Stochastic regularization effects of semi-martingales on random functions

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Abstract

In this paper we address an open question formulated in [16]. That is, we extend the Itô-Tanaka trick, which links the time-average of a deterministic function f depending on a stochastic process X and F the solution of the Fokker-Planck equation associated to X, to random mappings f. To this end we provide new results on a class of adapted and non-adapted Fokker-Planck SPDEs and BSPDEs.

Key words: Stochastic regularization; (Backward) Stochastic Partial Differential Equations; Malliavin calculus.

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1 Introduction

In [16], the authors analyzed the effects of a multiplicative stochastic perturbation on the well-posedness of a linear transport equation. One of the key tool in their analysis is the so-called *Itô-Tanaka trick* which links the time-average of a function f depending on a stochastic process and F the solution of the Fokker-Planck equation associated to the stochastic process. More precisely, the formula reads as

$$\int_0^T f(t, X_t^x)dt = -F(0,x) - \int_0^T \nabla F(t, X_t^x) \cdot dW_t, \ \mathbb{P} - a.s. \tag{1.1}$$

where $(X_t^x)_{t\geq 0}$ is a solution of the stochastic differential equation

$$X_t^x = x + \int_0^t b(s, X_s^x)ds + W_t, \tag{1.2}$$

and F is the solution of the backward Fokker-Planck equation

$$F(t,x) = \int_t^T \left(\frac{1}{2}\Delta + b(s,x)\cdot\nabla\right) F(s,x)ds - \int_t^T f(s,x)ds. \tag{1.3}$$

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In [25], by means of suitable regularity results for solutions of parabolic equations in $L^q(L^p)$ spaces, the authors showed, assuming $f, b \in E := L^q([0,T]; L^p(\mathbb{R}^d))$ with $2/q + d/p < 1$, that $F \in L^q([0,T]; W^{2,p}(\mathbb{R}^d))$. Hence, in the weak sense, F has 2 additional degrees of regularity compared to f in E. Thus, formula (1.1) tells us that the time-average of f with respect to the stochastic process $(X_t^x)_{t\geq 0}$ is more regular than f itself (it has 1 additional degree of regularity). This is what we call *a stochastic regularization effect* or *regularization by noise*. In this paper, we investigate the following open question stated in [16]:

"The generalization to nonlinear transport equations, where b depends on u itself, would be a major next step for applications to fluid dynamics but it turns out to be a difficult problem. Specifically there are already some difficulties in dealing with a vector field b which depends itself on the random perturbation W. There is no obvious extension of the Itô-Tanaka trick to integrals of the form $\int_0^T f(\omega, s, X_s^x(\omega))ds$ *with random f."*

A major "pathology" in the framework of stochastic regularization is the existence of random functions f for which the Itô-Tanaka trick should not improve the regularity of f. For instance, in [16], the authors consider a random function $\tilde{f}$ of the form

$$\tilde{f}(\omega, s, x) := f(x - W_s(\omega)),$$

where $(W_t)_{t\geq 0}$ is the Brownian motion from (1.2). This gives, for $b = 0$ in (1.2),

$$\int_0^T \tilde{f}(\omega, t, W_t + x)dt = \int_0^T f(t, x)dt,$$

which does not bring any additional regularity. It turns out that, when f is a random function, the solution F to (1.3) is not adapted anymore to $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$ the natural filtration of the Brownian motion, making the stochastic integral on the right-hand side of (1.1) ill-posed.

In this paper we tackle this difficulty by considering another equation which is the *adapted* version of the Fokker-Planck equation (1.3). More precisely, we show in Theorem 3.2 that given random functions b and f which depend in an adapted way, of a standard Brownian motion $(W_t)_{t\geq 0}$, the following formula holds

$$\int_0^T f(t, X_t^x)dt = -F(0,x) - \int_0^T \left(\nabla F(s, X_s^x) + Z(s, X_s^x)\right) dW_s - \int_0^T \mathrm{div} Z(s, X_s^x)ds, \ \mathbb{P}-a.s. \tag{1.4}$$

where (F, Z) is the *adapted* mild solution of the following backward stochastic partial differential equation (BSPDE)

$$F(t,x) = \int_t^T \left(\frac{1}{2}\Delta + b(s,x)\cdot\nabla\right) F(s,x)ds - \int_t^T f(s,x)ds - \int_t^T Z(s,x)dW_s, \tag{1.5}$$

and $(X_t^x)_{t\geq 0}$ together with a Brownian motion $(W_t)_{t\geq 0}$ is a weak solution of the stochastic differential equation

$$X_t^x = x + \int_0^t b(s, X_s^x)ds + W_t.$$

We name (1.4) the *Itô-Wentzell-Tanaka* trick as the derivation of (1.4) calls for the use of the Itô-Wentzell formula in place of the classical Itô formula which allows one to give a semimartingale type decomposition of $F(t, X_t^x)$ when $F(t,x)$ is itself a semimartingale random field. This contrasts with the classical Itô-Tanaka trick where both f and b must be deterministic mappings.

During the process of studying of the Fokker-Planck BSPDE, we incidentally prove new results as Theorem 3.1 on this equation in particular by allowing only $L^q(L^p)$ regularity on its coefficients together with a representation of its mild solution in terms of the solution to the non-adapted SPDE and of its Malliavin derivative. In addition, our methodology generalizes: the well-known *linearization technique* used for linear BSDEs and deterministic semigroups (see [12, Proposition 2.2]), and a Feynman-Kac formula for BSPDEs related to Forward-Backward SDEs as in [26, Corollary 6.2] by providing a unique (in the mild sense) solution to the BSPDE, which were, up to our knowledge, both unknown for this class of equations. We also prove that the F component of the solution is Malliavin differentiable. The study of the BSPDE relies on the one of the non-adapted Fokker-Planck equation in Section 4.2.

There are well-known results concerning the regularization effects of stochastic processes on *deterministic* functions (see the survey of Flandoli [14]) but, to our knowledge, there exists no extension of the Itô-Tanaka trick for random functions. Note however that pathwise regularization effect are obtained in [5] using a rough path analysis based on the class of so-called ρ-irregular functions. The phenomenon is widely used in the recovery of the strong uniqueness of solutions of stochastic differential equations (SDEs) with singular drifts [10, 19, 27, 25, 32, 35]. It has been generalized to SDEs in infinite dimension [8, 9, 28] and the conditions for the existence of a stochastic flow has also drawn attention (see [1, 15, 33]). Another direction of interest is the improvement of the well-posedness of stochastic partial differential equations (SPDEs). In particular, the stochastically perturbed linear transport equation has received a lot of interest [2, 4, 13, 16]. More recent works provide extensions to nonlinear SPDEs, see for instance [3, 17, 18] for models from fluid mechanics and [6, 7, 11] for dispersive equations. Let us also mention that the type of processes that yield a regularization effect is not restricted to Brownian semi-martingales. For instance, in [31, 34] where α-stable processes have been considered and, in [5], where the authors showed a regularization phenomenon using rough paths (in particular for the fractional Brownian motion).

The paper is organized as follows. In Section 2 we make precise the definitions, the notations and the material that will be used later on. Then, in Section 3 we state our main results: namely the existence, uniqueness (in the mild sense) and regularity of an adapted solution to Equation (1.5) in Theorems 3.1 on the one hand and the Itô-Wentzell-Tanaka trick in Theorem 3.2 on the other hand. We also include in this section a discussion about the regularization effect and some possible applications of the Itô-Wentzell-Tanaka trick. Finally, proofs of Theorems 3.1-3.2 are collected in Sections 4 and 5.

2 Notations and standing assumptions

2.1 Main notations

Throughout this paper T will be a fixed positive real number and d denotes a fixed positive integer. For any x in $\mathbb{R}^d$, we denote by $|x|$ the Euclidian norm of x. Let $(E, \|\cdot\|_E)$ be a Banach space, we set $\mathcal{B}(E)$ the Borelian σ-field on E. For given Banach spaces E, F and any $p \geq 0$, we set $L^p(E;F)$ the set of $\mathcal{B}(E)\backslash\mathcal{B}(F)$-measurable mappings

$f : E \to F$ such that

$$\|f\|^p_{L^p(E;F)} := \int \|f(x)\|^p_F \mu(dx) < +\infty,$$

where μ is a non-negative measure on $(E, \mathcal{B}(E))$. Naturally the norm depends on the choice of μ that will be made explicit in the context. If $F = \mathbb{R}^n$, $n \in \mathbb{N}$, then we simply set $L^p(E) := L^p(E; \mathbb{R}^n)$. We also denote by $\mathcal{C}^0(E)$ (resp. $\mathcal{C}^0_b(E)$) the set of continuous (resp. bounded continuous) real-valued mappings f on E. For any $p > 1$ we set $\bar{p}$ the Hölder conjugate of p.

For any mapping $\varphi : \mathbb{R}^d \to \mathbb{R}$ we denote by $\frac{\partial \varphi}{\partial x_i}$ the i-th partial derivative of φ, $(i = 1, \cdots, n)$, by $\nabla \varphi := (\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_d})$ the gradient of φ (when it is well-defined), and by $\Delta \varphi$ its Laplacian. For a multi index $k := (k_1, \cdots, k_d)$ in $\mathbb{N}^d$, we set $\nabla^k \varphi := \frac{\partial^{k_1 + \cdots + k_d} \varphi}{\partial x_1 \ldots \partial^{k_d}} \varphi$ and $|k| := \sum_{i=1}^d k_i$.

For $p, m \in \mathbb{R}$, we set

$$W^{m,p}(\mathbb{R}^d) = \left\{ \varphi \in L^p(\mathbb{R}^d); \mathfrak{F}^{-1}\left(([1 + |\xi|^2]^{m/2} \hat{\varphi} \right) \in L^p(\mathbb{R}^d)) \right\},$$

the usual Sobolev spaces equipped with its natural norm

$$\|\varphi\|_{W^{m,p}(\mathbb{R}^d)} := \left\| \mathfrak{F}^{-1}\left(([1 + |\xi|^2]^{m/2} \hat{\varphi} \right) \right\|_{L^p(\mathbb{R}^d))},$$

where $\hat{\varphi}(\xi) = \mathfrak{F}(\varphi)(\xi)$ and $\mathfrak{F}$ (resp. $\mathfrak{F}^{-1}$) denotes the Fourier transform (resp. the inverse Fourier transform). Let $n, k \in \mathbb{N}$ and $\alpha \in (0, 1)$. We set $\mathcal{C}^k_b(E)$ (resp. $\mathcal{C}^{k,\alpha}_b(E)$) the set of $\mathbb{R}^d$-valued bounded functions having bounded derivatives up to order k (resp. and with α-Hölder continuous kth partial derivatives). We also set:

$$\|\varphi\|_{\mathcal{C}^{k,\alpha}_b(E)} := \|\varphi\|_{\mathcal{C}^k_b(E)} + \sup_{|\ell| = k} \sup_{x \neq y} \frac{|\nabla^\ell f(x) - \nabla^\ell f(y)|}{|x - y|^\alpha},$$

where $\|\varphi\|_{\mathcal{C}^k_b(E)} := \sum_{|\ell| \leq k} \sup_{x \in E} |\nabla^\ell f(x)|$. Finally $\mathcal{C}^\infty_0(\mathbb{R}^n)$, $(n \in \mathbb{N}^*)$ stands for the set of infinitely continuously differentiable function with compact support.

Throughout this paper C will denote a non-negative constant which may differ from line to line.

Standing Assumption 2.1. *Unless stated otherwise, we always assume that the real numbers $p, q \in (2, \infty)$ verify*

$$\frac{d}{p} + \frac{2}{q} < 1.$$

Remark 2.1. *In the previous inequality, we exclude the cases $p = \infty$ or $q = \infty$. The latter can be handled without any difficulty since T is finite. However, the former calls for an analysis in other functional spaces that we do not consider in this paper.*

2.2 Malliavin-Sobolev spaces

In this section, we introduce the main notations about the Malliavin calculus for random fields. As we do not work, in the classical setting (that is we do not use a L^2 or Hilbert space structure), we provide some technical justifications in A.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $W := (W_t)_{t\in[0,T]}$ a Brownian motion on this space (to the price of heavier notations all the definitions and properties in this section and of the next one extend to a d-dimensional Brownian motion). We assume that $\mathcal{F} = \sigma(W_t,\ t \in [0,T])$.

Consider $\mathcal{S}$ be the set of cylindrical fields, that is the set of random fields $F : \Omega\times\mathbb{R}^d \to \mathbb{R}$ of the form:

$$F = \varphi(W_{t_1}, \cdots, W_{t_n}, x)$$

with $\varphi : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ in $\mathcal{C}_0^\infty(\mathbb{R}^{n+d})$. We fix p an integer with $p \geq 2$. For an element F in $\mathcal{S}$, we set DF the $L^p([0,T])$-valued random field as:

$$D_\theta F := \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(W_{t_1}, \cdots, W_{t_n}, x)\mathbf{1}_{[0,t_i]}(\theta), \quad \theta \in [0,T].$$

Note that for F in $\mathcal{S}$, $D\nabla^k F = \nabla^k DF$ for any multi index k. In addition, an integration by parts formula for the operators $D\nabla^k$ can be derived (see Lemma A.1). By Lemma A.2 the operators $D\nabla^k$ (and so $\nabla^k D$) are closable from $\mathcal{S}$ to $L^p(\Omega \times \mathbb{R}^d; L^p([0,T]))$. For a positive integer m, we set $\mathbb{D}^{1,m,p}$ the closure of $\mathcal{S}$ with respect to the norm:

$$\|F\|^p_{\mathbb{D}^{1,m,p}} := \mathbb{E}[\|F\|^p_{W^{m,p}}] + \int_0^T \mathbb{E}\left[\|D_\theta F\|^p_{W^{m,p}(\mathbb{R}^d)}\right] d\theta, \tag{2.1}$$

We also denote:

$$\|F\|^p_{\mathbb{W}^{m,p}} := \mathbb{E}[\|F\|^p_{W^{m,p}}], \tag{2.2}$$

We conclude this section on the Malliavin derivative by introducing the space $\mathbb{D}^{1,m,p}_q := L^q([0,T]; \mathbb{D}^{1,m,p})$ (with $p, q \geq 2$) which consists of mappings $F : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\|F\|^q_{\mathbb{D}^{1,m,p}_q} := \int_0^T \|F(t,\cdot)\|^q_{\mathbb{D}^{1,m,p}} dt < +\infty. \tag{2.3}$$

We also use the following notation:

$$\|F\|^q_{\mathbb{W}^{m,p}_q} := \int_0^T \|F(t,\cdot)\|^q_{\mathbb{W}^{m,p}} dt < +\infty.$$

2.3 An SDE with random drift

Before proceeding further, we introduce the following notation: for a continuous mapping $\psi \in \mathcal{C}([0,T])$ and $s \in [0,T]$, we set $\psi_{(s)}$ the element of $\mathcal{C}([0,T])$ defined by

$$\psi_{(s)} \mapsto \begin{cases} \psi_r \text{ if } r \in [0,s] \\ \psi_s \text{ if } r \in (s,T] \end{cases}.$$

We consider the following SDE:

$$X_t = X_0 + \int_0^t b(s, X_s, W_{(s)})ds + W_t, \quad t \in [0,T], \tag{2.4}$$

where b is assumed to be a $\mathcal{B}([0,T] \times \mathbb{R}^d \times \mathcal{C}([0,T]))$-measurable map, X_0 is in $\mathbb{R}^d$ and W is a d-dimensional Brownian motion. To begin with, let us recall the definition of a weak solution to Equation (2.4).

Definition 2.1. *A weak solution is a triple (X,W), $(\Omega, \mathcal{G}, \mathbb{P})$, $(\mathcal{G}_t)_{t\in[0,T]}$ where*

- $(\Omega, \mathcal{G}, \mathbb{P})$ *is a probability space equipped with some filtration* $(\mathcal{G}_t)_{t\in[0,T]}$ *that satisfies the usual conditions,*
- X *is a continuous,* $(\mathcal{G}_t)_{t\in[0,T]}$*-adapted* $\mathbb{R}^d$*-valued process,* W *is a d-dimensional* $(\mathcal{G}_t)_{t\in[0,T]}$*-Wiener process on the probability space,*
- $\mathbb{P}(X(0) = X_0) = 1$ *and* $\mathbb{P}(\int_0^t |b(s, X_s, W_{(s)})|ds < +\infty) = 1$, $\forall t \in [0, T]$,
- *Equation (2.4) holds for all* t *in* $[0, T]$ *with* $\mathbb{P}$*-probability one.*

Standing Assumption 2.2. *There exists a weak solution* (X, W), $(\Omega, \mathcal{G}, \mathbb{P})$, $(\mathcal{G}_t)_{t\in[0,T]}$ *to the SDE (2.4).*

By definition, W is a $(\mathcal{G}_t)_{t\in[0,T]}$-Brownian motion. So we denote by $(\mathcal{F}_t)_{t\in[0,T]}$ its natural completed right-continuous filtration which satisfies $\mathcal{F}_t \subset \mathcal{G}_t$ for any $t \in [0, T]$. In the following, the spaces $\mathbb{D}^{1,m,p}$ or $\mathbb{D}^{1,m,p}_q$ are understood to be defined with respect to $(\Omega, \mathcal{F}_T, \mathbb{P})$.

We now give a simple proof of existence and uniqueness of a weak solution to (2.4) under some non-optimal assumptions.

Proposition 2.1. *Let* $b \in L^q([0, T]; L^p(\mathbb{R}^d; \mathcal{C}^1_b(\mathbb{R}^d)))$. *Then there exists a unique weak solution to the SDE*

$$X_t = X_0 + \int_0^t b(s, X_s, W_s)ds + W_t, \quad t \in [0, T]. \tag{2.5}$$

Proof. The proof is based on Girsanov's theorem. Let us first remark that $b_0(t, x) := \sup_{y\in\mathbb{R}^d} |\nabla_y b(t, x, y)|$ and $b_1(t, x) := \sup_{y\in\mathbb{R}^d} |b(t, x, y)|$ belong in $L^q([0, T]; L^p(\mathbb{R}^d))$. Thus, since $2/q + d/p < 1$, by [25, Lemma 3.2] we have, $\forall \kappa \in \mathbb{R}^+$ and $k = 1, 2$,

$$\mathbb{E}\left[e^{\kappa \int_0^T b_0(s,W_s)^k ds}\right] + \mathbb{E}\left[e^{\kappa \int_0^T b_1(s,W_s)^k ds}\right] < +\infty, \tag{2.6}$$

where W is a standard Brownian motion.
Let $(X_t)_{t\geq 0}$ a standard Brownian motion on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ equipped with a filtration $(\mathcal{G}_t)_{t\in[0,T]}$. We consider the following SDE

$$Y_t = Y_0 - \int_0^t b(s, X_s, Y_s)ds + X_t, \quad t \in [0, T]. \tag{2.7}$$

In this step, we prove that there exists a unique solution to (2.7). Since b is Lipschitz, the uniqueness is obtain by a Gronwall lemma. Moreover, by using classical *a priori* estimates for Lipschitz SDE, we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right] \leq C\left(|Y_0|^2 + T + \mathbb{E}\left[\int_0^T \left(|b(s, X_s, 0)|^2 + b_0(s, W_s)^2\right) ds\right]\right),$$

which yields the existence of a strong solution.
By (2.6), we have, $\forall \kappa \in \mathbb{R}^+$,

$$\mathbb{E}\left[e^{\kappa \int_0^T |b(s,X_s,Y_s)|^2 ds}\right] \leq \mathbb{E}\left[e^{\kappa \int_0^T b_1(s,X_s)^2 ds}\right] < +\infty.$$

We deduce that

$$\rho(\cdot) := e^{\int_0^\cdot b(s,X_s,Y_s)dX_s - \frac{1}{2}\int_0^\cdot |b(s,X_s,Y_s)|^2 ds},$$

is a martingale under $\mathbb{P}$ by Novikov's criterion. Hence, by Girsanov's theorem, the process Y is a Brownian motion under the measure $\mathbb{Q}$ given by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \rho(T)$. Thus, by rewriting Y as W, the triple $(X, W), (\Omega, \mathcal{F}, \mathbb{Q}), (\mathcal{G}_t)_{t\in[0,T]}$ is a weak solution to the SDE (2.5). □

2.4 The adapted Fokker-Planck equation

Throughout this section, we consider (X, W) a weak solution to the SDE (2.4) and use the notations of the previous section. We say that a random field $\varphi : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is adapted if for any x in $\mathbb{R}^d$, $\varphi(\cdot, x)$ is $\mathcal{F}$-adapted. Note that by the definition of a weak solution (*cf.* Definition 2.1) as $\mathcal{F}_\cdot \subset \mathcal{G}_\cdot$, any $(\mathcal{F}_t)_{t\in[0,T]}$-adapted field is $(\mathcal{G}_t)_{t\in[0,T]}$-adapted. In order to derive our Itô-Wentzell-Tanaka trick, we consider f a random field and make the following standing assumption.

Standing Assumption 2.3. *f is an adapted stochastic field.*

We set the linear operator $\mathcal{L}_t^X$ on $\mathcal{C}_0^\infty(\mathbb{R}^d)$:

$$\mathcal{L}_t^X \vartheta(x) := \frac{1}{2}\Delta\vartheta(x) + b(t, x, W_{(t)}) \cdot \nabla\vartheta(x),\ \forall \vartheta \in \mathcal{C}_0^\infty(\mathbb{R}^d).$$

Now we consider the following BSPDE:

$$F(t,x) = \int_t^T \left(\mathcal{L}_r^X F(r,x) - f(r,x)\right) dr - \int_t^T Z(r,x) dW_r, \tag{2.8}$$

Before going further, we recall what is a solution to the BSPDE (2.8) in our context. We set for $m \in \mathbb{N}$:

$$\begin{aligned}
&\mathbb{W}_{\mathcal{P},q}^{m,p} := \{\varphi \text{ adapted field },\ \|\varphi\|_{\mathbb{W}_q^{m,p}} < +\infty\},\\
&\mathbb{M}^p := \left\{\varphi \text{ adapted field (and } \mathbb{R}^d\text{-valued)},\ \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\int_0^T |\varphi(s,x)|^2 dt\right)^{\frac{p}{2}}\right] dx < +\infty\right\}.
\end{aligned} \tag{2.9}$$

Definition 2.2 (Adapted strong solution to a BSPDE)**.** *We say that a pair of adapted random fields (F, Z) is strong solution to the BSPDE (2.8) if*

$$(F, Z) \in \mathbb{W}_{\mathcal{P},q}^{2,p} \times \mathbb{M}^p$$

with $\frac{d}{p} + \frac{2}{q} < 1$ and Relation (2.8) is satisfied for every t in $[0,T]$, for a.e. x in $\mathbb{R}^d$, $\mathbb{P}$-a.s..

Remark 2.2. *We warn the reader that in the previous definition, the adapted feature of the fields F, Z is crucial. In that sense we will speak of BSPDE. This differs from the SPDE (4.6) whose solution is not adapted (see Remark 4.1). In that case we will speak of a SPDE to emphasize that the measurability requirement is not present.*

3 Main results and discussion

3.1 Main results

In order to proceed further, we need some additional assumptions on the Malliavin derivatives of f and b.

Standing Assumption 3.1. *Let $\gamma \in \{0,1\}$, $\alpha > \gamma/2$ and $\ell, \bar{\ell} \in [p, \infty]$ such that $1/\ell + 1/\bar{\ell} = 1/p$. We assume that f and b belong to $\mathbb{D}_q^{1,0,p}$ and that:*

i) *there exist a function* $f' \in L^q([0,T];L^\ell(\Omega;W^{-\gamma,p}(\mathbb{R}^d)))$*, a function* $b' \in L^q([0,T];L^p(\Omega\times\mathbb{R}^d))$ *and two mappings* $\upsilon_f \in L^{\bar\ell}(\Omega;L^\infty([0,T]\times\mathbb{R}^d))$*,* $\upsilon_b \in L^\infty([0,T]\times\mathbb{R}^d)$ *such that*

$$D_\theta f(t,x) = f'(t,x)\upsilon_f(\theta,t), \quad \forall\theta\le t\le T,\ \mathbb{P}-a.s.,$$
$$\text{and} \quad D_\theta b(t,x,W_{(t)}) = b'(t,x,W_{(t)})\upsilon_b(\theta,t), \quad \forall\theta\le t\le T,\ \mathbb{P}-a.s.,$$

ii.a) *one of the following statement is in force*

- *there exist* $C_{1,f}, C_{2,f} > 0$ *such that,* $\forall 0\le\theta\le s\le t\le T$*,*

$$\begin{aligned}\|\upsilon_f(\theta,t)\|_{L^{\bar\ell}(\Omega)} &\le C_{1,f}|\theta-t|^\alpha,\\ \|\upsilon_f(\theta,s)-\upsilon_f(t,s)\|_{L^{\bar\ell}(\Omega)} &\le C_{2,f}|\theta-t|^\alpha.\end{aligned} \tag{3.1}$$

- $\gamma = 0$ *and* $\upsilon_f(\theta,t) = \mathbf{1}_{\{\theta\le\tau_f\}}$ *where* τ_f *is a random variable with values in* $[0,t]$*,*

ii.b) *one of the following statement is in force*

- *there exist* $C_{1,b}, C_{2,b} > 0$ *such that,* $\forall 0\le\theta\le s\le t\le T$*,*

$$\begin{aligned}|\upsilon_b(\theta,t)| &\le C_{1,b}|\theta-t|^\alpha,\\ |\upsilon_b(\theta,s)-\upsilon_b(t,s)| &\le C_{2,b}|\theta-t|^\alpha,\end{aligned} \tag{3.2}$$

- $\gamma = 0$ *and* $\upsilon_b(\theta,t) = \mathbf{1}_{\{\theta\le\tau_b\}}$ *where* τ_b *is a random variable with values in* $[0,t]$*,*

iii) *there exists a function* $\tilde b \in L^q([0,T])$ *such that*

$$\|b(t,\cdot,W_{(t)})\|_{L^p(\mathbb{R}^d)} + \|b'(t,\cdot,W_{(t)})\|_{L^p(\mathbb{R}^d)} \le \tilde b(t),\ \forall t\in[0,T],\ \mathbb{P}-a.s..$$

Moreover, if $\gamma = 1$*, we also assume that*

$$\|\nabla b(t,\cdot,W_{(t)})\|_{L^p(\mathbb{R}^d)} \le \tilde b(t), \quad \forall t\in[0,T],\ \mathbb{P}-a.s..$$

The conditions above are probably quite cumbersome at first glance. However, the counterpart of this formulation is that it allows one to consider quite general functional dependency of the Brownian motion in the coefficients f and b. In particular, they can depend of the past of W in a functional way and not only through the present value. To illustrate this fact, we give below some standard examples of random functionals f and b which satisfy Assumption 3.1.

Example 3.1. *Let* $f(t,x) = g_0(t, x+\int_0^t h_r dr)$*, where* $g_0 \in L^q([0,T];L^p(\mathbb{R}^d))$*,* h *is an adapted bounded stochastic process such that,* $\forall t\in[0,T]$*,* $h_t \in \mathbb{D}^{1,2}$ *and* $D_\theta h_t$ *is bounded uniformly in* (t,θ)*, and* $b(t,x) = g_1(t,x+W_t)$ *with* $g_1 \in L^q([0,T];W^{1,p}(\mathbb{R}^d))$*. In that case, the Malliavin derivatives of* f *and* b *are given by*

$$D_\theta f(t,x) = \nabla g_0\left(t, x+\int_0^t h_r dr\right)\int_\theta^t D_\theta h_r dr,$$
$$\text{and} \ \ D_\theta b(t,x) = \nabla g_1\left(t,x+W_t\right)\mathbf{1}_{\{\theta\le t\}}.$$

Example 3.2. *Let* $f(t,x) = g_1(t,x,\varphi_1(W_{(t)}))$*,* $b(t,x) = g_2(t,x,\varphi_2(W_{(t)}))$ *where* g_1 *and* g_2 *belong to* $L^q([0,T];L^p(\mathbb{R}^d;\mathcal{C}_b^1(\mathbb{R}^d)))$ *and:*

$$\varphi_i((r_s)_{0\le s\le t}) \in \left\{r_t,\ \max_{0\le s\le t} r_s,\ \min_{0\le s\le t} r_s\right\}, i=1,2.$$

In that cases, we have that (see e.g. [29, Exercice 1.2.11]):

$$D_\theta W_t = \mathbf{1}_{\{\theta \le t\}},\ D_\theta \max_{0\le s\le t} W_s = \mathbf{1}_{\{\theta \le \bar{\tau}\}},\ \textit{and}\ D_\theta \min_{0\le s\le t} W_s = \mathbf{1}_{\{\theta \le \underline{\tau}\}},$$

where $\bar{\tau} := \mathrm{argmax}_{0\le s\le t} W_s$ *and* $\underline{\tau} := \mathrm{argmin}_{0\le s\le t} W_s$.

We state below our first main theorem in which we provide existence, uniqueness (in the mild sense) and regularity results to Equation (2.8).

Theorem 3.1. *There exists a strong (adapted) solution to Equation* (2.8) *(recall notations* (2.9)*)*

$$(F, Z) \in \left(\mathbb{W}^{2,p}_{\mathcal{P},q}\right)^2.$$

Futhermore, we have the following representation of F

$$F(t,x) = \mathbb{E}\left[-\int_t^T P^X_{t,r} f(r,x)dr \Big| \mathcal{F}_t\right], \tag{3.3}$$

where P^X *denotes the propagator associated to* $\mathcal{L}^X$ *defined in* (4.16) *(see Section 4.2). In addition, for a.e.* (t,x)*,* $F(t,x)$ *is Malliavin differentiable* ($\|F\|_{\mathbb{D}^{1,2,p}_q} < +\infty$)*, and for a.e.* $x \in \mathbb{R}^d$*, a version of the process* $(Z(t,x))_{t\in[0,T]}$ *is given by*

$$Z(t,x) = \mathbb{E}\left[-\int_t^T D_t P^X_{t,r} f(r,x)dr \Big| \mathcal{F}_t\right]. \tag{3.4}$$

Finally, (F, Z) *is the unique mild solution of Equation* (2.8)*, that is*

$$F(t,x) = -\int_t^T P^X_{t,r} f(r,x)dr - \int_t^T P^X_{t,r} Z(r,x)dW_r. \tag{3.5}$$

Our second main result is the derivation of the Itô-Wentzell-Tanaka trick as follows:

Theorem 3.2. *Let* (F, Z) *be the unique strong solution to* (2.8)*. Then we have,*

$$\begin{aligned}\int_0^T f(s, X_s)ds =& -F(0, X_0) - \int_0^T \left(\nabla F(s, X_s) + Z(s, X_s)\right) dW_s \\ & - \int_0^T \mathrm{div} Z(s, X_s)ds,\ \mathbb{P} - a.s.\end{aligned} \tag{3.6}$$

We postpone the proof of Theorems 3.1-3.2 to Sections 4-5.

Remark 3.1. *If* f *and* b *are deterministic, then, the BSPDE* (2.8) *reduces to a PDE that is* $Z \equiv 0$*. Hence,* $\nabla Z \equiv 0$ *and we recover the formula of [25]. In particular, the regularity than one could obtain when* f *or* b *is random compared to the deterministic realm is completely contained in the regularity of* Z *and of its gradient.*

Remark 3.2. *Note that under Assumption 3.1, one can treat in a similar way the case where the original SDE* (2.4) *is replaced by:*

$$dX_t = b(t, X_t, W_{(t)})dt + \sigma(t, X_t)dW_t,$$

where $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *is Borel measurable, uniformly continuous in* $x \in \mathbb{R}^d$ *and such that,* $\forall \xi \in \mathbb{R}^d, \forall (t,x) \in [0,T] \times \mathbb{R}^d$,

$$\lambda^{-1}|\xi|^2 \le \sum_{1\le i,j\le d} \sigma_{i,j}(t,x)\xi_i\xi_j \le \lambda|\xi|^2,$$

for some $\lambda > 0$*. The parabolic estimates from Proposition 4.2 and Lemma 4.3 below can be obtained by following the same lines as in [21] with the additional assumption that* b *is uniformly bounded. Finally, we do not address here the question about existence of weak solutions in case of non-constant diffusion* σ.

3.2 Discussion on the results

3.2.1 Stochastic regularization effect in the case of stochastic perturbations

As stated in the introduction, the main application of the Itô-Wentzell-Tanaka trick is to yield a stochastic regularization effect. In the case of the Itô-Tanaka trick, the regularization is a direct consequence of the regularity of the solution of the Fokker-Planck equation since the terms in right-hand-side of (1.1) have an additional (weak) degree of regularity. Concerning the Itô-Wentzell-Tanaka trick, we know that there are examples of random functions f where there should not be any regularization effect even if f belongs to $\mathbb{W}_q^{0,p}$. In fact, one of the main pathology stems from the addition of what we call *stochastic perturbations* to X which can either be smooth (when there is a regularization effect) or singular (when there is no regularization effect). This problem is also investigated in [5] where the author identify a set of smooth perturbations in $\mathcal{C}([0,T])$ thanks to Girsanov's theorem.

Let $\tilde{f} \in L^q([0,T]; L^p(\mathbb{R}^d))$ and Y an adapted stochastic process defined on $[0,T]$. For simplicity, we assume that b is a deterministic function which belongs in $L^q([0,T]; W^{1,p}(\mathbb{R}^d))$. We set

$$f(t,x) := \tilde{f}(t, x + Y_t).$$

We notice that the Malliavin derivative of f implies, *a priori*, a loss of regularity compared to the case where f is deterministic since we have

$$D_\theta f(s,x) = \nabla \tilde{f}(x + Y_s) \cdot D_\theta Y_s. \tag{3.7}$$

However, even if $\nabla \tilde{f} \in L^q([0,T]; W^{-1,p}(\mathbb{R}^d))$, we can still use the Itô-Wentzell-Tanaka trick when $D_\theta Y_s$ verifies (3.1) of Assumption 3.1 with $\alpha > 1/2$. In this case, Y is a smooth perturbation. This is due to the fact that one can exchange the "time regularity" of $D_\theta Y_s$ to obtain "spatial regularity" through the heat semigroup. This is the underlying mechanism used in the proof of Proposition 4.3. Furthermore, we remark that if Y is given, as in Example 3.1, by

$$Y_t = \int_0^t h_r dr,$$

where h is an adapted bounded stochastic process, we could have used Girsanov's theorem to remove the shift $\int_0^t h_r dr$ from f under a new equivalent probability measure $\mathbb{Q}$, allowing one to apply the classical Itô-Tanaka trick with respect to $\mathbb{Q}$. On the contrary, if one choose for instance $Y_t = W_t$ or $\max_{s\in[0,t]} W_s$, Y becomes a singular perturbation since $D_\theta Y_s = \mathbf{1}_{\{\theta \le \tau_b\}}$ where τ_b is a random variable with values in $[0,s]$ (which obviously fails to verify (3.1) for $\alpha > 1/2$). In this case, we must have $\tilde{f} \in L^q([0,T]; W^{1,p}(\mathbb{R}^d))$ which implies that the regularity of the terms on the right-hand-side of (1.4) is the same as the one of $\tilde{f}$ (hence, there is, *a priori*, no regularization effect).

3.2.2 Strong uniqueness for SDEs with irregular stochastic drift

The aim of this section, is to provide a methodology to prove pathwise uniqueness to a particular case of SDEs of the form (2.5) that we recall below:

$$X_t = X_0 + \int_0^t b(s, X_s, W_s) ds + W_t, \quad t \in [0,T], \tag{3.8}$$

with $b \in L^q([0,T]; W^{1,p}(\mathbb{R}^d; \mathcal{C}_b^1(\mathbb{R}^d)))$. However, to keep the length of this paper within limits, we only sketch the different steps one has to go through to achieve this goal.

Thanks to Proposition 2.1, the existence of a weak solution (X, W) solution is guaranteed. Furthermore, we remark that b verifies Assumptions 2.3 and 3.1. Indeed, for Assumption 3.1, we have that (with the same notations as in the proof of Proposition 2.1)

$$D_\theta b(t,x,W_t) = \nabla_y b(t,x,W_t)\mathbf{1}_{\{\theta<t\}},$$

and, thus,

$$\begin{aligned}\|b(t,\cdot,W_t)\|^p_{L^p(\mathbb{R}^d)} + (T-t)\|\nabla_y b(t,\cdot,W_t)\|^p_{L^p(\mathbb{R}^d)} \\ \leq \max(T-t,1)\|b(t,\cdot,\cdot)\|_{L^p(\mathbb{R}^d;\mathcal{C}_b^1(\mathbb{R}^d))}.\end{aligned}$$

In fact, this computation also holds for $\nabla_x b$.

We now consider the following BSPDE (with $f = -b$ and where the equation is understood componentwise):

$$F_\lambda(t,x) = \int_t^T \left(\mathcal{L}_r^X F_\lambda(r,x) - \lambda F_\lambda(r,x) + b(r,x,W_r)\right) dr - \int_t^T Z_\lambda(r,x) dW_r. \quad (3.9)$$

It follows from Theorem 3.1 and a gauge change, that its mild solution $(F_\lambda, Z_\lambda) \in \left(\mathbb{W}^{2,p}_{\mathcal{P},q}\right)^2$ is given by

$$F_\lambda(t,x) = \mathbb{E}\left[-\int_t^T e^{-\lambda(r-t)} P^X_{t,r} b(r,x,W_r) dr \middle| \mathcal{F}_t\right],$$

and

$$Z_\lambda(t,x) = \mathbb{E}\left[-\int_t^T e^{-\lambda(r-t)} D_t P^X_{t,r} b(r,x,W_r) dr \middle| \mathcal{F}_t\right].$$

Denoting $\tilde{F}_\lambda(t,x) = -\int_t^T e^{-\lambda(r-t)} P^X_{t,r} b(r,x,W_r) dr$ and following the same lines as in the proof of Proposition 4.2, the fact that ∇b enjoys the requirement *iii)* in Assumption 3.1 for $\gamma = 1$, enables us to get that $\tilde{F}_\lambda \in \mathbb{D}_q^{1,3,p}$ by differentiation of Equation (4.10) (where f is replaced by $b + \lambda\tilde{F}_\lambda$) with respect to the space variable. By similar arguments as in the first step of the proof of Theorem 3.1, we deduce that $(F_\lambda, Z_\lambda) \in \left(\mathbb{W}^{3,p}_{\mathcal{P},q}\right)^2$. Finally, we obtain the next result which follows the lines of [17, Lemma 4].

Lemma 3.1. *There exists a deterministic mapping $\varphi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that $\varphi(\lambda) \underset{\lambda\to\infty}{\longrightarrow} 0$ and*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\nabla \tilde{F}_\lambda(t,x)| \leq \varphi(\lambda), \quad \mathbb{P}-a.s.$$

which implies that there exists $\lambda^* > 0$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\nabla F_{\lambda^*}(t,x)| \leq 1/4, \quad \mathbb{P}-a.s.$$

We are now in position to state and prove the following

Proposition 3.1. *Under the assumptions of this section, pathwise uniqueness holds for SDE* (3.8).

Proof. Mimicking what is done in [14, 16, 25, 31] for deterministic drifts, we rewrite Equation (3.8), thanks to the Itô-Wentzel-Tanaka trick, to get:

$$
\begin{aligned}
X_t = X_0 + F_\lambda(0, X_0) - F_\lambda(t, X_t) + \int_0^t \left(\nabla F_\lambda(s, X_s) + Z_\lambda(s, X_s) + Id\right) dW_s \\
+ \int_0^t \left(\sum_{i=1}^d \sum_{j=1}^d \partial_{x_i} (Z_\lambda)_j(s, X_s) - \lambda F_\lambda(s, X_s) \right) ds. \qquad (3.10)
\end{aligned}
$$

Let X_t^1, X_t^2 be two weak solutions defined on the same probability space and with the same Brownian motion W. For any $G \in \mathbb{W}^{2,p}_{\mathcal{P},q}$, a Sobolev embedding implies that

$$
\begin{aligned}
\mathbb{E}\left[|G(t, X_t^1) - G(t, X_t^2)|^2\right] &\le \mathbb{E}\left[\sup_{x \in \mathbb{R}^d} |\nabla G(t, x)|^2\right] \mathbb{E}\left[|X_t^1 - X_t^2|^2\right] \\
&\le \|G(t, \cdot)\|^2_{\mathbb{W}^{2,p}} \mathbb{E}\left[|X_t^1 - X_t^2|^2\right].
\end{aligned}
$$

For simplicity, we set $\delta X_t := X_t^1 - X_t^2$. Then choosing $\lambda = \lambda^*$, it follows from Equation (3.10) and Lemma 3.1 that

$$
\begin{aligned}
\mathbb{E}\left[|\delta X_t|^2\right] \le \frac{1}{2} \mathbb{E}\left[|\delta X_t|^2\right] + 4 \int_0^t \left(\|F_{\lambda^*}(s, \cdot)\|^2_{\mathbb{W}^{3,p}} + \|Z_{\lambda^*}(s, \cdot)\|^2_{\mathbb{W}^{2,p}}\right) \mathbb{E}\left[|\delta X_s|^2\right] ds \\
+ 4T(1 + \lambda^*) \int_0^t \left(\|Z_{\lambda^*}(s, \cdot)\|^2_{\mathbb{W}^{3,p}} + \|F_{\lambda^*}(s, \cdot)\|^2_{\mathbb{W}^{2,p}}\right) \mathbb{E}\left[|\delta X_s|^2\right] ds.
\end{aligned}
$$

As a consequence we have :

$$
\mathbb{E}\left[|\delta X_t|^2\right] \le 16(1 + T + \lambda^*) \int_0^t \mathbb{E}\left[|\delta X_s|^2\right] dA_s,
$$

where $A_t := \int_0^t \|F_{\lambda^*}(s, \cdot)\|^2_{\mathbb{W}^{3,p}} + \|Z_{\lambda^*}(s, \cdot)\|^2_{\mathbb{W}^{3,p}} ds$. We conclude by Gronwall Lemma that:

$$
\mathbb{E}\left[|\delta X_t|^2\right] = \mathbb{E}\left[|X_t^1 - X_t^2|^2\right] = 0, \quad \forall t \in [0, T].
$$

□

4 Proof of Theorem 3.1

4.1 Some estimates

We will need below several technical results that we present now. In the following, we denote by $(P_{t,s})_{s \ge t \ge 0}$ the heat semigroup. Adapting results from [22, 23] in the spaces $\mathbb{D}_q^{1,m,p}$ we have

Lemma 4.1. *Let $1 < q, p < +\infty$ and $\gamma \in \mathbb{R}$. Then, there exists a constant C such that, $\forall \phi \in \mathbb{D}_q^{1,\gamma,p}$,*

$$
\left\| \int_t^T P_{t,s} \phi(s, x) ds \right\|_{\mathbb{D}_q^{1,2+\gamma,p}} \le C \|\phi\|_{\mathbb{D}_q^{1,\gamma,p}}, \qquad (4.1)
$$

and, $\forall \varepsilon > 0$, there exists another constant $C_{\varepsilon,T} > 0$ such that, $\forall \varphi \in \mathbb{D}^{1,2+\gamma-2/q+\varepsilon,p}$,

$$
\|P_{t,T} \varphi\|_{\mathbb{D}_q^{1,2+\gamma,p}} \le C_{\varepsilon,T} \|\varphi\|_{\mathbb{D}^{1,2+\gamma-2/q+\varepsilon,p}}. \qquad (4.2)
$$

The next result gives a Schauder estimate on the solution of a backward heat equation with a source term in $\mathbb{D}_q^{1,0,p}$. Its proof is similar to the one from [22, Theorem 7.2] and the arguments can be directly extended to the norms $\mathbb{D}_q^{1,m,p}$.

Proposition 4.1. *Let $1 < q, p < +\infty$, $2/q < \beta \leq 2$ and $\phi \in \mathbb{D}_q^{1,0,p}$. Denote, for $(t,x) \in [0,T] \times \mathbb{R}^d$,*

$$u(t,x) := -\int_t^T P_{t,s}\phi(s,x)ds.$$

Then, there exists a constant $C > 0$ independent of T such that, for any $0 \leq s \leq t \leq T$,

$$\|u(t) - u(s)\|_{\mathbb{D}^{1,2-\beta,p}} \leq C(t-s)^{\beta/2-1/q}\|\phi\|_{\mathbb{D}_q^{1,0,p}}, \tag{4.3}$$

and, thus,

$$\|u\|_{\mathcal{C}_b^{0,\beta/2-1/q}([0,T];\mathbb{D}^{1,2-\beta,p})} \leq C\|\phi\|_{\mathbb{D}_q^{1,0,p}}. \tag{4.4}$$

A direct consequence of the previous result is the following

Corollary 4.1. *Let $\phi \in \mathbb{D}_q^{1,0,p}$. Denote, for $(t,x) \in [0,T] \times \mathbb{R}^d$,*

$$u(t,x) := -\int_t^T P_{t,s}\phi(s,x)ds.$$

Then, for any $\varepsilon \in (0,1)$ satisfying

$$\varepsilon + \frac{d}{p} + \frac{2}{q} < 1,$$

there exists a constant $C > 0$ and $\tilde{\varepsilon} > 0$ such that, $\forall t \in [0,T]$,

$$\left(\mathbb{E}\left[\|u(t,\cdot)\|^p_{C_b^{1,\varepsilon}(\mathbb{R}^d)}\right] + \mathbb{E}\left[\int_0^T \|D_\theta u(t,\cdot)\|^p_{C_b^{1,\varepsilon}(\mathbb{R}^d)} d\theta\right]\right)^{1/p} \leq C(T-t)^{\tilde{\varepsilon}/2}\|\phi\|_{\mathbb{D}_q^{1,0,p}}. \tag{4.5}$$

Proof. Let $\beta = \tilde{\varepsilon} + 2/q$ where $0 < \tilde{\varepsilon} < 1 - (\varepsilon + d/p + 2/q)$. The result follows by the Sobolev embedding $\mathcal{C}_b^{1,\alpha} \subset W^{2-\beta,p}$, with $\alpha = 1 - \beta - d/p = 1 - \tilde{\varepsilon} - q/2 - d/p > \varepsilon$, and Proposition 4.1. □

4.2 The non-adapted Fokker-Planck equation

Given $\varphi \in \mathbb{D}^{1,2-2/q+\varepsilon,p}$, with $\varepsilon > 0$, consider here the non-adapted Fokker-Planck equation

$$F(t,x) = \varphi(x) + \int_t^T \mathcal{L}_r^X F(r,x)dr - \int_t^T f(r,x)dr. \tag{4.6}$$

Definition 4.1. *A strong solution to Equation (4.6) is a function F in $\mathbb{D}_q^{1,2,p}$ such that, for all $t \in [0,T]$, we have*

$$F(t,x) = \varphi(x) + \int_t^T \mathcal{L}_r^X F(r,x)dr - \int_t^T f(r,x)dr. \tag{4.7}$$

Remark 4.1. *Note that each random variable $F(t,\cdot)$ solution to the previous SPDE is $\mathcal{F}_T$-measurable, and hence it is not adapted.*

We provide a Malliavin differentiability analysis for the solution the Fokker-Planck equation (4.6). We define, $\forall m \geq 0$,

$$\mathbb{G}_q^{1,m,p} := \left\{F \in \mathbb{D}_q^{1,m,p}; \partial_t F \in \mathbb{D}_q^{1,0,p}\right\},$$

and the associated norm

$$\|F\|_{\mathbb{G}_q^{1,m,p}} := \|F\|_{\mathbb{D}_q^{1,m,p}} + \|\partial_t F\|_{\mathbb{D}_q^{1,0,p}}.$$

We begin with a result concerning the existence and uniqueness of a solution to the non-adapted Fokker-Planck equation.

Lemma 4.2. *Let $u \in \mathbb{G}_q^{1,2,p}$ and denote*

$$\|u(t,\cdot)\|_{\mathbb{H}^{1,p}}^p := \mathbb{E}\left[\|u(t,x)\|_{C_b^1(\mathbb{R}^d)}^p\right] + \mathbb{E}\left[\int_0^T \|D_\theta u(t,x)\|_{C_b^1(\mathbb{R}^d)}^p d\theta\right].$$

Then

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{\mathbb{H}^{1,p}} \leq C_T \|u\|_{\mathbb{G}_q^{1,2,p}}, \tag{4.8}$$

where C_T is uniformly bounded with respect to T in compact sets of $\mathbb{R}^+$, and, $\forall t \in [0,T]$,

$$\|b(t,\cdot,W_{(t)}) \cdot \nabla u(t,\cdot)\|_{\mathbb{D}^{1,0,p}} \leq C\tilde{b}(t)\|u(t,\cdot)\|_{\mathbb{H}^{1,p}}. \tag{4.9}$$

Proof. Firstly, let us remark that we have, $\forall u \in \mathbb{G}_q^{1,2,p}$,

$$u(t,x) = -\int_t^T P_{t,r}\left[\partial_t u(r,x) + \frac{1}{2}\Delta u(r,x)\right] dr,$$

and then, by using Corollary 4.1, we obtain the estimate

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{\mathbb{H}^{1,p}} \leq C_T \|u\|_{\mathbb{G}_q^{1,2,p}}.$$

Secondly, we compute

$$\begin{aligned}\|b(t,\cdot,W_{(t)}) \cdot \nabla u(t,\cdot)\|_{\mathbb{D}^{1,0,p}}^p \leq\ & \mathbb{E}\left[\|b(t,\cdot,W_{(t)}) \cdot \nabla u(t,\cdot)\|_{L^p(\mathbb{R}^d)}^p\right] \\ &+ C\mathbb{E}\left[\int_0^T \|D_\theta b(t,\cdot,W_{(t)}) \cdot \nabla u(t,\cdot)\|_{L^p(\mathbb{R}^d)}^p d\theta\right] \\ &+ C\mathbb{E}\left[\int_0^T \|b(t,\cdot,W_{(t)}) \cdot D_\theta \nabla u(t,\cdot)\|_{L^p(\mathbb{R}^d)}^p d\theta\right].\end{aligned}$$

Since the Malliavin derivative commutes with the spatial derivative in L^p, we obtain

$$\begin{aligned}\|b(t,\cdot,W_{(t)}) \cdot \nabla u(t,\cdot)\|_{\mathbb{D}^{1,0,p}}^p \leq\ & \mathbb{E}\left[\|b(t,\cdot,W_{(t)})\|_{L^p(\mathbb{R}^d)}^p \sup_{x\in\mathbb{R}^d} |\nabla u(t,x)|^p\right] \\ &+ C\mathbb{E}\left[\int_0^T \|D_\theta b(t,\cdot,W_{(t)})\|_{L^p(\mathbb{R}^d)}^p d\theta \sup_{x\in\mathbb{R}^d} |\nabla u(t,x)|^p\right] \\ &+ C\mathbb{E}\left[\|b(t,\cdot,W_{(t)})\|_{L^p(\mathbb{R}^d)}^p \int_0^T \sup_{x\in\mathbb{R}^d} |\nabla D_\theta u(t,x)|^p d\theta\right].\end{aligned}$$

Thus, by Assumption 3.1, we have (4.9) as

$$\|b(t,\cdot,W_{(t)})\cdot\nabla u(t,\cdot)\|_{\mathbb{D}^{1,0,p}}\leq$$
$$C\tilde{b}(t)\left(\mathbb{E}\left[\sup_{x\in\mathbb{R}^d}|\nabla u(t,x)|^p\right]+\mathbb{E}\left[\int_0^T\sup_{x\in\mathbb{R}^d}|\nabla D_\theta u(t,x)|^p d\theta\right]\right)^{1/p}.$$

□

Proposition 4.2. *Let $\varphi\in\mathbb{D}^{1,2-2/q+\varepsilon,p}$, with $\varepsilon>0$. Then there exists a unique solution F in $\mathbb{G}_q^{1,2,p}$ to the equation*

$$F(t,x)=P_{t,T}\varphi(x)-\int_t^T P_{t,s}f(s,x)ds+\int_t^T P_{t,s}\left[b(s,x,W_{(s)})\cdot\nabla F(s,x)\right]ds. \quad (4.10)$$

Moreover, the following estimate on the solution holds

$$\|F\|_{\mathbb{G}_q^{1,2,p}}\leq C_T\left(\|\varphi\|_{\mathbb{D}^{1,2-2/q+\varepsilon,p}}+\|f\|_{\mathbb{D}_q^{1,0,p}}\right), \quad (4.11)$$

where $C_T>0$ depends on $\|\tilde{b}\|_{L^q([0,T])}$ and is uniformly bounded with respect to T on compact sets of $\mathbb{R}^+$.

Proof. **Step 1:** By using Corollary 4.1 and Relation (4.9), we have

$$\begin{aligned}\|F(t,\cdot)\|^q_{\mathbb{H}^{1,p}}\leq&C\|P_{t,T}\varphi\|^q_{\mathbb{H}^{1,p}}+C_T\|f\|^q_{\mathbb{D}_q^{1,0,p}}+C_T\|b\cdot\nabla F\|^q_{\mathbb{D}_q^{1,0,p}}\\ \leq&C\|\varphi\|^q_{\mathbb{H}^{1,p}}+C_T\|f\|^q_{\mathbb{D}_q^{1,0,p}}+C_T\int_t^T|\tilde{b}(s)|^q\|F(s,\cdot)\|^q_{\mathbb{H}^{1,p}}ds.\end{aligned}$$

Thanks to a Gronwall lemma and the Sobolev embedding $\mathcal{C}_b^{1,\varepsilon}\subset W^{2-2/q+\varepsilon,p}$, we deduce

$$\sup_{t\in[0,T]}\|F(t,\cdot)\|_{\mathbb{H}^{1,p}}\leq\left(C_T\|f\|_{\mathbb{D}_q^{1,0,p}}+C\|\varphi\|_{\mathbb{D}^{1,2-2/q+\varepsilon,p}}\right)e^{\frac{C_TT}{q}\|\tilde{b}\|^q_{L^q([0,T])}}. \quad (4.12)$$

We now turn to Estimate (4.11). We can apply the $\mathbb{D}_q^{1,2,p}$-norm to (4.10) and obtain, by using lemma 4.1,

$$\begin{aligned}\|F\|^q_{\mathbb{D}_q^{1,2,p}}\leq&C_T\|\varphi\|^q_{\mathbb{D}^{1,2-2/q+\varepsilon,p}}+C\|f\|^q_{\mathbb{D}_q^{1,0,p}}+C\int_t^T\|b(s,\cdot,W_{(s)})\cdot\nabla F(s,\cdot)\|^q_{\mathbb{D}^{1,0,p}}ds\\ \leq&C_T\|\varphi\|^q_{\mathbb{D}^{1,2-2/q+\varepsilon,p}}+C\|f\|^q_{\mathbb{D}_q^{1,0,p}}+C\int_t^T|\tilde{b}(s)|^q\|F(s,\cdot)\|^q_{\mathbb{H}^{1,p}}ds\end{aligned}$$

which yields, thanks to (4.12),

$$\|F\|^q_{\mathbb{D}_q^{1,2,p}}\leq C_T\left(1+\|\tilde{b}\|^q_{L^q([0,T])}e^{C_TT\|\tilde{b}\|^q_{L^q([0,T])}}\right)\left(\|\varphi\|^q_{\mathbb{D}^{1,2-2/q+\varepsilon,p}}+\|f\|^q_{\mathbb{D}_q^{1,0,p}}\right), \quad (4.13)$$

Then, we differentiate (4.10) with respect to the time variable and deduce the equation

$$\left\{\begin{array}{l}\partial_tF(t,x)=-\mathcal{L}_t^XF(t,x)+f(t,x),\\ F(T,x)=\phi(x).\end{array}\right. \quad (4.14)$$

By applying the $\mathbb{D}_q^{1,0,p}$-norm to (4.14) and by using the estimate (4.12), we obtain

$$\|\partial_tF\|_{\mathbb{D}_q^{1,0,p}}\leq\frac{1}{2}\|\Delta F\|_{\mathbb{D}_q^{1,0,p}}+\|f\|_{\mathbb{D}_q^{1,0,p}}+\|b\cdot\nabla F\|_{\mathbb{D}_q^{1,0,p}}$$

$$\leq C_T \left(\|\varphi\|_{\mathbb{D}^{1,2-2/q+\varepsilon,p}} + \|f\|_{\mathbb{D}_q^{1,0,p}} \right),$$

which, together with (4.13), gives Estimate (4.11).

Step 2: The last argument of the proof consists in using the so-called *continuity method*. For $\mu \in [0,1]$, we consider the equation

$$F_\mu(t,x) = P_{t,T}\varphi(x) - \int_t^T P_{t,s} f(s,x) ds + \int_t^T P_{t,s} \left[\mu b(s,x,W_{(s)}) \cdot \nabla F_\mu(s,x) \right] ds. \quad (4.15)$$

We wish to prove that the set $\nu \subset [0,1]$ of elements μ for which (4.15) admits a unique solution is $[0,1]$ (with $\mu = 1$ corresponding to the equation (4.10)). In the case where $\mu = 0$, the existence and uniqueness of a solution of (4.10) is straightforward and, thus, ν is not empty. Fix $\mu_0 \in \nu$ and denote $\mathcal{R}^{\mu_0}$ the mapping from $\mathbb{D}_q^{1,0,p}$ to $\mathbb{G}_q^{1,2,p}$ which maps f to the solution F_{μ_0} of (4.15) for $\varphi = 0$. Let $\mu \in [0,1]$ to be fix later. The existence and uniqueness of the solution of equation (4.15) relies on a fixed point argument. We consider the mapping Γ_μ given by

$$\Gamma_\mu(F) = P_{\cdot,T}\varphi + \mathcal{R}^{\mu_0} f + (\mu_0 - \mu)\mathcal{R}^{\mu_0}(b \cdot \nabla F),$$

and aim to prove that it is a contraction mapping from $\mathbb{G}_q^{1,2,p}$ to itself. It follows from the estimates (4.11) and (4.8) that, $\forall F_1, F_2 \in \mathbb{G}_q^{1,2,p}$,

$$\begin{aligned}
\|\Gamma_\mu(F_1) - \Gamma_\mu(F_2)\|_{\mathbb{G}_q^{1,2,p}} \leq & C|\mu - \mu_0| \|b \cdot \nabla(F_1 - F_2)\|_{\mathbb{D}_q^{1,0,p}} \\
\leq & C|\mu - \mu_0| \left(\int_0^T |\tilde{b}(s)|^q \|F_1(s,\cdot) - F_2(s,\cdot)\|_{\mathbb{H}^{1,p}}^q ds \right)^{1/q} \\
\leq & C|\mu - \mu_0| \|\tilde{b}\|_{L^q([0,T])} \|F_1 - F_2\|_{\mathbb{G}_q^{1,2,p}}.
\end{aligned}$$

Hence, by choosing μ such that $|\mu - \mu_0| < \frac{1}{C\|\tilde{b}\|_{L^q([0,T])}}$, we can conclude that there exists a unique solution to (4.15). Therefore, by repeating the argument a finite number of times, we prove that $\nu = [0,1]$ and that (4.10) admits a unique solution in $\mathbb{G}_q^{1,2,p}$. □

Using the regularity obtained above, we deduce the equality between the weak and the mild solution as stated below.

Corollary 4.2. *Let $\varphi \in \mathbb{D}^{1,2-2/q+\varepsilon,p}$, with $\varepsilon > 0$. There exists a unique solution F in $\mathbb{D}_q^{1,2,p}$ to the equation* (4.6).

From now on, we denote $(P_{s,t}^X)_{0 \leq s \leq t \leq T}$ the propagator associated to the solution of the Fokker-Planck equation determined by $\mathcal{L}^X$, that is, $P_{s,t}^X \varphi(x)$ is the solution to the SPDE

$$P_{s,t}^X \varphi(x) = \varphi(x) + \int_s^t \mathcal{L}_r^X P_{r,t}^X \varphi(x) dr, \quad 0 \leq s \leq t, \quad (4.16)$$

with φ a $\mathcal{F}_t$-measurable mapping in $\mathbb{D}^{1,2-2/q+\varepsilon,p}$. For $\varphi : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, we note:

$$P_{s,t}^X \varphi(r,x) := \left(P_{s,t}^X \varphi(r,\cdot) \right)(x).$$

We end this section by the following Lemma which gives some estimates on P^X.

Lemma 4.3. *Let $\varphi \in \mathbb{D}^{1,2-2/q+\varepsilon,p}$, with $\varepsilon > 0$. The following estimates hold*

$$\|P_{\cdot,T}^X \varphi\|_{\mathbb{G}_q^{1,2,p}} \leq C_{1,T} \|\varphi\|_{\mathbb{D}^{1,2-2/q+\varepsilon,p}}, \quad (4.17)$$

$$\left\| \int_{\cdot}^{T} P^X_{\cdot,r}\varphi(r,\cdot)dr \right\|_{\mathbb{G}_q^{1,2,p}} \leq C_{2,T}\|\varphi\|_{\mathbb{D}_q^{1,0,p}}, \tag{4.18}$$

and

$$\int_0^T \|\mathcal{L}^X_{\cdot} P^X_{\cdot,r}\varphi(r,\cdot)\|^q_{\mathbb{D}_q^{1,0,p}} dr \leq C\|\varphi\|^q_{\mathbb{D}_q^{1,2-2/q+\varepsilon,p}}. \tag{4.19}$$

Proof. The estimates (4.17) and (4.18) are direct consequences of Proposition 4.2. Concerning the third estimate, thanks to (4.8), (4.9), and (4.17), there exists a constant $C_r > 0$ uniformly bounded in $r \in [0,T]$ such that

$$\|b \cdot \nabla P^X_{\cdot,r}\varphi(r,\cdot)\|_{\mathbb{D}_q^{1,0,p}} \leq C_r\|\varphi(r,\cdot)\|_{\mathbb{D}^{1,2-2/q+\varepsilon,p}}.$$

Therefore, (4.19) follows from (4.17) since

$$\int_0^T \|\mathcal{L}^X_{\cdot} P^X_{\cdot,r}\varphi(r,\cdot)\|^q_{\mathbb{D}_q^{1,0,p}} dr \leq \int_0^T C_r^q\|\varphi(r,\cdot)\|^q_{\mathbb{D}^{1,2-2/q+\varepsilon,p}} dr.$$

□

We end this section with the following Proposition.

Proposition 4.3. *There exists a constant $C > 0$ such that*

$$\left\| \int_{\cdot}^{T} P^X_{\cdot,s} f'(s,\cdot)\upsilon_f(\cdot,s)ds \right\|_{\mathbb{W}_q^{2,p}} \leq C\|f'\|_{L^q([0,T];L^\ell(\Omega;W^{-\gamma,p}(\mathbb{R}^d)))},$$

and

$$\left\| \int_{\cdot}^{T} P^X_{\cdot,s} b'(s,\cdot)\upsilon_b(\cdot,s)ds \right\|_{\mathbb{W}_q^{2,p}} \leq C\|b'\|_{L^q([0,T];L^p(\Omega\times\mathbb{R}^d))}.$$

Proof. We only deal with the first estimate (the second one is derived in a similar fashion with $\gamma = 0$). Moreover, let us remark that the estimate is direct, by Lemma 4.3, if $\upsilon_f(\theta,t) = \mathbf{1}_{\{\theta\leq\tau_f\}}$. We consider equation (4.10) with $f = 0$ and $\varphi(x) = f'(s,x)$. Under Assumption 3.1 and by setting $F(t,x) = P^X_{t,s}f'(s,x)$, we obtain on one hand, thanks to Corollary 4.1 with deterministic norms (that is without the Malliavin derivative and the integration on $\omega \in \Omega$), the estimate

$$\sup_{x\in\mathbb{R}^d} |\nabla P^X_{t,s}f'(s,x)|^q \leq C \sup_{x\in\mathbb{R}^d} |\nabla P_{t,s}f'(s,x)|^q + C\int_t^s \tilde{b}(u)^q \sup_{x\in\mathbb{R}^d} |\nabla P^X_{u,s}f'(s,x)|^q du,$$

which, by a Gronwall lemma, leads to

$$\sup_{x\in\mathbb{R}^d} |\nabla P^X_{t,s}f'(s,x)| \leq C \sup_{x\in\mathbb{R}^d} |\nabla P_{t,s}f'(s,x)| e^{C_T\|\tilde{b}\|^q_{L^q([0,T])}}. \tag{4.20}$$

On another hand, we have, thanks to Fubini's theorem,

$$\begin{aligned}\int_t^T P^X_{t,s}f'(s,x)\upsilon_f(t,s)ds &= \int_t^T P_{t,s}f'(s,x)\upsilon_f(t,s)ds \\ &+ \int_t^T P_{t,u}\left[b(u,x,W_{(u)}) \cdot \nabla \int_u^T P^X_{u,s}f'(s,x)\upsilon_f(u,s)ds\right] du \\ &+ \int_t^T P_{t,u}\left[b(u,x,W_{(u)}) \cdot \nabla \int_u^T P^X_{u,s}f'(s,x)\delta_s\upsilon_f(t,u)ds\right] du,\end{aligned} \tag{4.21}$$

where we denote $\delta_s \upsilon_f(t,u) := \upsilon_f(t,s) - \upsilon_f(u,s)$. By a Sobolev embedding (as in the proof of Corollary 4.1) and a classical inequality on the heat semigroup, we deduce from (4.20) that

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} |\nabla P^X_{t,s} f'(s,x)| &\le C \sup_{x\in\mathbb{R}^d} |\nabla P_{t,s} f'(s,x)| \\ &\le C\|(1-\Delta)^{\gamma/2} P_{t,s}(1-\Delta)^{-\gamma/2} f'(s,\cdot)\|_{W^{2-\beta,p}(\mathbb{R}^d)} \\ &\le \frac{C}{|t-s|^{(2-\beta+\gamma)/2}} \|f'(s,\cdot)\|_{W^{-\gamma,p}(\mathbb{R}^d)}, \end{aligned} \tag{4.22}$$

where β is strictly greater than $2/q$. To conclude our proof, we need to provide adequate bounds on the $\mathbb{W}^{2,p}_q$-norm of each integrals from the right-hand-side of (4.21). Concerning the first integral, we have, by Hölder's inequality and an estimate on the heat semigroup,

$$\left\| \int_t^T P_{t,s} f'(s,x) \upsilon_f(t,s) ds \right\|_{\mathbb{W}^{2,p}} \le C \int_t^T \frac{\|f'(s,\cdot)\|_{L^\ell(\Omega; W^{-\gamma,p}(\mathbb{R}^d))}}{|s-t|^{1+\gamma/2-\alpha}} ds,$$

and, since $1+\gamma/2-\alpha < 1$, by a Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned}\left\| \int_t^T P_{t,s} f'(s,x) \upsilon_f(t,s) ds \right\|_{\mathbb{W}^{2,p}_q} &\le C \left\| \int_0^T \frac{\|f'(s,\cdot)\|_{L^\ell(\Omega; W^{-\gamma,p}(\mathbb{R}^d))}}{|s-\cdot|^{1+\gamma/2-\alpha}} ds \right\|_{L^q([0,T])} \\ &\le C \|f'\|_{L^q([0,T]; L^\ell(\Omega; W^{-\gamma,p}(\mathbb{R}^d)))}. \end{aligned}$$

We now turn to the second integral. If follows from Lemma 4.1, Inequality (4.22) and Assumption 3.1 that

$$\begin{aligned}&\left\| \int_\cdot^T P_{\cdot,u} \left[b(u,x,W_{(u)}) \cdot \nabla \int_u^T P^X_{u,s} f'(s,x) \upsilon_f(u,s) ds \right] du \right\|^q_{\mathbb{W}^{2,p}_q} \\ &\qquad \le C \int_t^T \tilde{b}(u)^q \left\| \int_u^T \sup_{x\in\mathbb{R}^d} \left| \nabla P^X_{u,s} f'(s,x) \upsilon_f(u,s) \right| ds \right\|^q_{L^p(\Omega)} du \\ &\qquad \le C \int_t^T \tilde{b}(u)^q \left(\int_u^T \frac{\|f'(s,\cdot)\|_{\mathbb{W}^{-\gamma,\ell}}}{|u-s|^{(2-\beta+\gamma-2\alpha)/2}} ds \right)^q du. \end{aligned}$$

Since $\varrho_1 := 1-\beta/2+\gamma/2-\alpha < 1/\bar{q}$, we have, by Hölder's inequality,

$$\int_u^T \frac{\|f'(s,\cdot)\|_{\mathbb{W}^{-\gamma,\ell}}}{|u-s|^{(2-\beta+\gamma-2\alpha)/2}} ds \le C \|f'\|_{\mathbb{W}^{-\gamma,p}_q} \int_0^T \frac{ds}{|s|^{\varrho_1 \bar{q}}},$$

which implies a bound on the second integral. Finally, we consider the third integral. We first assume that $\gamma = 0$. We have, since $\varrho_2 := 1-\beta/2 < 1/\bar{q}$,

$$\begin{aligned}&\left\| \int_t^T P_{t,u} \left[b(u,x,W_{(u)}) \cdot \nabla \int_u^T P^X_{u,s} f'(s,x) \delta_s \upsilon_f(t,u) ds \right] du \right\|_{\mathbb{W}^{2,p}} \\ &\qquad \le C \int_t^T \frac{\tilde{b}(u)}{|u-t|^{1-\alpha}} \left\| \int_u^T \sup_{x\in\mathbb{R}^d} \left| \nabla P^X_{u,s} f'(s,x) \right| ds \right\|_{L^p(\Omega)} du \\ &\qquad \le C \int_0^T \frac{\tilde{b}(u)}{|u-t|^{1-\alpha}} \left(\int_u^T \frac{\|f'(s,\cdot)\|_{L^\ell(\Omega; L^p(\mathbb{R}^d))}}{|u-s|^{1-\beta/2}} ds \right) du \end{aligned}$$

$$\leq C\left(\int_0^T \frac{\tilde{b}(u)}{|u-t|^{1-\alpha}}du\right)\|f'\|_{L^q([0,T];L^\ell(\Omega;L^p(\mathbb{R}^d)))}\left(\int_0^T \frac{ds}{|s|^{\varrho_2\bar{q}}}ds\right).$$

Applying the L^q-norm on the previous inequality and using the Hardy-Littlewood-Sobolev inequality, we bound the third integral.
We now suppose that $\gamma = 1$. We have

$$\begin{aligned}\int_t^T P_{t,u}\left[b(u,x,W_{(u)})\cdot\nabla\int_u^T P^X_{u,s}f'(s,x)\delta_s\upsilon_f(t,u)ds\right]du&\\ =\int_t^T \operatorname{div}\left(P_{t,u}\left[b(u,x,W_{(u)})\int_u^T P^X_{u,s}f'(s,x)\delta_s\upsilon_f(t,u)ds\right]\right)du&\\ -\int_t^T P_{t,u}\left[\operatorname{div} b(u,x,W_{(u)})\int_u^T P^X_{u,s}f'(s,x)\delta_s\upsilon_f(t,u)ds\right]du.&\end{aligned}$$

Following the same computations as in the case $\gamma = 0$, we obtain

$$\begin{aligned}\left\|\int_t^T \operatorname{div}\left(P_{t,u}\left[b(u,x,W_{(u)})\int_u^T P^X_{u,s}f'(s,x)\delta_s\upsilon_f(t,u)ds\right]\right)du\right\|_{\mathbb{W}^{2,p}}&\\ \leq C\left(\int_0^T \frac{\tilde{b}(u)}{|u-t|^{1+1/2-\alpha}}du\right)\|f'\|_{L^q([0,T];L^\ell(\Omega;W^{-1,p}(\mathbb{R}^d)))}\left(\int_0^T \frac{ds}{|s|^{\varrho_3\bar{q}}}ds\right),&\end{aligned}$$

and, since the L^p-norm of ∇b can be bounded by $\tilde{b}$,

$$\begin{aligned}\left\|\int_t^T P_{t,u}\left[\operatorname{div} b(u,x,W_{(u)})\int_u^T P^X_{u,s}f'(s,x)\delta_s\upsilon_f(t,u)ds\right]du\right\|_{\mathbb{W}^{2,p}}&\\ \leq C\left(\int_0^T \frac{\tilde{b}(u)}{|u-t|^{1-\alpha}}du\right)\|f'\|_{L^q([0,T];L^\ell(\Omega;W^{-1,p}(\mathbb{R}^d)))}\left(\int_0^T \frac{ds}{|s|^{\varrho_3\bar{q}}}ds\right).&\end{aligned}$$

We conclude our proof by applying the L^q-norm and using the Hardy-Littlewood-Sobolev inequality on the previous inequalities. □

We can also compute the Malliavin derivative of $(P^X_{s,t})_{0\leq s\leq t\leq T}$. This is the goal of the next lemma.

Lemma 4.4. *We have the following commutation formula between the Malliavin derivative and the operator P^X*

$$D_tP^X_{t,T}\varphi(x) = P^X_{t,T}D_t\varphi(x) + \int_t^T P^X_{t,r}\left(D_tb(r,x)\cdot\nabla P^X_{r,T}\varphi(x)\right)dr \tag{4.23}$$

Proof. Let $t \leq r \leq T$. Denote

$$\Phi(r,x) := D_tP^X_{r,T}\varphi(x),$$

then, a direct computation of the Malliavin derivative applied to the representation formula of P^X gives

$$\Phi(r,x) = \Phi(T,x) + \int_r^T \mathcal{L}^X_u\Phi(u,x)du + \int_r^T D_tb(u,x)\cdot\nabla P^X_{u,T}\varphi(x)du.$$

Hence, by the representation formula of P^X, we deduce the following mild formulation of Φ

$$\Phi(r,x) = P^X_{r,T}\Phi(T,x) + \int_r^T P^X_{r,u}\left(D_tb(u,x)\cdot\nabla P^X_{u,T}\varphi(x)\right)du,$$

and, thus, the desired result. □

4.3 Proof of Theorem 3.1

Throughout Step 1 and Step 2, we assume that f and f' are smooth with respect to x. Since the norms of F and Z in $\mathbb{W}_q^{2,p}$ are bounded by the norms of $f \in \mathbb{W}_q^{0,p}$ and $f' \in L^m([0,T]; L^\ell(\Omega; L^p(\mathbb{R}^d)))$ (see Step 1 and Step 2), we can consider two sequences of smooth approximations $(f_n)_{n\in\mathbb{N}}$ and $(f'_n)_{n\in\mathbb{N}}$ such that the limit $(F_n, Z_n) \underset{n\to\infty}{\longrightarrow} (F, Z)$ converges in $\mathbb{W}_q^{2,p}$. Moreover, thanks to the mild formulation (3.5), we obtain that (F, Z) is solution of the Equation (2.8).

Step 1: Set

$$F(t,x) := \mathbb{E}\left[-\int_t^T P^X_{t,r} f(r,x)dr \Big| \mathcal{F}_t\right]. \tag{4.24}$$

We start with proving that F belongs to $\mathbb{W}^{2,p}_{\mathcal{P},q}$. Indeed, by using (4.18) and Jensen's inequality, it holds that

$$\begin{aligned}
\|F(t,\cdot)\|^p_{\mathbb{D}^{1,2,p}} &= \left\|\mathbb{E}\left[-\int_t^T P^X_{t,s} f(s,\cdot)ds \Big| \mathcal{F}_t\right]\right\|^p_{\mathbb{D}^{1,2,p}} \\
&= \mathbb{E}\left[\left\|\mathbb{E}\left[-\int_t^T P^X_{t,s} f(s,\cdot)ds \Big| \mathcal{F}_t\right]\right\|^p_{W^{2,p}}\right] + \int_0^T \mathbb{E}\left[\left\|D_\theta \mathbb{E}\left[-\int_t^T P^X_{t,s} f(s,\cdot)ds \Big| \mathcal{F}_t\right]\right\|^p_{W^{2,p}}\right] d\theta \\
&\leq \mathbb{E}\left[\left\|\int_t^T P^X_{t,s} f(s,\cdot)ds\right\|^p_{W^{2,p}}\right] + \int_0^t \mathbb{E}\left[\left\|\int_t^T D_\theta P^X_{t,s} f(s,\cdot)ds\right\|^p_{W^{2,p}}\right] d\theta \\
&\leq \left\|\int_t^T P^X_{t,s} f(s,\cdot)ds\right\|^p_{\mathbb{D}^{1,2,p}} < +\infty.
\end{aligned} \tag{4.25}$$

We now turn to the derivation of Z. We have

$$\int_t^T \left(\mathcal{L}^X_s F(s,x) - f(x,s)\right) ds = -\int_t^T \mathbb{E}\left[\int_s^T \mathcal{L}^X_s P^X_{s,r} f(r,x)dr + f(s,x) \Big| \mathcal{F}_s\right] ds.$$

By denoting

$$m(s,x) := -\int_s^T \mathcal{L}^X_s P^X_{s,r} f(r,x)dr - f(s,x),$$

we have that, thanks to the representation (4.16),

$$\begin{aligned}
\int_t^T \mathbb{E}\left[m(s,x) \Big| \mathcal{F}_t\right] ds &= -\mathbb{E}\left[\int_t^T \int_s^T \mathcal{L}^X_s P^X_{s,r} f(r,x)drds + \int_t^T f(s,x)ds \Big| \mathcal{F}_t\right] \\
&= -\mathbb{E}\left[\int_t^T \int_t^r \mathcal{L}^X_s P^X_{s,r} f(r,x)dsdr + \int_t^T f(s,x)ds \Big| \mathcal{F}_t\right] \\
&= -\mathbb{E}\left[\int_t^T \left(P^X_{t,r} f(r,x) - f(r,x)\right) dr + \int_t^T f(s,x)ds \Big| \mathcal{F}_t\right] \\
&= F(t,x).
\end{aligned}$$

In the previous computations, we have used Fubini's theorem, which can be applied since, thanks to Lemma 4.3,

$$\begin{aligned}
\int_t^T \int_t^r \|\mathcal{L}^X_s P^X_{s,r} f(r,\cdot)\|_{\mathbb{D}^{1,0,p}} dsdr &\leq \left(\int_0^T \int_0^T \|\mathcal{L}^X_s P^X_{s,r} f(r,\cdot)\|^q_{\mathbb{D}^{1,0,p}} dsdr\right)^{1/q} \\
&\leq C\|f\|_{\mathbb{D}_q^{1,2-2/q+\varepsilon,p}}.
\end{aligned} \tag{4.26}$$

This enables us to conveniently express the martingale that we are looking for in terms of an adapted field Z. That is, we have

$$F(t,x) = \int_t^T \left(\mathcal{L}_s^X F(s,x) - f(s,x)\right) ds - M(T,x) + M(t,x),$$

where

$$M(t,x) := \int_0^t \mathbb{E}\left[m(s,x)\Big|\mathcal{F}_s\right] ds + \int_t^T \mathbb{E}\left[m(s,x)\Big|\mathcal{F}_t\right] ds.$$

Let us now check that M is indeed a $L^p(\mathbb{R}^d)$-valued martingale. Note first that by estimate (4.26), $M(T,\cdot)$ is integrable as

$$\begin{aligned}
\mathbb{E}\left[\|M(T,\cdot)\|^p_{L^p(\mathbb{R}^d)}\right] &= \mathbb{E}\left[\left\|\int_0^T \mathbb{E}\left[m(s,\cdot)\Big|\mathcal{F}_s\right] ds\right\|^p_{L^p(\mathbb{R}^d)}\right]\\
&\leq C\int_0^T \mathbb{E}\left[\|m(s,\cdot)\|^p_{L^p(\mathbb{R}^d)}\right] ds < +\infty,
\end{aligned}$$

since m belongs to $\mathbb{D}_q^{1,0,p}$ (by (4.26) and by our assumption on f). In addition, $\forall u \in [0,t]$, we have

$$\begin{aligned}
\mathbb{E}\left[M(t,\cdot) - M(u,\cdot)\Big|\mathcal{F}_u\right] &= \int_u^t \mathbb{E}\left[m(s,\cdot)\Big|\mathcal{F}_u\right] ds + \int_t^T \mathbb{E}\left[m(s,\cdot)\Big|\mathcal{F}_u\right] ds - \int_u^T \mathbb{E}\left[m(s,\cdot)\Big|\mathcal{F}_u\right] ds\\
&= 0,
\end{aligned}$$

therefore, M is indeed a martingale. It remains to represent M can be written as a stochastic integral against the Brownian motion W. To this end we apply a localization in space procedure. More precisely, for any $n \geq 1$, set:

$$M^n(t,x) := M(0,x) + M(t,x)\mathbf{1}_{|x|<n}\mathbf{1}_{t\in(0,T]}$$

which is a $L^2(\mathbb{R}^d)$-valued martingale. Hence, there exists an adapted random field Z^n such that

$$\mathbb{E}\left[\int_0^T \|Z^n(t,x)\|^2_{L^2(\mathbb{R}^d)} dt\right] < +\infty,$$

and

$$M^n(t,x) = M(0,x) + \int_0^t Z^n(s,x) dW_s, \quad (t,x) \in [0,T]\times\mathbb{R}^d.$$

Note that by definition, we have that:

$$Z^n(t,x) = Z^{n+1}(t,x), \quad \mathbb{P}-a.s. \text{ for } |x|<n.$$

Set $Z(t,x) := \lim_{n\to+\infty} Z^n(t,x)$, where the limit is pointwise and non-decreasing. We have using Fubini's theorem, and Doob's inequality for real-valued martingales:

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|M(t,\cdot) - M(0,\cdot) - \int_0^t Z(s,\cdot)dW_s\right\|^p_{L^p(\mathbb{R}^d)}\right]\\
&\leq \int_{\mathbb{R}^d} \mathbb{E}\left[\sup_{t\in[0,T]}\left|M(t,\cdot) - M(0,\cdot) - \int_0^t Z(s,\cdot)dW_s\right|^p\right] dx\\
&\leq \frac{p}{p-1}\int_{\mathbb{R}^d} \mathbb{E}\left[\left|M(T,\cdot) - M(0,\cdot) - \int_0^T Z(s,\cdot)dW_s\right|^p\right] dx
\end{aligned}$$

$$\leq \frac{p}{p-1} \lim_{n\to+\infty} \int_{|x|<n} \mathbb{E}\left[\left| M^n(T,\cdot) - M(0,\cdot) - \int_0^T Z^n(s,\cdot)dW_s \right|^p \right] dx = 0.$$

Hence, the representation $M(t,x) = M(0,x) - \int_0^t Z(s,x)dW_s$ holds for almost every (ω, t, x) and Z belongs to $\mathbb{M}^p$. Thus, we obtain that $(F,Z) \in \mathbb{W}^{2,p}_{\mathcal{P},q} \times \mathbb{M}^p$ solves Equation (2.8).

Step 2: Proof of (3.4).

Recall that by (4.25), $\|F(t,\cdot)\|_{\mathbb{D}^{1,2,p}} < +\infty$. In addition, following the same lines as in the computation of (4.25), we have that:

$$\begin{aligned}
\left\| \int_t^T \mathcal{L}_r^X F(r,\cdot)dr \right\|^q_{\mathbb{D}_q^{1,0,p}} &= \int_0^T \left\| \int_t^T \mathcal{L}_r^X F(r,\cdot)dr \right\|^q_{\mathbb{D}^{1,0,p}} dt \\
&\leq T \int_0^T \left\| \mathcal{L}_r^X \int_r^T P^X_{r,s} f(s,\cdot)ds \right\|^q_{\mathbb{D}^{1,0,p}} dr \\
&\leq T \int_0^T \int_0^s \| \mathcal{L}_r^X P^X_{r,s} f(r,\cdot)\|^q_{\mathbb{D}^{1,2,p}} drds, \text{ by Lemma 4.3} \\
&\leq CT \int_0^T \|f(r,\cdot)\|^q_{\mathbb{D}^{1,2-2/q+\varepsilon,p}} dr = CT\|f\|^q_{\mathbb{D}_q^{1,2-2/q+\varepsilon,p}}.
\end{aligned}$$

Combining this result with Relation (2.8), we obtain that for a.e. (t,x), $\int_t^T Z(s,x)dW_s$ belongs to $\mathbb{D}^{1,p}$ (see Remark A.2). Since $\mathbb{D}^{1,p} \subset \mathbb{D}^{1,2}$ (see A.1 for a definition of these spaces), by [30, Lemma 2.3], this is equivalent to for a.e. x, $Z(\cdot,x) \in L^2([t,T],\mathbb{D}^{1,2})$. As a consequence, for a.e. (t,x) and for any $0 \leq s \leq t$,

$$D_sF(t,x) = -\int_s^t \left(D_s\mathcal{L}_r^X F(r,x) - D_sf(r,x)\right)dr + Z(s,x) + \int_s^t D_sZ(r,x)dW_r, \ \mathbb{P}-a.s..$$

Hence taking $s = t$, in the previous relation, we have that for a.e. x, a version of the process $(Z(t,x))_{t\in[0,T]}$ is given by $Z(t,x) = D_tF(t,x)$. Representation (3.4) can then be deduced using [29, Proposition 1.2.8]. We are now in position to prove that Z belongs to $\mathbb{W}^{2,p}_q$. By using Lemma 4.4 and Assumption 3.1, we have

$$\begin{aligned}
D_tF(t,x) = \mathbb{E}\left[-\int_t^T D_tP^X_{t,r}f(r,x)dr \middle| \mathcal{F}_t \right] &= \mathbb{E}\left[-\int_t^T P^X_{t,r}f'(r,x)\upsilon_f(t,r)dr \middle| \mathcal{F}_t \right] \\
&+ \mathbb{E}\left[\int_t^T \int_t^r P^X_{t,u}\left(b'(u,x)\upsilon_b(t,u)\cdot\nabla P^X_{u,r}f(r,x)\right)dudr \middle| \mathcal{F}_t \right].
\end{aligned} \tag{4.27}$$

By Assumption 3.1 and Proposition 4.3, we estimate the first term on the rhs of (4.27)

$$\left\| \mathbb{E}\left[-\int_\cdot^T P^X_{\cdot,r} f'(r,\cdot)\upsilon_f(\cdot,r)dr \middle| \mathcal{F}_\cdot \right] \right\|_{\mathbb{W}^{2,p}_q} \leq C\|f'\|_{L^q([0,T];L^\ell(\Omega;W^{-\gamma,p}(\mathbb{R}^d)))}.$$

For the second term of (4.27), we remark that, thanks to Fubini's theorem,

$$\int_t^T \int_t^r P^X_{t,u}\left(b'(u,x,W_{(u)})\upsilon_b(t,u)\cdot\nabla P^X_{u,r}f(r,x)\right)dudr = \int_t^T P^X_{t,u}G(u,x)\upsilon_b(t,u)du,$$

where we denote $G(u,x) := b'(u,x,W_{(u)})\cdot\nabla\left[\int_u^T P^X_{u,r}f(r,x)dr\right]$. Hence, we can proceed by similar arguments as for the first term of the rhs of (4.27) since, by (4.9), (4.8) and

Lemma 4.3,

$$\|G\|_{L^q([0,T];L^p(\Omega\times\mathbb{R}^d))} \le C\left\|\int_u^T P^X_{u,r}f(r,x)dr\right\|_{\mathbb{G}^{0,2,p}_q} \le C\|f\|_{\mathbb{W}^{0,p}_q} < \infty.$$

Therefore, we conclude that $D_tF(t,x)$ belongs to $\mathbb{W}^{2,p}_q$ and, thus, Z itself belongs to $\mathbb{W}^{2,p}_q$.

Step 3: Proof of the mild representation (3.5).

Fix t in $[0,T]$. Set $\tilde{F}(t,x) := -\int_t^T P^X_{t,r}f(r,x)dr$. By Lemma 4.3, $\tilde{F}(t,x)$ belongs to $\mathbb{D}^{1,2}$ for a.e. $x\in\mathbb{R}^d$. Hence Clark-Ocone formula (see [29, Proposition 1.3.14]) implies that:

$$F(t,x) = \tilde{F}(t,x) - \int_t^T \mathbb{E}[D_s\tilde{F}(t,x)|\mathcal{F}_s]dW_s, \quad \mathbb{P}-a.s..$$

Recall that Z can be chosen as $Z(s,x) = D_sF(s,x)$. Thus Relation (3.5) follows if one proves that, for a.e. s in $[t,T]$,

$$\mathbb{E}[D_s\tilde{F}(t,x)|\mathcal{F}_s] = P^X_{t,s}D_s\mathbb{E}[\tilde{F}(s,x)|\mathcal{F}_s]. \tag{4.28}$$

Let $t\le u\le s$. Using Relation (4.16), we have:

$$\begin{aligned}
P^X_{u,s}D_s\mathbb{E}[\tilde{F}(s,x)|\mathcal{F}_s] &= D_s\mathbb{E}[\tilde{F}(s,x)|\mathcal{F}_s] + \int_u^s \mathcal{L}^X_a P^X_{a,s}D_s\mathbb{E}[\tilde{F}(s,x)|\mathcal{F}_s]da\\
&= D_s\mathbb{E}[\tilde{F}(u,x)|\mathcal{F}_s] + D_s\mathbb{E}[(\tilde{F}(s,x)-\tilde{F}(u,x))|\mathcal{F}_s] + \int_u^s \mathcal{L}^X_a P^X_{a,s}D_s\mathbb{E}[\tilde{F}(s,x)|\mathcal{F}_s]da.
\end{aligned}$$

We now compute the second term of the right-hand side above to get:

$$\begin{aligned}
&D_s\mathbb{E}[(\tilde{F}(s,x)-\tilde{F}(u,x))|\mathcal{F}_s]\\
&= D_s\mathbb{E}\left[-\int_s^T P^X_{s,r}f(r,x)dr\middle|\mathcal{F}_s\right] + D_s\mathbb{E}\left[\int_u^T P^X_{u,r}f(r,x)dr\middle|\mathcal{F}_s\right]\\
&= D_s\mathbb{E}\left[-\int_s^T P^X_{s,r}f(r,x)dr\middle|\mathcal{F}_s\right] + D_s\mathbb{E}\left[\int_s^T P^X_{u,r}f(r,x)dr\middle|\mathcal{F}_s\right], \quad \text{by [29, Proposition 1.2.8]}^*\\
&= D_s\mathbb{E}\left[-\int_s^T (P^X_{s,r}-P^X_{u,r})f(r,x)dr\middle|\mathcal{F}_s\right]\\
&= D_s\mathbb{E}\left[-\int_s^T\int_u^s \mathcal{L}^X_a P^X_{a,r}f(r,x)da\,dr\middle|\mathcal{F}_s\right]\\
&= D_s\mathbb{E}\left[\int_u^s \mathcal{L}^X_a P^X_{a,s}\left(-\int_s^T P^X_{s,r}f(r,x)dr\right)\,da\middle|\mathcal{F}_s\right].
\end{aligned}$$

Using similar arguments to those used in Lemma 4.4, one proves that:

$$D_s\mathbb{E}[P^X_{t,s}\Phi(x)|\mathcal{F}_s] = P^X_{t,s}D_s\mathbb{E}[\Phi(x)|\mathcal{F}_s],$$

for any $\mathcal{F}_r$-measurable random field Φ with $r\ge s$. Hence,

$$D_s\mathbb{E}[(\tilde{F}(s,x)-\tilde{F}(u,x))|\mathcal{F}_s] = \int_u^s \mathcal{L}^X_a P^X_{a,s}D_s\mathbb{E}\left[-\int_s^T P^X_{s,r}f(r,x)dr\middle|\mathcal{F}_s\right]da$$

which establishes (4.28) in light of the previous computations. We finally conclude the proof by addressing the uniqueness of the adapted mild solution (F,Z) which, by linearity, boils down to prove that $(0,0)$ is the unique solution to:

$$F(t,x) = -\int_t^T P^X_{t,r}Z(r,x)dW_r.$$

As F must be adapted, $F(t,x) = \mathbb{E}[F(t,x)|\mathcal{F}_t] = \mathbb{E}\left[-\int_t^T P_{t,r}^X Z(r,x)dW_r|\mathcal{F}_t\right] = 0$ if we prove that for almost every t, $\int_t^\cdot P_{t,r}^X Z(r,x)dW_r$ is a true martingale. In fact by Burkholder-Davis-Gundy's inequality for real-valued martingales and Lemma 4.3, it holds that

$$\begin{aligned}
\left\|\int_\cdot^T P_{\cdot,r}^X Z(r,\cdot)dW_r\right\|_{\mathbb{W}_q^{2,p}}^q &\le C\int_0^T \left(\left\|\int_t^T \left|P_{t,r}^X Z(r,\cdot)\right|^2 dr\right\|_{\mathbb{W}^{2,p/2}}^{1/2}\right)^q dt \\
&\le C\int_0^T \left(\int_t^T \left\|P_{t,r}^X Z(r,\cdot)\right\|_{\mathbb{W}^{2,p}}^2 dr\right)^{q/2} dt \\
&\le C\int_0^T \int_0^r \left\|P_{t,r}^X Z(r,\cdot)\right\|_{\mathbb{W}^{2,p}}^q dtdr \\
&\le C\|Z\|_{\mathbb{W}_q^{2,p}}^q < +\infty,
\end{aligned}$$

which proves the required property.

5 Proof of Theorem 3.2

5.1 The Itô-Wentzell formula

Let us recall the Itô-Wentzell formula in the context of processes with values in Sobolev spaces [24].

Proposition 5.1 (Itô-Wentzell formula)**.** *Let F in $\mathbb{W}_{\mathcal{P},q}^{2,p}$ be such that for any $\varphi \in L^{\bar{p}}(\mathbb{R}^d)$:*

$$\langle F(t,\cdot),\varphi\rangle = \langle F(0,\cdot),\varphi\rangle + \int_0^t \langle\Gamma(s,\cdot),\varphi\rangle dW_s + \int_0^t \langle A(s,\cdot),\varphi\rangle ds \tag{5.1}$$

with $F(0,\cdot)\in L^p(\mathbb{R}^d)$, A in $\mathbb{W}_{\mathcal{P},q}^{0,p}$ and Γ in $\mathbb{W}_{\mathcal{P},q}^{1,p}$. Then, $\forall t\in[0,T]$, $\forall\varphi\in L^{\bar{p}}(\mathbb{R}^d)$,

$$\begin{aligned}
\langle F(t,\cdot+X_t),\varphi\rangle =& \langle F(0,\cdot+X_0),\varphi\rangle + \int_0^t [\langle\Gamma(s,\cdot+X_s),\varphi\rangle + \langle\nabla F(s,\cdot+X_s),\varphi\rangle]dW_s \\
&+ \int_0^t [\langle \mathrm{div}\Gamma(s,\cdot+X_s),\varphi\rangle + \langle A(s,\cdot+X_s),\varphi\rangle]\, ds \\
&+ \int_0^t \langle\mathcal{L}_s^X F(s,\cdot+X_s),\varphi\rangle ds,\ \mathbb{P}-a.s..
\end{aligned} \tag{5.2}$$

Remark 5.1. *As noted earlier, elements of $\mathbb{W}_{\mathcal{P},q}^{2,p}$ are adapted with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration of W. However, by definition of a weak solution to the SDE, $(\mathcal{G}_t)_{t\in[0,T]}$-adapted processes are also $(\mathcal{F}_t)_{t\in[0,T]}$-adapted.*

Remark 5.2. *Note that for any φ in $L^{\bar{p}}(\mathbb{R}^d)$, the stochastic process $s\mapsto\langle\Gamma(s,\cdot+X_s),\varphi\rangle$ is square integrable so that the stochastic integral of this process against the Brownian motion is well-defined. The same comment implies that all the integrals involved in Relations* (5.1)-(5.2) *are well-defined. We also would like to point out that contrary to the original formula in [24] where the test functions φ are assumed to be infinitely differentiable, the regularity assumption on our processes allows us to consider only $L^{\bar{p}}$ test functions.*

5.2 Proof of Theorem 3.2

It follows from the Itô-Wentzell formula from Proposition 5.1 that, $\forall \varphi \in L^{\bar{p}}(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_0^T \langle f(s,\cdot+X_s),\varphi\rangle ds \\
&\qquad = -\langle F(0,\cdot+X_0),\varphi\rangle - \int_0^T \left(\langle \nabla F(s,\cdot+X_s),\varphi\rangle + \langle Z(s,\cdot+X_s),\varphi\rangle\right) dW_s \\
&\qquad\quad - \int_0^T \langle \mathrm{div} Z(s,\cdot+X_s),\varphi\rangle ds,\ \mathbb{P}-a.s.. \qquad (5.3)
\end{aligned}$$

Let us remark that by Theorem 3.1 and a Sobolev embedding, $F, Z \in L^q_{\mathcal{P}}([0,T];L^p(\Omega;\mathcal{C}^{1,\alpha}(\mathbb{R}^d)))$ for a certain $\alpha > 0$. We choose $\varphi = \theta^\varepsilon$, $\varepsilon > 0$ a mollifier in Equation (5.3). For any positive ε we have

$$\begin{aligned}
\int_0^T f^\varepsilon(s,X_s)ds = &-F^\varepsilon(0,X_0) - \int_0^T \left(\nabla F^\varepsilon(s,X_s) + Z^\varepsilon(s,X_s)\right) dW_s \\
&- \int_0^T \mathrm{div} Z^\varepsilon(s,X_s)ds,\ \mathbb{P}-a.s., \qquad (5.4)
\end{aligned}$$

where we denote $G^\varepsilon(t,x) = \langle G(t,\cdot),\theta^\varepsilon(x-\cdot)\rangle$ for $G = f, F, \nabla F, \sum_{i=1}^d \sum_{j=1}^d \partial_{x_i} Z_j$. We remark that, given a function $G \in L^q_{\mathcal{P}}([0,T];L^p(\Omega;\mathcal{C}_b^{0,\alpha}(\mathbb{R}^d)))$ it holds that

$$\begin{aligned}
\mathbb{E}\left[\int_0^T |G^\varepsilon(s,X_s) - G(s,X_s)|\, ds\right] \leq &\left(\int_0^T \left(\mathbb{E}\left[\int_{\mathbb{R}^d} |G(s,x+X_s) - G(s,X_s)|\theta^\varepsilon(x)dx\right]\right)^q ds\right)^{1/q} \\
\leq &\left(\int_0^T \mathbb{E}\left[\|G(s,\cdot)\|^p_{\mathcal{C}_b^{0,\alpha}(\mathbb{R}^d)}\right]^{q/p} ds\right)^{1/q} \left(\int_{\mathbb{R}^d} |x|^\alpha \theta^\varepsilon(x)dx\right) \\
\leq &C\|G\|_{L^q([0,T];L^p(\Omega;\mathcal{C}_b^{0,\alpha}(\mathbb{R}^d)))}\varepsilon^\alpha \underset{\varepsilon\to 0}{\longrightarrow} 0.
\end{aligned}$$

Thus, each term from the right-hand side of (5.4) converges to the corresponding value. In order to handle with the term in the left-hand side, we have to prove that the integral I defined by

$$I(x) := \int_0^T f(s,x+X_s)ds,$$

is continuous, $\mathbb{P}-a.s.$. This comes from the fact that I belongs to $\mathbb{W}^{1,p}$. Indeed, thanks to (5.3), Itô's isometry, a change of variable and Jensen's inequality, we have that

$$\begin{aligned}
\|I\|_{\mathbb{W}^{1,p}} \leq &\|F(0,\cdot+X_0)\|_{\mathbb{W}^{1,p}} \\
&+ 2\left(\int_0^T \|\nabla F(s,\cdot+X_s)\|^2_{\mathbb{W}^{1,p}} + \|Z(s,\cdot+X_s)\|^2_{\mathbb{W}^{1,p}} ds\right)^{1/2} \\
&+ \int_0^T \|\nabla Z(s,\cdot+X_s)\|_{\mathbb{W}^{1,p}} ds \\
\leq &\|F(0,\cdot)\|_{\mathbb{W}^{1,p}} + C\left(\int_0^T \|F(s,\cdot)\|^q_{\mathbb{W}^{2,p}} + \|Z(s,\cdot)\|^q_{\mathbb{W}^{1,p}} ds\right)^{1/q} \\
&+ \left(\int_0^T \|Z(s,\cdot)\|^q_{\mathbb{W}^{2,p}}\right)^{1/q} ds.
\end{aligned}$$

Since $F, Z \in \mathbb{W}_q^{1,2,p}$, we deduce that $I \in \mathbb{W}^{1,p}$. By the Sobolev embedding $\mathcal{C}^{0,\alpha}(\mathbb{R}^d) \subset W^{1,p}(\mathbb{R}^d)$, we deduce that I is $\mathbb{P}$-a.s. continuous. Thus, we have, by using Fubini's theorem,

$$\left|\int_0^T [f^\varepsilon(s, X_s) - f(s, X_s)]ds\right| = |\langle [I(\cdot) - I(0)], \theta^\varepsilon\rangle| \underset{\varepsilon\to 0}{\longrightarrow} 0, \ \mathbb{P} - a.s.,$$

which concludes the proof.

A Malliavin calculus for random fields

In this section we recall the classical definitions of Malliavin-Sobolev spaces presented in [29] and extended them to functional valued random variables that from now on we will refer as random fields. We start with some facts about Malliavin's calculus for random variables.

A.1 Malliavin calculus for random variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $W := (W_t)_{t\in[0,T]}$ a Brownian motion on this space (to the price of heavier notations all the definitions and properties in this section and of the next one extend to a d-dimensional Brownian motion). We assume that $\mathcal{F} = \sigma\left(W_t, \ t \in [0,T]\right)$.

Let $\mathcal{S}^{rv}$ be the set of cylindrical functionals, that is the set of random variable β of the form:

$$\beta = \varphi(W_{t_1}, \cdots, W_{t_n})$$

with $\mathbb{N}^*$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ in $\mathcal{C}_0^\infty(\mathbb{R}^n)$ and $0 \leq t_1 < \cdots < t_n \leq T$. For an element β in $\mathcal{S}^{rv}$, we set DF the $L^2([0,T])$-valued random variable as:

$$D_\theta\beta := \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(W_{t_1}, \cdots, W_{t_n})\mathbf{1}_{[0,t_i]}(\theta), \quad \theta \in [0,T].$$

For a positive integer $p \geq 1$, we set $\mathbb{D}^{1,p}$ the closure of $\mathcal{S}^{rv}$ with respect to the norm:

$$\|\beta\|_{\mathbb{D}^{1,p}}^p := \mathbb{E}[|\beta|^p] + \mathbb{E}\left[\left(\int_0^T |D_\theta\beta|^2 d\theta\right)^{p/2}\right].$$

To D is associated a dual operator denoted δ defined through the following integration by parts formula:

$$\mathbb{E}[\beta\delta(u)] = \mathbb{E}\left[\int_0^T D_t\beta \ u_t dt\right], \tag{A.1}$$

for any β in $\mathbb{D}^{1,2}$ and any $L^2([0,T])$-valued random variable u such that there exists a positive constant C such that $\left|\mathbb{E}\left[\int_0^T D_t\chi u_t dt\right]\right| \leq C\|\chi\|_{\mathbb{D}^{1,2}}, \ \forall \chi \in \mathbb{D}^{1,2}$. In particular if $u := (u_t)_{t\in[0,T]}$ is a adapted process then $\delta(u) = \int_0^T u_t dW_t$. In addition, according to [29, Proposition 1.3.4], for any β in $\mathcal{S}$ and any h in $L^p([0,T])$ (with $p \geq 2$), $\delta(h\beta)$ is well-defined and satisfies

$$\delta(h\beta) = \beta\delta(h) - \int_0^T h_t D_t\beta dt. \tag{A.2}$$

A.2 Malliavin calculus for random fields

We now extend these definitions to random fields that is to measurable mappings $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$. More precisely, we consider $\mathcal{S}$ be the set of cylindrical fields, that is the set of random fields F of the form:

$$F = \varphi(W_{t_1}, \cdots, W_{t_n}, x)$$

with $\varphi : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ in $\mathcal{C}_0^\infty(\mathbb{R}^{n+d})$. We fix p an integer with $p \geq 2$. For an element F in $\mathcal{S}$, we set DF the $L^p([0,T])$-valued random field as:

$$D_\theta F := \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(W_{t_1}, \cdots, W_{t_n}, x)\mathbf{1}_{[0,t_i]}(\theta), \quad \theta \in [0,T].$$

Note that for F in $\mathcal{S}$, $D\nabla^k F = \nabla^k DF$ for any multi index k. In addition, an integration by parts formula for the operators $D\nabla^k$ can be derived as follows.

Lemma A.1. *Let F in $\mathcal{S}$, h in $L^p([0,T])$ and G in $\mathcal{S}$. Let k be a multi-index in $\mathbb{N}^d$, then the following integration by parts formula holds true:*

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} D_t \nabla^k F(x) h_t G(x) dx dt\right] = \mathbb{E}\left[\int_{\mathbb{R}^d} F(x)\ \delta((\nabla^k)^* G(x) h) dx\right], \tag{A.3}$$

where $(\nabla^k)^*$ denotes the dual operator of ∇^k.

Proof. By the Malliavin-integration by parts formula (see *e.g.* [29, Lemma 1.2.1]) and by the classical integration by parts formula in $\mathbb{R}^d$ we have that:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} D_t \nabla^k F(x) h_t G(x) dx dt\right] \\
&\quad = \int_{\mathbb{R}^d} \mathbb{E}\left[\int_0^T D_t \nabla^k F(x) h_t G(x) dt\right] dx \\
&\quad = \int_{\mathbb{R}^d} \mathbb{E}\left[\nabla^k F(x) \delta(G(x)h)\right] dx, \text{ by (A.1)} \\
&\quad = \int_{\mathbb{R}^d} \mathbb{E}\left[\nabla^k F(x) G(x) \delta(h)\right] dx - \int_{\mathbb{R}^d} \mathbb{E}\left[\nabla^k F(x) \int_0^T D_t G(x) h_t dt\right] dx, \text{ by (A.2)} \\
&\quad = \mathbb{E}\left[\int_{\mathbb{R}^d} F(x) (\nabla^k)^* G(x) dx \delta(h)\right] - \mathbb{E}\left[\int_{\mathbb{R}^d} F(x) (\nabla^k)^* \int_0^T D_t G(x) h_t dt dx\right] \\
&\quad = \mathbb{E}\left[\int_{\mathbb{R}^d} F(x) \left((\nabla^k)^* G(x) \delta(h) - \int_0^T D_t (\nabla^k)^* G(x) h_t dt\right) dx\right] \\
&\quad = \mathbb{E}\left[\int_{\mathbb{R}^d} F(x)\ \delta((\nabla^k)^* G(x) h) dx\right], \text{ by (A.2).}
\end{aligned}$$

□

This integration by parts formula allows us to prove that the operators $D\nabla^k$ are closable.

Lemma A.2. *Let $p \geq 2$ and k be in $\mathbb{N}^d$. The operators $D\nabla^k$ (and so $\nabla^k D$) are closable from $\mathcal{S}$ to $L^p(\Omega \times \mathbb{R}^d; L^p([0,T]))$.*

Proof. Let $(F_n) \subset \mathcal{S}$ a sequence of random fields which converges in $L^p(\Omega\times\mathbb{R}^d; L^p(\mathbb{R}^d))$ to 0 and such that $(D\nabla^k F_n)_n$ converges in $L^p(\Omega \times \mathbb{R}^d; L^p([0,T]))$ to some element η in $L^p(\Omega \times \mathbb{R}^d; L^p([0,T]))$. Let h in $L^p([0,T])$ and $G : \mathbb{R}^d \to \mathbb{R}$ in $\mathcal{S}$. We recall that $\bar{p} := \frac{p}{p-1}$. For any $n \geq 1$, it holds that

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^d}\int_0^T \eta(t,x)h_t dt G(x)dx\right]\\
&= \mathbb{E}\left[\int_{\mathbb{R}^d}\int_0^T (\eta(t,x) - D_t\nabla^k F^n(x))h_t dt G(x)dx\right] + \mathbb{E}\left[\int_{\mathbb{R}^d}\int_0^T D_t\nabla^k F^n(x)h_t dt G(x)dx\right]\\
&= \mathbb{E}\left[\int_{\mathbb{R}^d}\int_0^T (\eta(t,x) - D_t\nabla^k F^n(x))h_t dt G(x)dx\right] + \mathbb{E}\left[\int_{\mathbb{R}^d} F^n(x)\ \delta((\nabla^k)^* G(x)h)dx\right],
\end{aligned}$$

where we have used the integration by parts formula (A.3). We estimate the two terms above separately. For the first one, using successive Hölder's Inequality, we have that

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_{\mathbb{R}^d}\int_0^T (\eta(t,x) - D_t\nabla^k F^n(x))h_t dt G(x)dx\right]\right|\\
&\leq \mathbb{E}\left[\int_{\mathbb{R}^d}\int_0^T |\eta(t,x) - D_t\nabla^k F^n(x)|^p dt dx\right]^{1/p} \mathbb{E}[\|G\|^{\bar{p}}_{L^{\bar{p}}(\mathbb{R}^d)}]^{1/\bar{p}}\|h\|_{L^{\bar{p}}([0,T])}\\
&\underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

The second term can be estimated as follows (using also Hölder's inequality and (A.2)).

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_{\mathbb{R}^d} F^n(x)\ \delta((\nabla^k)^* G(x)h)dx\right]\right|\\
&= \left|\mathbb{E}\left[\int_{\mathbb{R}^d} F^n(x)(\nabla^k)^* G(x)\delta(h)dx\right] - \mathbb{E}\left[\int_{\mathbb{R}^d} F^n(x)\int_0^T D_t(\nabla^k)^* G(x)h_t dt dx\right]\right|\\
&\leq C\mathbb{E}\left[\int_{\mathbb{R}^d} |F^n(x)|^p dx\right]^{1/p} \left(\mathbb{E}\left[\delta(h)^{2\bar{p}}\right]^{1/(2\bar{p})} \vee \|h\|_{L^2([0,T])}\right)\\
&\times \left(\mathbb{E}\left[\|(\nabla^k)^* G\|^{2\bar{p}}_{L^{2\bar{p}}(\mathbb{R}^d)}\right]^{1/(2\bar{p})} + \mathbb{E}\left[\int_{\mathbb{R}^d}\left(\int_0^T |D_t(\nabla^k)^* G(x)|^2 dt\right)^{\frac{\bar{p}}{2}} dx\right]^{1/\bar{p}}\right)\\
&\underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

Combining the previous estimates and relations we conclude that

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\int_0^T \eta(t,x)h_t dt G(x)dx\right] = 0.$$

The conclusion follows from the fact that the set of elements of the form Gh with h in $L^p([0,T])$ and G in $\mathcal{S}$ is dense in $L^p(\Omega \times \mathbb{R}^d; L^p([0,T]))$. □

Remark A.1. *Lemma A.1 and Lemma A.2 still holds if we replace the differential operator ∇^k with the Bessel potential $(1-\Delta)^{m/2}$ for any $m \in \mathbb{R}$.*

Remark A.2. *In particular, if a random field F belongs to $\mathbb{D}^{1,m,p}$ (with $m \geq 0$), then for a.e. (t,x), $\omega \mapsto \nabla^k F(t,x)(\omega)$ belongs to the classical Malliavin space $\mathbb{D}^{1,p}$ whose definition has been recalled in Section A.1 (for any k such that $|k| \leq m$) for random variables that depend only on ω and not on (t,x).*

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