Recall that the binomial model or CRR (Cox-Ross-Rubinstein) is a finite market model with finite time horizon $T < \infty$ and two assets: the bond $B_t$ and a stock with spot $S_t$.

$$B_t = (1 + r)^t$$

$$S_t = S_0 \prod_{i=1}^{t} \xi_i$$

The returns $\xi_i$ take two values $d < u$. And the natural filtration generated by the spot process is:

$$\mathcal{F}_t = \sigma(S_0, S_1, \ldots, S_t)$$

**Exercise 1** (Lesson question - 3 points)

- In the binomial model, explain what is the risk neutral measure $\mathbb{Q}$ and give the formula for:

  $$p = \mathbb{Q}(\xi_i = u)$$

- Let $\Phi_T$ be an $\mathcal{F}_T$-measurable random variable. What is the abstract formula giving the price $P_t$ of the European option with maturity $T$ and payoff $\Phi_T$.

**Exercise 2** (Binary option - 9 points)

For any time $t = 0, 1, \ldots, T$, consider the payoff:

$$\Phi_t = 1_{S_t \geq K}$$

**Pricing and hedging of the European option**

The European option with payoff $\Phi_T$ at time $T$ is called the European binary option.

1. Prove that the price of the option at time $t$ of the European binary option is $P_t = P(t, S_t)$ where:

   $$P(t, x) = \frac{1}{(1 + r)^{T-t}} \mathbb{P} \left( Bin(T-t, p) \geq \frac{\log \left( \frac{K}{x} \frac{d}{d-r} \right)}{\log(u/d)} \right)$$

   with $Bin(n, p)$ being a binomial random variable with parameters $n \in \mathbb{N}$ and $0 < p < 1$.

2. Describe a replicating (or hedging) strategy in terms of a predictable process $\phi_t = (\alpha_t, \beta_t), t = 1, 2, \ldots, T$ such that:

   $$\alpha_t = \alpha(t, S_{t-1})$$

   $$\beta_t = \beta(t, S_{t-1})$$

**The American option**
1. Let the price of the option at time $t$ of the American binary option be $P_{t}^{am} = f(t, S_t)$. Prove that it satisfies the backward equation:

$$f(T, x) = \mathbb{1}_{x \geq K}$$

$$f(t, x) = \mathbb{1}_{x \geq K} + \mathbb{1}_{x < K} \left( \frac{p}{1 + r} f(t + 1, xu) + \frac{1 - p}{1 + r} f(t + 1, xd) \right)$$

2. Describe the optimal times for exercising the option. (Hint: The two situations to consider are $S_t \geq K$ and $S_t < Ku^{T-t}$)

**Exercise 3** (4 points)
Consider an American call option with maturity $T$ and strike $K$. We assume $r \geq 0$.

1. Prove that the price of this option is $C_t = C(t, S_t)$ where

$$C(T, x) = (x - K)^{+}$$

$$C(t, x) = \max \left( (x - K)^{+}, \frac{p}{1 + r} C(t + 1, xu) + \frac{1 - p}{1 + r} C(t + 1, xd) \right)$$

2. Prove that:

$$\frac{p}{1 + r} C(t, xu) + \frac{1 - p}{1 + r} C(t, xd) \geq x - \frac{K}{1 + r}$$

3. Deduce that the optimal exercising time is always the maturity $T$.

**Exercise 4** (4 points)
We consider a two-period arbitrage-free financial market model consisting of a bond and a stock with discounted price process $S^* = (S_0^*, S_1^*, S_2^*)$. We consider the European option with discounted payoff:

$$\Phi^* = \max_{t=0,1,2} S_t^* - \min_{t=0,1,2} S_t^*$$

Assume that the processes evolve according to the following binomial tree, where each scenario has positive probability:

```
<table>
<thead>
<tr>
<th>$S_0^*$</th>
<th>$S_1^*$</th>
<th>$S_2^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>6</td>
</tr>
</tbody>
</table>
```

a) Determine the transition probabilities under $Q$ such that $S^*$ becomes a $Q$-martingale.

b) Compute the risk-neutral pricing of the option.