On Lie groups preserving certain structures

Exercise 1. [(7.3) in [1]]
Let $V \approx \mathbb{C}^n$ a complex vector space, and $H : V \times V \to \mathbb{C}$ a Hermitian form on $V$. Now, view $V \approx \mathbb{R}^{2n}$ as a real vector space. The multiplication by $i = \sqrt{-1}$ is then seen as an involutive automorphism on $\mathbb{R}^{2n}$, and elements in $GL_n(\mathbb{C})$ are seen as linear maps on $\mathbb{R}^{2n}$ commuting with the complex conjugation.

- Show that $Re(H)$, the real part of $H$ is a symmetric form on the underlying real vector space, and that $Im(H)$ the imaginary part of $H$ is skew-symmetric. They are related by:
  \[ C(v, w) = R(i v, w) \]
- Prove the invariance under $i$:
  \[ R(v, w) = R(i v, i w) \]
  \[ C(v, w) = C(i v, i w) \]
- Conversely, prove that every such $R$ is the real part of a unique Hermitian form $H$.
- If $H$ is the standard Hermitian form:
  \[ \forall (v, w) \in (\mathbb{C}^n)^2, H(v, w) = \sum_{i} \bar{v}_i w_i \]
  Then $R$ is standard, $R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$
  - Finally, deduce that the unitary group is a subgroup of $GL_{2n}(\mathbb{R})$:
  \[ U(n) = O_{2n}(\mathbb{R}) \cap Sp_{2n}(\mathbb{R}) \]

Exercise 2. [(7.4) in [1] - Optional]
Let $\mathbb{H}$ be the field of quaternions. One can consider the group of $\mathbb{H}$-linear automorphisms of $\mathbb{H}^n \approx \mathbb{C}^{2n}$ (as $\mathbb{C}$-vector spaces), giving rise to $GL_n(\mathbb{H}) \subset GL_{2n}(\mathbb{C})$.
Then, one can ask the question of describing the subgroup of transformations leaving invariant a quaternionic Hermitian form.

On coverings

Exercise 3. [(7.16) [1]] Let $M_2(\mathbb{C}) = \mathbb{C}^4$ be the space of $2 \times 2$ matrices, with symmetric form $Q(A, B) = \frac{1}{2} Trace(AB^3)$, where $B^3 = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is the adjoint of the matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In fact, the quadratic form associated to $Q$ is simply the determinant. Define the mapping:
\[ \varphi : SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to SO_4(\mathbb{C}) \quad (g, h) \mapsto (A \mapsto gAh^{-1}) \]
Prove that \( \varphi \) is a 2 : 1 covering and therefore realizes the universal cover of \( SO_4(\mathbb{C}) \), since \( SL_2(\mathbb{C}) \) is simply connected.

**Exercise 4.** [(7.17) [1]] Identify \( \mathbb{C}^3 \) with the space of traceless matrices in \( M_2(\mathbb{C}) \), and endow it with the non-degenerate symmetric bilinear form \( Q(A, B) = Trace(AB) \). Define the mapping:

\[
\varphi : SL_2(\mathbb{C}) \rightarrow SO_3(\mathbb{C}) \\
g \mapsto (A \mapsto gAg^{-1})
\]

Same question as before.

**References**