Exercise 1
Let \((X_t)_{t=0,..,T}\) a stochastic process on the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) and let \(\nu\) be a \((\mathcal{F}_t)\)-stopping time. The stopped process of \(X\) by \(\nu\) is defined as
\[
X_\nu^\nu(\omega) = \min\{\nu(\omega), t\}.
\]
i.e., on the set \(\{\nu = \tau\}\) we have
\[
X_\tau^\nu = \begin{cases} 
X_\tau & \text{if } \tau \leq t, \\
X_t & \text{if } \tau > t.
\end{cases}
\]
Prove that
i) If \(X\) is adapted, then \(X^\nu\) is adapted.
ii) If \(X\) is a martingale, then \(X^\nu\) is a martingale.

Exercise 2
Let \(Q\) be the risk neutral measure, the context of the usual binomial model. Let \(0 < q < 1\) be the underlying probability. Show that the price at time \(t\) of an American Put option on \(S\) with maturity \(T\) and strike \(K\) can be expressed as
\[
P_t = P(t,S_t),
\]
where:
\[
N \times \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a function, satisfying the recursive backward equation}
\]
\[
\begin{align*}
P(T,x) &= (K-x)^+ \\
P(t,x) &= \max\left\{ (K-x)^+, \frac{qP(t+1,x_u)+(1-q)P(t+1,x_d)}{1+r} \right\}, \quad t = 0, \ldots, t_{N-1}.
\end{align*}
\]
Show that the (super) replicating strategy of the American Put is characterized by a quantity \(\phi_t = \Delta(t,S_{t-1})\) at time \(t\), where \(\Delta\) can be expressed in terms of the function \(P\).

Exercise 3
Let \((X_t)_{t \in \{0,\ldots,T\}}\) be an adapted integrable process on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \{0,\ldots,T\}}, \mathbb{P})\) and let \((H_t)_{t \in \{0,\ldots,T\}}\) be its Snell envelope. Consider also the process \(V_t = \mathbb{E}[X_T|\mathcal{F}_t], \ t \in \{0, \ldots, T\}\).

i) Show that \(H_t \geq V_t\) a.s. for all \(t = 0, \ldots, T\).
ii) Show that if \(V_t \geq X_t\) for all \(t = 0, \ldots, T\), then \(H = V\) a.s.
iii) Show that if \(X\) is a submartingale, then \(H = V\) a.s.
iv) Use ii) to conclude that the prices of an American Call and a European Call in the arbitrage-free binomial model with \(r \geq 0\) are equal at each time \(t = 0, \ldots, T\).

Exercise 4
Consider an American Put option in a one-period binomial model with \(r > 0, q = \frac{1}{2}\), strike \(K > uS_0\) \((S_0\) current spot price of the underlying). When is it profitable to exercise the option?