Sheet 1

Exercise 1. Find all the one dimensional representations of the S_n, the symmetric group acting on n elements.

Exercise 2. [1.3 in the book by Fulton and Harris] Let $\rho : G \to GL(V)$ be any representation of a finite group G, V being n-dimensional. Moreover, suppose that for every $g \in G$, $\rho(g)$ has determinant 1. Show that the spaces $\wedge^k V$ and $\left(\wedge^{n-k} V\right)^*$ are isomorphic *as representations* of G.
Bonus : How would you modify the statement in order to get rid of the hypothesis "all $det\,(\rho(g))$ have determinant 1" ?

Exercise 3. [The (left)-regular representation of S_3] Recall that for every finite group G, the regular representation R corresponds to the action of G on its space of functions $Fun(G, \mathbb{C}) \approx \mathbb{C}^{|G|}$ by left translations. More precisely, when $g \in G$ acts on $\varphi \in Fun(G, \mathbb{C})$, it gives the function :

$$R(g)(\varphi) = g \cdot \varphi : \begin{array}{ccc} G & \longrightarrow & \mathbb{C} \\ h & \mapsto & \varphi(g^{-1}h) \end{array}$$

In the following, consider $G = S_3$ the symmetric group acting on 3 elements.

a) Give the matrices of $R(g)$, $g \in S_3$ in the basis $(\delta_g)_{g \in S_3}$ where :

$$\delta_g(h) = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{otherwise} \end{cases}$$

b) Give the spectrum of $R(g)$, $g \in S_3$.

c) Knowing that the irreducible representations of S_3 are (up to isomorphism) $\rho_{trivial}$, $\rho_{alternate}$ and $\rho_{standard}$; give the decomposition of the regular representation into irreducibles. You are asked to specify a basis adapted to this decomposition.