The critical velocity for vortex existence in a two dimensional rotating Bose-Einstein condensate

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Abstract

We investigate a model corresponding to the experiments for a two dimensional rotating Bose-Einstein condensate. It consists in minimizing a Gross-Pitaevskii functional defined in $\mathbb{R}^2$ under the unit mass constraint. We estimate the critical rotational speed Ω_1 for vortex existence in the bulk of the condensate and we give some fundamental energy estimates for velocities close to Ω_1.

1 Introduction

The phenomenon of Bose-Einstein condensation has given rise to an intense research, both experimentally and theoretically, since its first realization in alkali gases in 1995. One of the most beautiful experiments was carried out by the ENS group and consisted in rotating the trap holding the atoms [18, 19] (see also [1]). Since a Bose-Einstein condensate (BEC) is a quantum gas, it can be described by a single complex-valued wave function (order parameter) and it rotates as a superfluid: above a critical velocity, it rotates through the existence of vortices, i.e., zeroes of the wave function around which there is a circulation of phase. In an experiment where a harmonic trap strongly confines the atoms in the direction of the rotation axis, the mathematical analysis becomes two-dimensional by the decoupling of the wave function (see [10, 11, 24]). We restrict our study to this two-dimensional model used in [10, 11]. After the nondimensionalization of the energy (see [3]), the wave function u_ε minimizes the Gross-Pitaevskii energy

$$\int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2}\, V(x)|u|^2 + \frac{1}{4\varepsilon^2}|u|^4 - \Omega\, x^\perp \cdot (iu, \nabla u) \right\} dx \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 = 1, \tag{1.2}$$

where $\varepsilon > 0$ is small and represents a ratio of two characteristic lengths and $\Omega = \Omega(\varepsilon) \geq 0$ denotes the rotational velocity. We consider here the harmonic trapping case, that is $V(x) = |x|^2_\Lambda := x_1^2 + \Lambda^2 x_2^2$ for a fixed parameter $0 < \Lambda \leq 1$. In [11], the equilibrium configurations are studied by looking for the minimizers in a reduced class of functions and some numerical simulations are presented.

Our aim is to estimate the critical velocity above which the wave function has vortices, and in a future work [14] to analyze in more details the vortex patterns in the bulk of the condensate.

According to numerical and theoretical predictions (see [3, 11]), we expect to find the critical speed in the regime $\Omega = \mathcal{O}(|\ln\varepsilon|)$ so that we restrict our study to this situation.

Due to the constraint (1.2), we may rewrite the energy in the equivalent form

$$F_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[(|u|^2 - a(x))^2 - (a^-(x))^2\right] - \Omega x^\perp \cdot (iu, \nabla u) \right\} dx \tag{1.3}$$

where $a(x) = a_0 - |x|^2_\Lambda$ and a_0 is determined by $\int_{\mathbb{R}^2} a^+(x) = 1$ so that $a_0 = \sqrt{2\Lambda/\pi}$. Here a^+ and a^- represent respectively the positive and the negative part of a. Then we consider the wave function u_ε as a solution of the variational problem

$$\mathrm{Min}\left\{F_\varepsilon(u) \,:\, u \in \mathcal{H},\ \|u\|_{L^2(\mathbb{R}^2)} = 1\right\} \quad \text{where} \quad \mathcal{H} = \left\{u \in H^1(\mathbb{R}^2, \mathbb{C}) \,:\, \int_{\mathbb{R}^2} |x|^2|u|^2 < +\infty\right\}.$$

In the limit $\varepsilon \to 0$, the minimization of F_ε strongly forces $|u_\varepsilon|^2$ to be close to a^+ which means that the resulting density is asymptotically localized in the ellipsoidal region

$$\mathcal{D} := \left\{x \in \mathbb{R}^2 \,:\, a(x) > 0\right\} = \left\{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1^2 + \Lambda^2 x_2^2 < a_0\right\}.$$

We will also see that $|u_\varepsilon|$ decays exponentially fast outside $\mathcal{D}$. Actually, the domain $\mathcal{D}$ represents the region occupied by the condensate and consequently, vortices will be sought inside $\mathcal{D}$.

The main tools for studying vortices were developed by Bethuel, Brezis and Hélein [7] for "Ginzburg-Landau type" problems. We also refer to Sandier [20] and Sandier and Serfaty [21, 22, 23] for complementary techniques. In the case $a(x) \equiv 1$ and for a disc in $\mathbb{R}^2$, Serfaty proved the existence of local minimizers having vortices for different ranges of rotational velocity (see [25]). In [3], Aftalion and Du follow the strategy in [25] for the study of global minimizers of the Gross-Pitaevskii energy (1.3) where $\mathbb{R}^2$ is replaced by $\mathcal{D}$. In [2], Aftalion, Alama and Bronsard analyze the global minimizers of (1.3) for potentials of different nature leading to an annular region of confinement. We finally refer to [4, 5, 15] for mathematical studies on 3D models.

We emphasize that we tackle here the problem which corresponds exactly to the physical model. In particular, we minimize F_ε under the unit mass constraint and the admissible configurations are defined in the whole space $\mathbb{R}^2$. Several difficulties arise, especially in the proof of the existence results and the construction of test functions. We point out that we do not assume any implicit bound on the number of vortices. The singular and degenerate behavior of $\sqrt{a^+}$ near $\partial\mathcal{D}$ induces a cost of order $|\ln\varepsilon|$ in the energy and requires specific tools to detect vortices in the boundary region. Therefore we shall restrict our analysis to vortices lying down in the interior domain

$$\mathcal{D}_\varepsilon = \{x \in \mathcal{D} \,:\, a(x) > \nu_\varepsilon |\ln\varepsilon|^{-3/2}\} \tag{1.4}$$

where ν_ε is a chosen parameter in the interval $(1, 2)$ (see Proposition 4.1).

We now start to describe our main results. We prove that

$$\Omega_1 := \frac{\Lambda^2 + 1}{a_0}|\ln\varepsilon| = \frac{\sqrt{\pi}(\Lambda^2 + 1)}{\sqrt{2\Lambda}}|\ln\varepsilon|$$

is the asymptotic estimate as $\varepsilon \to 0$ of the critical angular speed for nucleation of vortices in $\mathcal{D}$. The critical angular velocity Ω_1 coincides with the one found in [3, 11]. We observe that a very stretched condensate, i.e., $\Lambda \ll 1$, yields a very large value of Ω_1 and that the smallest Ω_1 is reached for $\Lambda = 1/\sqrt{3}$ (and surprisingly not for the symmetric case, i.e., $\Lambda = 1$). For subcritical velocities, we will see that u_ε behaves as the "vortex-free" profile $\tilde{\eta}_\varepsilon e^{i\Omega S}$ where $\tilde{\eta}_\varepsilon$ is the positive minimizer of

$$E_\varepsilon(u) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left[(|u|^2 - a(x))^2 - (a^-(x))^2\right] \right\} dx$$

under the constraint (1.2) and the phase S is given by

$$S(x) = \frac{\Lambda^2 - 1}{\Lambda^2 + 1}\, x_1 x_2 \,. \tag{1.5}$$

For rotational speeds larger than Ω_1, we show the existence of vortices close to the origin. We also give some fundamental energy estimates in the regime $\Omega = \Omega_1 + \mathcal{O}(\ln|\ln\varepsilon|)$ which will allow to study the precise vortex structure of u_ε in [14].

Theorem 1.1. *Let u_ε be any minimizer of F_ε in $\mathcal{H}$ under the mass constraint* (1.2)*.*

(i) There exists a constant $\omega_1^\star < 0$ such that if $\Omega \leq \Omega_1 + \omega_1 \ln|\ln\varepsilon|$ with $\omega_1 < \omega_1^\star$ then $|u_\varepsilon| \to \sqrt{a^+}$ in $L^\infty_{\rm loc}(\mathbb{R}^2 \setminus \partial\mathcal{D})$ as $\varepsilon \to 0$. Moreover,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon\big(\tilde{\eta}_\varepsilon e^{i\Omega S}\big) + o(1) \tag{1.6}$$

and for any sequence $\varepsilon_n \to 0$, there exists a subsequence (still denoted by ε_n) and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $u_{\varepsilon_n} e^{-i\Omega S} \to \alpha\sqrt{a^+}$ in $H^1_{\rm loc}(\mathcal{D})$ as $n \to +\infty$.

(ii) If there exists some constant $\delta > 0$ such that $\Omega_1 + \delta\ln|\ln\varepsilon| \leq \Omega \leq \mathcal{O}(|\ln\varepsilon|)$, then u_ε has at least one vortex $x_\varepsilon \in \mathcal{D}$ such that $\mathrm{dist}(x^\varepsilon, \partial\mathcal{D}) \geq C > 0$ with C independent of ε. If in addition, $\Omega \leq \Omega_1 + \mathcal{O}(\ln|\ln\varepsilon|)$, then x^ε remains close to the origin, i.e., $|x^\varepsilon| \leq \mathcal{O}(|\ln\varepsilon|^{-1/6})$.

(iii) Set $v_\varepsilon = u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S})$ and assume that $\Omega \leq \Omega_1 + \omega_1\ln|\ln\varepsilon|$ for some $\omega_1 > 0$. Then there exist two positive constants $\mathcal{M}_1$ and $\mathcal{M}_2$ depending only on ω_1 such that

$$\int_{\mathcal{D}_\varepsilon} a(x)|\nabla v_\varepsilon|^2 + \frac{a^2(x)}{\varepsilon^2}(|v_\varepsilon|^2 - 1)^2 \leq \mathcal{M}_1|\ln\varepsilon|,$$

$$\int_{\mathcal{D}_\varepsilon\setminus\{|x|_\Lambda < 2|\ln\varepsilon|^{-1/6}\}} a(x)|\nabla v_\varepsilon|^2 + \frac{a^2(x)}{\varepsilon^2}(|v_\varepsilon|^2 - 1)^2 \leq \mathcal{M}_2\ln|\ln\varepsilon|.$$

From the estimates in *(iii)* in Theorem 1.1, we are going to determine in [14] the number and the location of vortices in function of the angular speed Ω as $\varepsilon \to 0$. More precisely, we will compute the asymptotic expansion of the energy $F_\varepsilon(u_\varepsilon)$ in order to estimate the critical velocity Ω_d for having d vortices in the bulk and to exhibit the configuration of vortices by a certain renormalized energy. We also mention that the techniques used in [14] will permit to prove that the best constant in *(i)* in Theorem 1.1 is $\omega_1^\star = 0$. The proof will rely mostly on the study of "bad discs" in [7].

Sketch of the proof. We now describe briefly the content of this paper.

Section 2 is devoted to the study of the density profile $\tilde{\eta}_\varepsilon$. We first introduce the real positive minimizer η_ε of E_ε, i.e.,

$$E_\varepsilon(\eta_\varepsilon) = \mathrm{Min}\,\big\{E_\varepsilon(\eta)\,:\,\eta \in \mathcal{H}\big\}. \tag{1.7}$$

We show the existence and uniqueness of η_ε (see Theorem 2.1) and we have that $E_\varepsilon(\eta_\varepsilon) \leq C|\ln\varepsilon|$ and $\eta_\varepsilon \to \sqrt{a^+}$ in $L^\infty(\mathbb{R}^2) \cap C^1_{\rm loc}(\mathcal{D})$ as $\varepsilon \to 0$ (see Proposition 2.1). Then we prove that there is a unique positive solution of the problem

$$\mathrm{Min}\,\big\{E_\varepsilon(\eta)\,:\,\eta \in \mathcal{H}\,,\ \|\eta\|_{L^2(\mathbb{R}^2)} = 1\big\} \tag{1.8}$$

called $\tilde{\eta}_\varepsilon$, which can be obtained from η_ε by a change of scale (see Theorem 2.2). This relationship yields an important estimate on the Lagrange multiplier k_ε associated to $\tilde{\eta}_\varepsilon$: $|k_\varepsilon| \leq \mathcal{O}(|\ln\varepsilon|)$, as well as the asymptotic properties of $\tilde{\eta}_\varepsilon$ from those of η_ε (see Proposition 2.2). In particular, we have $\tilde{\eta}_\varepsilon \to \sqrt{a^+}$ in $L^\infty(\mathbb{R}^2) \cap C^1_{\rm loc}(\mathcal{D})$ as $\varepsilon \to 0$.

In Section 3, we prove the existence of minimizers u_ε under the mass constraint (1.2) (see Proposition 3.1) and some general results about their behavior: $E_\varepsilon(u_\varepsilon) \le C|\ln\varepsilon|^2$, u_ε decreases exponentially quickly to 0 outside $\mathcal{D}$, $|\nabla u_\varepsilon| \le C_K\varepsilon^{-1}$ and $|u_\varepsilon| \lesssim \sqrt{a^+}$ in any compact $K \subset \mathcal{D}$ (see Proposition 3.2). Using a method introduced by Lassoued and Mironescu [16], we show that $F_\varepsilon(u_\varepsilon)$ splits into two independent pieces (see Lemma 3.2): the energy of the "vortex-free" profile $F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S})$ and the reduced energy of $v_\varepsilon = u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S})$:

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \tag{1.9}$$

where

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{R}}_\varepsilon(v_\varepsilon), \tag{1.10}$$

$$\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) = \int_{\mathbb{R}^2} \frac{\tilde{\eta}_\varepsilon^2}{2}|\nabla v_\varepsilon|^2 + \frac{\tilde{\eta}_\varepsilon^4}{4\varepsilon^2}(|v_\varepsilon|^2-1)^2\ , \quad \tilde{\mathcal{R}}_\varepsilon(v_\varepsilon) = \frac{\Omega}{\Lambda^2+1}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 \nabla^\perp a \cdot (iv_\varepsilon, \nabla v_\varepsilon)\ , \tag{1.11}$$

$$\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^2} \big(\Omega^2|\nabla S|^2 - 2\Omega^2 x^\perp \cdot \nabla S + k_\varepsilon\big)\tilde{\eta}_\varepsilon^2(|v_\varepsilon|^2-1). \tag{1.12}$$

The motivation of S is explained in [3]: S satisfies $\operatorname{div}\big(a^+(\nabla S - x^\perp)\big) = 0$ in $\mathbb{R}^2$ and corresponds to the limit as $\varepsilon \to 0$ of the phase (globally defined in $\mathbb{R}^2$) divided by Ω, of any solution of Min $\big\{F_\varepsilon(u) : u = \eta e^{i\varphi} \in \mathcal{H},\ \eta > 0\big\}$. The existence of the global limiting phase S is new in this type of variational problems related to the "Ginzburg-Landau" energy. We point out that the anisotropy carried by the phase S, leads to a negative term of order Ω^2 for $\Lambda \in (0,1)$ in the energy (see Remark 3.2):

$$F_\varepsilon\big(\tilde{\eta}_\varepsilon e^{i\Omega S}\big) = E_\varepsilon(\tilde{\eta}_\varepsilon) - \frac{\sqrt{2}(1-\Lambda^2)^2}{12\sqrt{\pi}(1+\Lambda^2)\Lambda^{3/2}}\Omega^2 + o(1).$$

We will prove that $|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)| = \mathcal{O}(\varepsilon|\ln\varepsilon|^3)$. Thus, we may focus on the reduced energy $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon)$. We study the vortex structure of u_ε via the map v_ε applying the Ginzburg-Landau techniques to the weighted energy $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon)$; the difficulty will arise in the region where $\tilde{\eta}_\varepsilon$ is small. We notice that v_ε inherits from u_ε and $\tilde{\eta}_\varepsilon$, the following properties (see Proposition 3.3): $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \le C|\ln\varepsilon|^2$, $|\nabla v_\varepsilon| \le C_K\varepsilon^{-1}$ and $|v_\varepsilon| \lesssim 1$ in any compact $K \subset \mathcal{D}$. Using $\tilde{\eta}_\varepsilon e^{i\Omega S}$ as a test function and (1.9), we obtain in Proposition 3.4, a crucial upper bound of the reduced energy inside $\mathcal{D}_\varepsilon$:

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le o(1). \tag{1.13}$$

Motivated by the behavior $\tilde{\eta}_\varepsilon^2 \sim a^+$ (see (3.32) and (3.33)), we will use in the sequel the energies $\mathcal{F}_\varepsilon$, $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ in the interior of $\mathcal{D}$ (see Notations below).

In Section 4, we compute a first lower bound of $\mathcal{E}_\varepsilon(v_\varepsilon)$ using a method due to Sandier and Serfaty (see [21, 23]). We start with the construction of small disjoint balls $\{B(p_i,r_i)\}_{i\in I_\varepsilon}$ in the domain $\mathcal{D}_\varepsilon$ (given by (1.4)): outside these balls $|v_\varepsilon|$ is close to 1, so that v_ε carries a degree d_i on $\partial B(p_i,r_i)$ (see Proposition 4.1) and

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \ge \sum_{i\in I_\varepsilon} \mathcal{E}_\varepsilon(v_\varepsilon, B(p_i,r_i)) \gtrsim \pi \sum_{i\in I_\varepsilon} a(p_i)|d_i|\,|\ln\varepsilon|. \tag{1.14}$$

Then we prove an asymptotic expansion of the rotational energy outside the balls $\{B(p_i,r_i)\}_{i\in I_\varepsilon}$ (see Proposition 4.2),

$$\mathcal{R}_\varepsilon\big(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i\big) \approx -\frac{\pi\Omega}{\Lambda^2+1}\sum_{i\in I_\varepsilon} a^2(p_i)\, d_i. \tag{1.15}$$

The presence of $a^2(p_i)$ is due to the harmonic type of the potential. In fact, for slightly more general potentials $a(x)$, we compute the solution ξ of the problem (see [3])

$$\operatorname{div}\Big(\frac{1}{a}\nabla\xi\Big) = -2 \text{ in } \mathcal{D} \quad \text{and} \quad \xi = 0 \text{ on } \partial\mathcal{D} \tag{1.16}$$

and the rotational energy will exhibit the terms $\xi(p_i)$ in (1.15). For our harmonic potential $a(x)$, an easy computation leads to $\xi = \frac{a^2}{2(\Lambda^2+1)}$. By (1.14) and (1.15), the first term in the lower expansion of the energy is

$$\pi \sum_{i\in I_\varepsilon} a(p_i)\bigg(|d_i||\ln\varepsilon| - d_i\Omega\frac{2\xi(p_i)}{a(p_i)}\bigg). \tag{1.17}$$

For having a vortex ball B_i with nonzero degree, Ω has to be larger than $\Omega_1 = \frac{1+\Lambda^2}{a_0}|\ln\varepsilon|$, p_i maximizes ξ/a and d_i is positive. Indeed, we obtain the subcritical case *(i)* in Theorem 1.1 matching (1.13) with (1.17). For velocities larger than Ω_1, we use an improvement of the upper estimate (1.13) using a test function having a single vortex at the origin. From here, we deduce *(ii)* in Theorem 1.1. We also prove that for $\Omega \leq \Omega_1 + \mathcal{O}(\ln|\ln\varepsilon|)$, the number of vortex balls with nonzero degree is uniformly bounded in ε and they appear close to the origin (see Proposition 4.4). We conclude by the two fundamental energy estimates stated in *(iii)* in Theorem 1.1.

Our analysis deals with vortices inside $\mathcal{D}$. However, we believe that for Ω small ($\Omega = \mathcal{O}(1)$), the solution should not have any vortices in $\mathbb{R}^2$. For Ω larger ($\Omega \sim \Omega_1$), vortices may exist in the region where u_ε is small. The study of the vortex structure in the region where $|u_\varepsilon|$ is small requires the development of other tools than energy estimates.

We recall that the choice of the harmonic potential is motivated by the physical experiments. For some other potentials a such that ξ/a has a unique maximum point at the origin, our method can be applied and the critical speed is given by

$$\Omega_1 = \frac{a(0)}{2\xi(0)}|\ln\varepsilon|.$$

If the set of maximum points of $\frac{\xi}{a}$ is not finite (it can be a curve, see Remark 4.1), the techniques are different and it will be the topic of a future work.

Notations. Throughout the paper, we denote by C a positive constant independent of ε and we use the subscript to point out a possible dependence on the argument. For $x = (x_1, x_2) \in \mathbb{R}^2$, we write

$$x^\perp = (-x_2, x_1),\ |x|_\Lambda = \sqrt{x_1^2 + \Lambda^2 x_2^2} \quad \text{and} \quad B_R^\Lambda = \big\{x \in \mathbb{R}^2 \,:\, |x|_\Lambda < R\big\}$$

and for $\mathcal{A} \subset \mathbb{R}^2$,

$$\begin{aligned}
\tilde{\mathcal{E}}_\varepsilon(v,\mathcal{A}) &= \int_{\mathcal{A}} \frac{1}{2}\tilde{\eta}_\varepsilon^2|\nabla v|^2 + \frac{\tilde{\eta}_\varepsilon^4}{4\varepsilon^2}(1-|v|^2)^2\,, \quad \mathcal{E}_\varepsilon(v,\mathcal{A}) = \int_{\mathcal{A}} \frac{1}{2}a|\nabla v|^2 + \frac{a^2}{4\varepsilon^2}(1-|v|^2)^2,\\
\tilde{\mathcal{R}}_\varepsilon(v,\mathcal{A}) &= \frac{\Omega}{1+\Lambda^2}\int_{\mathcal{A}} \tilde{\eta}_\varepsilon^2\nabla^\perp a\cdot(iv,\nabla v)\,, \quad \mathcal{R}_\varepsilon(v,\mathcal{A}) = \frac{\Omega}{1+\Lambda^2}\int_{\mathcal{A}} a\nabla^\perp a\cdot(iv,\nabla v),\\
\tilde{\mathcal{F}}_\varepsilon(v,\mathcal{A}) &= \tilde{\mathcal{E}}_\varepsilon(v,\mathcal{A}) + \tilde{\mathcal{R}}_\varepsilon(v,\mathcal{A})\,, \quad \mathcal{F}_\varepsilon(v,\mathcal{A}) = \mathcal{E}_\varepsilon(v,\mathcal{A}) + \mathcal{R}_\varepsilon(v,\mathcal{A}).
\end{aligned} \tag{1.18}$$

We do not write the dependence on $\mathcal{A}$ when $\mathcal{A} = \mathbb{R}^2$.

2 Analysis of the density profiles

In this section, we establish some preliminary results on η_ε and $\tilde{\eta}_\varepsilon$ defined respectively by (1.7) and (1.8). We will show that the shapes of η_ε and $\tilde{\eta}_\varepsilon$ are similar.

We notice that the space $\mathcal{H}$ in which we perform the minimization, is exactly the set of finiteness for E_ε. In the sequel, we endow $\mathcal{H}$ with the scalar product

$$\langle u, v\rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} \nabla u \cdot \nabla v + (1 + |x|^2)(u \cdot v) \quad \text{for } u, v \in \mathcal{H};$$

obviously, $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a Hilbert space.

2.1 The free profile

We start by proving the existence and uniqueness for small ε of η_ε defined as the real positive solution of (1.7). Hence η_ε has to satisfy the associated Euler-Lagrange equation

$$\begin{cases} \varepsilon^2 \Delta \eta_\varepsilon + (a(x) - \eta_\varepsilon^2)\eta_\varepsilon = 0 & \text{in } \mathbb{R}^2, \\ \eta_\varepsilon > 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{2.1}$$

We denote by λ, the first eigenvalue of the elliptic operator $-\Delta + |x|^2_\Lambda$ in $\mathbb{R}^2$, i.e.,

$$\lambda = \mathrm{Inf} \left\{ \int_{\mathbb{R}^2} |\nabla \phi|^2 + |x|^2_\Lambda |\phi|^2 \ : \ \phi \in \mathcal{H} \ , \ \|\phi\|_{L^2(\mathbb{R}^2)} = 1 \right\}.$$

We have the following result:

Theorem 2.1. *If $0 < \varepsilon < \frac{a_0}{\lambda}$, there exists a unique classical solution η_ε of (2.1). Moreover, $\eta_\varepsilon \leq \sqrt{a_0}$ and η_ε is the unique minimizer of E_ε in $\mathcal{H}$ up to a complex multiplier of modulus one. If $\varepsilon \geq \frac{a_0}{\lambda}$, then zero is the unique critical point of E_ε in $\mathcal{H}$.*

The method that we use for solving (2.1) involves several classical arguments generally used for a bounded domain. The main difficulty here is due to the fact that the equation is posed in the entire space $\mathbb{R}^2$ without any condition at infinity. We start with the construction of the *minimal solution*: we consider the solution $\eta_{R,\varepsilon}$ of the same equation posed in a ball of large radius R with homogeneous Dirichlet boundary condition and then we pass to the limit in R. We prove the uniqueness by estimating the ratio between the constructed solution and any other solution. A crucial point in the proof is an L^∞-bound of any weak solution.

Before proving Theorem 2.1, we present the asymptotic properties of η_ε as $\varepsilon \to 0$. We show that η_ε decays exponentially fast outside $\mathcal{D}$ and that η_ε^2 tends uniformly to a^+. The following estimates will be essential at several steps of our analysis.

Proposition 2.1. *For ε sufficiently small, we have*

2.1.a) $E_\varepsilon(\eta_\varepsilon) \leq C|\ln \varepsilon|$,

2.1.b) $0 < \eta_\varepsilon(x) \leq C\varepsilon^{1/3} \exp\left(\dfrac{a(x)}{4\varepsilon^{2/3}}\right)$ *in* $\mathbb{R}^2 \setminus \mathcal{D}$,

2.1.c) $0 \leq \sqrt{a(x)} - \eta_\varepsilon(x) \leq C\varepsilon^{1/3}\sqrt{a(x)}$ *for* $x \in \mathcal{D}$ *with* $|x|_\Lambda < \sqrt{a_0} - \varepsilon^{1/3}$,

2.1.d) $\|\nabla \eta_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{-1}$,

2.1.e) $\|\eta_\varepsilon - \sqrt{a}\,\|_{C^1(K)} \leq C_K \varepsilon^2$ *for any compact subset* $K \subset \mathcal{D}$.

Remark 2.1. We observe that *2.1.a)* in Proposition 2.1 implies

$$\int_{\mathbb{R}^2\setminus\mathcal{D}} |\eta_\varepsilon|^4 + 2a^-(x)|\eta_\varepsilon|^2 + \int_{\mathcal{D}} (a(x) - |\eta_\varepsilon|^2)^2 \le C\varepsilon^2 |\ln\varepsilon|. \tag{2.2}$$

Proof of Theorem 2.1. Step 1: Existence for $0 < \varepsilon < \frac{a_0}{\lambda}$. For $R > 0$, we consider the equation

$$\begin{cases} \varepsilon^2\Delta\eta_R + (a(x) - \eta_R^2)\eta_R = 0 & \text{in } B_R, \\ \eta_R > 0 & \text{in } B_R, \\ \eta_R = 0 & \text{on } \partial B_R. \end{cases} \tag{2.3}$$

By a result of Brezis and Oswald (see [9]), we have the existence and uniqueness of weak solutions of (2.3) if and only if the following first eigenvalue condition holds

$$\text{Inf}\left\{\int_{B_R} |\nabla\phi|^2 - \frac{a(x)|\phi|^2}{\varepsilon^2} \;:\; \phi \in H_0^1(B_R)\;,\; \|\phi\|_{L^2(B_R)} = 1\right\} < 0, \text{ i.e.,}$$

$$\lambda_1(L_\varepsilon, B_R) = \text{Inf}\left\{\int_{B_R} |\nabla\phi|^2 + \frac{|x|_\Lambda^2|\phi|^2}{\varepsilon^2} \;:\; \phi \in H_0^1(B_R)\;,\; \|\phi\|_{L^2(B_R)} = 1\right\} < \frac{a_0}{\varepsilon^2} \tag{2.4}$$

where we denoted the elliptic operator $L_\varepsilon = -\Delta + \frac{|x|_\Lambda^2}{\varepsilon^2}$. We claim that for R sufficiently large, (2.4) is fulfilled. Indeed, let ψ be an eigenfunction of L_ε in $\mathbb{R}^2$ associated to the first eigenvalue $\lambda_1(L_\varepsilon, \mathbb{R}^2)$ with $\|\psi\|_{L^2(\mathbb{R}^2)} = 1$ (the existence of ψ is a direct consequence of the compact embedding $\mathcal{H} \hookrightarrow L^2(\mathbb{R}^2)$ proved in Lemma 2.1). For any integer $n \ge 1$, set $\psi_n(x) = c_n\,\zeta\left(\frac{|x|}{n}\right)\psi(x)$, where $\zeta : \mathbb{R} \to \mathbb{R}$ is the "cut-off" type function given by

$$\zeta(t) = \begin{cases} 1 & \text{if } t \le 1, \\ 2 - t & \text{if } t \in (1,2), \\ 0 & \text{if } t \ge 2 \end{cases} \tag{2.5}$$

and the constant c_n is chosen such that $\|\psi_n\|_{L^2(\mathbb{R}^2)} = 1$. We easily check that

$$\lambda_1(L_\varepsilon, B_{2n}) \le \int_{B_{2n}} \left(|\nabla\psi_n|^2 + \frac{|x|_\Lambda^2}{\varepsilon^2}|\psi_n|^2\right) \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2} \left(|\nabla\psi|^2 + \frac{|x|_\Lambda^2}{\varepsilon^2}|\psi|^2\right) = \lambda_1(L_\varepsilon, \mathbb{R}^2)$$

and we deduce that the sequence $\big\{\lambda_1(L_\varepsilon, B_R)\big\}_{R>0}$ (which is decreasing in R) tends to $\lambda_1(L_\varepsilon, \mathbb{R}^2)$ as $R \to \infty$. Since

$$\lambda_1(L_\varepsilon, \mathbb{R}^2) = \frac{\lambda}{\varepsilon},$$

we conclude that there exists $R_\varepsilon > 0$ such that for every $R > R_\varepsilon$, condition (2.4) is fulfilled and equation (2.3) admits a unique weak solution $\eta_{R,\varepsilon}$.

By standard methods, it results that $\eta_{R,\varepsilon}$ is a smooth classical solution of (2.3). We notice that, for any $R_\varepsilon < R < \tilde{R}$, $\eta_{\tilde{R},\varepsilon}$ is a supersolution of (2.3) in B_R and thus $\eta_{R,\varepsilon} \le \eta_{\tilde{R},\varepsilon}$ in B_R by the uniqueness of $\eta_{R,\varepsilon}$. By the maximum principle, we infer that $\eta_{R,\varepsilon} \le \sqrt{a_0}$ in $\mathbb{R}^2$. For every $R > R_\varepsilon$, we extend $\eta_{R,\varepsilon}$ by 0 in $\mathbb{R}^2 \setminus B_R$. Since the function $R \to \eta_{R,\varepsilon}(x)$ is non-decreasing for any $x \in \mathbb{R}^2$, we may define for $x \in \mathbb{R}^2$, $\eta_\varepsilon(x) = \lim\limits_{R\to+\infty} \eta_{R,\varepsilon}(x)$. It results that η_ε satisfies $0 < \eta_\varepsilon \le \sqrt{a_0}$ and

$$\varepsilon^2\Delta\eta_\varepsilon + (a(x) - \eta_\varepsilon^2)\eta_\varepsilon = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{2.6}$$

Since $\eta_\varepsilon \in L^\infty(\mathbb{R}^2)$, we derive by standard methods that η_ε is a smooth classical solution of (2.1).

Step 2. L^∞-bound for solutions of (2.1). The method we use in this step is due to Farina (see [12]) and relies on a result of Brezis (see [8]). We present the proof for convenience. Let η be any weak solution of (2.1) in $L^3_{\rm loc}(\mathbb{R}^2)$. We claim that

$$\eta \le \sqrt{a_0} \quad \text{a.e. in } \mathbb{R}^2.$$

Indeed, if we consider $w = \varepsilon^{-1}(\eta - \sqrt{a_0})$, then $w \in L^3_{\rm loc}(\mathbb{R}^2)$ and since η satisfies (2.1), we infer that $\Delta w \in L^1_{\rm loc}(\mathbb{R}^2)$. By Kato's inequality, we have

$$\Delta(w^+) \ge \mathrm{sgn}^+(w)\Delta w \ge \frac{\mathrm{sgn}^+(w)}{\varepsilon^3}(\eta^2 - a_0)\eta = \frac{1}{\varepsilon^2}\, w^+(\varepsilon w + 2\sqrt{a_0})(\varepsilon w + \sqrt{a_0}) \ge (w^+)^3.$$

Therefore $w^+ \in L^3_{\rm loc}(\mathbb{R}^2)$ and w^+ satisfies

$$-\Delta(w^+) + (w^+)^3 \le 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

By Lemma 2 in [8], it leads to $w^+ \le 0$ a.e. in $\mathbb{R}^2$ and thus $w^+ \equiv 0$.

Step 3. Uniqueness for $0 < \varepsilon < \frac{a_0}{\lambda}$. Let η_ε be the solution constructed at Step 1 and let η be any weak solution of (2.1) in $L^3_{\rm loc}(\mathbb{R}^2)$. By the previous step, $\eta \in L^\infty(\mathbb{R}^2)$ and using standard arguments, we derive that η is smooth and defines a classical solution of (2.1). We observe that η is a supersolution of (2.3) for every $R > R_\varepsilon$. Since $\eta_{R,\varepsilon}$ is extended by 0 outside B_R, $\eta_{R,\varepsilon} \le \eta$ in $\mathbb{R}^2$. Passing to the limit in R, we get that $0 < \eta_\varepsilon \le \eta$ in $\mathbb{R}^2$. Hence the function $\rho : \mathbb{R}^2 \to \mathbb{R}$ defined by $\rho = \eta_\varepsilon/\eta$ is smooth and takes values in $(0,1]$. We easily check that ρ satisfies

$$\mathrm{div}(\eta^2\nabla\rho) + \frac{\eta^4}{\varepsilon^2}(1-\rho^2)\rho = 0 \quad \text{in } \mathbb{R}^2. \tag{2.7}$$

For every integer $n \ge 1$, we set $\zeta_n(x) = \zeta\left(n^{-1}|x|\right)$, where ζ is given by (2.5). Multiplying (2.7) by $(1-\rho)\zeta_n^2$ and integrating by parts, we derive

$$\int_{\mathbb{R}^2}\left(\frac{\eta^4}{\varepsilon^2}\,\rho(1-\rho)^2(1+\rho)\zeta_n^2 + \eta^2\zeta_n^2|\nabla\rho|^2\right) = 2\int_{\mathbb{R}^2}\eta^2(1-\rho)\zeta_n(\nabla\rho\cdot\nabla\zeta_n). \tag{2.8}$$

Since ρ is bounded, the Cauchy-Schwarz inequality yields

$$\begin{aligned}\int_{\mathbb{R}^2}\eta^2(1-\rho)\zeta_n(\nabla\rho\cdot\nabla\zeta_n) &= \int_{B_{2n}\setminus B_n}\eta^2(1-\rho)\zeta_n(\nabla\rho\cdot\nabla\zeta_n)\\ &\le \left(\int_{B_{2n}}\eta^2(1-\rho)^2|\nabla\zeta_n|^2\right)^{1/2}\left(\int_{B_{2n}\setminus B_n}\eta^2\zeta_n^2|\nabla\rho|^2\right)^{1/2}\\ &\le 2\sqrt{\pi}\,\|\eta\|_{L^\infty(\mathbb{R}^2)}\left(\int_{\mathbb{R}^2\setminus B_n}\eta^2\zeta_n^2|\nabla\rho|^2\right)^{1/2}.\end{aligned}$$

Using (2.8) and the L^∞-bound on η obtained in Step 2, we infer that

$$\int_{\mathbb{R}^2}\eta^2\zeta_n^2|\nabla\rho|^2 \le 4\sqrt{\pi a_0}\left(\int_{\mathbb{R}^2\setminus B_n}\eta^2\zeta_n^2|\nabla\rho|^2\right)^{1/2}. \tag{2.9}$$

It follows

$$16\pi a_0 \ge \int_{\mathbb{R}^2}\eta^2\zeta_n^2|\nabla\rho|^2 \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2}\eta^2|\nabla\rho|^2$$

by monotone convergence. Since $\eta^2|\nabla\rho|^2 \in L^1(\mathbb{R}^2)$, the right hand side in (2.9) tends to 0 as $n \to +\infty$ and we finally deduce that $\int_{\mathbb{R}^2} \eta^2|\nabla\rho|^2 = 0$. Hence ρ is constant in $\mathbb{R}^2$ and by (2.8), we necessarily have $\rho = 1$, i.e., $\eta = \eta_\varepsilon$.

Step 4. End of the proof. The existence of a minimizer η of E_ε in $\mathcal{H}$ is standard. Since $E_\varepsilon(|\hat\eta|) \leq E_\varepsilon(\hat\eta)$ for any $\hat\eta \in \mathcal{H}$, we infer that $\hat\eta := |\eta|$ is also a minimizer and therefore $\hat\eta$ satisfies the equation

$$\begin{cases} \varepsilon^2\Delta\hat\eta + (a(x) - \hat\eta^2)\hat\eta = 0 & \text{in } \mathbb{R}^2, \\ \hat\eta \geq 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{2.10}$$

By the maximum principle, it follows that either $\hat\eta > 0$ in $\mathbb{R}^2$ or $\hat\eta \equiv 0$.

If $0 < \varepsilon < \frac{a_0}{\lambda}$, we claim that $\hat\eta > 0$. Indeed, for $R > 0$ sufficiently large, we consider the unique solution $\eta_{R,\varepsilon}$ of (2.3). By [9], $\eta_{R,\varepsilon}$ is the unique non-negative minimizer of $E_\varepsilon(\cdot, B_R)$ in $H^1_0(B_R,\mathbb{R})$. Since $\eta_{R,\varepsilon}$ is extended by 0 outside B_R, we have

$$E_\varepsilon(\hat\eta) \leq E_\varepsilon(\eta_{R,\varepsilon}) = E_\varepsilon(\eta_{R,\varepsilon}\,, B_R) < E_\varepsilon(0, B_R) = E_\varepsilon(0)$$

which implies that $\hat\eta$ is not identically equal to 0. Then $\hat\eta$ solves (2.1) and by Step 3, we conclude that $|\eta| = \hat\eta = \eta_\varepsilon$. From the equality $E_\varepsilon(|\eta|) = E_\varepsilon(\eta)$, we easily deduce that there exists a real constant α such that $\eta = |\eta|e^{i\alpha} = \eta_\varepsilon e^{i\alpha}$.

If $\varepsilon \geq \frac{a_0}{\lambda}$, we prove that $\hat\eta \equiv 0$. Multiplying (2.10) by $\hat\eta$, it results

$$\int_{\mathbb{R}^2} |\nabla\hat\eta|^2 + \frac{|x|^2_\Lambda}{\varepsilon^2}\hat\eta^2 + \frac{1}{\varepsilon^2}\hat\eta^4 = \frac{a_0}{\varepsilon^2}\int_{\mathbb{R}^2} \hat\eta^2 \leq \frac{\lambda}{\varepsilon}\int_{\mathbb{R}^2} \hat\eta^2.$$

On the other hand,

$$\int_{\mathbb{R}^2} |\nabla\hat\eta|^2 + \frac{|x|^2_\Lambda}{\varepsilon^2}\hat\eta^2 \geq \lambda_1(L_\varepsilon, \mathbb{R}^2)\int_{\mathbb{R}^2} \hat\eta^2 = \frac{\lambda}{\varepsilon}\int_{\mathbb{R}^2} \hat\eta^2.$$

It follows that $\int_{\mathbb{R}^2} \hat\eta^4 = 0$, i.e., $\hat\eta \equiv 0$. Thus, in this range of ε, zero is the unique minimizer of E_ε.

Now it remains to show that zero is the unique critical point of E_ε when $\varepsilon \geq \frac{a_0}{\lambda}$. Indeed, let $\tilde\eta$ be any critical point of E_ε in $\mathcal{H}$, i.e., $\tilde\eta$ satisfies the equation (2.6). Then

$$\int_{\mathbb{R}^2} |\nabla\tilde\eta|^2 = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} a(x)\tilde\eta^2 - \tilde\eta^4. \tag{2.11}$$

Since zero is the global minimizer, we have that $E_\varepsilon(\tilde\eta) \geq E_\varepsilon(0)$, so that

$$\int_{\mathbb{R}^2} |\nabla\tilde\eta|^2 + \frac{1}{2\varepsilon^2}\int_{\mathbb{R}^2} \tilde\eta^4 - 2a(x)\tilde\eta^2 \geq 0. \tag{2.12}$$

Combining (2.11) and (2.12), we derive that $\int_{\mathbb{R}^2} \tilde\eta^4 = 0$, i.e., $\tilde\eta \equiv 0$. ∎

We recall the following classical result:

Lemma 2.1. *The embedding $\mathcal{H} \hookrightarrow L^2(\mathbb{R}^2,\mathbb{C})$ is compact.*

Proof. Let $u_n \rightharpoonup 0$ weakly in $\mathcal{H}$ as $n \to \infty$. Extracting a subsequence if necessary, by the Sobolev embedding theorem, we may assume that $u_n \to 0$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. Obviously, $\int_{\mathbb{R}^2} |x|^2|u_n|^2 \leq C$. For any $R > 0$, we have

$$R^2 \limsup_{n\to\infty} \int_{\mathbb{R}^2\setminus B_R} |u_n|^2 \leq \limsup_{n\to\infty} \int_{\mathbb{R}^2} |x|^2|u_n|^2 \leq C.$$

Letting $R \to +\infty$ in this inequality, we conclude that $u_n \to 0$ strongly in $L^2(\mathbb{R}^2)$. ∎

Remark 2.2. We emphasize that from the proof of Theorem 2.1, it follows that any smooth function η satisfying

$$\begin{cases} -\varepsilon^2\Delta\eta \geq (a(x) - |\eta|^2)\eta & \text{in } \mathbb{R}^2, \\ \eta > 0 & \text{in } \mathbb{R}^2, \end{cases}$$

verifies $\eta \geq \eta_\varepsilon$ in $\mathbb{R}^2$.

Proof of Proposition 2.1. Proof of 2.1.a). We construct an explicit test function $\varphi \in H^1(\mathbb{R}^2)$ such that $E_\varepsilon(\varphi) \leq C|\ln\varepsilon|$. Since η_ε minimizes E_ε, we deduce $E_\varepsilon(\eta_\varepsilon) \leq E_\varepsilon(\varphi) \leq C|\ln\varepsilon|$. The function φ is defined as in [15]: let

$$\gamma(s) = \begin{cases} \sqrt{s} & \text{if } s \geq \varepsilon^{2/3}, \\ \dfrac{s}{\varepsilon^{1/3}} & \text{otherwise} \end{cases}$$

and set $\varphi(x) = \gamma(a^+(x))$ for $x \in \mathbb{R}^2$. It results that

$$\int_{\mathbb{R}^2} |\nabla\varphi|^2 \leq C|\ln\varepsilon| \quad \text{and} \quad \int_{\mathbb{R}^2} (a^+ - \varphi^2)^2 \leq C\varepsilon^2 \tag{2.13}$$

for a positive constant C independent of ε.

Proof of 2.1.b). We construct a supersolution $\overline{\eta}$ of (2.1) of the form:

$$\overline{\eta}(x) = \begin{cases} \sqrt{a(x)} & \text{if } |x|_\Lambda \leq \sqrt{a_0 - \delta}\,, \\ \dfrac{-|x|_\Lambda\sqrt{a_0-\delta} + a_0}{\sqrt{\delta}} & \text{if } \sqrt{a_0-\delta} \leq |x|_\Lambda \leq r_\delta, \\ \beta\exp(-|x|^2_\Lambda/2\sigma) & \text{otherwise,} \end{cases} \tag{2.14}$$

where $\delta > 0$ will be determined later,

$$r_\delta = \frac{a_0}{2\sqrt{a_0-\delta}} + \frac{\sqrt{a_0}}{2}$$

and β, σ are chosen such that $\overline{\eta} \in C^1(\mathbb{R}^2)$, i.e.,

$$\beta = \frac{a_0 - \sqrt{a_0(a_0-\delta)}}{2\sqrt{\delta}}\exp(r_\delta^2/2\sigma) \quad \text{and} \quad \sigma = \frac{a_0\delta}{4(a_0-\delta)}.$$

A straightforward computation shows that for $\delta = 4a_0^{1/3}\varepsilon^{2/3}$, $\overline{\eta}$ is a supersolution of (2.1) and we also have

$$r_\delta - \sqrt{a_0} = \mathcal{O}(\varepsilon^{2/3}), \quad \sigma = \mathcal{O}(\varepsilon^{2/3}) \quad \text{and} \quad \beta = \mathcal{O}(\varepsilon^{1/3}e^{a_0/2\sigma}).$$

By Remark 2.2, it results that $\eta_\varepsilon \leq \overline{\eta}$ in $\mathbb{R}^2$ which leads to *2.1.b)*. Notice that we also obtain

$$\begin{cases} \eta_\varepsilon(x) \leq \sqrt{a(x)} & \text{for } |x|_\Lambda \leq \sqrt{a_0-\delta}\,, \\ \eta_\varepsilon(x) \leq C\varepsilon^{1/3} & \text{for } \sqrt{a_0-\delta} \leq |x|_\Lambda \leq \sqrt{a_0}\,. \end{cases} \tag{2.15}$$

Proof of 2.1.c). The estimate *2.1.c)* follows exactly as in Proposition 2.1 in [2] and we shall omit it.

Proof of 2.1.d). Taking $x_0 \in \mathbb{R}^2$ arbitrarily, it suffices to show that $|\nabla\eta_\varepsilon| \leq C\varepsilon^{-1}$ in $B(x_0,\varepsilon)$ with a constant C independent of x_0. We define the re-scaled function $\phi_\varepsilon : B_2(0) \to \mathbb{R}$ by $\phi_\varepsilon(y) = \eta_\varepsilon(x_0 + \varepsilon y)$. From estimates *2.1.b)* and *2.1.c)*, we derive that $|\Delta\phi_\varepsilon| = \left|\left(a(x_0+\varepsilon y) - \phi_\varepsilon^2\right)\phi_\varepsilon\right| \leq C$ in

$B_2(0)$ for a constant C independent of x_0. By elliptic regularity, we deduce that for any $1 \le p < \infty$, $\|\phi_\varepsilon\|_{W^{2,p}(B_1(0))} \le C_p$ for a constant C_p independent of ε and x_0. Taking some $p > 2$, it implies that $\|\nabla\phi_\varepsilon\|_{L^\infty(B_1(0))} \le C$ for a constant C independent of ε and x_0 which yields the result.

Proof of 2.1.e). The idea of the proof is due to Shafrir [26]. First we prove that $|\nabla\eta_\varepsilon|$ remains bounded with respect to ε in any compact set $K \subset \mathcal{D}$. We choose some radii $0 < r < R < \sqrt{a_0}$ such that $K \subset B_r^\Lambda \subset B_R^\Lambda \subset \mathcal{D}$. We claim that

$$|\eta_\varepsilon - \sqrt{a}\,| \le C_R\,\varepsilon^2 \quad \text{in } B_r^\Lambda. \tag{2.16}$$

Indeed, we infer from (2.1) that

$$-\varepsilon^2\Delta(\sqrt{a} - \eta_\varepsilon) + \eta_\varepsilon(\eta_\varepsilon + \sqrt{a}\,)(\sqrt{a} - \eta_\varepsilon) = -\varepsilon^2\Delta(\sqrt{a}\,) = \mathcal{O}(\varepsilon^2) \quad \text{in } B_R^\Lambda.$$

By estimate *2.1.c)*, we have $|\sqrt{a} - \eta_\varepsilon| \le \frac{\sqrt{a}}{2}$ in B_R^Λ for ε small. Thus $\eta_\varepsilon(\eta_\varepsilon + \sqrt{a}) \ge A_R > 0$ in B_R^Λ for some positive constant A_R which only depends on R. Then (2.16) follows from Lemma 2.2 below (which is a slight modification of Lemma 2 in [6]).

Lemma 2.2. *Assume that $A > 0$ and $0 < r < R$. Let w_ε be a smooth function satisfying*

$$\begin{cases} -\varepsilon^2\Delta w_\varepsilon + Aw_\varepsilon \le B\varepsilon^2 & \text{in } B_R^\Lambda, \\ w_\varepsilon \le 1 & \text{on } \partial B_R^\Lambda, \end{cases}$$

for some constant $B \in \mathbb{R}$. Then $w_\varepsilon \le C\varepsilon^2$ in B_r^Λ with $C = C(R, r, A, B)$.

Proof of 2.1.e) completed. By (2.1) and (2.16), we deduce that η_ε is uniformly bounded in $W^{2,p}(B_r^\Lambda)$ for any $1 \le p < \infty$. In particular, it implies

$$\|\nabla\eta_\varepsilon\|_{L^\infty(K)} \le C_K\,. \tag{2.17}$$

We repeat the above argument with the functions $z_\varepsilon = \frac{\partial\eta_\varepsilon}{\partial x_j}$ and $z_0 = \frac{\partial\sqrt{a}}{\partial x_j}$, $j = 1, 2$. Obviously, we can assume that (2.16) and (2.17) hold in B_R^Λ. Using (2.16), we easily check that

$$-\varepsilon^2\Delta(z_\varepsilon - z_0) + (3\eta_\varepsilon^2 - a)(z_\varepsilon - z_0) = \mathcal{O}(\varepsilon^2).$$

By (2.17), we can apply Lemma 2.2 which yields the announced result. ■

We now state a result that we will require in Section 2.2. We follow here a technique introduced by Struwe (see [27]).

Lemma 2.3. *Let $I : (0,\infty) \mapsto \mathbb{R}_+$ defined by*

$$I(\varepsilon) = \text{Min}\,\big\{E_\varepsilon(\eta) \,:\, \eta \in \mathcal{H}\big\}. \tag{2.18}$$

Then $I(\cdot)$ is locally Lipschitz continuous and non-increasing in $(0,\infty)$. Moreover,

$$|I'(\varepsilon)| \le C\left(\frac{|\ln\varepsilon|}{\varepsilon} + 1\right) \quad \textit{for almost every } \varepsilon \in (0,\infty). \tag{2.19}$$

Proof. For every $\varepsilon \ge \frac{a_0}{\lambda}$, we know by Theorem 2.1 that $I(\varepsilon) = E_\varepsilon(0) = \frac{C}{\varepsilon^2}$ and $|I'(\varepsilon)| = \frac{C}{\varepsilon^3}$. Hence it remains to prove that the conclusion holds for $0 < \varepsilon < \frac{a_0}{\lambda} + 1$. By convention, we set $\eta_\varepsilon \equiv 0$ if $\varepsilon \ge \frac{a_0}{\lambda}$. Naturally, we have

$$I(\varepsilon) = E_\varepsilon(\eta_\varepsilon) \le E_\varepsilon(0) = \frac{C}{\varepsilon^2} \ \text{ for every}\,\varepsilon > 0. \tag{2.20}$$

If ε is small, we infer from *2.1.b)* in Proposition 2.1 that we can find some radius $R > \frac{\sqrt{a_0}}{\Lambda}$ such that

$$\int_{\mathbb{R}^2\setminus B_R} |\eta_\varepsilon|^4 + 2a^-(x)|\eta_\varepsilon|^2 \leq C\varepsilon^3. \tag{2.21}$$

Using (2.20), we deduce that (2.21) holds for $0 < \varepsilon < \frac{a_0}{\lambda} + 1$. Let us now fix some $\varepsilon_0 \in (0, \frac{a_0}{\lambda} + 1)$ and $0 < h \ll 1$. We have

$$E_{\varepsilon_0+h}(\eta_{\varepsilon_0+h}) = I(\varepsilon_0 + h) \leq E_{\varepsilon_0+h}(\eta_{\varepsilon_0-h}) \leq E_{\varepsilon_0-h}(\eta_{\varepsilon_0-h}) = I(\varepsilon_0 - h) \leq E_{\varepsilon_0-h}(\eta_{\varepsilon_0+h}).$$

Hence, I is a non-increasing function and

$$E_{\varepsilon_0-h}(\eta_{\varepsilon_0-h}) - E_{\varepsilon_0+h}(\eta_{\varepsilon_0-h}) \leq I(\varepsilon_0 - h) - I(\varepsilon_0 + h) \leq E_{\varepsilon_0-h}(\eta_{\varepsilon_0+h}) - E_{\varepsilon_0+h}(\eta_{\varepsilon_0+h}).$$

By (2.21), it leads to

$$\frac{I(\varepsilon_0 + h) - I(\varepsilon_0 - h)}{2h} \geq \frac{-\varepsilon_0}{2(\varepsilon_0 + h)^2(\varepsilon_0 - h)^2}\left(\int_{B_R} (a(x) - |\eta_{\varepsilon_0+h}|^2)^2 - (a^-(x))^2\right) - C \tag{2.22}$$

and

$$\frac{I(\varepsilon_0 + h) - I(\varepsilon_0 - h)}{2h} \leq \frac{-\varepsilon_0}{2(\varepsilon_0 + h)^2(\varepsilon_0 - h)^2}\int_{B_R} \left[(a(x) - |\eta_{\varepsilon_0-h}|^2)^2 - (a^-(x))^2\right] \tag{2.23}$$

which proves with (2.20) that $I(\cdot)$ is locally Lipschitz continuous in $(0, \frac{a_0}{\lambda} + 1)$. Therefore $I(\cdot)$ is differentiable almost everywhere in $(0, \frac{a_0}{\lambda} + 1)$. We easily check using standard arguments that $\eta_{\varepsilon_0-h} \to \eta_{\varepsilon_0}$ and $\eta_{\varepsilon_0+h} \to \eta_{\varepsilon_0}$ in $L^4(B_R)$ as $h \to 0$. Assuming that ε_0 is a point of differentiability of $I(\cdot)$, we obtain letting $h \to 0$ in (2.22) and (2.23),

$$I'(\varepsilon_0) = \frac{-1}{2\varepsilon_0^3}\int_{B_R} \left[(a(x) - |\eta_{\varepsilon_0}|^2)^2 - (a^-(x))^2\right] + \mathcal{O}(1). \tag{2.24}$$

Then we deduce (2.19) combining (2.2) and (2.24). ∎

2.2 The profile under the mass constraint

In this section, we study the minimization problem (1.8). The motivation is to define the "vortex-free" profile

$$\tilde{\eta}_\varepsilon e^{i\Omega S} \tag{2.25}$$

and to construct admissible test functions for the model. Existence and uniqueness results for general potentials a are also presented in [17]. Our contribution consists in proving the identity (2.27) between η_ε and $\tilde{\eta}_\varepsilon$. By this formula, we obtain a precise information about the asymptotic behavior of the profile $\tilde{\eta}_\varepsilon$.

Theorem 2.2. *For every $\varepsilon > 0$, problem* (1.8) *admits a unique solution $\tilde{\eta}_\varepsilon$ up to a complex multiplier of modulus one. Moreover, there exists $k_\varepsilon \in \mathbb{R}$ such that*

$$-\Delta\tilde{\eta}_\varepsilon = \frac{1}{\varepsilon^2}(a(x) - |\tilde{\eta}_\varepsilon|^2)\tilde{\eta}_\varepsilon + k_\varepsilon\tilde{\eta}_\varepsilon \quad \textit{in } \mathbb{R}^2 \tag{2.26}$$

and $\tilde{\eta}_\varepsilon$ is characterized by

$$\tilde{\eta}_\varepsilon(x) = \frac{\sqrt{a_0 + k_\varepsilon\varepsilon^2}}{\sqrt{a_0}}\,\eta_{\tilde{\varepsilon}}\Big(\frac{\sqrt{a_0}\,x}{\sqrt{a_0 + k_\varepsilon\varepsilon^2}}\Big) \quad \textit{with} \quad \tilde{\varepsilon} = \frac{a_0\varepsilon}{a_0 + k_\varepsilon\varepsilon^2} \in (0, \frac{a_0}{\lambda})\,. \tag{2.27}$$

In addition, for small $\varepsilon > 0$,

$$|k_\varepsilon| \le C|\ln \varepsilon| \tag{2.28}$$

and

$$\big|E_\varepsilon(\tilde{\eta}_\varepsilon) - E_\varepsilon(\eta_\varepsilon)\big| \le C\varepsilon^2 |\ln \varepsilon|^2. \tag{2.29}$$

Identity (2.27) gives us automatically the asymptotic properties of $\tilde{\eta}_\varepsilon$ from those of η_ε by a change of scale and hence we obtain the analogue of Proposition 2.1 for $\tilde{\eta}_\varepsilon$:

Proposition 2.2. *For ε sufficiently small, we have*

2.2.a) $E_\varepsilon(\tilde{\eta}_\varepsilon) \le C|\ln \varepsilon|$,

2.2.b) $0 < \tilde{\eta}_\varepsilon(x) \le C\varepsilon^{1/3} \exp\left(\dfrac{a(x)}{4\varepsilon^{2/3}}\right)$ *for* $|x|_\Lambda \ge \sqrt{a_0} + \varepsilon$,

2.2.c) $\big|\sqrt{a(x)} - \tilde{\eta}_\varepsilon(x)\big| \le C\varepsilon^{1/3}\sqrt{a(x)}$ *for* $x \in \mathcal{D}$ *with* $|x|_\Lambda < \sqrt{a_0} - 2\varepsilon^{1/3}$,

2.2.d) $\|\nabla \tilde{\eta}_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{-1}$,

2.2.e) $\|\tilde{\eta}_\varepsilon - \sqrt{a}\,\|_{C^1(K)} \le C_K \varepsilon^2 |\ln \varepsilon|$ *for any compact subset* $K \subset \mathcal{D}$.

Remark 2.3. We observe that *2.2.a)* in Proposition 2.2 implies for small $\varepsilon > 0$,

$$\int_{\mathbb{R}^2\setminus\mathcal{D}} |\tilde{\eta}_\varepsilon|^4 + 2a^-(x)|\tilde{\eta}_\varepsilon|^2 + \int_{\mathcal{D}} (a(x) - |\tilde{\eta}_\varepsilon|^2)^2 \le C\varepsilon^2 |\ln \varepsilon| \tag{2.30}$$

Proof of Theorem 2.2. Step 1: Existence. Let $(\eta_n)_{n\in\mathbb{N}}$ be a minimizing sequence for (1.8). Extracting a subsequence if necessary, by Lemma 2.1, we may assume that $\eta_n \rightharpoonup \tilde{\eta}_\varepsilon$ weakly in $\mathcal{H}$ and strongly in $L^2(\mathbb{R}^2)$ as $n \to \infty$. Then we derive from (1.2) that $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. We easily check that E_ε is lower semi-continuous on $\mathcal{H}$ with respect to the weak $\mathcal{H}$-topology and therefore $E_\varepsilon(\tilde{\eta}_\varepsilon) \le \liminf_{n\to\infty} E_\varepsilon(\eta_n)$, i.e., $\tilde{\eta}_\varepsilon$ is a minimizer of (1.8). Since $E_\varepsilon(|\tilde{\eta}_\varepsilon|) = E_\varepsilon(\tilde{\eta}_\varepsilon)$, we infer that $\tilde{\eta}_\varepsilon = |\tilde{\eta}_\varepsilon| e^{i\alpha}$ for some constant α. Hence we may assume that $\tilde{\eta}_\varepsilon \ge 0$ in $\mathbb{R}^2$.

Step 2: Proof of (2.27). Let $\tilde{\eta}_\varepsilon$ be a solution of (1.8). As in *Step 1*, we may assume that $\tilde{\eta}_\varepsilon \ge 0$. Since $\tilde{\eta}_\varepsilon$ is a minimizer of E_ε under the constraint $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, there exists $k_\varepsilon \in \mathbb{R}$ such that $\tilde{\eta}_\varepsilon$ satisfies (2.26) and we necessarily have $\tilde{\eta}_\varepsilon > 0$ in $\mathbb{R}^2$ by the maximum principle. We rewrite equation (2.26) as

$$-\Delta \tilde{\eta}_\varepsilon = \frac{1}{\varepsilon^2}(a_\varepsilon(x) - |\tilde{\eta}_\varepsilon|^2)\tilde{\eta}_\varepsilon \quad \text{in } \mathbb{R}^2, \tag{2.31}$$

with

$$a_\varepsilon(x) = a_0 + k_\varepsilon \varepsilon^2 - |x|_\Lambda^2. \tag{2.32}$$

Multiplying (2.31) by $\tilde{\eta}_\varepsilon$, integrating by parts and using that $\int_{\mathbb{R}^2} |\tilde{\eta}_\varepsilon|^2 = 1$, we obtain that

$$\frac{a_0 + k_\varepsilon \varepsilon^2}{\varepsilon^2} = \int_{\mathbb{R}^2} |\nabla \tilde{\eta}_\varepsilon|^2 + \frac{|x|_\Lambda^2}{\varepsilon^2}|\tilde{\eta}_\varepsilon|^2 + \frac{1}{\varepsilon^2}|\tilde{\eta}_\varepsilon|^4 > \lambda_1(L_\varepsilon, \mathbb{R}^2) = \frac{\lambda}{\varepsilon}$$

and therefore, $\tilde{\varepsilon} = \frac{a_0\varepsilon}{a_0 + k_\varepsilon\varepsilon^2} \in \big(0, \frac{a_0}{\lambda}\big)$. Setting

$$\vartheta_\varepsilon(x) = \frac{\sqrt{a_0}}{\sqrt{a_0 + k_\varepsilon \varepsilon^2}}\, \tilde{\eta}_\varepsilon\Big(\frac{\sqrt{a_0 + k_\varepsilon\varepsilon^2}\, x}{\sqrt{a_0}}\Big), \tag{2.33}$$

a straightforward computation shows that

$$\begin{cases} -\tilde{\varepsilon}^2 \Delta \vartheta_\varepsilon = (a(x) - |\vartheta_\varepsilon|^2)\vartheta_\varepsilon & \text{in } \mathbb{R}^2, \\ \vartheta_\varepsilon > 0 & \text{in } \mathbb{R}^2. \end{cases}$$

By Theorem 2.1, it leads to

$$\vartheta_\varepsilon \equiv \eta_{\tilde{\varepsilon}}. \tag{2.34}$$

Combining this identity with (2.33) we obtain (2.27).

Step 3: Uniqueness. Let $\hat{\eta}_\varepsilon$ be another solution of (1.8). As for $\tilde{\eta}_\varepsilon$, we may assume that $\hat{\eta}_\varepsilon$ is a real positive function. Let $\hat{k}_\varepsilon$ be the Lagrange multiplier associated to $\hat{\eta}_\varepsilon$, i.e., $\hat{\eta}_\varepsilon$ satisfies

$$-\Delta \hat{\eta}_\varepsilon = \frac{1}{\varepsilon^2}(a(x) - |\hat{\eta}_\varepsilon|^2)\hat{\eta}_\varepsilon + \hat{k}_\varepsilon \hat{\eta}_\varepsilon \quad \text{in } \mathbb{R}^2.$$

By *Step 2*, the solution $\hat{\eta}_\varepsilon$ is characterized by

$$\hat{\eta}_\varepsilon(x) = \frac{\sqrt{a_0 + \hat{k}_\varepsilon \varepsilon^2}}{\sqrt{a_0}}\, \eta_{\hat{\varepsilon}}\Big(\frac{\sqrt{a_0}\, x}{\sqrt{a_0 + \hat{k}_\varepsilon \varepsilon^2}}\Big) \quad \text{with} \quad \hat{\varepsilon} = \frac{a_0 \varepsilon}{a_0 + \hat{k}_\varepsilon \varepsilon^2} \in (0, \frac{a_0}{\lambda})\,.$$

Hence it suffices to prove that $\hat{k}_\varepsilon = k_\varepsilon$. We proceed by contradiction. Assume for instance that $k_\varepsilon < \hat{k}_\varepsilon$. Then $\hat{\eta}_\varepsilon$ satisfies

$$-\Delta \hat{\eta}_\varepsilon \geq \frac{1}{\varepsilon^2}(a(x) - |\hat{\eta}_\varepsilon|^2)\hat{\eta}_\varepsilon + k_\varepsilon \hat{\eta}_\varepsilon \quad \text{in } \mathbb{R}^2. \tag{2.35}$$

We consider the function

$$\hat{\vartheta}_\varepsilon(x) = \frac{\sqrt{a_0}}{\sqrt{a_0 + k_\varepsilon \varepsilon^2}}\, \hat{\eta}_\varepsilon\Big(\frac{\sqrt{a_0 + k_\varepsilon \varepsilon^2}\, x}{\sqrt{a_0}}\Big), \tag{2.36}$$

which satisfies by (2.35),

$$\begin{cases} -\tilde{\varepsilon}^2 \Delta \hat{\vartheta}_\varepsilon \geq (a(x) - |\hat{\vartheta}_\varepsilon|^2)\hat{\vartheta}_\varepsilon & \text{in } \mathbb{R}^2, \\ \hat{\vartheta}_\varepsilon > 0 & \text{in } \mathbb{R}^2. \end{cases}$$

Therefore $\hat{\vartheta}_\varepsilon$ is a supersolution of (2.1) with $\tilde{\varepsilon}$ instead of ε. By Remark 2.2 we infer that $\hat{\vartheta}_\varepsilon \geq \eta_{\tilde{\varepsilon}}$ in $\mathbb{R}^2$. By (2.27) and (2.36), it leads to $\hat{\eta}_\varepsilon \geq \tilde{\eta}_\varepsilon$ in $\mathbb{R}^2$. Since $\|\hat{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = \|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we conclude that $\hat{\eta}_\varepsilon \equiv \tilde{\eta}_\varepsilon$ and hence $k_\varepsilon = \hat{k}_\varepsilon$, contradiction.

Step 4: Energy bound for small $\varepsilon > 0$. We now prove that for small $\varepsilon > 0$,

$$E_\varepsilon(\tilde{\eta}_\varepsilon) \leq C|\ln \varepsilon|. \tag{2.37}$$

Let φ be the test function constructed in the proof of *2.1.a)* in Proposition 2.1. Setting $\hat{\varphi} = \|\varphi\|^{-1}_{L^2(\mathbb{R}^2)}\varphi$, it suffices to check that $E_\varepsilon(\hat{\varphi}) \leq C|\ln \varepsilon|$ by the minimizing property of $\tilde{\eta}_\varepsilon$. First we show that $\|\varphi\|_{L^2(\mathbb{R}^2)}$ remains close to 1 as $\varepsilon \to 0$. Since $\int_{\mathbb{R}^2} a^+ = 1$, we have $\int_{\mathbb{R}^2} |\varphi|^2 = 1 + \int_{\mathcal{D}} (|\varphi|^2 - a^+(x))$ and by (2.13),

$$\int_{\mathcal{D}} \big||\varphi|^2 - a^+(x)\big| \leq C \left(\int_{\mathcal{D}} (|\varphi|^2 - a^+(x))^2 \right)^{1/2} \leq C\varepsilon.$$

Hence $\|\varphi\|^2_{L^2(\mathbb{R}^2)} = 1 + \mathcal{O}(\varepsilon)$. Then we derive from (2.13),

$$\int_{\mathbb{R}^2} |\nabla \hat{\varphi}|^2 = \|\varphi\|^{-2}_{L^2(\mathbb{R}^2)} \int_{\mathbb{R}^2} |\nabla \varphi|^2 \leq \int_{\mathbb{R}^2} |\nabla \varphi|^2 + C\varepsilon|\ln \varepsilon| \leq C|\ln \varepsilon|$$

and

$$
\begin{aligned}
\frac{1}{\varepsilon^2}\int_{\mathcal{D}}(a(x)-|\hat{\varphi}|^2)^2 &= \frac{1}{\varepsilon^2}\int_{\mathcal{D}}(a(x)-|\varphi|^2)^2 + \frac{2(1-\|\varphi\|^{-2}_{L^2(\mathbb{R}^2)})}{\varepsilon^2}\int_{\mathcal{D}}(a(x)-|\varphi|^2)|\varphi|^2 \\
&\quad + \frac{(1-\|\varphi\|^{-2}_{L^2(\mathbb{R}^2)})^2}{\varepsilon^2}\int_{\mathcal{D}}|\varphi|^4 \\
&\le C + C\left(\frac{1}{\varepsilon^2}\int_{\mathcal{D}}(a-|\varphi|^2)^2\right)^{1/2} \le C.
\end{aligned}
$$

Therefore $E_\varepsilon(\hat{\varphi}) \le C|\ln\varepsilon|$ and (2.37) holds.

Step 5: First bound on the Lagrange multiplier for small $\varepsilon > 0$. Let $\tilde{\eta}_\varepsilon$ be the positive solution of (1.8) and let $k_\varepsilon \in \mathbb{R}$ be such that $\tilde{\eta}_\varepsilon$ satisfies (2.26). Multiplying (2.26) by $\tilde{\eta}_\varepsilon$, integrating by parts and using that $\int_{\mathbb{R}^2}|\tilde{\eta}_\varepsilon|^2 = 1$, we obtain that

$$
k_\varepsilon = \int_{\mathbb{R}^2}|\nabla\tilde{\eta}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\big(|\tilde{\eta}_\varepsilon|^2 - a(x)\big)|\tilde{\eta}_\varepsilon|^2. \tag{2.38}
$$

From (2.37) we derive

$$
\left|\int_{\mathbb{R}^2}|\nabla\tilde{\eta}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2\setminus\mathcal{D}}\big(|\tilde{\eta}_\varepsilon|^2 - a(x)\big)|\tilde{\eta}_\varepsilon|^2\right| \le C|\ln\varepsilon|
$$

and

$$
\begin{aligned}
\left|\frac{1}{\varepsilon^2}\int_{\mathcal{D}}(|\tilde{\eta}_\varepsilon|^2 - a(x))|\tilde{\eta}_\varepsilon|^2\right| &\le \frac{1}{\varepsilon^2}\int_{\mathcal{D}}(|\tilde{\eta}_\varepsilon|^2 - a(x))^2 + \frac{1}{\varepsilon^2}\int_{\mathcal{D}}a(x)\big||\tilde{\eta}_\varepsilon|^2 - a(x)\big| \\
&\le C|\ln\varepsilon| + \frac{C}{\varepsilon^2}\left(\int_{\mathcal{D}}(|\tilde{\eta}_\varepsilon|^2 - a(x))^2\right)^{1/2} \le C\varepsilon^{-1}|\ln\varepsilon|^{1/2}.
\end{aligned}
$$

Hence, by (2.38), we have

$$
|k_\varepsilon| \le C\varepsilon^{-1}|\ln\varepsilon|^{1/2}. \tag{2.39}
$$

Step 6: Proof of (2.28). We define the functional $\tilde{E}_\varepsilon : \mathcal{H} \to \mathbb{R}$ by

$$
\tilde{E}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2}(a_\varepsilon(x) - |u|^2)^2 - (a_\varepsilon^-(x))^2 \tag{2.40}
$$

where $a_\varepsilon(x)$ is given by (2.32). Then, by (2.27), we get

$$
\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) = \frac{a_0 + k_\varepsilon\varepsilon^2}{a_0}\,E_{\tilde{\varepsilon}}(\eta_{\tilde{\varepsilon}}) = \frac{a_0 + k_\varepsilon\varepsilon^2}{a_0}\,I(\tilde{\varepsilon}). \tag{2.41}
$$

Since $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we have

$$
\begin{aligned}
\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) &= E_\varepsilon(\tilde{\eta}_\varepsilon) - \frac{k_\varepsilon}{2} + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2}(a_\varepsilon^+(x))^2 - (a^+(x))^2 && (2.42)\\
&\ge I(\varepsilon) - \frac{k_\varepsilon}{2} + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2}(a_\varepsilon^+(x))^2 - (a^+(x))^2. && (2.43)
\end{aligned}
$$

Using the fact that $\int_{\mathbb{R}^2}a^+ = 1$, a simple computation leads to

$$
-\frac{k_\varepsilon}{2} + \frac{1}{4\varepsilon^2}\int_{\mathbb{R}^2}(a_\varepsilon^+(x))^2 - (a^+(x))^2 = \frac{\pi a_0 k_\varepsilon^2\varepsilon^2}{4\Lambda} + \frac{\pi k_\varepsilon^3\varepsilon^4}{12\Lambda}\,. \tag{2.44}
$$

Combining (2.41), (2.43) and (2.44), we infer that

$$\frac{\pi a_0 k_\varepsilon^2 \varepsilon^2}{4\Lambda} \leq |I(\tilde{\varepsilon}) - I(\varepsilon)| + \frac{|k_\varepsilon|\varepsilon^2}{a_0} I(\tilde{\varepsilon}) + \frac{\pi |k_\varepsilon|^3 \varepsilon^4}{12\Lambda} . \tag{2.45}$$

For small $\varepsilon > 0$, we obtain using (2.19), (2.39) and *2.1.a)* in Proposition 2.1,

$$\big|I(\tilde{\varepsilon}) - I(\varepsilon)\big| \leq C\varepsilon^{-1} |\ln \varepsilon| |\tilde{\varepsilon} - \varepsilon| \leq C|k_\varepsilon|\varepsilon^2 |\ln \varepsilon| \tag{2.46}$$

and

$$\frac{|k_\varepsilon|\varepsilon^2}{a_0} I(\tilde{\varepsilon}) \leq C|k_\varepsilon|\varepsilon^2|\ln\varepsilon|, \qquad \frac{\pi|k_\varepsilon|^3\varepsilon^4}{12} \leq C|k_\varepsilon|\varepsilon^2|\ln\varepsilon|.$$

Inserting this estimates in (2.45), we deduce that $|k_\varepsilon| \leq C|\ln \varepsilon|$.

Step 7: Proof of (2.29). From (2.28), (2.41), (2.46) and *2.1.a)* in Proposition 2.1, we derive that $\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + \mathcal{O}(\varepsilon^2|\ln\varepsilon|^2)$. On the other hand, (2.28), (2.42) and (2.44) yield $\tilde{E}_\varepsilon(\tilde{\eta}_\varepsilon) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \mathcal{O}(\varepsilon^2|\ln\varepsilon|^2)$ and (2.29) follows. ∎

3 Minimizing F_ε under the mass constraint

Our aim in this section is to make a first description of minimizers u_ε of F_ε under the mass constraint. We prove the existence of u_ε and some asymptotic properties of u_ε (in particular, we show that $|u_\varepsilon|$ is concentrated in $\mathcal{D}$). We also present some tools that we will require in the sequel, in particular the splitting of energy (1.9).

3.1 Existence and first properties of minimizers

First, we seek minimizers u_ε of F_ε under the constraint $\|u_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. We perform the minimization in $\mathcal{H}$ and we shall see that F_ε is well defined on $\mathcal{H}$:

Lemma 3.1. *For any $u \in \mathcal{H}$, $\sigma > 0$ and $R > \sqrt{a_0}$, we have*

$$\left|\Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu, \nabla u)\right| \leq \sigma \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{\Omega^2 R^2}{8\Lambda^2 \sigma (R^2 - a_0)} \int_{\mathbb{R}^2} \left[(a(x) - |u|^2)^2 - (a^-(x))^2\right] + C_{R,\sigma}\,\Omega^2.$$

In particular, the functional F_ε is well defined on $\mathcal{H}$.

Proposition 3.1. *Assume that $\Omega < \Lambda\varepsilon^{-1}$. Then there exists at least one minimizer u_ε of F_ε in $\big\{u \in \mathcal{H} : \|u\|_{L^2(\mathbb{R}^2)} = 1\big\}$. Moreover, u_ε is smooth and there exists $\ell_\varepsilon \in \mathbb{R}$ such that u_ε satisfies*

$$-\Delta u_\varepsilon + 2i\Omega x^\perp \cdot \nabla u_\varepsilon = \frac{1}{\varepsilon^2}(a(x) - |u_\varepsilon|^2) u_\varepsilon + \ell_\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}^2. \tag{3.1}$$

We emphasize that the result is stated for an angular velocity Ω strictly less than Λ/ε but we only consider in this paper the case of an rotational speed Ω at most of order $|\ln\varepsilon|$, i.e.,

$$\Omega \leq \omega_0 |\ln \varepsilon| \tag{3.2}$$

for some positive constant ω_0.

Before proving Lemma 3.1 and Proposition 3.1, we present some basic properties of any minimizer u_ε. We point out that the exponential decay of $|u_\varepsilon|$ outside the domain $\mathcal{D}$ (see *3.2.c)* in Proposition 3.2) shows that almost all the mass of u_ε is concentrated in $\mathcal{D}$.

Proposition 3.2. *Assume that* (3.2) *holds for some* $\omega_0 > 0$. *For* ε *sufficiently small, we have*

3.2.a) $E_\varepsilon(u_\varepsilon) \leq C_{\omega_0} |\ln \varepsilon|^2$,

3.2.b) $|\ell_\varepsilon| \leq C_{\omega_0}\, \varepsilon^{-1} |\ln \varepsilon|$,

3.2.c) $|u_\varepsilon(x)| \leq C_{\omega_0}\, \varepsilon^{1/3} |\ln \varepsilon|^{1/2} \exp\left(\dfrac{a(x)}{4\varepsilon^{2/3}}\right)$ *for* $x \in \mathbb{R}^2 \setminus \mathcal{D}$ *with* $|x|_\Lambda \geq \sqrt{a_0 + 2\varepsilon^{1/3}}$,

3.2.d) $|u_\varepsilon(x)| \leq \sqrt{a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2 \Omega^2 |x|^2}$ *for* $x \in \mathcal{D}$ *with* $|x|_\Lambda \leq \sqrt{a_0} - \varepsilon^{1/8}$,

3.2.e) $|u_\varepsilon| \leq \sqrt{a_0} + C_{\omega_0}\, \varepsilon |\ln \varepsilon|$ *in* $\mathbb{R}^2$,

3.2.f) $\|\nabla u_\varepsilon\|_{L^\infty(K)} \leq C_{\omega_0, K}\, \varepsilon^{-1}$ *for any compact set* $K \subset \mathbb{R}^2$.

Remark 3.1. We observe that *3.2.a)* in Proposition 3.2 implies

$$\int_{\mathbb{R}^2 \setminus \mathcal{D}} \left(|u_\varepsilon|^4 + 2a^-(x)|u_\varepsilon|^2\right) + \int_{\mathcal{D}} (|u_\varepsilon|^2 - a(x))^2 \leq C_{\omega_0}\, \varepsilon^2 |\ln \varepsilon|^2. \tag{3.3}$$

Proof of Lemma 3.1. Let $u \in \mathcal{H}$ and $\sigma \in (0,1)$. We have

$$4\sigma \left|\Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu, \nabla u)\right| \leq 4\sigma^2 \int_{\mathbb{R}^2} |\nabla u|^2 + \Omega^2 \int_{\mathbb{R}^2} |x|^2 |u|^2 \leq 4\sigma^2 \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{\Omega^2}{\Lambda^2} \int_{\mathbb{R}^2} |x|_\Lambda^2 |u|^2.$$

For $R > \sqrt{a_0}$, we easily check that $|x|_\Lambda^2 \leq -\frac{R^2}{R^2 - a_0}\, a(x)$ whenever $|x|_\Lambda \geq R$. Then we derive

$$4\sigma \left|\Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu, \nabla u)\right| \leq 4\sigma^2 \int_{\mathbb{R}^2} |\nabla u|^2 - \frac{\Omega^2 R^2}{2\Lambda^2 (R^2 - a_0)} \int_{\mathbb{R}^2 \setminus B_R^\Lambda} 2a(x)|u|^2 + \frac{\Omega^2}{\Lambda^2} \int_{B_R^\Lambda} |x|_\Lambda^2 |u|^2. \tag{3.4}$$

Now we notice that

$$\begin{aligned}\int_{B_R^\Lambda} |x|_\Lambda^2 |u|^2 &= \frac{R^2}{2(R^2 - a_0)} \int_{B_R^\Lambda} -2a(x)|u|^2 - \frac{a_0}{R^2 - a_0} \int_{B_R^\Lambda} |x|_\Lambda^2 |u|^2 + \frac{a_0 R^2}{R^2 - a_0} \int_{B_R^\Lambda} |u|^2 \\ &\leq \frac{R^2}{2(R^2 - a_0)} \int_{B_R^\Lambda} -2a(x)|u|^2 + \frac{R^2}{2(R^2 - a_0)} \int_{B_R^\Lambda} |u|^4 + \frac{\pi R^4 a_0^2}{2\Lambda(R^2 - a_0)}.\end{aligned}$$

Inserting this estimate in (3.4), we obtain

$$\begin{aligned}\left|\Omega \int_{\mathbb{R}^2} x^\perp \cdot (iu, \nabla u)\right| \leq \sigma \int_{\mathbb{R}^2} |\nabla u|^2 &+ \frac{\Omega^2 R^2}{8\Lambda^2 \sigma (R^2 - a_0)} \int_{\mathbb{R}^2} \left[(a(x) - |u|^2)^2 - (a^-(x))^2\right] \\ &+ \frac{\pi \Omega^2 R^4 a_0^2}{8\Lambda^3 \sigma (R^2 - a_0)}\end{aligned}$$

and the proof is complete. ■

Proof of Proposition 3.1. Since $\Omega < \Lambda\varepsilon^{-1}$, we can find $0 < \delta < 1$ such that $\Omega \leq \delta\Lambda\varepsilon^{-1}$. Taking in Lemma 3.1,

$$\sigma = \frac{\delta^2 + 1}{4} \quad \text{and} \quad R = \sqrt{\frac{2(1+\delta^2)a_0}{1 - \delta^2}},$$

we infer that for any $u \in \mathcal{H}$,

$$\frac{1 - \delta^2}{4}\, E_\varepsilon(u) - C_\delta\, \Omega^2 \leq F_\varepsilon(u) \leq 2\, E_\varepsilon(u) + C_\delta\, \Omega^2. \tag{3.5}$$

We easily check that E_ε is coercive in $\mathcal{H}$ (i.e., there exists a positive constant C such that $E_\varepsilon(u) \geq C(\|u\|^2_{\mathcal{H}} - 1)$ for any $u \in \mathcal{H}$) and by (3.5), F_ε is coercive, too. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ be a minimizing sequence of F_ε in $\{u \in \mathcal{H} : \|u\|_{L^2(\mathbb{R}^2)} = 1\}$. From the coerciveness of F_ε, we get that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$ and therefore, there exists $u_\varepsilon \in \mathcal{H}$ such that up to a subsequence,

$$u_n \rightharpoonup u_\varepsilon \text{ weakly in } \mathcal{H} \text{ and } u_n \to u_\varepsilon \text{ in } L^4_{\text{loc}}(\mathbb{R}^2). \tag{3.6}$$

By Lemma 2.1, it results that $u_n \to u_\varepsilon$ in $L^2(\mathbb{R}^2)$ and consequently, $\|u_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. We write for $u \in \mathcal{H}$,

$$\begin{aligned} F_\varepsilon(u) =& \frac{1}{2}\int_{\mathbb{R}^2} \left|(\nabla - i\Omega x^\perp)u\right|^2 + \frac{1}{2\varepsilon^2}\int_{\{a^-(x)\geq\Omega^2\varepsilon^2|x|^2\}} \left[\frac{1}{2}|u|^4 + \left(a^-(x) - \varepsilon^2\Omega^2|x|^2\right)|u|^2\right] \\ &+ \frac{1}{4\varepsilon^2}\int_{\{a^-(x)\leq\Omega^2\varepsilon^2|x|^2\}} \left[(a(x) - |u|^2)^2 - (a^-(x))^2 - 2\Omega^2\varepsilon^2|x|^2\,|u|^2\right]. \end{aligned}$$

We observe that the functional

$$u \in \mathcal{H} \mapsto \frac{1}{2}\int_{\mathbb{R}^2} \left|(\nabla - i\Omega x^\perp)u\right|^2 + \frac{1}{2\varepsilon^2}\int_{\{a^-(x)\geq\Omega^2\varepsilon^2|x|^2\}} \left[\frac{1}{2}|u|^4 + (a^-(x) - \varepsilon^2\Omega^2|x|^2)\,|u|^2\right]$$

is convex continuous on $\mathcal{H}$ for the strong topology. Then from (3.6), it follows that $F_\varepsilon(u_\varepsilon) \leq \liminf_{n\to\infty} F_\varepsilon(u_n)$. Hence u_ε minimizes F_ε in $\{u \in \mathcal{H} : \|u\|_{L^2(\mathbb{R}^2)} = 1\}$ and by the Lagrange multiplier rule, there exists $\ell_\varepsilon \in \mathbb{R}$ such that (3.1) holds. By standard elliptic regularity, we deduce that u_ε is smooth in $\mathbb{R}^2$. ■

Proof of Proposition 3.2. Proof of 3.2.a). Let $\tilde{\eta}_\varepsilon$ be the positive real minimizer of E_ε under the constraint $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$. Since $\tilde{\eta}_\varepsilon$ is real valued, we have $(i\tilde{\eta}_\varepsilon, \nabla\tilde{\eta}_\varepsilon) \equiv 0$ and we derive from (2.37),

$$F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(\tilde{\eta}_\varepsilon) = E_\varepsilon(\tilde{\eta}_\varepsilon) \leq C|\ln\varepsilon|. \tag{3.7}$$

By (3.5) (with $\delta = \frac{1}{\sqrt{2}}$), we infer that for ε small enough,

$$\frac{1}{8}\,E_\varepsilon(u_\varepsilon) - C\Omega^2 \leq F_\varepsilon(u_\varepsilon). \tag{3.8}$$

Combining (3.2), (3.7) and (3.8), we obtain *3.2.a)*.

Proof of 3.2.b). Multiplying equation (3.1) by u_ε and using $\int_{\mathbb{R}^2}|u_\varepsilon|^2 = 1$, we infer that

$$\ell_\varepsilon = \int_{\mathbb{R}^2}|\nabla u_\varepsilon|^2 - 2\Omega\int_{\mathbb{R}^2} x^\perp\cdot(iu_\varepsilon, \nabla u_\varepsilon) + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}(|u_\varepsilon|^2 - a(x))|u_\varepsilon|^2. \tag{3.9}$$

From *3.2.a)* and Lemma 3.1, we derive

$$\left|\int_{\mathbb{R}^2}|\nabla u_\varepsilon|^2 - 2\Omega\int_{\mathbb{R}^2} x^\perp\cdot(iu_\varepsilon, \nabla u_\varepsilon) + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2\setminus\mathcal{D}}(|u_\varepsilon|^2 - a(x))|u_\varepsilon|^2\right| \leq C_{\omega_0}|\ln\varepsilon|^2 \tag{3.10}$$

and arguing as in the proof of (2.39), we obtain by (3.3),

$$\left|\frac{1}{\varepsilon^2}\int_{\mathcal{D}}(|u_\varepsilon|^2 - a(x))|u_\varepsilon|^2\right| \leq C_{\omega_0}\,\varepsilon^{-1}|\ln\varepsilon|. \tag{3.11}$$

Using (3.9), (3.10) and (3.11), we conclude that $|\ell_\varepsilon| \leq C_{\omega_0}\varepsilon^{-1}|\ln\varepsilon|$.

Proof of 3.2.c). We argue as in [2], Proposition 2.5. Setting $U_\varepsilon := |u_\varepsilon|^2$, we deduce from equation (3.1),

$$\frac{1}{2}\,\Delta U_\varepsilon = |\nabla u_\varepsilon|^2 - 2\Omega\, x^\perp\cdot(iu_\varepsilon,\nabla u_\varepsilon) - \frac{1}{\varepsilon^2}(a(x)-U_\varepsilon)U_\varepsilon - \ell_\varepsilon U_\varepsilon$$

and hence

$$\Delta U_\varepsilon \geq \frac{2}{\varepsilon^2}\left(U_\varepsilon - (a(x)+\varepsilon^2|\ell_\varepsilon| + \varepsilon^2\Omega^2|x|^2)\right)U_\varepsilon \quad \text{in } \mathbb{R}^2. \tag{3.12}$$

Let $\Theta_\varepsilon = \big\{x\in\mathbb{R}^2\setminus\mathcal{D} \,:\, a^-(x) > 2(\varepsilon^2|\ell_\varepsilon| + \varepsilon^2\Omega^2|x|^2)\big\}$. From (3.12), we infer that

$$\Delta U_\varepsilon \geq \frac{1}{\varepsilon^2}\,a^-(x)U_\varepsilon \geq 0 \quad \text{in } \Theta_\varepsilon \tag{3.13}$$

and thus U_ε is subharmonic in $\Theta_\varepsilon\subset\mathbb{R}^2\setminus\mathcal{D}$. Note that by (3.3),

$$\int_{\mathbb{R}^2\setminus\mathcal{D}} U_\varepsilon^2 \leq C_{\omega_0}\varepsilon^2|\ln\varepsilon|^2. \tag{3.14}$$

By *3.2.b)*, for ε small enough we have $\partial\Theta_\varepsilon\subset\big\{x\in\mathbb{R}^2 \,:\, |x|^2_\Lambda \leq a_0 + \frac{\varepsilon^{1/3}}{2}\big\}$. Consider now for $r_\varepsilon = \sqrt{a_0+\varepsilon^{1/3}}$, the set $\Xi_\varepsilon = \mathbb{R}^2\setminus B^\Lambda_{r_\varepsilon} = \big\{x\in\mathbb{R}^2 \,:\, |x|^2_\Lambda > a_0+\varepsilon^{1/3}\big\}\subset\Theta_\varepsilon$. Then for ε small and any $x_0\in\Xi_\varepsilon$, we have $B(x_0,\frac{\varepsilon^{1/3}}{2})\subset\Theta_\varepsilon$. We infer from the subharmonicity of U_ε in Θ_ε and (3.14),

$$0\leq U_\varepsilon(x_0) \leq \frac{4}{\pi\varepsilon^{2/3}}\int_{B(x_0,\frac{\varepsilon^{1/3}}{2})} U_\varepsilon \leq \frac{C}{\varepsilon^{1/3}}\left(\int_{B(x_0,\frac{\varepsilon^{1/3}}{2})} U_\varepsilon^2\right)^{1/2} \leq C^\star_{\omega_0}\,\varepsilon^{2/3}|\ln\varepsilon| \quad \text{for } x_0\in\Xi_\varepsilon,$$

with a constant $C^\star_{\omega_0}$ independent of x_0. Hence we conclude that $U_\varepsilon\to 0$ locally uniformly in $\mathbb{R}^2\setminus\overline{\mathcal{D}}$ as $\varepsilon\to 0$. It also follows that $u_\varepsilon\in L^\infty(\mathbb{R}^2)$ and then $U_\varepsilon\in H^1(\mathbb{R}^2)$. By (3.13), U_ε is a subsolution of

$$\begin{cases} -\varepsilon^2\Delta w + a^-(x)w = 0 & \text{in } \Xi_\varepsilon,\\ w = C^\star_{\omega_0}\,\varepsilon^{2/3}|\ln\varepsilon| & \text{on } \partial\Xi_\varepsilon. \end{cases} \tag{3.15}$$

We easily check that for ε small enough,

$$v_{\rm out}(x) = C^\star_{\omega_0}\varepsilon^{2/3}|\ln\varepsilon|\exp\Big(\frac{a_0+\varepsilon^{1/3}-|x|^2_\Lambda}{\varepsilon^{2/3}}\Big)$$

is a supersolution of (3.15). Therefore

$$U_\varepsilon(x) = |u_\varepsilon(x)|^2 \leq v_{\rm out}(x) \leq C^\star_{\omega_0}\varepsilon^{2/3}|\ln\varepsilon|\exp\left(\frac{a_0-|x|^2_\Lambda}{2\varepsilon^{2/3}}\right) \quad \text{for } |x|^2_\Lambda \geq a_0+2\varepsilon^{1/3}.$$

Proof of 3.2.d) and 3.2.e). We set $\tilde r_\varepsilon = \sqrt{a_0}-\varepsilon^{1/8}$ (recall that $r_\varepsilon = \sqrt{a_0+\varepsilon^{1/3}}$). We define in $B^\Lambda_{r_\varepsilon}$, the function

$$v_{\rm in}(x) = \begin{cases} a(x) + |\ell_\varepsilon|\varepsilon^2 + \dfrac{\varepsilon^2\Omega^2}{\Lambda^2}\,|x|^2_\Lambda & \text{if} \quad |x|_\Lambda\leq\tilde r_\varepsilon,\\[2ex] a_0 - (1-\dfrac{\varepsilon^2\Omega^2}{\Lambda^2})\tilde r_\varepsilon(2|x|_\Lambda-\tilde r_\varepsilon) + |\ell_\varepsilon|\varepsilon^2 & \text{if} \quad \tilde r_\varepsilon\leq|x|_\Lambda\leq r_\varepsilon. \end{cases}$$

We easily verify that for ε sufficiently small, $v_{\rm in}$ satisfies

$$\begin{cases} -\varepsilon^2\Delta v_{\rm in} \geq 2\left(a(x)+|\ell_\varepsilon|\varepsilon^2+\varepsilon^2\Omega^2|x|^2 - v_{\rm in}\right)v_{\rm in} & \text{in } B^\Lambda_{r_\varepsilon},\\ v_{\rm in}(x)\geq C^\star_{\omega_0}\varepsilon^{2/3}|\ln\varepsilon| & \text{on } \partial B^\Lambda_{r_\varepsilon} \end{cases} \tag{3.16}$$

and

$$v_{\rm in}(x) \geq a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2|x|^2 > 0 \quad \text{in } B^\Lambda_{r_\varepsilon}.$$

Setting $V_\varepsilon = U_\varepsilon - v_{\rm in}$, we deduce from (3.12) and (3.16),

$$\begin{cases} -\varepsilon^2\Delta V_\varepsilon + b(x)V_\varepsilon \leq 0 & \text{in } B^\Lambda_{r_\varepsilon}, \\ V_\varepsilon \leq 0 & \text{on } \partial B^\Lambda_{r_\varepsilon}, \end{cases}$$

with

$$b(x) = 2\big(U_\varepsilon + v_{\rm in} - (a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2|x|^2)\big) \geq 0 \quad \text{in } B^\Lambda_{r_\varepsilon}.$$

Hence $V_\varepsilon \leq 0$ which gives us *3.2.d)*. Then estimate *3.2.e)* directly follows from the construction of $v_{\rm in}$ and $v_{\rm out}$ and from *3.2.b)*.

Proof of 3.2.f). Without loss of generality, we may assume that $K = B_R$ with $R > 0$. Consider the re-scaled function $\tilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ defined for $x \in B_{3+R\varepsilon^{-1}}$. From (3.1), we obtain

$$-\Delta\tilde{u}_\varepsilon = (a(\varepsilon x) - |\tilde{u}_\varepsilon|^2)\tilde{u}_\varepsilon - 2i\Omega\varepsilon^2 x^\perp \cdot \nabla\tilde{u}_\varepsilon + \ell_\varepsilon\varepsilon^2\tilde{u}_\varepsilon \quad \text{in} \quad B_{3+R\varepsilon^{-1}}.$$

Taking an arbitrary $x_0 \in B_{R\varepsilon^{-1}}$, it suffices to prove that exists a constant $C_R > 0$ independent of x_0 and ε such that

$$\|\nabla\tilde{u}_\varepsilon\|_{L^\infty(B(x_0,1))} \leq C_{\omega_0,R}. \tag{3.17}$$

By *3.2.c)*, we know that $a(x)u_\varepsilon$ is uniformly bounded in $\mathbb{R}^2$. Using *3.2.a)*, *3.2.b)* and *3.2.e)*, we derive that

$$\begin{aligned} \|\Delta\tilde{u}_\varepsilon\|_{L^2(B(x_0,3))} &\leq C\big(\|(a(x) + \ell_\varepsilon\varepsilon^2 - |u_\varepsilon|^2)u_\varepsilon\|_{L^\infty(\mathbb{R}^2)} + \Omega\varepsilon^2\|x^\perp \cdot \nabla\tilde{u}_\varepsilon\|_{L^2(B(x_0,3))}\big) \\ &\leq C_{\omega_0}(1 + \Omega\varepsilon\|x^\perp \cdot \nabla u_\varepsilon\|_{L^2(B_{R+1})}) \leq C_{\omega_0,R}. \end{aligned}$$

Since $\|\tilde{u}_\varepsilon\|_{L^\infty(B(x_0,3))} \leq C_{\omega_0}$ by *3.2.e)*, it follows that $\|\tilde{u}_\varepsilon\|_{H^2(B(x_0,2))} \leq C_{\omega_0,R}$. From Sobolev imbedding, we deduce that $\|\nabla\tilde{u}_\varepsilon\|_{L^4(B(x_0,2))} \leq C_{\omega_0,R}$. We now repeat the above argument and it follows $\|\Delta\tilde{u}_\varepsilon\|_{L^4(B(x_0,2))} \leq C_{\omega_0,R}(1 + \Omega\varepsilon^{3/2}\|\nabla\tilde{u}_\varepsilon\|_{L^4(B(x_0,2))}) \leq C_{\omega_0,R}$. It finally yields $\|\tilde{u}_\varepsilon\|_{W^{2,4}(B(x_0,1))} \leq C_{\omega_0,R}$ which implies (3.17) by Sobolev imbedding. ■

3.2 Splitting the energy

In this section, we prove the splitting of the energy (1.9). The splitting technique has been introduced by Lassoued and Mironescu in [16]. The goal is to decouple the energy $F_\varepsilon(u)$ into two independent parts: the energy of the "vortex-free" profile $\tilde{\eta}_\varepsilon e^{i\Omega S}$ and the reduced energy of $u/(\tilde{\eta}_\varepsilon e^{i\Omega S})$ where the function S is defined in (1.5). For $\varepsilon > 0$, we introduce the class

$$\mathcal{G}_\varepsilon = \left\{ v \in H^1_{\rm loc}(\mathbb{R}^2, \mathbb{C}) \,:\, \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla v|^2 + \tilde{\eta}_\varepsilon^4(1 - |v|^2)^2 < +\infty \right\}.$$

We have the following result (valid for any rotational speed Ω):

Lemma 3.2. *Let $u \in \mathcal{H}$ and $\varepsilon > 0$. Then $v = u/(\tilde{\eta}_\varepsilon e^{i\Omega S})$ is well defined, belongs to $\mathcal{G}_\varepsilon$ and*

$$F_\varepsilon(u) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(v) + \tilde{\mathcal{T}}_\varepsilon(v) \tag{3.18}$$

where the functionals $\tilde{\mathcal{F}}_\varepsilon$ and $\tilde{\mathcal{T}}_\varepsilon$ are defined in (1.10) *and* (1.12).

Before proving Lemma 3.2, we are going to translate some of the properties of the map u_ε to $u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S})$. To this aim, we define the subclass $\tilde{\mathcal{G}}_\varepsilon \subset \mathcal{G}_\varepsilon$ by

$$\tilde{\mathcal{G}}_\varepsilon = \big\{ v \in \mathcal{G}_\varepsilon \,:\, \tilde{\eta}_\varepsilon v \in \mathcal{H} \text{ and } \|\tilde{\eta}_\varepsilon v\|_{L^2(\mathbb{R}^2)} = 1 \big\}.$$

Proposition 3.3. *Assume that* (3.2) *holds for some* $\omega_0 > 0$. *Let* u_ε *be a minimizer of* F_ε *in* $\big\{u \in \mathcal{H} \,:\, \|u\|_{L^2(\mathbb{R}^2)} = 1\big\}$. *Then* $v_\varepsilon = u_\varepsilon/(\tilde{\eta}_\varepsilon e^{i\Omega S})$ *minimizes the functional* $\tilde{\mathcal{F}}_\varepsilon + \tilde{\mathcal{T}}_\varepsilon$ *in* $\tilde{\mathcal{G}}_\varepsilon$. *Moreover, for* $\varepsilon > 0$ *sufficiently small, we have*

3.3.a) $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \le C_{\omega_0} |\ln \varepsilon|^2$,

3.3.b) $\big|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)\big| \le C_{\omega_0}\, \varepsilon |\ln \varepsilon|^3$,

3.3.c) $|v_\varepsilon(x)| \le 1 + C_{\omega_0}\, \varepsilon^{1/3}$ *for* $x \in \mathcal{D}$ *with* $|x|_\Lambda \le \sqrt{a_0} - \varepsilon^{1/8}$,

3.3.d) $\|\nabla v_\varepsilon\|_{L^\infty(K)} \le C_{\omega_0,K}\, \varepsilon^{-1}$ *for any compact subset* $K \subset \mathcal{D}$.

Proof of Lemma 3.2: Step 1. For $u \in \mathcal{H}$, we set $\tilde{v} = u/\tilde{\eta}_\varepsilon \in H^1_{\rm loc}(\mathbb{R}^2)$. We want to prove that $\tilde{v} \in \mathcal{G}_\varepsilon$ and

$$E_\varepsilon(u) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \tilde{\mathcal{E}}_\varepsilon(\tilde{v}) + \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2(|\tilde{v}|^2 - 1). \tag{3.19}$$

We consider the sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ defined by $u_n(x) = \zeta\big(n^{-1}|x|\big)\, u(x)$ where ζ is the "cut-off" type function defined in (2.5). We easily check that $u_n \to u$ a.e. and $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^2$. Setting $\tilde{v}_n = u_n/\tilde{\eta}_\varepsilon$, then we have $\tilde{v}_n \to \tilde{v}$ a.e. and $\nabla \tilde{v}_n \to \nabla \tilde{v}$ a.e. in $\mathbb{R}^2$. Since u_n has a compact support, we get that $\tilde{v}_n \in \mathcal{G}_\varepsilon$ for any $n \in \mathbb{N}$. We have

$$|\nabla u_n|^2 = |\nabla \tilde{\eta}_\varepsilon|^2 + \tilde{\eta}_\varepsilon^2 |\nabla \tilde{v}_n|^2 + (|\tilde{v}_n|^2 - 1)|\nabla \tilde{\eta}_\varepsilon|^2 + \tilde{\eta}_\varepsilon \nabla \tilde{\eta}_\varepsilon \cdot \nabla(|\tilde{v}_n|^2 - 1),$$

and therefore,

$$\begin{aligned} E_\varepsilon(u_n) = E_\varepsilon(\tilde{\eta}_\varepsilon) &+ \frac{1}{2}\int_{\mathbb{R}^2} \Big(\tilde{\eta}_\varepsilon^2 |\nabla \tilde{v}_n|^2 + \frac{\tilde{\eta}_\varepsilon^4}{2\varepsilon^2}(|\tilde{v}_n|^2 - 1)^2\Big) \\ &+ \frac{1}{2}\int_{\mathbb{R}^2} \Big((|\tilde{v}_n|^2 - 1)|\nabla \tilde{\eta}_\varepsilon|^2 + \tilde{\eta}_\varepsilon \nabla \tilde{\eta}_\varepsilon \cdot \nabla(|\tilde{v}_n|^2 - 1) + \frac{1}{\varepsilon^2}\tilde{\eta}_\varepsilon^2(|\tilde{v}_n|^2 - 1)(\tilde{\eta}_\varepsilon^2 - a(x))\Big). \end{aligned}$$

As in [16], the main idea is to multiply the equation (2.26) by $\tilde{\eta}_\varepsilon(|\tilde{v}_n|^2 - 1)$ and then to integrate by parts. It leads to

$$\int_{\mathbb{R}^2} \bigg\{(|\tilde{v}_n|^2 - 1)|\nabla \tilde{\eta}_\varepsilon|^2 + \tilde{\eta}_\varepsilon \nabla \tilde{\eta}_\varepsilon \nabla(|\tilde{v}_n|^2 - 1) + \frac{\tilde{\eta}_\varepsilon^2}{\varepsilon^2}(|\tilde{v}_n|^2 - 1)(\tilde{\eta}_\varepsilon^2 - a(x))\bigg\} = k_\varepsilon \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2(|\tilde{v}_n|^2 - 1)$$

and we conclude that for every $n \in \mathbb{N}$,

$$E_\varepsilon(u_n) = E_\varepsilon(\tilde{\eta}_\varepsilon) + \tilde{\mathcal{E}}_\varepsilon(\tilde{v}_n) + \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2(|\tilde{v}_n|^2 - 1).$$

Now we observe that

$$|u_n| \le |u| \quad \text{and} \quad |\nabla u_n| \le |\nabla u| + |u| \quad \text{a.e. in } \mathbb{R}^2, \tag{3.20}$$

and by the dominated convergence theorem, it results that $E_\varepsilon(u_n) \to E_\varepsilon(u)$ and

$$\frac{k_\varepsilon}{2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2(|\tilde{v}_n|^2 - 1) = \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2} (|u_n|^2 - \tilde{\eta}_\varepsilon^2) \longrightarrow \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2} (|u|^2 - \tilde{\eta}_\varepsilon^2) = \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2(|\tilde{v}|^2 - 1).$$

Applying Fatou's lemma, we obtain

$$\begin{aligned}\tilde{\mathcal{E}}_\varepsilon(\tilde v) \le \lim_{n\to+\infty} \tilde{\mathcal{E}}_\varepsilon(\tilde v_n) &= \lim_{n\to+\infty} \left\{ E_\varepsilon(u_n) - E_\varepsilon(\tilde\eta_\varepsilon) - \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2} (|u_n|^2 - \tilde\eta_\varepsilon^2)\right\} \\ &= E_\varepsilon(u) - E_\varepsilon(\tilde\eta_\varepsilon) - \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2} \tilde\eta_\varepsilon^2(|\tilde v|^2-1) < +\infty,\end{aligned}$$

and we conclude that $\tilde v \in \mathcal{G}_\varepsilon$. Since $|\tilde v_n||\nabla\tilde\eta_\varepsilon| \le |\nabla u| + \tilde\eta_\varepsilon|\nabla\tilde v|$, we infer from (3.20) that $\tilde\eta_\varepsilon^2|\nabla\tilde v_n|^2 \le C(|\nabla u|^2+|u|^2+\tilde\eta_\varepsilon^2|\nabla\tilde v|^2)$ and $\tilde\eta_\varepsilon^4(|\tilde v_n|^2-1)^2 \le 2(|u|^4+\tilde\eta_\varepsilon^4)$. By the dominated convergence theorem, we finally get that

$$\tilde{\mathcal{E}}_\varepsilon(\tilde v) = \lim_{n\to+\infty}\tilde{\mathcal{E}}_\varepsilon(\tilde v_n) = E_\varepsilon(u) - E_\varepsilon(\tilde\eta_\varepsilon) - \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2}\tilde\eta_\varepsilon^2(|\tilde v|^2-1).$$

Step 2. Consider now $\tilde u = u/e^{i\Omega S}$. Then $\tilde u \in \mathcal{H}$ and we have the decomposition

$$F_\varepsilon(u) = E_\varepsilon(\tilde u) + \frac{\Omega}{1+\Lambda^2}\int_{\mathbb{R}^2}\nabla^\perp a\cdot(i\tilde u,\nabla\tilde u) + \frac{\Omega^2}{2}\int_{\mathbb{R}^2}\big(|\nabla S|^2 - 2x^\perp\cdot\nabla S\big)|\tilde u|^2. \tag{3.21}$$

Indeed, we use that

$$|\nabla u|^2 - 2\Omega x^\perp\cdot(iu,\nabla u) = |\nabla\tilde u|^2 + \frac{2\Omega}{1+\Lambda^2}\nabla^\perp a\cdot(i\tilde u,\nabla\tilde u) + \Omega^2\big(|\nabla S|^2-2x^\perp\cdot\nabla S\big)|\tilde u|^2 \text{ a.e. in } \mathbb{R}^2.$$

Since $|\nabla S| \le C|x|$, $|\nabla a| \le C|x|$, we infer that (3.21) holds.
Step 3. We show that (3.18) takes place. Let $u\in\mathcal{H}$. Set $\tilde u = u/e^{i\Omega S}$ and $v = \tilde u/\tilde\eta_\varepsilon$. By Step 1 and Step 2, it results that $\tilde u\in\mathcal{H}$ and $v\in\mathcal{G}_\varepsilon$. By (3.19), we have

$$E_\varepsilon(\tilde u) = E_\varepsilon(\tilde\eta_\varepsilon) + \tilde{\mathcal{E}}_\varepsilon(v) + \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2}\tilde\eta_\varepsilon^2(|v|^2-1). \tag{3.22}$$

Since $\nabla^\perp a\cdot(i\tilde u,\nabla\tilde u) = \tilde\eta_\varepsilon^2\nabla^\perp a\cdot(iv,\nabla v)$ and $|\tilde u|^2 = \tilde\eta_\varepsilon^2|v|^2$ a.e. in $\mathbb{R}^2$, we infer from (3.21) and (3.22) that

$$F_\varepsilon(u) = E_\varepsilon(\tilde\eta_\varepsilon) + \tilde{\mathcal{E}}_\varepsilon(v) + \tilde{\mathcal{R}}_\varepsilon(v) + \frac{\Omega^2}{2}\int_{\mathbb{R}^2}\big(|\nabla S|^2-2x^\perp\cdot\nabla S\big)\tilde\eta_\varepsilon^2|v|^2 + \frac{k_\varepsilon}{2}\int_{\mathbb{R}^2}\tilde\eta_\varepsilon^2(|v|^2-1). \tag{3.23}$$

On the other hand, (3.21) yields

$$F_\varepsilon(\tilde\eta_\varepsilon e^{i\Omega S}) = E_\varepsilon(\tilde\eta_\varepsilon) + \frac{\Omega^2}{2}\int_{\mathbb{R}^2}\big(|\nabla S|^2-2x^\perp\cdot\nabla S\big)\tilde\eta_\varepsilon^2 \tag{3.24}$$

and the conclusion follows combining (3.23) and (3.24). ∎

Remark 3.2. The energy of the "vortex-free" profile is given by

$$F_\varepsilon(\tilde\eta_\varepsilon e^{i\Omega S}) = E_\varepsilon(\tilde\eta_\varepsilon) - \frac{\pi a_0^3(1-\Lambda^2)^2}{24(1+\Lambda^2)\Lambda^3}\,\Omega^2 + o(1). \tag{3.25}$$

It directly follows from (3.24) and Proposition 2.2.

Proof of Proposition 3.3. The minimizing property of v_ε follows directly from Proposition 3.1 and Lemma 3.2.

Proof of 3.3.a) and 3.3.b). Since u_ε minimizes F_ε in $\big\{u\in\mathcal{H} : \|u\|_{L^2(\mathbb{R}^2)} = 1\big\}$, we have using Lemma 3.2,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(\tilde\eta_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{R}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \le F_\varepsilon(\tilde\eta_\varepsilon e^{i\Omega S}),$$

and it yields

$$\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \leq |\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon)| + |\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)|. \tag{3.26}$$

Arguing as in the proof of Lemma 3.1 with $\sigma = 1/4$ and $R = \sqrt{2a_0}$, we infer from *3.2.e)* in Proposition 3.2 and (3.3),

$$\begin{aligned}
\left|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon)\right| &\leq \frac{1}{4}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{4\Omega^2}{(\Lambda^2+1)^2}\int_{\mathbb{R}^2}|x|_\Lambda^2|u_\varepsilon|^2 \\
&\leq \frac{1}{4}\int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{4\Omega^2}{(\Lambda^2+1)^2}\int_{\mathbb{R}^2\setminus B^\Lambda_{\sqrt{2a_0}}} 2a^-(x)|u_\varepsilon|^2 + \frac{8a_0\Omega^2}{(\Lambda^2+1)^2}\int_{B^\Lambda_{\sqrt{2a_0}}}|u_\varepsilon|^2 \\
&\leq \frac{1}{2}\,\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) + C_{\omega_0}|\ln\varepsilon|^2.
\end{aligned} \tag{3.27}$$

We obtain from (2.28), (2.30) and (3.3) that

$$\begin{aligned}
|\tilde{\mathcal{T}}_\varepsilon(v_\varepsilon)| &= \Big|\frac{1}{2}\int_{\mathbb{R}^2}\big(\Omega^2|\nabla S|^2 - 2\Omega^2 x^\perp\cdot\nabla S + k_\varepsilon\big)(|u_\varepsilon|^2 - \tilde{\eta}_\varepsilon^2)\Big| \\
&\leq C_{\omega_0}|\ln\varepsilon|^2\left[\int_{\mathbb{R}^2\setminus B^\Lambda_{\sqrt{2a_0}}} 2a^-(x)(|u_\varepsilon|^2 + \tilde{\eta}_\varepsilon^2) + \left(\int_{B^\Lambda_{\sqrt{2a_0}}}(|u_\varepsilon|^2 - a^+)^2 + (\tilde{\eta}_\varepsilon^2 - a^+)^2\right)^{1/2}\right] \\
&\leq C_{\omega_0}\,\varepsilon|\ln\varepsilon|^3.
\end{aligned} \tag{3.28}$$

According to (3.26), (3.27) and (3.28), we conclude that $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) \leq C_{\omega_0}|\ln\varepsilon|^2$.

Proof of 3.3.c). From *2.2.c)* in Proposition 2.2, *3.2.b)* and *3.2.d)*, we infer that

$$|v_\varepsilon(x)| = \frac{|u_\varepsilon(x)|}{\tilde{\eta}_\varepsilon(x)} \leq \frac{\sqrt{a(x) + |\ell_\varepsilon|\varepsilon^2 + \varepsilon^2\Omega^2|x|^2}}{(1 - C\varepsilon^{1/3})\sqrt{a(x)}} \leq 1 + C_{\omega_0}\varepsilon^{1/3} \quad \text{for } x \in B^\Lambda_{\sqrt{a_0}-\varepsilon^{1/8}}.$$

Proof of 3.3.d). Let $K \subset B^\Lambda_{\sqrt{a_0}}$ be any compact set. We denote $\tilde{v}_\varepsilon = e^{i\Omega S}v_\varepsilon = \frac{u_\varepsilon}{\tilde{\eta}_\varepsilon}$. By *2.2.c)* in Proposition 2.2, we know that there exists $C_K > 0$ independent of ε such that $\tilde{\eta}_\varepsilon \geq (1 - C\varepsilon^{1/3})\sqrt{a} \geq C_K$ in K. Since $\nabla\tilde{v}_\varepsilon = \tilde{\eta}_\varepsilon^{-1}\nabla u_\varepsilon - (\tilde{\eta}_\varepsilon^{-2}\nabla\tilde{\eta}_\varepsilon)u_\varepsilon$, using Proposition 2.2 and Proposition 3.2, it follows $\|\nabla\tilde{v}_\varepsilon\|_{L^\infty(K)} \leq C_{\omega_0,K}\varepsilon^{-1}$. Hence we deduce (using *3.3.c)*) that

$$\|\nabla v_\varepsilon\|_{L^\infty(K)} \leq \|\nabla\tilde{v}_\varepsilon\|_{L^\infty(K)} + \Omega\|\tilde{v}_\varepsilon\nabla S\|_{L^\infty(K)} \leq C_{\omega_0,K}\,\varepsilon^{-1}$$

and the proof is complete. ∎

3.3 Splitting the domain

The main goal in this section is to show that we can excise the region of $\mathbb{R}^2$ where the density $|u_\varepsilon|$ is very small (which corresponds to the exterior of $\mathcal{D}$) without modifying the relevant part in the energy.

Proposition 3.4. *Assume that* (3.2) *holds. For small $\varepsilon > 0$ and $\nu \in [1,2]$, we set $\mathcal{D}_\varepsilon^\nu = \big\{x \in \mathbb{R}^2 : a(x) > \nu|\ln\varepsilon|^{-3/2}\big\}$. We have*

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \leq C_{\omega_0}|\ln\varepsilon|^{-1}.$$

Proof. Since u_ε minimizes F_ε on $\big\{u \in \mathcal{H} : \|u\|_{L^2(\mathbb{R}^2)} = 1\big\}$, we have for ε sufficiently small that $F_\varepsilon(u_\varepsilon) \leq F_\varepsilon\big(\tilde{\eta}_\varepsilon e^{i\Omega S}\big)$. Then Lemma 3.2 yields $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(v_\varepsilon) \leq 0$ and we derive from *3.3.b)* in Proposition 3.3,

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \leq C_{\omega_0}\varepsilon|\ln\varepsilon|^3. \tag{3.29}$$

We now set $\mathcal{N}_\varepsilon^\nu = \mathbb{R}^2 \setminus \mathcal{D}_\varepsilon^\nu$. From the previous inequality, it suffices to prove that

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathcal{N}_\varepsilon^\nu) \geq -C_{\omega_0}|\ln\varepsilon|^{-1} \tag{3.30}$$

for a constant $C_{\omega_0} > 0$ independent of ε and ν. Arguing as in the proof of Lemma 3.1 with $\sigma = 1/4$ and $R = \sqrt{2a_0}$, we infer from (3.3),

$$\begin{aligned}\Big|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon, \mathcal{N}_\varepsilon^\nu)\Big| \leq& \frac{1}{4}\int_{\mathcal{N}_\varepsilon^\nu} \tilde{\eta}_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{4\Omega^2}{(1+\Lambda^2)^2}\int_{\mathcal{N}_\varepsilon^\nu}|x|_\Lambda^2|u_\varepsilon|^2\\ \leq& \frac{1}{4}\int_{\mathcal{N}_\varepsilon^\nu} \tilde{\eta}_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{4\Omega^2}{(1+\Lambda^2)^2}\int_{\mathbb{R}^2\setminus B^\Lambda_{\sqrt{2a_0}}} 2a^-(x)|u_\varepsilon|^2 + \frac{8a_0\Omega^2}{(1+\Lambda^2)^2}\int_{B^\Lambda_{\sqrt{2a_0}}\setminus\mathcal{D}_\varepsilon^\nu}|u_\varepsilon|^2\\ \leq& \frac{1}{4}\int_{\mathcal{N}_\varepsilon^\nu} \tilde{\eta}_\varepsilon^2|\nabla v_\varepsilon|^2 + \frac{8a_0\Omega^2}{(1+\Lambda^2)^2}\int_{B^\Lambda_{\sqrt{2a_0}}\setminus\mathcal{D}_\varepsilon^\nu}|u_\varepsilon|^2 + C_{\omega_0}\varepsilon^2|\ln\varepsilon|^4.\end{aligned}$$

By (3.3), we may also estimate

$$\begin{aligned}\int_{B^\Lambda_{\sqrt{2a_0}}\setminus\mathcal{D}_\varepsilon^\nu}|u_\varepsilon|^2 =& \int_{B^\Lambda_{\sqrt{2a_0}}\setminus B^\Lambda_{\sqrt{a_0}}}|u_\varepsilon|^2 + \int_{B^\Lambda_{\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu}(|u_\varepsilon|^2 - a(x)) + \int_{B^\Lambda_{\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu} a(x)\\ \leq& C\Big(\int_{B^\Lambda_{\sqrt{2a_0}}\setminus B^\Lambda_{\sqrt{a_0}}}|u_\varepsilon|^4\Big)^{1/2} + C\Big(\int_{B^\Lambda_{\sqrt{a_0}}\setminus\mathcal{D}_\varepsilon^\nu}(|u_\varepsilon|^2 - a(x))^2\Big)^{1/2} + C|\ln\varepsilon|^{-3}\\ \leq& C_{\omega_0}(|\ln\varepsilon|^{-3} + \varepsilon|\ln\varepsilon|).\end{aligned}$$

Then it follows that

$$|\tilde{\mathcal{R}}_\varepsilon(v_\varepsilon, \mathcal{N}_\varepsilon^\nu)| \leq \frac{1}{2}\,\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon, \mathcal{N}_\varepsilon^\nu) + C_{\omega_0}|\ln\varepsilon|^{-1} \tag{3.31}$$

which leads to (3.30). ■

For some technical reasons, it will be easier to deal with a^+ instead of $\tilde{\eta}_\varepsilon^2$ in the energies. To replace $\tilde{\eta}_\varepsilon^2$ by a^+, we shall prove that the energy estimates inside $\mathcal{D}_\varepsilon^\nu$ remain unchanged.

Proposition 3.5. *Assume that* (3.2) *holds for some* $\omega_0 > 0$. *We have*

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \leq C_{\omega_0}|\ln\varepsilon|^2 \quad \textit{and} \quad \mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \leq C_{\omega_0}|\ln\varepsilon|^{-1}$$

where $\mathcal{E}_\varepsilon$ *and* $\mathcal{F}_\varepsilon$ *are defined in* (1.18).

Proof. From *2.2.c)* in Proposition 2.2, we infer that

$$\left\|\frac{a - \tilde{\eta}_\varepsilon^2}{\tilde{\eta}_\varepsilon^2}\right\|_{L^\infty(\mathcal{D}_\varepsilon^\nu)} \leq C\varepsilon^{1/3} \quad \text{and} \quad \left\|\frac{a^2 - \tilde{\eta}_\varepsilon^4}{\tilde{\eta}_\varepsilon^4}\right\|_{L^\infty(\mathcal{D}_\varepsilon^\nu)} \leq C\varepsilon^{1/3}$$

and then *3.3.a)* in Proposition 3.3 yields

$$\Big|\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) - \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu)\Big| \leq C\varepsilon^{1/3}\,\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) \leq C_{\omega_0}\varepsilon^{1/3}|\ln\varepsilon|^2. \tag{3.32}$$

Using *3.2.a)* and *3.2.e)* in Proposition 3.2, we derive

$$\Big|\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) - \tilde{\mathcal{R}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu)\Big| \leq \Omega\int_{\mathcal{D}_\varepsilon^\nu}\frac{a - \tilde{\eta}_\varepsilon^2}{\tilde{\eta}_\varepsilon^2}\,|u_\varepsilon|\,|\nabla u_\varepsilon| \leq C\varepsilon^{1/3}\Omega(E_\varepsilon(u_\varepsilon, \mathcal{D}_\varepsilon^\nu))^{1/2} \leq C_{\omega_0}\varepsilon^{1/3}|\ln\varepsilon|^2.$$

Therefore, it follows that

$$\Big|\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu) - \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon^\nu)\Big| \leq C_{\omega_0}\varepsilon^{1/3}|\ln\varepsilon|^2. \tag{3.33}$$

Then the conclusion comes immediately from *3.3.a)* in Proposition 3.3 and Proposition 3.4. ■

4 Energy and degree estimates

This section is devoted to the proof of Theorem 1.1. The method we use is inspired from [21, 23] and provides some information about the location and the number of vortices inside $\mathcal{D}$.

4.1 Construction of vortex balls and expansion of the rotation energy

We start with the construction of vortex balls by a method due to Sandier [20] and Sandier and Serfaty [22]; it permits to localize the vorticity set of v_ε.

Proposition 4.1. *Assume that* (3.2) *holds for some* $\omega_0 > 0$. *Then there exists a positive constant* $\mathcal{K}_{\omega_0}$ *such that for* ε *sufficiently small, there exist* $\nu_\varepsilon \in (1,2)$ *and a finite collection of disjoint balls* $\{B_i\}_{i\in I_\varepsilon} := \{B(p_i,r_i)\}_{i\in I_\varepsilon}$ *satisfying the conditions:*

(i) *for every* $i \in I_\varepsilon, B_i \subset\subset \mathcal{D}_\varepsilon = \{x \in \mathbb{R}^2 \,:\, a(x) > \nu_\varepsilon |\ln\varepsilon|^{-3/2}\}$,

(ii) $\{x \in \mathcal{D}_\varepsilon \,:\, |v_\varepsilon(x)| < 1 - |\ln\varepsilon|^{-5}\} \subset \cup_{i\in I_\varepsilon} B_i$,

(iii) $\displaystyle\sum_{i\in I_\varepsilon} r_i \le |\ln\varepsilon|^{-10}$,

(iv) $\displaystyle\frac{1}{2}\int_{B_i} a(x)|\nabla v_\varepsilon|^2 \ge \pi a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_{\omega_0}\ln|\ln\varepsilon|\big)$,

where $d_i = \deg\left(\dfrac{v_\varepsilon}{|v_\varepsilon|}, \partial B_i\right)$ *for every* $i \in I_\varepsilon$.

Proof. According to the technique presented in [20] and [22], we construct as in [2] (using Proposition 3.5 with $\nu = 1$) a finite collection of disjoint balls $\{B_i\}_{i\in\tilde I_\varepsilon} = \{B(p_i,r_i)\}_{i\in\tilde I_\varepsilon}$ such that

$$\{x \in \mathcal{D} \,:\, a(x) > |\ln\varepsilon|^{-3/2} \text{ and } |v_\varepsilon(x)| < 1 - |\ln\varepsilon|^{-5}\} \subset \cup_{i\in\tilde I_\varepsilon} B_i\,,$$

(iii) is fulfilled and

$$\int_{B_i} \frac{a(x)}{2}|(\nabla - i\Omega x^\perp)v_\varepsilon|^2 \ge \pi a(p_i)|d_i|\big(|\ln\varepsilon| - \mathcal{K}_{\omega_0}\ln|\ln\varepsilon|\big) \quad \text{for each } i \in \tilde I_\varepsilon.$$

By *(iii)*, we can find $\nu_\varepsilon \in (1,2)$ such that $\partial\{x \in \mathcal{D} \,:\, a(x) > \nu_\varepsilon|\ln\varepsilon|^{-3/2}\} \cap \cup_{i\in\tilde I_\varepsilon} B_i = \emptyset$. By cancelling the balls B_i that are not included in $\{x \in \mathcal{D} \,:\, a(x) > \nu_\varepsilon|\ln\varepsilon|^{-3/2}\}$, it remains a finite collection $\{B_i\}_{i\in I_\varepsilon}$ that satisfies *(i), (ii)* and *(iii)*. Notice now that *(iv)* takes place since we have

$$\Omega^2\int_{B_i}\frac{a(x)}{2}|x|^2|v_\varepsilon|^2 \le \Omega^2\int_{B_i}|x|^2|u_\varepsilon|^2 \le C_{\omega_0}|\ln\varepsilon|^2 r_i^2,$$

$$\Big|\Omega\int_{B_i} a(x)x^\perp\cdot(iv_\varepsilon,\nabla v_\varepsilon)\Big| \le C\Omega\int_{B_i}\frac{a(x)}{\tilde\eta_\varepsilon}|u_\varepsilon|\,|\nabla v_\varepsilon| \le C\Omega\|\sqrt{a}\nabla v_\varepsilon\|_{L^2(B_i)}r_i \le C_{\omega_0}|\ln\varepsilon|^2 r_i \qquad (4.1)$$

(here we used Proposition 3.5). Hence these terms can be absorbed by $\mathcal{K}_{\omega_0}\ln|\ln\varepsilon|$ (up to a different constant $\mathcal{K}_{\omega_0}+1$). ∎

We are now in a position to compute an asymptotic expansion of the rotation energy according to the center of each vortex ball B_i and the associated degree d_i:

Proposition 4.2. *Assume that* (3.2) *holds for some* $\omega_0 > 0$. *For* ε *sufficiently small, we have*

$$\mathcal{R}_\varepsilon\big(v_\varepsilon, \mathcal{D}_\varepsilon\big) = \frac{-\pi\Omega}{1+\Lambda^2} \sum_{i\in I_\varepsilon} (a^2(p_i) - \nu_\varepsilon^2 |\ln\varepsilon|^{-3})\, d_i \; + \; o(|\ln\varepsilon|^{-5}).$$

Proof. By Proposition 4.1, $\mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i \subset \mathcal{D}_\varepsilon \setminus \{|v_\varepsilon| < 1/2\}$ whenever ε is small enough. For $x \in \mathcal{D}_\varepsilon$ such that $|v_\varepsilon(x)| \geq 1/2$, we set

$$w_\varepsilon(x) = \frac{v_\varepsilon(x)}{|v_\varepsilon(x)|}\,.$$

Since $(iv_\varepsilon, \nabla v_\varepsilon) = |v_\varepsilon|^2 (iw_\varepsilon, \nabla w_\varepsilon)$ in $\mathcal{D}_\varepsilon \setminus \{|v_\varepsilon| < 1/2\}$, we have

$$\begin{aligned} \mathcal{R}_\varepsilon\left(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i\right) = &\frac{\Omega}{1+\Lambda^2} \int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon} B_i} a(x)\nabla^\perp a \cdot (iw_\varepsilon, \nabla w_\varepsilon) \\ &+ \frac{\Omega}{1+\Lambda^2} \int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon} B_i} a(x)(|v_\varepsilon|^2 - 1)\nabla^\perp a \cdot (iw_\varepsilon, \nabla w_\varepsilon). \end{aligned} \tag{4.2}$$

Then we estimate using Proposition 3.5,

$$\begin{aligned} \left| \int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon} B_i} a(x)(|v_\varepsilon|^2 - 1)\, \nabla^\perp a \cdot (iw_\varepsilon, \nabla w_\varepsilon) \right| &\leq C\varepsilon \left(\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon)\right)^{1/2} \|\nabla w_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon\setminus\{|v_\varepsilon|<1/2\})} \\ &\leq C\varepsilon |\ln\varepsilon| \|\nabla w_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon\setminus\{|v_\varepsilon|<1/2\})}\,. \end{aligned} \tag{4.3}$$

In $\mathcal{D}_\varepsilon \setminus \{|v_\varepsilon| < 1/2\}$, we have $|\nabla w_\varepsilon| \leq 2(|\nabla v_\varepsilon| + |\nabla |v_\varepsilon||) \leq 4|\nabla v_\varepsilon|$. We deduce that

$$\int_{\mathcal{D}_\varepsilon\setminus\{|v_\varepsilon|<1/2\}} |\nabla w_\varepsilon|^2 \leq 16 \int_{\mathcal{D}_\varepsilon} |\nabla v_\varepsilon|^2 \leq 16|\ln\varepsilon|^{3/2} \int_{\mathcal{D}_\varepsilon} a(x)|\nabla v_\varepsilon|^2 \leq C|\ln\varepsilon|^{7/2} \tag{4.4}$$

and hence we obtain combining (4.2), (4.3) and (4.4),

$$\mathcal{R}_\varepsilon\left(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i\right) = \frac{\Omega}{1+\Lambda^2} \int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon} B_i} a(x)\nabla^\perp a \cdot (iw_\varepsilon, \nabla w_\varepsilon) + \mathcal{O}(\varepsilon|\ln\varepsilon|^4). \tag{4.5}$$

Since $(iw_\varepsilon, \nabla w_\varepsilon) = w_\varepsilon \wedge \nabla w_\varepsilon$ and $a(x)\nabla^\perp a = \nabla^\perp \mathcal{P}_\varepsilon(x)$ with

$$\mathcal{P}_\varepsilon(x) = \frac{a^2(x) - \nu_\varepsilon^2|\ln\varepsilon|^{-3}}{2}, \tag{4.6}$$

we derive that

$$\begin{aligned} \int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon} B_i} a(x)\nabla^\perp a \cdot (iw_\varepsilon, \nabla w_\varepsilon) &= \int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon} B_i} \nabla^\perp \mathcal{P}_\varepsilon(x) \cdot (w_\varepsilon \wedge \nabla w_\varepsilon) \\ &= -\sum_{i\in I_\varepsilon} \int_{\partial B_i} \mathcal{P}_\varepsilon(x) \left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau}\right) \end{aligned}$$

where τ denotes the counterclockwise oriented unit tangent vector to ∂B_i. The smoothness of v_ε implies the existence of $\alpha_\varepsilon \in (\frac{1}{2}, \frac{2}{3})$ such that $\mathcal{U} = \{x \in \mathbb{R}^2 \,:\, |v_\varepsilon| < \alpha_\varepsilon\}$ is a smooth open set. Then we set for $i \in I_\varepsilon$, $\mathcal{U}_i = B_i \cap \mathcal{U}$ (notice that by Proposition 4.1, $\mathcal{U}_i \subset\subset B_i$ for small ε). Using (4.4), we derive

$$\begin{aligned} \left| \int_{\partial B_i} \mathcal{P}_\varepsilon(x) \left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial\tau}\right) - \int_{\partial\mathcal{U}_i} \mathcal{P}_\varepsilon(x) \left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial\tau}\right) \right| &= \left| \int_{B_i\setminus\mathcal{U}_i} \nabla^\perp \mathcal{P}_\varepsilon(x) \cdot (w_\varepsilon \wedge \nabla w_\varepsilon) \right| \\ &\leq C\, r_i\, \|\nabla w_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon\setminus\{|v_\varepsilon|<1/2\})} \\ &\leq C\, r_i\, |\ln\varepsilon|^{7/4} \end{aligned}$$

and since $|v_\varepsilon| \le \alpha_\varepsilon$ in $\mathcal{U}_i$ and $|\mathcal{P}_\varepsilon(x) - \mathcal{P}_\varepsilon(p_i)| \le r_i \|\nabla \mathcal{P}_\varepsilon\|_{L^\infty(\mathcal{D})}$, $\forall x \in B(p_i, r_i)$, it results from Proposition 3.5,

$$\begin{aligned}\left|\int_{\partial \mathcal{U}_i} (\mathcal{P}_\varepsilon(x) - \mathcal{P}_\varepsilon(p_i)) \left(w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau}\right)\right| &= \alpha_\varepsilon^{-2} \left|\int_{\partial \mathcal{U}_i} (\mathcal{P}_\varepsilon(x) - \mathcal{P}_\varepsilon(p_i)) \left(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial \tau}\right)\right| \\ &\le \alpha_\varepsilon^{-2} \left|\int_{\mathcal{U}_i} a(x) \nabla^\perp a \cdot (iv_\varepsilon, \nabla v_\varepsilon)\right| \\ &\quad + 2\alpha_\varepsilon^{-2} \left|\int_{\mathcal{U}_i} (\mathcal{P}_\varepsilon(x) - \mathcal{P}_\varepsilon(p_i)) \det(\nabla v_\varepsilon)\right| \\ &\le C\left(r_i \, \|\sqrt{a}\, \nabla v_\varepsilon\|_{L^2(\mathcal{D}_\varepsilon)} + r_i \, |\ln\varepsilon|^{3/2} \, \|\sqrt{a}\, \nabla v_\varepsilon\|^2_{L^2(\mathcal{U}_i)}\right) \\ &\le C\, r_i \, |\ln \varepsilon|^{7/2} \,.\end{aligned}$$

Therefore we conclude by *(iii)* in Proposition 4.1 that

$$\begin{aligned}\mathcal{R}_\varepsilon\left(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i\right) &= \frac{-\Omega}{1+\Lambda^2} \sum_{i\in I_\varepsilon} \mathcal{P}_\varepsilon(p_i) \int_{\partial \mathcal{U}_i} w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial \tau} \; + \; o(|\ln\varepsilon|^{-5}) \\ &= \frac{-2\pi\Omega}{1+\Lambda^2} \sum_{i\in I_\varepsilon} \mathcal{P}_\varepsilon(p_i)\, d_i \; + \; o(|\ln\varepsilon|^{-5}).\end{aligned}$$

On the other hand, we infer from (4.1) and *(iii)* in Proposition 4.1 that

$$\left|\mathcal{R}_\varepsilon(v_\varepsilon, \cup_{i\in I_\varepsilon} B_i)\right| \le C|\ln\varepsilon|^2 \sum_{i\in I_\varepsilon} r_i \le C|\ln\varepsilon|^{-8}.$$

According to (4.6), the proof is completed. ∎

4.2 Asymptotic behavior for subcritical velocities. Proof of *(i)* in Theorem 1.1.

In this section, we prove *(i)* in Theorem 1.1. We will distinguish different types of vortex balls through the partition $I_\varepsilon = I_0 \cup I_* \cup I_-$ where

$$\begin{aligned} I_0 &= \left\{ i \in I_\varepsilon \, : \, d_i \ge 0 \text{ and } |p_i|_\Lambda < |\ln\varepsilon|^{-1/6} \right\}, \\ I_* &= \left\{ i \in I_\varepsilon \, : \, d_i \ge 0 \text{ and } |p_i|_\Lambda \ge |\ln\varepsilon|^{-1/6} \right\}, \\ I_- &= \left\{ i \in I_\varepsilon \, : \, d_i < 0 \right\} \end{aligned}$$

in order to improve the lower bound for $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon)$ (see (4.12)). In the sequel, we assume that

$$\Omega \le \Omega_1 + \omega_1 \ln|\ln\varepsilon| \tag{4.7}$$

for some constant $\omega_1 \in \mathbb{R}$. Therefore, if ε is small, we have $\Omega \le \frac{3}{a_0}|\ln\varepsilon|$ and we will use the constant $\mathcal{K}_{\frac{3}{a_0}}$ given by Proposition 4.1. In fact, one can choose instead of $\frac{3}{a_0}$ any other constant ω_0 such that $\omega_0 > \frac{1+\Lambda^2}{a_0}$. First, we show the following:

Proposition 4.3. *Assume that* (4.7) *holds with* $\omega_1 < \omega_1^\star := \frac{-(1+\Lambda^2)\mathcal{K}_{\frac{3}{a_0}}}{a_0}$. *Then for* ε *sufficiently small, we have* $\sum_{i\in I_\varepsilon} |d_i| = 0$ *and*

$$|v_\varepsilon| \to 1 \quad \textit{in } L^\infty_{\mathrm{loc}}(\mathcal{D}) \;\; \textit{as } \varepsilon \to 0. \tag{4.8}$$

Moreover,

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = o(1) \quad \textit{and} \quad \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) = o(1). \tag{4.9}$$

Proof. From Proposition 3.5 and Proposition 4.1, we get that

$$\mathcal{O}(|\ln\varepsilon|^{-1}) \geq \mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) \geq \frac{1}{2}\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)|\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_{\mathcal{D}_\varepsilon} a^2(x)(1-|v_\varepsilon|^2)^2 \\ + \pi\sum_{i\in I_\varepsilon} a(p_i)|d_i|\Big(|\ln\varepsilon| - \mathcal{K}_{\frac{3}{a_0}}\ln|\ln\varepsilon|\Big) + \mathcal{R}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon). \tag{4.10}$$

Combining Proposition 4.2 and (4.7), it results that

$$\begin{aligned}\mathcal{R}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) &\geq \frac{-\pi a_0\Omega}{1+\Lambda^2}\sum_{i\in I_0} a(p_i)|d_i| - \frac{\pi(a_0-|\ln\varepsilon|^{-1/3})\Omega}{1+\Lambda^2}\sum_{i\in I_*} a(p_i)|d_i| + o(|\ln\varepsilon|^{-5})\\ &\geq -\pi\sum_{i\in I_0\cup I_*} a(p_i)|d_i||\ln\varepsilon| - \frac{\pi a_0\omega_1}{1+\Lambda^2}\sum_{i\in I_0} a(p_i)|d_i|\ln|\ln\varepsilon| \\ &\quad + \frac{\pi}{2a_0}\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{2/3} + o(|\ln\varepsilon|^{-5})\end{aligned} \tag{4.11}$$

(here we used that

$$\frac{(a_0-|\ln\varepsilon|^{-1/3})\Omega}{1+\Lambda^2} \leq |\ln\varepsilon| - \frac{1}{a_0}|\ln\varepsilon|^{2/3} + \frac{a_0\omega_1}{1+\Lambda^2}\ln|\ln\varepsilon| \leq |\ln\varepsilon| - \frac{1}{2a_0}|\ln\varepsilon|^{2/3}$$

for ε small). Then we deduce from (4.10) and (4.11) that for ε small enough,

$$\frac{1}{2}\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)|\nabla v_\varepsilon|^2 + \int_{\mathcal{D}_\varepsilon}\frac{a^2(x)}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2 - \pi\big(\frac{a_0\omega_1}{1+\Lambda^2}+\mathcal{K}_{\frac{3}{a_0}}\big)\sum_{i\in I_0} a(p_i)|d_i|\ln|\ln\varepsilon| \\ + \frac{\pi}{4a_0}\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{2/3} + \frac{\pi}{2}\sum_{i\in I_-} a(p_i)|d_i||\ln\varepsilon| + o(|\ln\varepsilon|^{-5}) \leq \mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) \leq \mathcal{O}(|\ln\varepsilon|^{-1}). \tag{4.12}$$

Since $\frac{a_0\omega_1}{1+\Lambda^2} < -\mathcal{K}_{\frac{3}{a_0}}$ and $a(p_i) \geq a_0/2$ for $i\in I_0$, we derive from (4.12) that $\sum_{i\in I_0}|d_i| = o(|\ln\varepsilon|^{-1})$. Now since $a(p_i) \geq |\ln\varepsilon|^{-3/2}$ in $\mathcal{D}_\varepsilon$, we also obtain from (4.12) that $\sum_{i\in I_*}|d_i| = \mathcal{O}(|\ln\varepsilon|^{-1/6})$ and $\sum_{i\in I_-}|d_i| = \mathcal{O}(|\ln\varepsilon|^{-1/2})$. Hence $\sum_{i\in I_\varepsilon}|d_i| \equiv 0$ for ε sufficiently small. Coming back to (4.12), we infer that for any $0 < R < \sqrt{a_0}$,

$$\frac{1}{\varepsilon^2}\int_{B_R^\Lambda}(1-|v_\varepsilon|^2)^2 \leq \frac{C_R}{\varepsilon^2}\int_{\mathcal{D}_\varepsilon} a^2(x)(1-|v_\varepsilon|^2)^2 \leq o(1). \tag{4.13}$$

Then the proof of (4.8) follows as in [6] using the estimate *3.3.d)* in Proposition 3.3 on $|\nabla v_\varepsilon|$.

Since $\sum_{i\in I_\varepsilon}|d_i| = 0$, we derive from Proposition 4.2 that $\mathcal{R}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$. Using that $\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) \leq o(1)$, we deduce that $\mathcal{E}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$ and hence we have $\mathcal{F}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$. By (3.32) and (3.33), it leads to

$$\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1) \tag{4.14}$$

and $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) = o(1)$. Using (3.29) and (3.30), we get that

$$o(1) \leq \tilde{\mathcal{F}}_\varepsilon(v_\varepsilon,\mathcal{N}_\varepsilon^{\nu_\varepsilon}) \leq -\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon,\mathcal{D}_\varepsilon) + o(1) \leq o(1) \tag{4.15}$$

and therefore $\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) = o(1)$. By (3.31), we have

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon,\mathcal{N}_\varepsilon^{\nu_\varepsilon}) = \tilde{\mathcal{E}}_\varepsilon(v_\varepsilon,\mathcal{N}_\varepsilon^{\nu_\varepsilon}) + \tilde{\mathcal{R}}_\varepsilon(v_\varepsilon,\mathcal{N}_\varepsilon^{\nu_\varepsilon}) \geq \frac{1}{2}\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon,\mathcal{N}_\varepsilon^{\nu_\varepsilon}) + o(1)$$

and it results from (4.15) that $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon, \mathcal{N}_\varepsilon^{\nu_\varepsilon}) = o(1)$. By (4.14), we conclude that $\tilde{\mathcal{E}}_\varepsilon(v_\varepsilon) = o(1)$. ■

Proof of (i) in Theorem 1.1. By *2.2.c)* in Proposition 2.2 and (4.8), it follows that $|u_\varepsilon| \to \sqrt{a^+}$ in $L^\infty_{\rm loc}(\mathcal{D})$. According to *3.2.c)* in Proposition 3.2, it turns out that $|u_\varepsilon| \to \sqrt{a^+}$ in $L^\infty_{\rm loc}(\mathbb{R}^2 \setminus \partial\mathcal{D})$. Moreover, by (4.9), for any sequence $\varepsilon_n \to 0$ we can extract a subsequence (still denoted (ε_n)) such that $v_{\varepsilon_n} \to \alpha$ in $H^1_{\rm loc}(\mathcal{D})$ for some constant $\alpha \in S^1$. We obtain that $u_{\varepsilon_n} e^{-i\Omega S} \to \alpha\sqrt{a^+}$ in $H^1_{\rm loc}(\mathcal{D})$ by *2.2.e)* in Proposition 2.2. By Lemma 3.2, *3.3.b)* in Proposition 3.3 and (4.9), we conclude that (1.6) holds. ■

4.3 Vortex existence near the critical velocity. Proof of *(ii)* in Theorem 1.1.

We now prove *(ii)* in Theorem 1.1. We will use an appropriate test function in order to improve the upper bound of the energy $F_\varepsilon(u_\varepsilon)$.

Proof of (ii) in Theorem 1.1. Step1: Construction of a test function. Assume that $\Omega_1 + \delta \ln|\ln\varepsilon| \leq \Omega \leq \omega_0 |\ln\varepsilon|$ for some positive constants δ and ω_0 (thus, $\omega_0 > \frac{\Lambda^2+1}{a_0}$). We consider the map $\tilde{v}_\varepsilon$ defined by

$$\tilde{v}_\varepsilon(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } |x| \geq \varepsilon, \\[2mm] \dfrac{x}{\varepsilon} & \text{otherwise} \end{cases}$$

and we set $\hat{u}_\varepsilon = \tilde{\eta}_\varepsilon e^{i\Omega S} \tilde{v}_\varepsilon$. We easily check that $\hat{u}_\varepsilon \in \mathcal{H}$. Lemma 3.2 yields

$$F_\varepsilon(\hat{u}_\varepsilon) = F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) + \tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) + \tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon).$$

Then we estimate

$$\big|\tilde{\mathcal{T}}_\varepsilon(\tilde{v}_\varepsilon)\big| \leq \frac{1}{2}\int_{B_\varepsilon} \Big|\Omega^2|\nabla S|^2 - 2\Omega^2 x^\perp \cdot \nabla S + k_\varepsilon\Big| \tilde{\eta}_\varepsilon^2 (1 - |\tilde{v}_\varepsilon|^2) = o(1).$$

A straightforward computation (using Proposition 2.2) leads to

$$\tilde{\mathcal{F}}_\varepsilon(\tilde{v}_\varepsilon) \leq -\frac{\pi a_0^2 \delta}{1+\Lambda^2} \ln|\ln\varepsilon| + \mathcal{O}(1)$$

and consequently

$$F_\varepsilon(\hat{u}_\varepsilon) \leq F_\varepsilon(\tilde{\eta}_\varepsilon e^{i\Omega S}) - \frac{\pi a_0^2 \delta}{1+\Lambda^2} \ln|\ln\varepsilon| + \mathcal{O}(1). \tag{4.16}$$

We now set $\tilde{u}_\varepsilon = m_\varepsilon^{-1} \hat{u}_\varepsilon$ with $m_\varepsilon = \|\hat{u}_\varepsilon\|_{L^2(\mathbb{R}^2)}$ (so that $\|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$). Since $\|\tilde{\eta}_\varepsilon\|_{L^2(\mathbb{R}^2)} = 1$, we have

$$m_\varepsilon^2 = \int_{\mathbb{R}^2} \tilde{\eta}_\varepsilon^2 |\tilde{v}_\varepsilon|^2 = 1 + \int_{B_\varepsilon} \tilde{\eta}_\varepsilon^2 (|\tilde{v}_\varepsilon|^2 - 1) = 1 + \mathcal{O}(\varepsilon^2).$$

From this estimate, we easily check that

$$F_\varepsilon(\tilde{u}_\varepsilon) = F_\varepsilon(\hat{u}_\varepsilon) + o(1). \tag{4.17}$$

Step 2. By the minimizing property of u_ε, we know that $F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(\tilde{u}_\varepsilon)$. In view of *3.3.b)* in Proposition 3.3, (4.16) and (4.17), it yields

$$\tilde{\mathcal{F}}_\varepsilon(v_\varepsilon) \leq -\frac{\pi a_0^2 \delta}{1+\Lambda^2} \ln|\ln\varepsilon| + \mathcal{O}(1).$$

Using (3.30) and then (3.33), we derive that

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le -\frac{\pi a_0^2 \delta}{1+\Lambda^2} \ln|\ln\varepsilon| + \mathcal{O}(1). \tag{4.18}$$

On the other hand, by Proposition 4.2, we have

$$\begin{aligned}\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) &\ge -\frac{\pi\omega_0}{1+\Lambda^2} \sum_{i\in I_\varepsilon,\, d_i>0} a^2(p_i)\, d_i |\ln\varepsilon| \;+\; o(1) \\ &\ge -\frac{\pi\omega_0\, a_0}{1+\Lambda^2} \sum_{i\in \hat{I}_\varepsilon,\, d_i>0} a(p_i)\, d_i |\ln\varepsilon| - \frac{\pi}{2} \sum_{i\in I_\varepsilon\setminus \hat{I}_\varepsilon,\, d_i>0} a(p_i)\, d_i |\ln\varepsilon| \;+\; o(1)\end{aligned}$$

where we denoted

$$\hat{I}_\varepsilon = \Big\{ i \in I_\varepsilon \,:\, a(p_i) \ge \frac{\Lambda^2+1}{2\omega_0} \Big\}.$$

Then, by Proposition 4.1, we deduce that

$$\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \ge \mathcal{E}_\varepsilon(v_\varepsilon, \cup_{i\in I_\varepsilon} B_i) + \mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \ge -C_{\omega_0} \sum_{i\in\hat{I}_\varepsilon,\, d_i>0} a(p_i)\, d_i |\ln\varepsilon| \;+\; o(1)$$

for some constant $C_{\omega_0} > 0$. Therefore, by (4.18), it results that for small $\varepsilon > 0$,

$$\sum_{i\in\hat{I}_\varepsilon,\, d_i>0} d_i > 0.$$

We conclude that there exists $i_0 \in \hat{I}_\varepsilon$ such that $d_{i_0} > 0$, so that there exists at least one vortex inside the bulk $\mathcal{D}$ which remains at a positive distance (independent of ε) from $\partial\mathcal{D}$. If in addition, (4.7) holds, we claim that u_ε has at least one vortex close to the origin. Indeed, by (4.12) and (4.18), we obtain

$$-\pi\Big(\frac{a_0\omega_1}{1+\Lambda^2} + \mathcal{K}_{\frac{3}{a_0}}\Big) \sum_{i\in I_0} a(p_i)|d_i| \ln|\ln\varepsilon| \le -\frac{\pi a_0^2\delta}{1+\Lambda^2} \ln|\ln\varepsilon| + \mathcal{O}(1)$$

which implies for ε small enough that $\sum_{i\in I_0} |d_i| \ge C > 0$ for a constant C independent of ε. Hence, for ε small, there exists a ball B_{j_0} ($j_0 \in I_0$) that carries a vortex x^ε with $|x^\varepsilon| \le \mathcal{O}(|\ln\varepsilon|^{-1/6})$. ∎

4.4 Energy estimates near the critical velocity. Proof of *(iii)* in Theorem 1.1.

In this section, we prove the energy estimates stated in *(iii)* in Theorem 1.1 in the regime (4.7). First, we shall prove that the number of vortex balls with nonzero degree lying in a slightly smaller domain than $\mathcal{D}_\varepsilon$, is bounded.

Proposition 4.4. *Assume that* (4.7) *holds. Then*

$$N_0 := \sum_{i\in I_0} |d_i| \le C_{\omega_1} \tag{4.19}$$

and setting $\mathcal{B}_\varepsilon = \big\{ x \in \mathbb{R}^2 \,:\, a(x) \ge |\ln\varepsilon|^{-1/2} \big\}$*, we have for* ε *sufficiently small,*

$$\sum_{i\in I_*\cup I_-,\, p_i\in\mathcal{B}_\varepsilon} |d_i| = 0. \tag{4.20}$$

Proof. Arguing as for (4.12), we derive that for ε small enough,

$$\begin{aligned}\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)|\nabla v_\varepsilon|^2 + \sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{2/3} + \sum_{i\in I_-} a(p_i)|d_i||\ln\varepsilon| \leq \\ \leq C\Big|\frac{a_0\omega_1}{1+\Lambda^2} + \mathcal{K}_{\frac{3}{a_0}}\Big|\sum_{i\in I_0} a(p_i)|d_i|\ln|\ln\varepsilon| + \mathcal{O}(|\ln\varepsilon|^{-1}) \\ \leq C_0 N_0 \ln|\ln\varepsilon| + \mathcal{O}(|\ln\varepsilon|^{-1}) \qquad (4.21)\end{aligned}$$

for some positive constant C_0 independent of ε. We set

$$\tilde{I}_* = \{i\in I_* \,:\, p_i\in\mathcal{B}_\varepsilon\}\ , \quad N_* = \sum_{i\in\tilde{I}_*}|d_i|,$$

and

$$\tilde{I}_- = \{i\in I_- \,:\, p_i\in\mathcal{B}_\varepsilon\}\ , \quad N_- = \sum_{i\in\tilde{I}_-}|d_i|.$$

Since $a(p_i)\geq|\ln\varepsilon|^{-1/2}$ for any $i\in\tilde{I}_*\cup\tilde{I}_-$, we obtain from (4.21),

$$\int_{\mathcal{D}_\varepsilon\setminus\cup_{i\in I_\varepsilon}B_i} a(x)|\nabla v_\varepsilon|^2 + N_*|\ln\varepsilon|^{1/6} + N_-|\ln\varepsilon|^{1/2} \leq C_0N_0\ln|\ln\varepsilon| + \mathcal{O}(|\ln\varepsilon|^{-1}) \qquad (4.22)$$

which implies in particular that

$$\max\{N_*,N_-\}\leq\frac{N_0}{2} \qquad (4.23)$$

for ε sufficiently small. We now show that N_0 is uniformly bounded in ε. Consider the sets

$$\mathcal{I}_\varepsilon = \Big[\,|\ln\varepsilon|^{-1/6}, \frac{\sqrt{a_0}}{2}\,\Big] \quad\text{and}\quad \mathcal{J}_\varepsilon = \big\{r\in\mathcal{I}_\varepsilon \,:\, \partial B_r^\Lambda\cap(\cup_{i\in I_\varepsilon}\overline{B}_i)=\emptyset\big\}\,.$$

Notice that $\mathcal{J}_\varepsilon$ is a finite union of intervals verifying $|\mathcal{I}_\varepsilon\setminus\mathcal{J}_\varepsilon|\leq|\ln\varepsilon|^{-10}$. For $r\in\mathcal{J}_\varepsilon$ and ε small, we have $|v_\varepsilon|\geq\frac{1}{2}$ on ∂B_r^Λ and therefore, we can define

$$D(r) = \deg\left(\frac{v_\varepsilon}{|v_\varepsilon|},\partial B_r^\Lambda\right).$$

By (4.23), we obtain that for small ε,

$$|D(r)| = \Big|\sum_{|p_i|_\Lambda<r} d_i\Big| \geq N_0 - N_- \geq \frac{N_0}{2} \quad\text{for any } r\in\mathcal{J}_\varepsilon.$$

We have (using elliptic coordinates $x_1 = r\cos\theta$, $x_2=\Lambda^{-1}r\sin\theta$)

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}}\setminus\cup_{i\in I_\varepsilon}B_i} a(x)|\nabla v_\varepsilon|^2 \geq \frac{3a_0}{4\Lambda}\int_{\mathcal{J}_\varepsilon}\left(\int_0^{2\pi}|\nabla v_\varepsilon|^2 r\,d\theta\right)dr \geq C\int_{\mathcal{J}_\varepsilon}\frac{1}{r}\left(\int_0^{2\pi}\Big|v_\varepsilon\wedge\frac{\partial v_\varepsilon}{\partial\tau}\Big|^2 r^2\,d\theta\right)dr.$$

We set $w_\varepsilon = \frac{v_\varepsilon}{|v_\varepsilon|}$ in $B^\Lambda_{\frac{\sqrt{a_0}}{2}}\setminus\cup_{i\in I_\varepsilon}B_i$. Since $|v_\varepsilon\wedge\dfrac{\partial v_\varepsilon}{\partial\tau}| = |v_\varepsilon|^2|w_\varepsilon\wedge\dfrac{\partial w_\varepsilon}{\partial\tau}|\geq\dfrac{1}{4}|w_\varepsilon\wedge\dfrac{\partial w_\varepsilon}{\partial\tau}|$ in $B^\Lambda_{\frac{\sqrt{a_0}}{2}}\setminus\cup_{i\in I_\varepsilon}B_i$, we infer that

$$\begin{aligned}\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}}\setminus\cup_{i\in I_\varepsilon}B_i} a(x)|\nabla v_\varepsilon|^2 &\geq C\int_{\mathcal{J}_\varepsilon}\frac{1}{r}\left(\int_0^{2\pi}\Big|w_\varepsilon\wedge\frac{\partial w_\varepsilon}{\partial\tau}\Big|^2 r^2\,d\theta\right)dr \\ &\geq C\int_{\mathcal{J}_\varepsilon}\frac{1}{r}\left(\int_0^{2\pi}w_\varepsilon\wedge\frac{\partial w_\varepsilon}{\partial\tau}\,r\,d\theta\right)^2 dr \geq C\int_{\mathcal{J}_\varepsilon}\frac{D(r)^2}{r}\,dr \geq CN_0^2\int_{\mathcal{J}_\varepsilon}\frac{dr}{r}.\end{aligned}$$

Notice now that

$$\left| \int_{\mathcal{I}_\varepsilon} \frac{dr}{r} - \int_{\mathcal{J}_\varepsilon} \frac{dr}{r} \right| \le |\ln\varepsilon|^{1/6} |\mathcal{I}_\varepsilon \setminus \mathcal{J}_\varepsilon| = o(1)$$

and since $\displaystyle\int_{\mathcal{I}_\varepsilon} \frac{dr}{r} = C \ln|\ln\varepsilon| + \mathcal{O}(1)$, we finally get that

$$\int_{B^\Lambda_{\frac{\sqrt{a_0}}{2}} \setminus \cup_{i\in I_\varepsilon} B_i} a(x)|\nabla v_\varepsilon|^2 \ge C_1 \ln|\ln\varepsilon| N_0^2$$

for some positive constant C_1 independent of ε. From (4.22), we derive

$$\big(C_1 N_0^2 - C_0 N_0\big) \ln|\ln\varepsilon| \le \mathcal{O}(|\ln\varepsilon|^{-1})$$

which implies that N_0 is uniformly bounded in ε. Then it follows by (4.22) that

$$N_* \le \mathcal{O}\Big(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^{1/6}}\Big) \quad \text{and} \quad N_- \le \mathcal{O}\Big(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^{1/2}}\Big).$$

Therefore, $N_- = N_* = 0$ for ε sufficiently small. ■

Proof of (iii) in Theorem 1.1. From Proposition 4.2, (4.7) and (4.20), we infer that for ε small,

$$\begin{aligned}
\mathcal{R}_\varepsilon\big(v_\varepsilon, \mathcal{D}_\varepsilon\big) &\ge \frac{-\pi a_0 \Omega}{1+\Lambda^2} \sum_{i\in I_0} a(p_i)|d_i| - \frac{\pi\Omega}{1+\Lambda^2} |\ln\varepsilon|^{-1/2} \sum_{i\in I_*\setminus \tilde{I}_*} a(p_i)|d_i| + o(|\ln\varepsilon|^{-5}) \\
&\ge -\pi \sum_{i\in I_0} a(p_i)|d_i| \big(|\ln\varepsilon| + \frac{a_0\omega_1}{1+\Lambda^2} \ln|\ln\varepsilon|\big) - \frac{2\pi}{a_0} \sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{1/2} + o(|\ln\varepsilon|^{-5}).
\end{aligned}$$

We now inject this estimate in (4.10) to derive that $\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon| \le C N_0 \ln|\ln\varepsilon| + o(1)$ and hence, by (4.19), $\sum_{i\in I_*} a(p_i)|d_i||\ln\varepsilon|^{1/2} = o(1)$. It yields

$$\mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = \mathcal{R}_\varepsilon\left(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i\right) + o(1) \ge -\pi \sum_{i\in I_0} a(p_i)|d_i| \big(|\ln\varepsilon| + \frac{a_0\omega_1}{1+\Lambda^2} \ln|\ln\varepsilon|\big) + o(1).$$

Since $\mathcal{F}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) + \mathcal{R}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le \mathcal{O}(|\ln\varepsilon|^{-1})$, it follows

$$\begin{aligned}
\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) &\le \pi \sum_{i\in I_0} a(p_i)|d_i| \big(|\ln\varepsilon| + \frac{a_0\omega_1}{1+\Lambda^2} \ln|\ln\varepsilon|\big) + o(1) \\
&\le C_{\omega_1} N_0 |\ln\varepsilon| + o(1) \le C_{\omega_1} |\ln\varepsilon|.
\end{aligned} \tag{4.24}$$

Set $\mathcal{A}_\varepsilon = \mathcal{D}_\varepsilon \setminus B^\Lambda_{2|\ln\varepsilon|^{-1/6}}$. Matching *(iv)* in Proposition 4.1 with (4.24), we finally obtain

$$\begin{aligned}
\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{A}_\varepsilon) \le \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_0} B_i) &\le \pi\Big(\frac{a_0\omega_1}{1+\Lambda^2} + \mathcal{K}_{\frac{3}{a_0}}\Big) \sum_{i\in I_0} a(p_i)|d_i| \ln|\ln\varepsilon| + o(1) \\
&\le C_{\omega_1} N_0 \ln|\ln\varepsilon| \le C_{\omega_1} \ln|\ln\varepsilon|
\end{aligned}$$

and the proof is complete. ■

Remark 4.1. For general potentials $a(x)$, the analysis becomes rather delicate when the set of maximum points of the quotient $\frac{\xi}{a}$ in $\mathcal{D} = \{x \in \mathbb{R}^2 : a(x) > 0\}$ is not finite. Recall that ξ is the solution of the problem (1.16). An example is given by the following perturbation at the origin of the harmonic potential $1-|x|^2$:

$$a(x) = \begin{cases} \frac{1}{1+|x|^2} & \text{if } |x|<1, \\ \frac{2-|x|}{2} & \text{if } |x|\geq 1. \end{cases}$$

Here, the set of maximum points of the quotient $\frac{\xi}{a}$ is a circle centered in the origin.

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References

[1] Abo-Shaeer J.R., Raman C., Vogels J.M, Ketterle W., *Observation of vortex lattices in Bose-Einstein condensate*, Science **292** (2001).

[2] Aftalion A., Alama S., Bronsard L., *Giant vortex and the breakdown of strong pinning in a rotating Bose–Einstein condensate*, to appear in Arch. Ration. Mech. Anal.

[3] Aftalion A., Du Q., *Vortices in a rotating Bose-Einstein condensate: Critical angular velocities and energy diagrams in the Thomas-Fermi regime*, Phys. Rev. A **64** (2001).

[4] Aftalion A., Jerrard R.L., *Shape of vortices for a rotating Bose-Einstein condensate*, Phys. Rev. A **66** (2002).

[5] Aftalion A., Rivière T., *Vortex energy and vortex bending for a rotating Bose-Einstein condensate*, Phys. Rev. A **64** (2001).

[6] Bethuel F., Brezis H., Hélein F., *Asymptotics for the minimization of a Ginzburg-Landau functional*, Calc. of Var. and Partial Differential Equations **1** (1993), 123–148.

[7] Bethuel F., Brezis H., Hélein F., *Ginzburg-Landau Vortices*, Birkhäuser, 1993.

[8] Brezis H., *Semilinear equations in $\mathbb{R}^N$ without conditions at infinity,* Appl. Math. Optim. **12** (1984), 271–282

[9] Brezis H., Oswald L., *Remarks on sublinear elliptic equations,* Nonlinear Anal. **10** (1986), 55–64.

[10] Butts D., Rokhsar D., *Predicted signatures of rotating Bose-Einstein condensates*, Nature **397** (1999).

[11] Castin Y., Dum R., *Bose-Einstein condensates with vortices in rotating traps*, Eur. Phys. J. D **7** (1999), 399–412.

[12] Farina A., *From Ginzburg-Landau to Gross-Pitaevskii,* Monatsh. Math. **139** (2003), 265–269.

[13] Ignat R., Millot V., *Vortices in a 2d rotating Bose-Einstein condensate*, C. R. Acad. Sci. Paris Série I **340** (2005), 571–576.

[14] Ignat R., Millot V., *Energy expansion and vortex location for a two dimensional rotating Bose-Einstein condensate*, to appear in Rev. Math. Phys.

[15] Jerrard R.L., *More about Bose-Einstein condensate*, preprint (2004).

[16] LASSOUED L., MIRONESCU P., *Ginzburg-Landau type energy with discontinuous constraint*, J. Anal. Math. **77** (1999), 1–26.

[17] LIEB E. H., SEIRINGER R., YNGVASON J.,*A rigorous derivation of the Gross-Pitaevskii energy functional for a two-dimensional Bose gas*, Comm. Math. Phys. **224** (2001), 17–31.

[18] MADISON K., CHEVY F., DALIBARD J., WOHLLEBEN W., *Vortex formation in a stirred Bose-Einstein condensate*, Phys. Rev. Lett. **84** (2000).

[19] MADISON K., CHEVY F., DALIBARD J., WOHLLEBEN W., *Vortices in a stirred Bose-Einstein condensate*, J. Mod. Opt. **47** (2000).

[20] SANDIER E., *Lower bounds for the energy of unit vector fields and applications*, J. of Funct. Anal. **152** (1998), 119–145.

[21] SANDIER E., SERFATY S., *Global minimizers for the Ginzburg-Landau functional below the first critical magnetic field*, Ann. Inst. H. Poincaré Anal. Nonlinéaire **17** (2000), 119–145.

[22] SANDIER E., SERFATY S., *A rigorous derivation of a free boundary problem arising in superconductivity*, Ann. Sci. École Norm. Sup. (4) **33** (2000), 561–592.

[23] SANDIER E., SERFATY S., *Ginzburg-Landau minimizers near the first critical field have bounded vorticity*, Calc Var. and Partial Differential Equations **17** (2003), 17–28.

[24] SCHNEE K., YNGVASON J., *Bosons in disc-shape traps: from 3D to 2D*, preprint (2004).

[25] SERFATY S., *On a model of rotating superfluids*, ESAIM: Control, Optim., Calc. Var. **6** (2001), 201–238.

[26] SHAFRIR I., personal communication.

[27] STRUWE M., *An asymptotic estimate for the Ginzburg-Landau model*, C. R. Acad. Sci. Paris Sr. I Math. **317** (1993), 677–680.