Sheet 3

Standard facts

Exercise 1. [The unitary trick - (1.14) in [1]] Let $V$ be an irreducible representation of the finite group $G$. Show that, up to scalars, there is a unique Hermitian inner product $\langle , \rangle$ on $V$ invariant under $G$:

$$\forall g \in G, \forall (x, y) \in V^2, \langle g \cdot x, g \cdot y \rangle = \langle x, y \rangle$$

Solution of exercise 1.
The existence of an invariant scalar product $\langle , \rangle$ is given by the usual unitary trick, i.e. by averaging any scalar product $\langle , \rangle_0$:

$$\forall (x, y) \in V^2, \langle x, y \rangle := \frac{1}{|G|} \sum_{g \in G} \langle g \cdot x, g \cdot y \rangle_0$$

Now for the uniqueness part: two invariant scalar products $\langle , \rangle$ and $( , )$ give rise to $G$-isomorphisms:

$$\varphi : V \to V^* \quad x \mapsto (x, )$$

$$\psi : V \to V^* \quad x \mapsto (x, )$$

Therefore $\psi^{-1} \circ \varphi$ is a $G$-automorphism of $V$, which by Schur’s lemma must be a multiple of the identity $\lambda \text{id}$. This can be rewritten:

$$\forall (x, y) \in V^2, \langle x, y \rangle = \lambda(x, y)$$

Exercise 2. [(2.34) in [1]] Let $V$ and $W$ be irreducible representations of $G$ and $L_0 : V \to W$ be any linear mapping. Define $L : V \to W$ by:

$$L(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot L_0(g \cdot v)$$

Show that $L = 0$ if $V$ and $W$ are not isomorphic, and that $L$ is a multiplication by $\frac{\text{trace}(L_0)}{\dim V}$ if $W = V$.

Solution of exercise 2. It is easy to check that $L$ is $G$-linear. Since $V$ and $W$ are irreducible, we can apply Schur’s lemma.

- If $V$ and $W$ are not isomorphic, $L = 0$.
- If $V = W$, $L = \lambda \text{id}$ with:

$$\lambda = \frac{\text{Tr}(L)}{\dim V} = \frac{1}{|G|} \sum_{g \in G} \text{Tr}(gL_0g^{-1}) = \frac{\text{Tr}(L_0)}{\dim V}$$
On the dihedral group

Exercise 3. [(3.7) and (3.8) in [1]] The dihedral group $G = D_{2n}$ is the group of isometries of a regular $n$-gon in the plane. It is made of $n$ rotations and $n$ reflections. The purpose of this exercise is to understand the representation theory of this group.

- Method 1: Consider an arbitrary representation $V$. Notice that $\Gamma$, the subgroup of rotations is abelian. The space $V$ breaks up into eigenspaces for the action of $\Gamma$. Analyze the action of reflections on these eigenspaces, and explain how to decompose the representation $V$.

- Method 2: Using character theory, give a character table.

Solution of exercise 3. Let $\omega = e^{\frac{2\pi i}{n}}$ and denote by $r$ rotation of angle $\frac{2\pi}{n}$ and $s$ a reflection. Clearly:

$$D_{2n} = \langle r, s \rangle$$

More precisely, $D_{2n}$ can be presented as the group generated by $r$ and $s$ subject to the relations:

$$r^n = 1$$

$$s^2 = 1$$

$$srs = r^{-1}$$

- Method 1: Let $\rho : G \to GL(V)$ be a representation. As $\Gamma$ is abelian, the elements in $\rho(\Gamma) \subset GL(V)$ can be simultaneously diagonalized, breaking $V$ into a direct of eigenspaces for say $r$:

$$V = \bigoplus_{k=0}^{n-1} E(\omega^k)$$

Now notice that if $x \in E(\omega^k)$ meaning that $rx = \omega^k x$, then let $y = sx$ and:

$$ry = rsx = sr^{-1}x = \omega^{-k}y$$

Hence the two cases:

- $\omega^{2k} \neq 1$: $(x, sx)$ is free and generate an irreducible subrepresentation of $V$. In this representation, we can write:

$$\rho(r) = \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{pmatrix}$$

$$\rho(r) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- $\omega^{2k} = 1$: If $\omega^k = 1$, either $(x, sx)$ is free and generates an invariant subspace isomorphic to a sum of trivial and determinant representations; either $(x, sx)$ is not free and generates already one of those representations depending on $sx = \pm x$.

If $\omega^k = -1$, which is possible only when $n$ is even, the space generated by $(x, sx)$ if two dimensional breaks into one dimensional spaces. Two representations are possible, depending on whether $s$ acts as 1 or $-1$.

- Method 2:

- $n = 2p + 1$ odd: All reflections are conjugate and rotations with opposite angles are in the same conjugacy class. Hence $p + 2$ conjugacy classes.

The irreducible representations are the trivial one, the determinant and the $p$ two-dimensional representations described in “Method 1” where the character of $r$ is $\omega^k + \omega^{-k}$. 

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– $n = 2p$ even: $id$ and $-id$ form two conjugation classes. Reflections break into two conjugacy classes. Add to this $p - 1$ classes of two rotations with opposite (different) angles. Hence $p + 3$ conjugacy classes.

The irreducible representations are four one dimensional ones, and $p - 1$ two dimensional ones (also described in in “Method 1”).

That is sufficient to draw a character table, while using character theory to back-check our results.

References