Exercise 1
Find an example of a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) and an adapted process \((M_t)_{t \in [0,T]}\) with 
\[E[M_t] = E[M_0] \] for all \(t = 0, \ldots, T\) such that \((M_t)_{t \in [0,T]}\) is not a martingale.

Exercise 2
Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of i.i.d. random variables. Set \(\mathcal{F}_n = \sigma(X_1, \ldots, X_n)\) and define 
\[S_n := X_1 + \cdots + X_n, \quad n \in \mathbb{N} \]

i) Assume that for some \(\alpha \in [0, \frac{1}{2}]\),
\[
\mathbb{P}(X_1 = 1) = \frac{1}{2} + \alpha, \quad \mathbb{P}(X_1 = -1) = \frac{1}{2} - \alpha.
\]
Show that for all \(k, n \in \mathbb{N}\), with \(k \leq n\),
\[
E[S_n | \mathcal{F}_k] = S_k + 2\alpha(n-k).
\]

ii) Assume that \(X_1 \sim \mathcal{N}(0, \sigma^2)\), with positive variance \(0 < \sigma^2 < \infty\). Set 
\[Z_n^u := \exp\left(uS_n - \frac{nu^2\sigma^2}{2}\right).\]
Show that for every \(u \in \mathbb{R}\), the process \((Z_n^u)_{n \in \mathbb{N}}\) is an \((\mathcal{F}_n)\)-martingale

Note: The relation \(E(XY | \mathcal{G}) = XE(Y | \mathcal{G})\) holds for \(X \mathcal{G}\)-measurable, as soon as both \(XY\) and \(Y\) are integrable.

Exercise 3
Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a filtered probability space with \(\mathcal{F}_0 = \{\emptyset, \Omega\}\). Assume that \((X_t)_{t \in [0,T]}\) is an \(\mathbb{R}^d\)-valued adapted stochastic process and \(\phi = (\phi_t)_{t \in [0,T]}\) is an \(\mathbb{R}^d\)-valued predictable process. We set 
\[V_0 = \phi_1 \cdot X_0, \quad V_t = \phi_t \cdot X_t, \quad t = 1, \ldots, T \]
and
\[G_0 = 0, \quad G_t = \sum_{k=1}^t \phi_k \cdot (X_k - X_{k-1}), \quad t = 1, \ldots, T. \]

In the following assume that \(X\) is a martingale. Show that

i) If \((\phi_t)_{1 \leq t \leq T}\) is bounded, then \((G_t)_{0 \leq t \leq T}\) is a martingale.

ii) If \((\phi_t)_{1 \leq t \leq T}\) is bounded, then
\[
\phi_t \cdot X_t = \phi_{t+1} \cdot X_t, \quad \forall t = 1, \ldots, T - 1.
\]
implies that \((V_t)_{0 \leq t \leq T}\) is a martingale.
Note: This exercise has the following economical interpretation: a market model consists of $d$ risky assets whose discounted prices are denoted by $X$. Moreover, $\phi$ is a trading strategy with associated value process $V$ and a gain process $G$, that is the sum of the variation of the portfolio’s values

$$G_t = \sum_{k=1}^{t} V_t - V_{t-1}.$$ 

Condition (1) is the self-financing condition which means that no funds are added or removed after time $t = 0$. If the probability measure is risk neutral, then $X$ will be a martingale and the above exercise shows that also the gain and value process will be martingales under the self-financing condition.

Exercise 4

Let $(Y_t)_{1 \leq t \leq T}$ be a sequence of i.i.d. normal random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[Y_t] = 0$, $\text{Var}(Y_t) = \sigma^2 > 0$. For the sake of simplicity, interest rate is zero. We define the filtration

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_t = \sigma(Y_1, \ldots, Y_t), \quad t = 1, \ldots, T$$

and the (discounted) price process

$$S_0 = 1, \quad S_t = S_0 + \sum_{k=1}^{t} Y_k.$$ 

We ignore the fact that $S$ can take negative values. Furthermore we define another filtration $(\mathcal{G}_t)_{0 \leq t \leq T}$ by including the insider information about the final stock price $S_T$, i.e. $\mathcal{G}_t = \sigma(\mathcal{F}_t, S_T), \quad t = 0, \ldots, T$.

i) Show that $(S_t)$ is a $(\mathcal{F}_t)$-martingale, but not a $(\mathcal{G}_t)$-martingale. Moreover prove that the process

$$S_t^* := S_t - \sum_{k=0}^{t-1} \frac{S_T - S_k}{T - k}, \quad t = 0, \ldots, T,$$

is a $(\mathcal{G}_t)$-martingale.

**Hint:** Use properties of the Gaussian to see that $Y_{t+1} - \alpha (S_T - S_t)$ is independent from $\mathcal{G}_t$ if and only if $\alpha = \frac{1}{T-t}$.

ii) Having additional information about $S_T$ allows one to create portfolios with positive expected profit. Compute a trading strategy $(\hat{\phi}_t)_{t=1,\ldots,T}$ which maximizes the expected discounted profit $\mathbb{E}[G_T]$ for

$$G_T := \sum_{k=1}^{T} \phi_k (S_k - S_{k-1}),$$

in the class of all $(\mathcal{G}_t)$-predictable trading strategy $\phi$ (describing the holdings in the stock) satisfying $|\phi_t| \leq 1$ for all $t = 1, \ldots, T$.

**Hint:** begin using the Exercise 3 to deduce that

$$\mathbb{E} \left[ \sum_{k=1}^{T} \phi_k (S_k - S_{k-1}) \right] = \mathbb{E} \left[ \sum_{k=1}^{T} \phi_k \cdot \frac{S_T - S_{k-1}}{T - (k-1)} \right].$$