On a stochastic Hardy-Littlewood-Sobolev inequality with application to Strichartz estimates for a noisy dispersion

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Abstract
In this paper, we investigate a stochastic Hardy-Littlewood-Sobolev inequality. Due to the non-homogenous nature of the potential in the inequality, a constant proportional to the length of the interval appears on the right-hand-side. As a direct application, we derive local Strichartz estimates for randomly modulated dispersions and solve the Cauchy problem of the critical nonlinear Schrödinger equation.

Key words: Stochastic regularization; Stochastic Partial Differential Equations; Nonlinear Schrödinger equation, Hardy-Littlewood-Sobolev inequality.

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1 Introduction

Let $(\Omega, \mathbb{P})$ be the standard probability space endowed with the Wiener filtration $(\mathcal{F}_t)_{t\geq 0}$. We consider the stochastic process W^H, a fractional brownian motion with Hurst parameter $H \in (0,1)$, given by, $\forall t \in \mathbb{R}^+$,

$$W_t^H = \int_{-\infty}^t ((t-s)_+^H - (-s)_+^H) dW_s,$$

where W is a standard wiener process. The main objective of this paper is to derive a stochastic counter-part to the classical Hardy-Littlewood-Sobolev inequality [21, 22, 28]. To be more specific, we obtain the following result.

Theorem 1.1. *Let $(W_t^H)_{t\geq 0}$ be a fractional brownian motion of Hurst index $H \in (0,1)$, $\beta \in (0, 1-H)$, $p, q \in (1, \infty)$ and $\alpha \in (0,1)$ such that*

$$2 - \alpha = \frac{1}{p} + \frac{1}{q}.$$

Then, there exist $T > 0$ and $C_{1.1} > 0$ such that, $\mathbb{P}$-a.s., $\forall f \in L^p([0,T])$, $\forall g \in L^q([0,T])$ the following inequality holds

$$\left| \int_0^T \int_0^T f(t) |W_t^H - W_s^H|^{-\alpha} g(s) ds dt \right| \leq C_{1.1} T^{\alpha\beta} \|f\|_{L^p([0,T])} \|g\|_{L^q([0,T])}. \tag{1.1}$$

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Our motivation to prove such result stems from the Cauchy problem of nonlinear evolution equations with a randomly modulated dispersion. Such equations are for instance: the nonlinear Schrödinger equation [13, 14, 8, 3], the Korteweg-de Vries equation [8] and the Benjamin-Bona-Mahony [7]. They have recently raised an interest due to the effects of the stochastic modulation. Here, we address the local Cauchy problem for the following nonlinear Schrödinger equation with noisy dispersion in its mild formulation

$$\psi(t,x) = P_{0,t}\psi_0(x) + \lambda \int_0^t P_{s,t}|\psi|^{2\sigma}\psi(s,x)ds, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \tag{1.2}$$

where $\lambda \in \mathbb{C}$ and, $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$P_{s,t}\varphi := \mathcal{F}^{-1}\left(e^{-i|\xi|^2(W_t^H - W_s^H)}\hat{\varphi}\right).$$

For $d = 1$, $H = 1/2$ and $\sigma = 1$, this equation arises in the field of nonlinear optics as a limit model for the propagation of light pulse in an optical fiber where the dispersion varies along the fiber [2, 1]. These variations in the dispersion accounts for the so-called *dispersion management* which aims to improve the transmission of a light signal by constructing a zero-mean dispersion fiber in order to avoid the problem of the chromatic dispersion of the light signal. When the variations are assumed to be random, a noisy dispersion can be derived (see [26, 13]) which leads, in the white noise case, to Equation (1.2).

As part of the problems concerning the propagation of waves in random media, there is a vast literature around random Schrödinger equations. Let us mention in particular the cases of random potentials [15, 16] and noisy potentials [10, 11, 12]. In these works, the effects of the stochastic potential greatly affect the dynamic of the Schrödinger equation and are, in a broader context, a motivation to introduce randomness in PDEs. Specifically, there is a well known effect which attracted a lot of attention: the so-called *regularization by noise* phenomenon (see [18] for a survey). This phenomenon can be summarized as an improvement, due to the presence of noise, of the well-posedness of differential equations and has been studied in the context of SDEs [31, 29, 24, 27, 5], transport equation [19, 17, 4], SPDEs [9] and scalar conservation laws [20]. We remark that obtaining a regularization by noise in the context of nonlinear random PDE is a challenging task and most of the results are obtained in a linear setting. For instance, an open problem is to obtain a regularization by noise for the Euler or Navier-Stokes equations.

As mentioned previously, we are not the first one to investigate the Cauchy problem of Equation (1.2). It was first studied in [13] where the global Cauchy problem was solved for $H = 1/2$ and $\sigma < 2/d$ (which corresponds to a classical L^2-subcritical nonlinearity). In [14], the authors proved that, in the L^2-critical case, when $d = 1$, $H = 1/2$ and $\sigma = 5$, the solutions are globally well-posed, which is not the case for the deterministic nonlinear Schrödinger equation and, thus, hints for a regularization by noise effect. In [8], the authors study the case for $d = 1$, $\sigma = 2$, H small enough and $d = 2$, $\sigma = 1$, $H \in (0,1)$. By a simple scaling argument on the space and time variables of (1.2) and thanks to the scaling invariance of the Wiener process, it was conjectured in [3] that, in fact, the critical nonlinearity should be $\sigma = 4/d$ for $H = 1/2$, a L^2-supercritical nonlinearity, which is twice as large as the deterministic L^2-critical nonlinearity. Furthermore, this fact was supported by numerical simulations in 1D and leads to believe that the white noise dispersion has a strong stabilizing property.

In this paper, we prove the global Cauchy problem (1.2) for $d \in \mathbb{N}$, $\sigma \leq 2/d$ and $H \in (0,1)$. To be more specific, we obtain the following result.

Theorem 1.2. *Let $\sigma \leq \frac{2}{d}$, $\psi_0 \in L^2(\mathbb{R}^d)$ and $a \in (2,\infty)$ such that $2/a = d\,(1/2 - 1/(2\sigma+2))$. Then, $\mathbb{P}$-a.s., there exists a unique solution $\psi \in L^a([0,+\infty[; L^{2\sigma+2}(\mathbb{R}^d))$ to (1.2).*

Remark 1.1. *Thus, the modulation by a random noise of the dispersion operator leads to a regularizing effect since we are able to prove the global existence and uniqueness of solutions in the critical case $\sigma = 2/d$. The other interesting fact of our result is that, no matter how close to 1 the Hurst parameter is, we still reach the critical case. This problem was left open in [8] where H needs to be large enough.*

The classical approach to investigate the Cauchy problem for nonlinear Schrödinger equations is to derive local Strichartz estimates [6]. These estimates are a direct consequence of the dispersive property of the linear operator $i\Delta$. However, as pointed out in [14], it is much harder to obtain such estimates in the case of a white noise dispersion because of the presence of the Wiener process. We recall from [26, 13] that the propagator associated to the linear part of (1.2) is explicitly given by, $\forall t,s \in (0,\infty)$ and $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$P_{s,t}\varphi(x) = \frac{1}{(4\pi(W_t^H - W_s^H))^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4(W_t^H - W_s^H)}} \varphi(y)dy. \tag{1.3}$$

Following the classical proof of Strichartz estimates (see for instance [23]), a fundamental tool is the Hardy-Littlewood-Sobolev inequality. This is where Theorem 1.1 comes at hand since the classical potential $|t-s|^{-\alpha}$ is replaced by $|W_t - W_s|^{-\alpha}$. As a direct consequence, we obtain the following stochastic Strichartz estimates.

Definition 1.1. *For any $(q,p) \in (2,\infty)^2$, we say that (q,p) is an admissible pair if*

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{p}\right).$$

Proposition 1.1. *Let $(W_t^H)_{t\geq 0}$ be a fractional brownian motion of Hurst index $H \in (0,1)$, $\beta \in (0, 1-H)$ and (q,p) an admissible pair. Then, $\mathbb{P}$-a.s., there exist $T > 0$ and $C_{1,1.1}, C_{2,1.1} > 0$ such that, $\forall f \in L^2(\mathbb{R}^d)$ and $\forall g \in L^{r'}([0,T]; L^{l'}(\mathbb{R}^d))$, the following inequalities holds*

$$\|P_{0,\cdot}f\|_{L^q([0,T];L^p(\mathbb{R}^d))} \leq C_{1,1.1} T^{\alpha\beta} \|f\|_{L^2}, \tag{1.4}$$

$$\left\| \int_0^T P_{s,\cdot}g(s)ds \right\|_{L^q([0,T];L^p(\mathbb{R}^d))} \leq C_{2,1.1} T^{\alpha\beta} \|g\|_{L^{r'}([0,T];L^{l'}(\mathbb{R}^d))}, \tag{1.5}$$

for any (r,l) admissible pair.

These Strichartz estimates are more powerful due to the presence of the term $T^{\alpha\beta}$. Indeed, in the fixed-point argument, this term will be necessary to obtain the contraction of the mapping in the critical case $\sigma = 2/d$. This will be the main argument to prove Theorem 1.2. The rest of the paper is devoted to the proof of Theorem 1.1 in section 2 and the proofs of Proposition 1.1 and Theorem 1.2 in section 3.

2 Proof of Theorem 1.1

Before proceeding any further, let us remark that we can, without loss of generality, assume that $f \in L^p([0,T])$ and $g \in L^q([0,T])$ are non-negative functions and, furthermore, by a scaling argument, we can assume that $\|f\|_{L^p([0,T])} = \|g\|_{L^q([0,T])} = 1$. Our strategy follows the proof of Lieb-Loss [25, Section 4.3]. It is based on the following layer cake representation

$$f(t) = \int_0^{+\infty} \mathbf{1}_{f(t)>a} da, \quad g(s) = \int_0^{+\infty} \mathbf{1}_{g(s)>b} db$$
$$\text{and} \quad |W_t^H - W_s^H|^{-\alpha} = \alpha \int_0^{+\infty} c^{-1-\alpha} \mathbf{1}_{|W_t^H - W_s^H|<c} dc.$$

Then, by Fubini's theorem, the left-hand-side of (1.1) can be rewritten as

$$\int_0^T \int_0^T f(t)|W_t^H - W_s^H|^{-\alpha} g(s) ds dt =$$
$$\alpha \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} c^{-1-\alpha} \left(\int_0^T \int_0^T \mathbf{1}_{f(t)>a} \mathbf{1}_{g(s)>b} \mathbf{1}_{|W_t^H - W_s^H|<c} ds dt \right) dc db da. \quad (2.1)$$

By denoting

$$\check{f}(a) := \int_0^T \mathbf{1}_{f(t)>a} dt, \quad \check{g}(b) := \int_0^T \mathbf{1}_{g(s)>a} ds \quad \text{and} \quad \check{W}(c, T) := \sup_{t \in [0,T]} \int_0^T \mathbf{1}_{|W_t^H - W_s^H|<c} ds,$$

we have the following result whose proof is postponed.

Lemma 2.1. *Let $H \in (0, 1)$ and $\beta \in (0, 1 - H)$. There exists $T > 0$ and $C_{2.1} > 0$ such that, $\mathbb{P}$-a.s., $\forall c \in \mathbb{R}^{+*}$,*

$$\check{W}(c, T) \le C_{2.1} T^\beta c.$$

We now set $p, q \in (2, +\infty)$ such that

$$\frac{1}{p} + \frac{1}{q} + \alpha = 2.$$

We see that we can bound each characteristic function by 1 in (2.1) and, thus, we deduce that

$$\int_0^T \int_0^T f(t)|W_t^H - W_s^H|^{-\alpha} g(s) ds dt \le \alpha \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} c^{-\alpha-1} I(a, b, c) da db dc,$$

with

$$I(a, b, c) := \begin{cases} \check{f}(a)\check{g}(b), & \text{if} \quad \mathfrak{m}(a, b, c) = \check{h}(c), \\ \check{f}(a)\check{W}(c, T), & \text{if} \quad \mathfrak{m}(a, b, c) = \check{g}(b), \\ \check{W}(c, T)\check{g}(b), & \text{if} \quad \mathfrak{m}(a, b, c) = \check{f}(a), \end{cases}$$

where

$$\mathfrak{m}(a, b, c) = \lim_{\iota \to 0} \max\{\check{f}(a), \check{g}(b), \check{h}(c) + \iota\} \quad \text{and} \quad \check{h}(c) := C_{2.1} T^\beta c.$$

Remark 2.1. *In the previous definition of $\mathfrak{m}$, we choose to have, in the case where* $\max\{\check{f}(a), \check{g}(b), \check{h}(c)\} = \check{f}(a) = \check{h}(c)$, $\mathfrak{m}(a, b, c) = \check{h}(c)$ *(and similarly for $\check{g}$).*

From here, we essentially follow the arguments from [25]. We first assume that $\check{f}(a) \ge \check{g}(b)$. We deduce that

$$\int_0^{+\infty} c^{-\alpha-1} I(a, b, c) dc \le$$
$$\check{g}(b) \int_0^{+\infty} c^{-\alpha-1} \check{h}(c) \mathbf{1}_{\check{h}(c) \le \check{f}(a)} dc + \check{f}(a)\check{g}(b) \int_0^{+\infty} c^{-\alpha-1} \mathbf{1}_{\check{f}(a) \le \check{h}(c)} dc.$$

We denote $\kappa_T = C_{2.1}T^{\beta}$. Since $\check{h}(c) \le \check{f}(a)$ is equivalent to

$$c \le \check{f}(a)/\kappa_T,$$

the first integral on the right-hand-side is estimated as

$$\int_0^{+\infty} c^{-\alpha-1}\check{h}(c)\mathbf{1}_{\check{h}(c)\le\check{f}(a)}dc \le \kappa_T \int_0^{\check{f}(a)/\kappa_T} c^{-\alpha}dc$$
$$\le (1-\alpha)^{-1}\kappa_T\left(\check{f}(a)/\kappa_T\right)^{1-\alpha} = (1-\alpha)^{-1}\kappa_T^{\alpha}\check{f}(a)^{1-\alpha}.$$

The second integral is bounded as

$$\int_0^{+\infty} c^{-\alpha-1}\mathbf{1}_{\check{f}(a)\le\check{h}(c)}dc \le \int_{\check{f}(a)/\kappa_T}^{+\infty} c^{-\alpha-1}dc \le \alpha^{-1}\kappa_T^{\alpha}\check{f}(a)^{-\alpha}.$$

Hence, since by assumption $\check{g}(b)^{-\alpha} \le \check{f}(a)^{-\alpha}$, it follows that

$$\int_0^{+\infty} c^{-\alpha-1}I(a,b,c)dc \lesssim_{\alpha} \kappa_T^{\alpha}\check{g}(b)\check{f}(a)^{1-\alpha} = \kappa_T^{\alpha}\min\{\check{g}(b)\check{f}(a)^{1-\alpha}, \check{f}(a)\check{g}(b)^{1-\alpha}\}.$$

By assuming that $\check{f}(a) \le \check{g}(b)$ and following the same arguments, we obtain

$$\int_0^{+\infty} c^{-\alpha-1}I(a,b,c)dc \lesssim_{\alpha} \kappa_T^{\alpha}\min\{\check{g}(b)\check{f}(a)^{1-\alpha}, \check{f}(a)\check{g}(b)^{1-\alpha}\}.$$

We then proceed by integrating with respect to b and a. We have

$$\int_0^{+\infty}\int_0^{+\infty} \min\{\check{g}(b)\check{f}(a)^{1-\alpha}, \check{f}(a)\check{g}(b)^{1-\alpha}\}dbda =$$
$$\int_0^{+\infty}\int_0^{a^{p/q}} \check{f}(a)\check{g}(b)^{1-\alpha}dbda + \int_0^{+\infty}\int_{a^{p/q}}^{+\infty} \check{g}(b)\check{f}(a)^{1-\alpha}dbda.$$

Thanks to Hölder' inequality, we have, for $r = (q-1)(1-\alpha)$,

$$\int_0^{a^{p/q}} \check{g}(b)^{1-\alpha}db = \int_0^{a^{p/q}} \check{g}(b)^{1-\alpha}b^{-r}b^{r}db$$
$$\le \left(\int_0^{a^{p/q}} \check{g}(b)b^{q-1}db\right)^{1-\alpha}\left(\int_0^{a^{p/q}} b^{-r/\alpha}db\right)^{\alpha/\beta_2}.$$

The norms of f and g are such that

$$\|f\|^p_{L^p([0,T])} = p\int_0^{+\infty} a^{-p-1}\check{f}(a)da = 1 \quad\text{and}\quad \|g\|^q_{L^q([0,T])} = q\int_0^{+\infty} b^{-q-1}\check{g}(b)db = 1.$$

Thus, since

$$\frac{p}{q}\left(1-\frac{r}{\alpha}\right)\alpha = \frac{p}{q}\left(\alpha-(q-1)(1-\alpha)\right)$$
$$= \frac{p}{q}\left(1-q(1-\alpha)\right) = p\left(\frac{1}{q}-1+\alpha\right) = p-1,$$

we obtain

$$\int_0^{+\infty}\check{f}(a)\int_0^{a^{p/q}} \check{g}(b)^{1-\alpha}dbda \lesssim \|g\|^{q(1-\alpha)}_{L^q([0,T])}\int_0^{+\infty}\check{f}(a)a^{p-1}da = \|g\|^{q(1-\alpha)}_{L^q([0,T])}\|f\|^p_{L^p([0,T])} \le 1.$$

By similar arguments, we deduce that

$$\int_0^{+\infty}\int_{a^{p/q}}^{+\infty}\check{g}(b)\check{f}(a)^{1-\alpha}dbda \lesssim 1,$$

which concludes the proof of Theorem 1.1.

It remains to prove Lemma 2.1. We have that, $\forall c>0$ and $\forall t\in[0,T]$,

$$\int_0^T \mathbf{1}_{|W_t^H-W_s^H|<c}ds = \int_{\mathbb{R}} \mathbf{1}_{|y|<c}\ell_{[0,T]}^{W_t^H-y}dy = \int_{-c}^{c}\ell_{[0,T]}^{W_t^H-y}dy,$$

where ℓ is the local time of W^H given as

$$\ell_{[s,t]}^z := \lim_{\varepsilon\to 0}\int_{[s,t]} P_\varepsilon\delta_z(W_u^H)du,$$

where $(P_t)_{t\geq 0}$ is the heat semigroup. We need the following result from [30].

Theorem 2.1. *Let $(W_t^H)_{t\geq 0}$ be a fractional brownian motion with Hurst parameter $H\in(0,1)$. Then, for any interval $I\subset\mathbb{R}$, there exists a positive finite constant K such that, $\mathbb{P}$-a.s.,*

$$\limsup_{r\to 0}\sup_{t\in I}\frac{\sup_{x\in\mathbb{R}}\ell_{[t-r,t+r]}^x}{r^{1-H}\log(1/r)^H}\leq K.$$

We deduce from the previous result that for any $\beta\in(0,1-H)$ and $T>0$ small enough, there exists a constant $C_{2.1}>0$ such that, $\mathbb{P}$-a.s., we have

$$\sup_{x\in\mathbb{R}}\ell_{[0,T]}^x\leq C_{2.1}T^\beta.$$

It follows that, for T small enough,

$$\int_0^T \mathbf{1}_{|W_t^H-W_s^H|<c}ds\leq 2C_{2.1}cT^\beta,$$

which is exactly the desired result.

3 Proof of Proposition 1.1 and Theorem 1.2

3.1 Stochastic Strichartz estimates

Since $(P_{s,t})_{0\leq s\leq t}$ is an isometry from L^2 to itself, we deduce by the Hausdorff-Young inequality and an interpolation argument, that, $\forall p\in[2,\infty]$,

$$\|P_{s,t}\varphi\|_{L^p(\mathbb{R}^d)}\lesssim\frac{1}{|W_t-W_s|^{d(1/2-1/p)}}\|\varphi\|_{L^{p'}(\mathbb{R}^d)}, \tag{3.1}$$

where p' is the Hölder conjugate of p. We denote $(P_{s,t}^*)_{0\leq s\leq t}$ the adjoint of $(P_{s,t})_{0\leq s\leq t}$, that is

$$P_{s,t}^*\varphi(x) := \mathcal{F}^{-1}\left(e^{-\frac{i}{2}|\xi|^2(W_t^H-W_s^H)}\hat{\varphi}(\xi)\right) = P_{t,s}.$$

This leads, in particular, to the fact that

$$P_{s,t}^* = P_{t,s},\quad P_{0,s}^*P_{0,t} = P_{s,t}\quad\text{and}\quad P_{s,t}P_{r,t}^* = P_{s,r},\ \forall r\in[s,t].$$

The proof of Proposition 1.1 is based on the TT^* argument. We set $\alpha = d(1/2 - 1/p)$ and consider the integral, $\forall f, g \in \mathcal{C}([0,T], \mathcal{C}_0^\infty(\mathbb{R}^d))$,

$$I(f,g) := \left| \int_0^T \int_0^T \langle P_{0,t} f(s), P_{0,s} g(t) \rangle_{L^2} ds dt \right| = \left| \int_0^T \int_0^T \langle P_{0,s}^* f(s), P_{0,t}^* g(t) \rangle_{L^2} ds dt \right|$$
$$= \left| \int_0^T \int_0^T \langle P_{s,t} f(s), g(t) \rangle_{L^2} ds dt \right|$$

By Hölder's inequality, (3.1) and Theorem 1.1, we deduce that

$$\begin{aligned} I(f,g) &\leq \int_0^T \int_0^T \|P_{s,t} f(s)\|_{L^p(\mathbb{R}^d)} \|g(t)\|_{L^{p'}(\mathbb{R}^d)} ds dt \\ &\lesssim \int_0^T \int_0^T |W_t - W_s|^{-\alpha} \|f(t)\|_{L^{p'}(\mathbb{R}^d)} \|g(s)\|_{L^{p'}(\mathbb{R}^d)} ds dt \\ &\lesssim T^{\alpha\beta} \|f\|_{L^{q'}([0,T], L^{p'}(\mathbb{R}^d))} \|g\|_{L^{q'}([0,T], L^{p'}(\mathbb{R}^d))}, \end{aligned}$$

since

$$2 - d\left(\frac{1}{2} - \frac{1}{p}\right) = \frac{1}{q'} + \frac{1}{q'} = 2 - \frac{2}{q}.$$

This yields, on one hand, that

$$\left\| \int_0^T P_{0,s}^* f(s) ds \right\|_{L^2(\mathbb{R}^d)}^2 = I(f,f) \lesssim T^{\alpha\beta} \|f\|^2_{L^{q'}([0,T], L^{p'}(\mathbb{R}^d))}, \tag{3.2}$$

and, on another hand, by a duality argument,

$$\left\| \int_0^T P_{s,\cdot} f(s) ds \right\|_{L^q([0,T], L^p(\mathbb{R}^d))} \lesssim T^{\alpha\beta} \|f\|_{L^{q'}([0,T], L^{p'}(\mathbb{R}^d))} \tag{3.3}$$

We are now in position to prove (1.4) and (1.5). It follows from (3.2) that, $\forall f \in L^2(\mathbb{R}^d)$ and $\forall g \in L^{q'}([0,T]; L^{p'}(\mathbb{R}^d))$,

$$\begin{aligned} \int_0^T \langle P_{0,t} f, g(t) \rangle_{L^2} dt = \left\langle f, \int_0^T P_{0,t}^* g(t) \right\rangle_{L^2} &\leq \|f\|_{L^2(\mathbb{R}^d)} \left\| \int_0^T P_{0,t}^* g(t) ds \right\|^2_{L^2(\mathbb{R}^d)} \\ &\lesssim T^{\alpha\beta} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^{q'}([0,T], L^{p'}(\mathbb{R}^d))}, \end{aligned}$$

we obtain (1.4) by a duality argument. We now turn to (1.5). We have, by (3.2),

$$\begin{aligned} \left\| \int_0^T P_{s,\cdot} f(s) ds \right\|_{L^q([0,T]; L^p(\mathbb{R}^d))} &\leq \int_0^T \|P_{s,\cdot} f(s)\|_{L^q([0,T]; L^p(\mathbb{R}^d))} ds \\ &\lesssim T^{\alpha\beta} \int_0^T \|f(s)\|_{L^2(\mathbb{R}^d))} ds = T^{\alpha\beta} \|f\|_{L^1([0,T]; L^2(\mathbb{R}^d))}. \end{aligned}$$

Thanks to this estimate, by an interpolation argument with (3.3), we deduce (1.5).

3.2 Well-posedness of equation (1.2)

We can now apply the previous result to solve the local Cauchy problem of (1.2) The strategy is based on a fixed-point argument of the mapping Γ from $L^q([0,T]; L^p(\mathbb{R}^d))$ to itself given by

$$\Gamma(\psi)(t,x) = P_{0,t}\psi_0(x) - i\lambda \int_0^T P_{s,t} |\psi|^{2\sigma} \psi(s,x) ds. \tag{3.4}$$

We denote a closed ball of $L^q([0,T];L^p(\mathbb{R}^d))$

$$B_{R,L^q([0,T];L^p(\mathbb{R}^d))} := \left\{\psi \in L^q([0,T];L^p(\mathbb{R}^d)); \|\psi\|_{L^q([0,T];L^p(\mathbb{R}^d))} \le R\right\}.$$

Fix $R > 0$ that will be set later. For any $\psi \in B_{R,L^q([0,T];L^p(\mathbb{R}^d))}$, we apply the $L^q([0,T];L^p(\mathbb{R}^d))$ norm to (3.4) and deduce, thanks to (1.4) and (1.5),

$$\|\Gamma(\psi)\|_{L^q([0,T];L^p(\mathbb{R}^d))} \le C_1\|\psi_0\|_{L^2(\mathbb{R}^d)} + C_2|\lambda|T^{\alpha\beta}\,\|\psi\|^{2\sigma+1}_{L^{r'(2\sigma+1)}([0,T];L^{l'(2\sigma+1)}(\mathbb{R}^d))}.$$

for any (r,l) admissible. By choosing $(q,p) = (r,l) = (a, 2\sigma+2)$, we have

$$l' = \frac{l}{l-1} = \frac{2\sigma+2}{2\sigma+1}.$$

Hence, we obtain, by Hölder's inequality,

$$\|\psi\|^{2\sigma+1}_{L^{r'(2\sigma+1)}([0,T];L^{l'(2\sigma+1)}(\mathbb{R}^d))} = \|\psi\|^{2\sigma+1}_{L^{r'(2\sigma+1)}([0,T];L^{2\sigma+2}(\mathbb{R}^d))} \le T^{1-\frac{2}{d\sigma}}\,\|\psi\|^{2\sigma+1}_{L^a([0,T];L^{2\sigma+2}(\mathbb{R}^d))},$$

which gives us

$$\|\Gamma(\psi)\|_{L^q([0,T];L^p(\mathbb{R}^d))} \le C_1\|\psi_0\|_{L^2(\mathbb{R}^d)} + C_2|\lambda|T^{1+\alpha\beta-\frac{2}{d\sigma}}\,\|\psi\|^{2\sigma+1}_{L^a([0,T];L^{2\sigma+2}(\mathbb{R}^d))}. \tag{3.5}$$

By similar computations, we obtain that, $\forall\psi_1,\psi_2 \in B_{R,L^q([0,T];L^p(\mathbb{R}^d))}$,

$$\|\Gamma(\psi_1)-\Gamma(\psi_2)\|_{L^q([0,T];L^p(\mathbb{R}^d))} \le C_2|\lambda|T^{1+\alpha\beta-\frac{2}{d\sigma}}R^{2\sigma}\,\|\psi_1-\psi_2\|_{L^a([0,T];L^{2\sigma+2}(\mathbb{R}^d))}. \tag{3.6}$$

We remark that, since $2/d\sigma \le 1$ and $\alpha\beta > 0$, we have

$$1+\alpha\beta-\frac{2}{d\sigma} > 0.$$

Hence, by setting

$$R = 2C_1\|\psi_0\|_{L^2(\mathbb{R}^d)},$$

and taking $T > 0$ such that

$$C_2|\lambda|T^{1+\alpha\beta-\frac{2}{d\sigma}}R^{2\sigma} < 1,$$

we can see that Γ is a contraction from $B_{R,L^a([0,T];L^{2\sigma+2}(\mathbb{R}^d))}$ to itself. It follows from a Banach fixed point theorem that there exists a unique solution to (1.2). The proof of Theorem 1.2 then follows by iterating this argument on time intervals of length T since we have $\|\psi(T)\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}$.

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