Final examination (3h)

- Only exercise 2 is in continuous time.
- In exercises 3 and 4, the candidate will be asked at certain questions to provide both a mathematical answer and a financial answer. Typically, a mathematical answer consists in invoking tools from mathematical finance such as the risk neutral measure. A financial answer requires exhibiting an arbitrage or hedging strategy.
- Exercises can be done in any order. Therefore, it is advised that you read all the subject before starting.
- You will not get the correction if you leave earlier.
- You are specifically asked to write with a black or blue pen. No pencils or red pen.
- For identification purposes, please display your ID card on the corner of the table.

Good luck! - and happy Valentine’s day...

Exercise 1 (Lesson question - 3 points)
Consider a finite market model i.e finite time horizon and finite $\Omega$. Define equivalent martingale measures and then state as clearly as possible the two fundamental theorems of asset pricing.

Exercise 2 (On some stochastic processes - 8 points)
In this exercise, all stochastic integrals with respect to Brownian motion will be assumed to be real martingales. Let $W_t$ be a real standard Brownian motion.

A martingale
Let $\lambda \in \mathbb{R}$ and

$$M_t := e^{-\frac{1}{2} \lambda^2 t} \cosh(\lambda W_t)$$

where $\cosh$ is the hyperbolic cosine:

$$\cosh(x) = \frac{1}{2} (e^x + e^{-x})$$

Prove $M$ is a martingale.

On a certain affine process
Let $X_0 \in \mathbb{R}$ be a deterministic value. Define the stochastic process $X_t$ to be the unique process starting at $X_0$ and solution to the following SDE (stochastic differential equation):

$$dX_t = (a - bX_t) dt + \sigma \sqrt{X_t} dW_t$$

with $b > 0$ and $\sigma > 0$. Existence and uniqueness are assumed.

1. Let $f(t) = \mathbb{E}(X_t)$ be the expectation at time $t$ of our process. Show that $f$ is a smooth function solving the ODE (ordinary differential equation):

$$f'(t) = a - bf(t)$$
2. Compute \( \mathbb{E}(X_t) \) as a function of \( X_0, a, b \) and time \( t \).

3. Thanks to the Itô formula, give the decomposition of \( X_t^2 \) as an Itô process.

4. Let \( g(t) = \text{Var}(X_t) \) be the variance of our process at time \( t \). Prove that it solves the ODE:
   \[
g'(t) = -2b \, g(t) + \sigma^2 \, f(t)
   \]

5. Solve the ODE. One can look for a solution of the form:
   \[
g(t) = C_0 + C_1 e^{-bt} + C_2 e^{-2bt}
   \]

**Remark 0.1.** For your own culture, such processes are part of the so-called “affine processes” and are often used in order to model short-term interest rates or default intensities. These are tailored to have a mean-reverting property. For example, in the exercise, you see that \( \mathbb{E}(X_t) \) and \( \text{Var}(X_t) \) converge to a long term value, independent of \( X_0 \).

**Exercise 3** (Spread option - 5 points)
Consider a complete finite market model with absence of opportunity of arbitrage. It has a single stock \( S \). Its natural filtration is:

\[
\mathcal{F}_t = \sigma(S_0, S_1, S_2, \ldots, S_t)
\]

Time is discrete with finite time horizon \( t \in \{0, 1, 2, \ldots, T\} \). Interest rate is \( r \).

The spread option on the stock \( S \), with maturity \( T \) and strikes \( K_1 < K_2 \) has payoff:

\[
\Phi_T = (S_T - K_1)^+ - (S_T - K_2)^+
\]

1. Prove that the price of this option is bounded by \( B^0_T (K_2 - K_1) \), where \( B^0_T \) is a zero coupon with maturity \( T \). Give both a mathematical and a financial argument. Recall that the zero coupon is the amount that insures a payoff of 1$ at maturity.

2. If \( C(T, K) \) is the price of a call option with maturity \( T \) and strike \( K \), prove that the price of the spread option is given by:
   \[
   C(T, K_1) - C(T, K_2)
   \]
   Again, give both a mathematical and financial derivation of this result.

3. Describe situations where the spread option is desirable for an investor, and for the bank selling it.

**Exercise 4** (Bullet option - 8 points)
Consider a binomial model with one stock \( S \) and a bond \( B \).

\[
S_t = S_0 \prod_{i=0}^{t} \xi_i
\]

\[
B_t = (1 + r)^t
\]

where the \( \xi_i \) are independent and identically distributed. \( r \) is the interest rate. The natural filtration of \( S \) is denoted:

\[
\mathcal{F}_t = \sigma(S_0, S_1, S_2, \ldots, S_t)
\]

Under the risk neutral measure \( \mathbb{Q} \):

\[
\mathbb{Q}(\xi_i = u) = p = 1 - \mathbb{Q}(\xi_i = d)
\]
The “bullet option” with strikes $K_1 < K_2$ is an option with payoff at time $t$:

$$\Phi_t = 1_{\{K_1 \leq S_t \leq K_2\}}$$

**General question**
Describe the condition for absence of opportunity of arbitrage in terms of $r$, $u$ and $d$. Give both a financial and mathematical argument for this inequality.

**Pricing and hedging of the European option**
The European option with payoff $\Phi_T$ at time $T$ is called the European bullet option.

1. Prove that the price of the option at time $t$ of the European bullet option is $P_t = P(t, S_t)$ where:

$$P(t, x) = \frac{1}{(1 + r)^{T-t}} \mathbb{P}\left( \log\left( \frac{K_1}{d} \frac{x}{d} \right) \leq Bin(T - t, p) \leq \log\left( \frac{K_2}{d} \frac{x}{d} \right) \right)$$

with $Bin(n, p)$ being a binomial random variable with parameters $n \in \mathbb{N}$ and $0 < p < 1$.

2. Describe a replicating (or hedging) strategy in terms of a predictable process $\phi_t = (\alpha_t, \beta_t), t = 1, 2, \ldots, T$ such that:

$$\alpha_t = \alpha(t, S_{t-1})$$
$$\beta_t = \beta(t, S_{t-1})$$

**The American option**

1. We assume $r > 0$. Let the price at time $t$ of the American bullet option be $P_{t}^{\text{am}} = f(t, S_t)$.

Prove that it satisfies the backward equation:

$$f(T, x) = 1_{\{K_1 \leq x \leq K_2\}}$$

$$f(t, x) = 1_{\{K_1 \leq x \leq K_2\}} + 1_{\{x < K_1 \text{ or } x > K_2\}} \left( \frac{p}{1 + r} f(t + 1, xu) + \frac{1 - p}{1 + r} f(t + 1, xd) \right)$$

2. Describe the optimal stopping times for exercising the option, and discuss uniqueness.

**Exercise 5 ( 4 points )**

We consider a three-period arbitrage-free financial market model consisting of a bond and a stock with discounted price process $S^* = (S^*_0, S^*_1, S^*_2, S^*_3)$. We consider the call option with up and out barrier at 10$ and strike 7$:

$$\Phi^* = (S^*_3 - 7)^+ 1_{\{\max_{t=0,1,2,3} S^*_t \leq 10\}}$$

Assume that the processes evolve according to the following binomial tree, where each scenario has positive probability:

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S^*_0 = 6

S^*_1 = 9
S^*_2 = 8
S^*_3 = 7

S^*_1 = 9
S^*_2 = 8
S^*_3 = 7
S^*_4 = 9
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S^*_1 = 5
S^*_2 = 4
S^*_3 = 6
S^*_4 = 2
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S^*_2 = 7
S^*_3 = 8
S^*_4 = 2
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S^*_2 = 8
S^*_3 = 9
S^*_4 = 8
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S^*_2 = 11
S^*_3 = 12
S^*_4 = 11
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Compute the risk-neutral probability transitions and the price of the option.