ν_0 and ν and passing through 0 and y', lies outside the ball B'_r; denoting by $\theta = \angle(\nu, \nu_0) \in (0, \frac{\pi}{2}]$ the angle between ν and ν_0, it follows that

$$\frac{y' \cdot \nu}{\sin\theta} = |z'| \geq r \text{ and } \cos\frac{\theta}{2} \geq \frac{\sqrt{2}}{2},$$

that is,

$$\delta \geq \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \geq \sqrt{2}\sin\frac{\theta}{2}|\nu - \nu_0|\frac{1}{\sqrt{2}}.$$

Hence,

$$|\nu - \nu_0| \leq \frac{2\sqrt{2}}{(1-\frac{\pi}{4})^2}|y'|.$$

It remains to prove that (77) implies

$$\nu = m'(y') \quad \text{and} \quad \nu_0 = m'(0). \tag{105}$$

W.l.o.g. we establish $\nu_0 = m'(0)$. Indeed, in the view of (102), (77) takes the form

$$\frac{1}{2}\int_{\{x'\cdot\nu_0<0\}} m' \cdot \nabla'\zeta\, dx' - \frac{1}{2}\int_{\{x'\cdot\nu_0>0\}} m' \cdot \nabla'\zeta\, dx' = \int_{\{x'\cdot\nu_0=0\}} \zeta\, d\mathcal{H}^1, \tag{106}$$

for all $\zeta \in C_c^\infty(B'_{1-\frac{\pi}{4}})$. We now fix a $\zeta_1 \in C_c^\infty(B'_{1-\frac{\pi}{4}})$ such that $\int_{\{x'\cdot\nu_0=0\}} \zeta_1\, d\mathcal{H}^1 = 1$ and for $r < 1$, consider $\zeta_r \in C_c^\infty(B'_{r(1-\frac{\pi}{4})})$ given by

$$\zeta_r(x') = \frac{1}{r}\zeta_1(\frac{x'}{r}).$$

Since

$$\int_{\mathbb{R}^2} |\nabla'\zeta_r|\, dx' = \int_{\mathbb{R}^2} |\nabla'\zeta_1|\, dx'$$

and 0 is a Lebesgue point of m', we have

$$\begin{aligned}
&\lim_{r\to 0}\left(\frac{1}{2}\int_{\{x'\cdot\nu_0<0\}} m' \cdot \nabla'\zeta_r\, dx' - \frac{1}{2}\int_{\{x'\cdot\nu_0>0\}} m' \cdot \nabla'\zeta_r\, dx'\right) \\
&\quad = m'(0) \cdot \lim_{r\to 0}\left(\frac{1}{2}\int_{\{x'\cdot\nu_0<0\}} \nabla'\zeta_r\, dx' - \frac{1}{2}\int_{\{x'\cdot\nu_0>0\}} \nabla'\zeta_r\, dx'\right) \\
&\quad = (m'(0)\cdot\nu_0)\lim_{r\to 0}\int_{\{x'\cdot\nu_0=0\}} \zeta_r\, d\mathcal{H}^1.
\end{aligned} \tag{107}$$

Since

$$\int_{\{x'\cdot\nu_0=0\}} \zeta_r\, d\mathcal{H}^1 = \int_{\{x'\cdot\nu_0=0\}} \zeta_1\, d\mathcal{H}^1 = 1,$$

we obtain from (106) and (107), $m'(0)\cdot\nu_0 = 1$, which implies (105) because of $|m'(0)| = 1$. □

The last lemma establishes the principle of characteristics for m' in B'_1. Because of translation and scaling invariance and a continuity argument, it suffices to prove the following:

Lemma 6 *Let m' be as in Lemma 1 and Lipschitz continuous. Then*

$$m'(tm'(0)^\perp) = m'(0) \quad \textit{for all } |t| < 1 - \frac{\pi}{4}. \tag{108}$$

Proof of Lemma 6. Let χ be the characteristic function associated to 0 according to Lemma 1. From Lemmas 2 and 3 we gather that there exists $\nu \in S^1$ such that

$$\chi = \left\{ \begin{array}{cc} \frac{1}{2} & \text{for } x' \cdot \nu > 0 \\ -\frac{1}{2} & \text{else} \end{array} \right\} \ \text{ in } B'_{1-\frac{\pi}{4}}.$$

As in Lemma 5 we deduce from (77) and the continuity of m':

$$m' = \nu \ \text{ on } \{x' \cdot \nu = 0\} \cap B'_{1-\frac{\pi}{4}}.$$

This is a reformulation of (108). $\square$

5 Optimality of the straight walls

In this section, we prove Theorem 1:

Proof of Theorem 1. Let $m' : \mathbb{R}^2 \to S^1$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy the hypothesis of Theorem 1. Using the same argument as in the proof of Theorem 4 and because of (1), we identify the center line of the transition layer: There exists a set $G' \subset \mathbb{R}^2$ with the inner normal ν' such that

$$\begin{aligned} & G' \text{ is } L\text{−periodic in } x_2, \\ & (1, +\infty) \times \mathbb{R} \subset G', \ (-\infty, -1) \times \mathbb{R} \subset \mathbb{R}^2 \setminus G' \\ & m' = \nu' \text{ on } \partial G' \end{aligned} \tag{109}$$

(see Figure 15). This set was already introduced in [5]. We consider the related characteristic function

$$\chi \begin{cases} \frac{1}{2} & \text{in } \ G', \\ -\frac{1}{2} & \text{in } \ \mathbb{R}^2 \setminus G'. \end{cases} \tag{110}$$

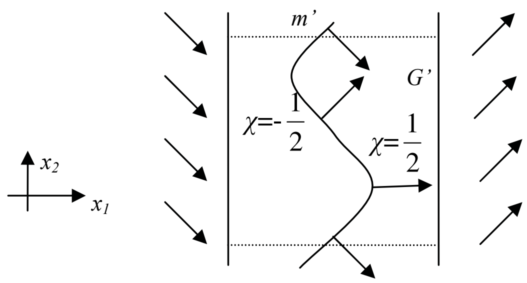

Figure 15: Center line of the wall

Then (109) translates into

$$\chi \text{ is } L\text{−periodic in } x_2, \tag{111}$$

$$\chi = \pm\frac{1}{2} \text{ for } \ \pm x_1 \geq 1, \tag{112}$$

$$\int_{\mathbb{R}\times[0,L)} \eta^2 \chi \nabla' \cdot m' \, dx' = -\int_{\mathbb{R}\times[0,L)} \nabla'(\eta^2) \cdot m' \chi \, dx' - \int_{\mathbb{R}\times[0,L)} \eta^2 |D'\chi|, \tag{113}$$

where $\eta \in C^\infty(\mathbb{R}^3)$ is a $L-$periodic function in x_2 that satisfies (55). Like in (63), we introduce the energy density e as a non-negative measure on $\mathbb{R}^3$ via

$$\int_{\mathbb{R}^3} \zeta\, de = \frac{2}{\pi}|\ln\varepsilon|\Big(\varepsilon\int_{\mathbb{R}^2}\zeta|\nabla'\cdot m'|^2\,dx' + \int_{\mathbb{R}^3}\zeta|h|^2\,dx\Big),\ \forall\zeta\in C_c(\mathbb{R}^3). \tag{114}$$

Step 1. *We have an a priori bound on $L^{-1}\int_{\mathbb{R}\times[0,L)}|D'\chi|$ in terms of $L^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R})$: for any $\alpha\in(0,1)$,*

$$\begin{aligned}(1-\alpha)L^{-1}\int_{\mathbb{R}\times[0,L)}|D'\chi| &\le m_{1,\infty} + \frac{1}{4\alpha}L^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R})\\ &\quad + C(L)\Big(|\ln\varepsilon|^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R})\Big)^{1/2}\Big(1+\int_{\mathbb{R}\times[0,L)}|D'\chi|\Big).\end{aligned} \tag{115}$$

Indeed, with the above choices and notations, (57) turns into

$$\begin{aligned}\Big|\int_{\mathbb{R}\times[0,L)}\eta^2\chi\nabla'\cdot m'\,dx'\Big| &\overset{(112)}{\le} \Big(\int_{\mathbb{R}\times[0,L)\times\mathbb{R}}\eta^2\,de\int_{\mathbb{R}\times[0,L)}\eta^2|D'\chi|\Big)^{1/2}\\ &\quad + C(L)\Big(|\ln\varepsilon|^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R})\Big)^{1/2}\\ &\quad \times(\sup_{\mathbb{R}^3}|\eta|+\sup_{\mathbb{R}^3}|\nabla\eta|)\cdot\Big(\sup_{\mathbb{R}^3}|\eta|+\int_{\mathbb{R}\times[0,L)}|\eta||D'\chi|\Big).\end{aligned}$$

Using (113) on the left-hand side and Young's inequality on the first term of the right-hand side yields for any $\alpha\in(0,1)$,

$$\begin{aligned}(1-\alpha)\int_{\mathbb{R}\times[0,L)}\eta^2|D'\chi| &\le -\int_{\mathbb{R}\times[0,L)}\nabla'(\eta^2)\cdot m'\chi\,dx' + \frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}}\eta^2\,de\\ &\quad + C(L)\Big(|\ln\varepsilon|^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R})\Big)^{1/2}\\ &\quad \times(\sup_{\mathbb{R}^3}|\eta|+\sup_{\mathbb{R}^3}|\nabla\eta|)\cdot\Big(\sup_{\mathbb{R}^3}|\eta|+\int_{\mathbb{R}\times[0,L)}|\eta||D'\chi|\Big).\end{aligned} \tag{116}$$

We select $\eta:\mathbb{R}^3\to\mathbb{R}$ such that

$$\begin{aligned}&\eta=\eta(x_1,x_3),\ \eta=1\ \text{ on } (-1,1)\times\mathbb{R}\times\{0\},\\ &\operatorname{supp}\eta\subset(-2,2)\times\mathbb{R}\times(-1,1),\ |\eta|\le1,\ |\nabla\eta|\le C.\end{aligned} \tag{117}$$

We consider the terms in (116) one-by-one:

$$\begin{aligned}\int_{\mathbb{R}\times[0,L)}\eta^2|D'\chi| &\overset{(112,117)}{=} \int_{\mathbb{R}\times[0,L)}|D'\chi|,\\ -\int_{\mathbb{R}\times[0,L)}\nabla'(\eta^2)\cdot m'\chi\,dx' &\overset{(1,112,117)}{=} -\int_{(-\infty,-1)\times[0,L)}\begin{pmatrix}\partial_1\eta^2\\0\end{pmatrix}\cdot\begin{pmatrix}m_{1,\infty}\\-\sqrt{1-{m_{1,\infty}}^2}\end{pmatrix}\frac{-1}{2}\,dx'\\ &\qquad -\int_{(1,+\infty)\times[0,L)}\begin{pmatrix}\partial_1\eta^2\\0\end{pmatrix}\cdot\begin{pmatrix}m_{1,\infty}\\\sqrt{1-{m_{1,\infty}}^2}\end{pmatrix}\frac{1}{2}\,dx'\\ &= Lm_{1,\infty},\\ \int_{\mathbb{R}\times[0,L)\times\mathbb{R}}\eta^2\,de &\le e(\mathbb{R}\times[0,L)\times\mathbb{R}).\end{aligned} \tag{118}$$

Using (117) to estimate the $\eta-$terms in (116), we then obtain

$$(1-\alpha)\int_{\mathbb{R}\times[0,L)}|D'\chi| \le Lm_{1,\infty} + \frac{1}{4\alpha}e(\mathbb{R}\times[0,L)\times\mathbb{R}) \\ + C(L)\Big(|\ln\varepsilon|^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R})\Big)^{1/2}\Big(1+\int_{\mathbb{R}\times[0,L)}|D'\chi|\Big).$$

Dividing by L yields (115).

Step 2. *Sketch of the proof of Theorem 1.* We give an argument by contradiction. To this purpose, we consider sequences $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \downarrow 0$, $\{m'_k : \mathbb{R}^2 \to S^1\}_{k\uparrow\infty}$ and $\{h_k : \mathbb{R}^3 \to \mathbb{R}^3\}_{k\uparrow\infty}$ that satisfy the first three hypothesis in Theorem 1 and

$$\limsup_{k\to\infty} L^{-1}e_k(\mathbb{R}\times[0,L)\times\mathbb{R}) \le (1-m_{1,\infty})^2, \tag{119}$$

which corresponds to (12) (here, e_k is the energy density (114) associated to m'_k and h_k). Because of periodicity of e_k, (119) implies that the energy is locally bounded, so that we may apply Theorem 4. Hence there exists a measurable $m' : \mathbb{R}^2 \to S^1$ with

$$m'_k \to m' \quad \text{ in } L^1_{loc}(\mathbb{R}^2), \tag{120}$$

after passage to a subsequence. Properties (1) and (4) are preserved under (120) while in addition (see Theorem 4),

$$\nabla'\cdot m' = 0 \text{ distributionally in } \mathbb{R}^2. \tag{121}$$

Because of (1) and (4), (120) yields

$$\int_{\mathbb{R}\times[0,L)} |m'_k - m'|\,dx' \to 0.$$

We thus have to argue that m' has the form (14). Because of periodicity of e, (119) implies that there exists a non-negative measure e on $\mathbb{R}^3$ such that

$$e_k \overset{w^*}{\rightharpoonup} e \quad \text{ weakly}^* \text{ in } \mathcal{M}(\mathbb{R}^3), \tag{122}$$

after passage to a subsequence. Notice that (119) is preserved under (122):

$$L^{-1}e(\mathbb{R}\times[0,L)\times\mathbb{R}) \le (1-m_{1,\infty})^2. \tag{123}$$

We shall argue that there exists an $x_1^* \in [-1,1]$ such that

$$\operatorname{supp} e \cap \Big((-2,2)\times\mathbb{R}\times(-1,1)\Big) \subseteq \{x_1^*\}\times\mathbb{R}\times\{0\}. \tag{124}$$

We then apply Theorem 5 on balls in $(-2,x_1^*)\times\mathbb{R}\times(-1,1)$ and $(x_1^*,2)\times\mathbb{R}\times(-1,1)$ respectively. This yields that m' is locally Lipschitz and satisfies the principle of characteristics in both $(-2,x_1^*)\times\mathbb{R}\times(-1,1)$ and $(x_1^*,2)\times\mathbb{R}\times(-1,1)$. In view of the form (1), this indeed implies that m' is of the form (14). Hence it suffices to show (124).

Step 3. *Proof of* (124). We first address the function χ_k defined as in (110) for m'_k. In view of (115) (applied to χ_k and e_k) and (119), we have

$$\Big\{L^{-1}\int_{\mathbb{R}\times[0,L)}|D'\chi_k|\Big\}_{k\uparrow\infty} \text{ is bounded.} \tag{125}$$

Because of periodicity (111), there exists a measurable function $\chi : \mathbb{R}^2 \to \{-\frac{1}{2}, \frac{1}{2}\}$ of locally bounded variation such that

$$\chi_k \to \chi \text{ in } L^1_{loc}(\mathbb{R}^2) \tag{126}$$

up to a subsequence. Notice that periodicity (111) and the boundary conditions (112) are preserved by (126). We shall argue in Step 4 that χ is of the form

$$\chi = \pm\frac{1}{2} \text{ for } \pm x_1 > \pm x_1^*, \tag{127}$$

for some $x_1^* \in [-1, 1]$. Now we give the argument how (127) implies (124). For this we turn back to (116). Again, because of the convergences (120), (122), (126) and the boundedness expressed in (119) and (125), inequality (116) (applied for χ_k, m'_k and e_k) yields in the limit as $k \to \infty$,

$$(1-\alpha)\int_{\mathbb{R}\times[0,L)} \eta^2 |D'\chi| \le -\int_{\mathbb{R}\times[0,L)} \nabla'(\eta^2)\cdot m'\chi\, dx' + \frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2\, de, \tag{128}$$

for any $\eta \in C^\infty(\mathbb{R}^3)$ that is $L-$ periodic in x_2 and satisfies (55). We choose

$$\alpha = \frac{(1-m_{1,\infty})}{2}. \tag{129}$$

In view of (128),

$$\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \zeta\, d\lambda = \frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \zeta\, de - \int_{\mathbb{R}\times[0,L)} \nabla'\zeta\cdot m'\chi\, dx' - (1-\alpha)\int_{\mathbb{R}\times[0,L)} \zeta|D'\chi|$$

defines a non-negative distribution λ in $(-2,2)\times\mathbb{R}\times(-1,1)$ for functions $\zeta : \mathbb{R}^3 \to \mathbb{R}$ which are $L-$periodic in x_2 and satisfy (55). Because of (127), λ simplifies to

$$\begin{aligned}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \zeta\, d\lambda &= \frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \zeta\, de \\ &\quad + \frac{1}{2}\int_{(-\infty,x_1^*)\times[0,L)} \nabla'\zeta\cdot m'\, dx' - \frac{1}{2}\int_{(x_1^*,+\infty)\times[0,L)} \nabla'\zeta\cdot m'\, dx' \\ &\quad - (1-\alpha)\int_{[0,L)} \zeta(x_1^*, x_2, 0)\, dx_2.\end{aligned} \tag{130}$$

In fact, λ is a non-negative measure: Because of $|m'| = 1$ and the divergence-free property (121), we have

$$\left|\frac{1}{2}\int_{(-\infty,x_1^*)\times[0,L)} \nabla'\zeta\cdot m'\, dx' - \frac{1}{2}\int_{(x_1^*,+\infty)\times[0,L)} \nabla'\zeta\cdot m'\, dx'\right| \le \int_{[0,L)} |\zeta(x_1^*, x_2, 0)|\, dx_2. \tag{131}$$

Estimate (131) formally follows from integration by parts and can be rigorously established by approximating m' with smooth m''s while preserving $|m'| \le 1$, (121) and the periodicity in x_2. We now consider $\zeta = \eta^2$ in (130) such that (55) holds and

$$\eta = \eta(x_1, x_3),\ \eta = 1 \text{ on } (-1,1)\times\mathbb{R}\times\{0\},\ |\eta| \le 1.$$

Using the same arguments as in (118), we learn that (130) turns into

$$\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2\, d\lambda = \frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2\, de + Lm_{1,\infty} - L(1-\alpha).$$

Since (123) implies that $\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2\, de \le e(\mathbb{R}\times[0,L)\times\mathbb{R}) \le L(1-m_{1,\infty})^2$, this yields

$$\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \eta^2\, d\lambda \le L\left[\frac{1}{4\alpha}(1-m_{1,\infty})^2 + m_{1,\infty} - (1-\alpha)\right] \overset{(129)}{=} 0.$$

We let η^2 converge monotonically to one in $(-2,2)\times\mathbb{R}\times(-1,1)$ and obtain

$$\lambda((-2,2)\times[0,L)\times(-1,1)) \le 0$$

and thus, $\lambda\equiv 0$ in $(-2,2)\times[0,L)\times(-1,1)$. Hence, (130) simplifies to

$$\begin{aligned}\frac{1}{4\alpha}\int_{\mathbb{R}\times[0,L)\times\mathbb{R}} \zeta\, de \;&= (1-\alpha)\int_{[0,L)} \zeta(x_1^*,x_2,0)\,dx_2 \\ &\quad - \frac{1}{2}\int_{(-\infty,x_1^*)\times[0,L)} \nabla'\zeta\cdot m'\,dx' + \frac{1}{2}\int_{(x_1^*,+\infty)\times[0,L)} \nabla'\zeta\cdot m'\,dx' \\ &\overset{(131)}{\le} (1-\alpha)\int_{[0,L)} \zeta(x_1^*,x_2,0)\,dx_2 + \int_{[0,L)} |\zeta(x_1^*,x_2,0)|\,dx_2,\end{aligned}$$

for every $\zeta\in C^\infty(\mathbb{R}^3)$ that is $L-$periodic in x_2 and satisfies (55). This implies (124) by periodicity of e. Thus, it remains to prove (127).

Step 4. *Proof of* (127). We first notice that because of (119), (125) and the lower semicontinuity of $\int_{\mathbb{R}\times[0,L)}|D'\chi_k|$ under (126), inequality (115) (applied for χ_k and e_k) yields in the limit as $k\to\infty$,

$$(1-\alpha)L^{-1}\int_{\mathbb{R}\times[0,L)} |D'\chi| \le m_{1,\infty} + \frac{(1-m_{1,\infty})^2}{4\alpha}.$$

As before, the choice (129) gives

$$L^{-1}\int_{\mathbb{R}\times[0,L)} |D'\chi| \le 1. \tag{132}$$

Now the boundary conditions (112) and inequality (132) enforce the form (127). For the convenience of the reader, we display this standard argument. Let μ and ν' be the measure-theoretic *line measure* $|D'\chi|$ and *normal* $\dfrac{D'\chi}{|D'\chi|}$ related to the function χ of bounded variation. Both inherit the periodicity of χ and are characterized by

$$-\int_{\mathbb{R}\times[0,L)} \nabla'\cdot\zeta'\chi\,dx' = \int_{\mathbb{R}\times[0,L)} \nu'\cdot\zeta'\,d\mu \tag{133}$$

for all $\zeta':\mathbb{R}^2\to\mathbb{R}^2$ which are $L-$periodic in x_2 and compactly supported in x_1. Now we show that (112) yields

$$\int_{\mathbb{R}\times[0,L)} \nu_1\,d\mu = L. \tag{134}$$

Indeed, (134) can be seen by selecting a function $\eta=\eta(x_1)$ with $\eta=1$ for $|x_1|\le 1$ and $\operatorname{supp}\eta\subset(-2,2)\times\mathbb{R}$ so that

$$\begin{aligned}\int_{\mathbb{R}\times[0,L)} \nu_1\,d\mu = \int_{\mathbb{R}\times[0,L)} \eta^2\nu_1\,d\mu &\overset{(133)}{=} -\int_{\mathbb{R}\times[0,L)} \frac{d\eta^2}{dx_1}\chi\,dx' \\ &= -\int_{(-\infty,-1)\times[0,L)} \frac{-1}{2}\frac{d\eta^2}{dx_1}\,dx' - \int_{(1,+\infty)\times[0,L)} \frac{1}{2}\frac{d\eta^2}{dx_1}\,dx' = L.\end{aligned}$$

Now (132) (i.e., $\int_{\mathbb{R}\times[0,L)} d\mu \le L$) and (134) combine to $\int_{\mathbb{R}\times[0,L)}(1-\nu_1)\,d\mu \le 0$. But since $1-\nu_1 \ge 0$ we must have $1-\nu_1 = 0$ $\mu-$a.e., that is, $\nu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\mu-$a.e. Hence (133) turns into

$$-\int_{\mathbb{R}\times[0,L)} \nabla'\cdot\zeta'\chi\,dx' = \int_{\mathbb{R}\times[0,L)} \zeta_1\,d\mu. \tag{135}$$

Choosing ζ' with $\zeta_1 \equiv 0$, we deduce that χ has a representative with $\chi = \chi(x_1)$. In particular, (135) then yields

$$-\int_{\mathbb{R}} \frac{d\eta^2}{dx_1}\chi\,dx_1 \ge 0,$$

for all $\eta = \eta(x_1)$ with compact support. Hence χ has a representative with $\chi = \chi(x_1)$ that is monotone non-decreasing. Since $\chi \in \{-\frac{1}{2}, \frac{1}{2}\}$, this yields (127). Now the proof of the theorem is completed. $\square$

Remark 4 One can improve (124) to $\operatorname{supp} e \subset \{x_1^*\}\times\mathbb{R}\times\{0\}$ using Corollary 4 for trial functions η with support in $(-a,a)\times\mathbb{R}\times(-a,a)$, where a is arbitrarily large.

6 The case of 1-d magnetizations

In the framework of Theorem 1, we will now focus on 1-d magnetizations $m' = (m_1(x_1), m_2(x_1))$. As in [5], we consider the minimal stray field energy corresponding to m' in the strip $\mathbb{R}\times[0,1)$. (Here, we fix the width $L = 1$ of the strip.) Let $U \in H_0^1(\mathbb{R}\times(0,1)\times\mathbb{R})$ be the unique solution of the variational problem:

$$\int_{\mathbb{R}\times(0,1)\times\mathbb{R}} \nabla U\cdot\nabla\zeta\,dx = \int_{\mathbb{R}\times(0,1)} \zeta\,\nabla'\cdot m'\,dx', \quad \forall\zeta\in C_c^\infty(\mathbb{R}\times(0,1)\times\mathbb{R}). \tag{136}$$

(That is a direct application of the Lax-Milgram Theorem.) The function U is the unique symmetric harmonic map in $H_0^1(\mathbb{R}\times(0,1)\times\mathbb{R})$ such that

$$\begin{cases} \Delta U = 0 & \text{in} \quad \mathbb{R}\times(0,1)\times(\mathbb{R}\setminus\{0\}), \\ \left[\frac{\partial U}{\partial x_3}\right] = -\nabla'\cdot m' & \text{on} \quad \mathbb{R}\times(0,1), \end{cases}$$

where $[\xi]$ denotes the jump of a quantity ξ across the plane $\mathbb{R}^2\times\{0\}$. We extend $U:\mathbb{R}^3\to\mathbb{R}$ by $1-$periodicity in x_2-direction. Then (2) holds for $h = \nabla U$. An elementary computation yields that the stray field energy is given by the homogeneous $H^{-1/2}$ norm of the divergence of m':

$$\int_{\mathbb{R}\times(0,1)\times\mathbb{R}} |\nabla U|^2\,dx = \frac{1}{2}\int_{\mathbb{R}\times(0,1)} \left||\nabla'|^{-1/2}\nabla'\cdot m'\right|^2 dx'. \tag{137}$$

Since m' is 1-d, then

$$\int_{\mathbb{R}\times(0,1)} \left||\nabla'|^{-1/2}\nabla'\cdot m'\right|^2 dx' = \int_{\mathbb{R}} \left||\frac{d}{dx_1}|^{1/2}m_1\right|^2 dx_1$$

and therefore, (137) explains the expression of the energy $E^{1d}_\varepsilon(m')$ given in (11). Also observe that the chosen stray field energy is minimal because for any $h:\mathbb{R}^3\to\mathbb{R}^3$ that is $1-$periodic in x_2 and satisfies (2) for $\nabla'\cdot m'$, we have

$$\int_{\mathbb{R}\times(0,1)\times\mathbb{R}} |\nabla U|^2\,dx \le \int_{\mathbb{R}\times(0,1)\times\mathbb{R}} |h|^2\,dx.$$

We now present the proof of Theorems 2 and 3:

Proof of Theorem 2. We proceed in several steps:
Step 1. *We show that*

$$m_{1,k}-m_{1,\infty}\to 0 \quad in \quad L^1(\mathbb{R}) \quad as \quad k\to\infty.$$

Indeed, by (1) and (15), we deduce that

$$\begin{aligned}\int_{\mathbb{R}} |m_{1,k}-m_{1,\infty}|^2\,dt = \int_{-1}^{1} |m_{1,k}-m_{1,\infty}|^2\,dt &= \int_{-1}^{1}\int_2^3 |m_{1,k}(t)-m_{1,k}(t+s)|^2\,dt\,ds\\ &\le 9\int_{-1}^{1}\int_2^3 \frac{|m_{1,k}(t)-m_{1,k}(t+s)|^2}{s^2}\,dt\,ds\\ &\le 9\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|m_{1,k}(t)-m_{1,k}(s)|^2}{|t-s|^2}\,dt\,ds \ \to 0 \quad \text{as} \quad k\to\infty\end{aligned}$$

and the conclusion follows by (1).
Step 2. *We locate the regions where $m_{1,k}$ (and $m_{2,k}$) have large variations.* For that, we choose the intervals (a^k_n,b^k_n), $n=1,\dots,N_k$ in the following way (see Figure 16): We set $b^k_0=-\infty$ and we recursively define for $n=1,\dots,N_k$, $b^k_n\in(b^k_{n-1},1]$ to be the smallest number such that

$$m_{2,k}(b^k_n)=\frac{(-1)^{n-1}\sqrt{1-{m_{1,\infty}}^2}}{2}$$

and respectively, $a^k_n\in[b^k_{n-1},b^k_n]$ be the biggest number such that

$$m_{2,k}(a^k_n)=\frac{(-1)^{n}\sqrt{1-{m_{1,\infty}}^2}}{2}.$$

By (1), we have that

$$-1<a^k_1<b^k_1\le a^k_2<b^k_2\le\dots\le a^k_{N_k}<b^k_{N_k}<1 \quad \text{and} \quad N_k\le\frac{2}{1-{m_{1,\infty}}^2}\int_{\mathbb{R}}\left|\frac{dm_{2,k}}{dt}\right|^2 dt.$$

Indeed,

$$\frac{1-{m_{1,\infty}}^2}{b^k_n-a^k_n}=\frac{1}{b^k_n-a^k_n}\left(\int_{a^k_n}^{b^k_n}\frac{dm_{2,k}}{dt}\,dt\right)^2\le\int_{a^k_n}^{b^k_n}\left|\frac{dm_{2,k}}{dt}\right|^2 dt\le\int_{\mathbb{R}}\left|\frac{dm_{2,k}}{dt}\right|^2 dt$$

and therefore,

$$N_k\le\frac{\sum_{n=1}^{N_k}(b^k_n-a^k_n)}{1-{m_{1,\infty}}^2}\int_{\mathbb{R}}\left|\frac{dm_{2,k}}{dt}\right|^2 dt\le\frac{2}{1-{m_{1,\infty}}^2}\int_{\mathbb{R}}\left|\frac{dm_{2,k}}{dt}\right|^2 dt.$$

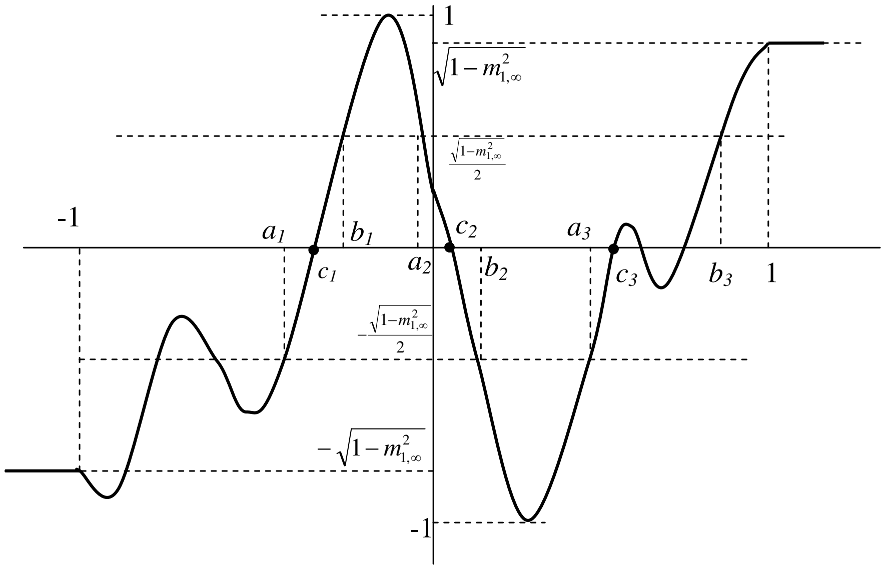

Figure 16: The variations of m_2

We also notice that N_k is an odd integer (because of (1)),

$$|m_{2,k}| \leq \frac{\sqrt{1-{m_{1,\infty}}^2}}{2} \text{ in any interval } (a_n^k, b_n^k) \tag{138}$$

$$\text{and} \quad (-1)^{n-1} m_{2,k} \leq \frac{\sqrt{1-{m_{1,\infty}}^2}}{2} \text{ in } (b_{n-1}^k, b_n^k)\,,\ n = 1, \ldots, N_k. \tag{139}$$

Step 3. *We prove that the sequence $\{N_k\}_{k\uparrow\infty}$ is bounded.* The idea is to define a good step function with $2N_k$ jumps and to apply Corollary 4. Set

$$\chi_k = \begin{cases} \operatorname{sgn}(m_{1,k}) & \text{in} \quad (a_n^k, c_n^k) \quad \text{for } n = 1, \ldots, N_k, \\ 0 & \text{elsewhere,} \end{cases}$$

where $c_n^k \in [a_n^k, b_n^k]$ is the smallest number such that $m_{2,k}(c_n^k) = 0$. Since (138) implies that $m_{1,k}$ does not change sign in (a_n^k, c_n^k), we obtain:

$$\int_{\mathbb{R}} \Big|\frac{d\chi_k}{dt}\Big| = 2N_k \tag{140}$$

$$\begin{aligned} \int_{-1}^{1} \chi_k \, \frac{dm_{1,k}}{dt}\, dt &= \sum_{n=1}^{N_k} \int_{a_n^k}^{c_n^k} \operatorname{sgn}(m_{1,k}) \, \frac{dm_{1,k}}{dt}\, dt \\ &= \sum_{n=1}^{N_k} \big(|m_{1,k}|(c_n^k) - |m_{1,k}|(a_n^k)\big) = N_k \bigg(1 - \frac{\sqrt{3+{m_{1,\infty}}^2}}{2}\bigg). \end{aligned} \tag{141}$$

Now we apply Corollary 4 for the harmonic extension U_k given by (136) associated to m'_k and for the test function $\eta = \eta(x_1, x_3) : \mathbb{R}^3 \to [-1,1]$ satisfying (117):

$$\Big| \int_{\mathbb{R}\times[0,1)} \eta^2 \chi_k \frac{dm_{1,k}}{dx_1}\, dx' \Big| \le \bigg(\frac{4}{\pi} |\ln \varepsilon_k| \int_{\mathbb{R}\times[0,1)} \eta^2 |D'\chi_k| \int_{\mathbb{R}\times[0,1)\times\mathbb{R}} \eta^2 |\nabla U_k|^2\, dx \bigg)^{1/2}$$
$$+ C\bigg(\varepsilon_k \int_{\mathbb{R}\times[0,1)} |\frac{dm_{1,k}}{dx_1}|^2\, dx' + \int_{\mathbb{R}\times[0,1)\times\mathbb{R}} |\nabla U_k|^2\, dx \bigg)^{1/2}$$
$$\times \bigg(1 + \int_{\mathbb{R}\times[0,1)} |D'\chi_k| \bigg),$$

that is,

$$\Big| \int_{-1}^{1} \chi_k \frac{dm_{1,k}}{dt}\, dt \Big| \overset{(137)}{\le} C\bigg(|\ln \varepsilon_k|\, E^{1d}_{\varepsilon_k}(m_k') \int_{\mathbb{R}} |\frac{d\chi_k}{dt}| \bigg)^{1/2}$$
$$+ \frac{C}{\sqrt{|\ln \varepsilon_k|}} \bigg(|\ln \varepsilon_k| E^{1d}_{\varepsilon_k}(m_k') \bigg)^{1/2} \times \bigg(1 + \int_{\mathbb{R}} |\frac{d\chi_k}{dt}| \bigg).$$

Therefore, by (15), (140) and (141), we deduce that $N_k \le C$ for some absolute constant $C > 0$.
Step 4. *We show that the sequence $\{m_{2,k}\}_{k\uparrow\infty}$ is relatively compact in L^1_{loc}.* We consider the step function

$$\psi_k = \sum_{n=1}^{N_k+1} (-1)^n \sqrt{1 - m_{1,\infty}{}^2}\ 1_{(b^k_{n-1}, b^k_n)},$$

where $b^k_{N_k+1} = +\infty$. Observe that

$$\int_{\mathbb{R}} |\frac{d\psi_k}{dt}| = 2N_k \sqrt{1 - m_{1,\infty}{}^2}.$$

It follows by Step 3 that the sequence $\{\psi_k\}$ is bounded in $BV_{loc}(\mathbb{R})$. Therefore, any accumulation point $\psi : \mathbb{R} \to \{\pm\sqrt{1 - m_{1,\infty}{}^2}\}$ of $\{\psi_k\}_{k\uparrow\infty}$ in L^1_{loc} is of bounded variation and has the form

$$\psi = \sum_{n=1}^{2N} (-1)^n \sqrt{1 - m_{1,\infty}{}^2}\ 1_{(b_{n-1}, b_n)},$$

where $-\infty = b_0 < b_1 < \cdots < b_{2N-1} < b_{2N} = +\infty$ and $b_n \in [-1,1]$ for $n = 1, \dots, 2N-1$. Finally, by (139), we have that

$$|\psi_k + m_{2,k}| \ge \frac{\sqrt{1 - m_{1,\infty}{}^2}}{2} \quad \text{in } \mathbb{R}, \tag{142}$$

and therefore,

$$\int_{\mathbb{R}} |\psi_k - m_{2,k}|\, dt = \int_{-1}^{1} |\psi_k - m_{2,k}|\, dt \overset{(142)}{\le} \frac{2}{\sqrt{1 - m_{1,\infty}{}^2}} \int_{-1}^{1} |\psi_k^2 - m_{2,k}^2|\, dt$$
$$\le \frac{2}{\sqrt{1 - m_{1,\infty}{}^2}} \int_{-1}^{1} |(1 - m_{1,\infty}{}^2) - m_{2,k}^2|\, dt$$
$$\le \frac{4}{\sqrt{1 - m_{1,\infty}{}^2}} \int_{-1}^{1} |m_{1,k} - m_{1,\infty}|\, dt.$$

We conclude by Step 1 that up to a subsequence, $m_{2,k} - \psi \to 0$ in $L^1(\mathbb{R})$, i.e.,

$$m_k' - \begin{pmatrix} m_{1,\infty} \\ \psi \end{pmatrix} \to 0 \quad \text{in } L^1(\mathbb{R})$$

as $k \to \infty$. $\square$

Since the asymptotic limits of the sequence $\{m'_k\}_{k\uparrow\infty}$ in L^1_{loc} belong to BV, one may ask whether the sequence $\{m'_k\}$ is bounded in BV. The answer is negative according to Theorem 3. The idea is that m'_k may have small variations on a large number of intervals (that have not been taken into account in the construction of the trial functions χ_k in the previous proof).

Proof of Theorem 3. For simplicity, we assume that $m_{1,\infty} = 0$. Set $\delta = \varepsilon^{1/4}$, $\omega = \varepsilon^{1/2}$ and $\eta = \varepsilon|\ln\varepsilon|$. For small $\varepsilon > 0$, we consider the following sample in $(-\omega, \omega)$:

$$f_\varepsilon(t) = \begin{cases} \frac{\delta}{|\ln\varepsilon|} \ln \frac{\omega}{\sqrt{t^2+\varepsilon^2}} & \text{if} \quad |t| \le \sqrt{\omega^2-\varepsilon^2}\,, \\ 0 & \text{if} \quad t \in (-\omega,\omega) \setminus (-\sqrt{\omega^2-\varepsilon^2}, \sqrt{\omega^2-\varepsilon^2}). \end{cases}$$

This type of function was already used in [5]. We define $m_{1,\varepsilon}$ in $\mathbb{R}$ as follows: We fill in the intervals $(-1,-\frac{1}{2})$, respectively $(\frac{1}{2},1)$ by at most $\frac{1}{4\omega}$ samples of length 2ω where $m_{1,\varepsilon}$ is given via f_ε, so that $m_{1,\varepsilon}$ is symmetric with respect to 0. In the interval $(-\sqrt{\frac{1}{2}-\eta^2}, \sqrt{\frac{1}{2}-\eta^2})$, set

$$m_{1,\varepsilon}(t) = \frac{1}{|\ln(\sqrt{2}\eta)|} \ln \frac{1}{\sqrt{2(t^2+\eta^2)}}.$$

Otherwise, we set $m_{1,\varepsilon} = 0$. Hence, $m_{1,\varepsilon}$ is an even function in $H^1(\mathbb{R})$, $0 \le m_{1,\varepsilon} \le \delta/2$ in $\mathbb{R} \setminus (-\frac{1}{2}, \frac{1}{2})$ and $m_{1,\varepsilon}(0) = 1$. We then define

$$m_{2,\varepsilon}(t) = \pm\sqrt{1-m_{1,\varepsilon}^2(t)} \quad \text{if} \quad \pm\, t \ge 0;$$

hence, $m_{2,\varepsilon} \in H^1_{loc}(\mathbb{R})$ and (1) is satisfied. We compute the energy $E^{1d}_\varepsilon(\,(m_{1,\varepsilon}, m_{2,\varepsilon})\,)$. We have for $\varepsilon \ll 1$,

$$\begin{aligned} \int_{(-1,-\frac{1}{2})\cup(\frac{1}{2},1)} \Big|\frac{dm_{1,\varepsilon}}{dt}\Big|^2 dt &\le \frac{C}{\omega} \int_{-\omega}^{\omega} \frac{\delta^2}{|\ln\varepsilon|^2} \frac{t^2}{(t^2+\varepsilon^2)^2}\, dt \\ &\le \frac{C}{\varepsilon|\ln\varepsilon|^2} \int_0^{\frac{\omega}{\varepsilon}} \frac{y^2}{(y^2+1)^2}\, dy \\ &\le \frac{C}{\varepsilon|\ln\varepsilon|^2} \Big(\int_0^1 dy + \int_1^{\varepsilon^{-1/2}} \frac{dy}{y^2} \Big) \le \frac{C}{\varepsilon|\ln\varepsilon|^2}. \end{aligned} \tag{143}$$

Similarly, we compute that

$$\int_{(-\frac{1}{2},\frac{1}{2})} \Big|\frac{dm_{1,\varepsilon}}{dt}\Big|^2 dt \le \frac{C}{\eta|\ln\eta|^2} \int_0^\infty \frac{y^2}{(y^2+1)^2}\, dy \le \frac{C}{\varepsilon|\ln\varepsilon|^3}.$$

Now we compute the homogeneous $H^{1/2}-$norm of $m_{1,\varepsilon}$. For that, we extend the function $m_{1,\varepsilon}$ to $\mathbb{R}^2$ by

$$\tilde{m}_{1,\varepsilon}(t,s) = m_{1,\varepsilon}(\sqrt{t^2+s^2})\,,\ \forall (t,s) \in \mathbb{R}^2.$$

According to the trace estimate in $H^{1/2}$, it follows by the same argument as in (143),

$$\begin{aligned} \int_{\mathbb{R}} \Big| \Big|\frac{d}{dt}\Big|^{1/2} m_{1,\varepsilon} \Big|^2 dt &\le \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \tilde{m}_{1,\varepsilon}(t,s)|^2\, dt\, ds \\ &\le \frac{C}{\omega} \int_0^\omega \frac{\delta^2}{|\ln\varepsilon|^2} \frac{t^3}{(t^2+\varepsilon^2)^2}\, dt + \frac{C}{|\ln\eta|^2} \int_0^{1/2} \frac{t^3}{(t^2+\varepsilon^2)^2}\, dt \le \frac{C}{|\ln\varepsilon|}. \end{aligned}$$

Hence, $|\ln\varepsilon|E_\varepsilon^{1d}(\,m'_\varepsilon\,)\le C$ where $C>0$ is a universal constant. On the other hand, we have

$$\int_{\mathbb{R}}|\frac{dm_{1,\varepsilon}}{dt}|\,dt\ge\int_{(-1,-\frac{1}{2})\cup(\frac{1}{2},1)}|\frac{dm_{1,\varepsilon}}{dt}|\,dt\ge\frac{C\delta}{\omega|\ln\varepsilon|}\int_0^\omega\frac{t}{(t^2+\varepsilon^2)}\,dt\ge\frac{C}{\varepsilon^{1/4}}\to\infty\quad\text{as }\varepsilon\to 0.$$

$\square$

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