The Whittaker process as weakly non-intersecting particles

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2 Two particles conditioned to never intersect

3 Two particles with weak interaction

4 $n$ particles conditioned to never intersect

5 $n$ particles with weak interaction

6 Conclusion and ouverture
Generalities

There are two ways of doing mathematics:

- Exact computations in a world of rigid structures ("Algebra").
- Comparisons and variational approaches in a more fluid world ("Analysis").

Even probability theory does not escape such a dichotomy, and these two approaches can work hand in hand.
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Even probability theory does not escape such a dichotomy, and these two approaches can work hand in hand.

Example: Lindberg’s proof of CLT

- Proving the CLT for the “integrable case” of Gaussians is virtually trivial. Gaussian calculus is exact.
- Lindberg’s swapping trick: Sums of i.i.d with matching moments will necessarily give the same result.
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Example 2: Wigner matrices
- Integrable case: Gaussian Unitary Ensemble (GUE). Using the rigid tool of determinantal point processes, one can prove the semi-circular law, sine-kernel at the edge and Tracy-Widom distribution for GUE.
- Tao and Vu’s fourth moment theorem: Local statistics match with GUE if four first moments match.

In this talk, we will introduce an integrable process of weakly non-intersecting particles. The integrability finds its source in representation theory and will only be hinted to.
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Analytic construction (1)

Consider two independent Brownian particles. The center of mass is a Brownian motion and is used for centering. Conditioning the particles to never intersect tantamounts to constructing Brownian motion conditioned to remain in $\mathbb{R}_+$. This conditioning is singular and gives the Bessel three process $BES^3$. 

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Approach using regular conditioning: $W^{(\mu)}$ BM with drift $\mu > 0$ killed upon touching 0.

- Infinitesimal generator with Dirichlet boundary conditions:
  \[ L^{(\mu)} = \frac{1}{2} \partial_x^2 + \mu \partial_x \]

- Special harmonic function for $L^{(\mu)}$:
  \[ \mathbb{P}_x \left( W^{(\mu)} \text{ survives} \right) \]
Analytic construction (2)

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$$h_{\mu}(x) = \frac{1}{\mu} e^{\mu x} \mathbb{P}_x \left( W^{(\mu)} \text{ survives} \right) \overset{\text{thm}}{=} \frac{e^{\mu x} - e^{-\mu x}}{\mu}$$

Notice that this normalisation gives analytic extension and symmetry.

- The process conditioned to survive has generator:

$$G^{(\mu)} = h_{\mu}(x)^{-1} \left( \frac{1}{2} \Delta - \frac{1}{2} \mu^2 \right) h_{\mu}(x) = \frac{1}{2} \partial_x^2 + \mu \frac{\cosh(\mu x)}{\sinh(\mu x)} \partial_x$$
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Consider two independent Brownian particles. The center of mass is a Brownian motion and is used for centering. Conditioning the particles to never intersect tantamounts to constructing Brownian motion conditioned to remain in $\mathbb{R}_+$. This conditioning is singular and gives the Bessel three process $BES^3$.

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By letting $\mu \to 0$, we recover the $BES^3 = $ BM conditioned to stay positive.
Consider the $2 \times 2$ Hermitian Brownian motion:

$$GUE_t := \begin{pmatrix} B^1_t & B^2_t + iB^3_t \\ B^2_t - iB^3_t & -B^1_t \end{pmatrix}$$

Its spectrum is $\{\Lambda_t, -\Lambda_t\}$ where $(\Lambda_t; t \geq 0) \overset{\mathcal{L}}{=} BES^3$:

$$\mathbb{P} (\Lambda_t \in dx) = \frac{1}{Z} x^2 e^{-\frac{x^2}{2t}}$$
Let $W$ a standard random walk on $\mathbb{Z}$. Then:

$$\Lambda_n := W_n - 2 \inf_{0 \leq k \leq n} W_k$$

is Markov with transition kernel on $\mathbb{N}$:

$$Q(x, x + 1) = \frac{1}{2} \frac{x + 2}{x + 1} \quad Q(x, x - 1) = \frac{1}{2} \frac{x}{x + 1}$$
Theorem (Discrete Pitman(1975))

Let \( W \) a standard random walk on \( \mathbb{Z} \). Then:

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After the diffusive scaling:

Theorem (Continuous Pitman(1975))

Let \( W \) a standard BM on \( \mathbb{R} \). Then \( \Lambda_t = W_t - 2 \inf_{0 \leq s \leq t} W_s \) is a \( BES^3 \).

Comments

- Very strong rigidity. No other coefficient but 2 works.
- The Pitman transform is of representation theoretic significance.
- The transition probabilities reflect structure constants of the representation theory of \( \mathfrak{sl}_2 \).
Representation theoretic explanation (1):

There is a representation-theoretic story to give here ($2 = \alpha(\alpha^\vee)$).
Consider the Lie algebra $\mathfrak{sl}_2$. For any $n \in \mathbb{N}$, highest weight, there is an irreducible representation $V(n)$ of dimension $n + 1$.

$V(n) \sim B(n)$ a crystal = a combinatorial object that can be realized as paths thanks to the Littelmann path model.

Figure: $\mathfrak{sl}_2$ path crystal with highest weight $n = 4$
Representation theoretic explanation (2)

The Pitman transform

\[ \mathcal{P} : \pi \mapsto \pi(t) - 2 \inf_{0 \leq s \leq t} \pi(s) \]

has a special interpretation in the context of the Littelmann path model: It gives the dominant path in a crystal.

Let \( V(1) = \mathbb{C}^2 \) be the standard representation of \( \mathfrak{sl}_2 \).

- Looking at the standard random walk \( B_n \) can be seen as following a weight vector in \( V(1)^{\otimes n} \).

- Looking at its Pitman transform \( X_n \) means following a highest weight in a decomposition of \( V(1)^{\otimes n} \) into irreducibles. The transition probabilities are given by the Clebsch-Gordan rule:

\[
V(n) \otimes V(1) \approx V(n + 1) \oplus V(n - 1)
\]

Conclusion: Pitman's theorem is about the Markov property of a highest weight process and transition probabilities are expressed in terms of structure constants.
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The exponential potential (1)

In order to have a weak repulsion from zero, an idea is to consider $W$ a BM “slowly killed” when being negative. The framework of submarkovian generators fits the bill.

- Infinitesimal generator:
  \[ \mathcal{L}^{(\mu)} = \frac{1}{2} \partial_x^2 + \mu \partial_x - 2e^{-2x} \]

- Special harmonic function for $\mathcal{L}^{(\mu)}$:
  \[
  \mathbb{P}_x \left( W^{(\mu)} \text{ survives} \right) = \mathbb{E}_x \left( \exp \left( -2 \int_0^\infty e^{-2W_s^{(\mu)}} ds \right) \right)
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The exponential potential (2)

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- Special harmonic function for $\frac{1}{2} \Delta - 2e^{-2x} - \frac{1}{2} \mu^2$:

$$\psi_{\mu}(x) = \Gamma(\mu) e^{\mu x} P_x \left( W^{(\mu)} \text{ survives} \right)$$

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This normalisation gives analytic extension and symmetry ($2x - t \longleftrightarrow t$ changes $\mu$ to $-\mu$).

- The process conditioned to survive has generator:

$$G^{(\mu)} = \psi_{\mu}(x)^{-1} \left( \frac{1}{2} \Delta - 2e^{-2x} - \frac{1}{2} \mu^2 \right) \psi_{\mu}(x) = \frac{1}{2} \partial_x^2 + \partial_x \log \psi_{\mu}(x) \partial_x$$
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The limit $\mu \to 0$ makes sense.
Let $W^{(\mu)}$ a Brownian motion with drift $\mu$. Then:

$$\Lambda_t^{(\mu)} = W_t^{(\mu)} + \log \left( \int_0^t e^{-2W_s^{(\mu)}} \, ds \right)$$

is Markov with inf. generator

$$\psi^{-1}_\mu \left( \frac{1}{2} \frac{d^2}{dx^2} - 2e^{-2x} - \frac{\mu^2}{2} \right) \psi_\mu$$
Theorem (Matsumoto-Yor(2000))

Let $W^{(\mu)}$ a Brownian motion with drift $\mu$. Then:

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$$\psi_{\mu}^{-1} \left( \frac{1}{2} d^2 x^2 - 2e^{-2x} - \frac{\mu^2}{2} \right) \psi_{\mu}$$

By Brownian rescaling and the Laplace method:

$$W_t^{(\mu)} + h \log \left( \int_0^t e^{-2\frac{W_s^{(\mu)}}{h}} ds \right) \xrightarrow{h \to 0} W_t^{(\mu)} - 2 \inf_{0 \leq s \leq t} W_s^{(\mu)}$$

is Markov with inf. generator

$$\psi_{h,\mu}^{-1} \left( \frac{1}{2} d^2 x^2 - 2e^{-2\frac{x}{h}} - \frac{\mu^2}{2} \right) \psi_{h,\mu} \xrightarrow{h \to 0} h_{\mu}^{-1} \left( \frac{1}{2} d^2 x^2 - \frac{\mu^2}{2} \right) h_{\mu}$$
Geometric construction

Such process appears in a curved version of the Hermitian Brownian motion. Consider, a left-invariant SDE on the lower triangular $2 \times 2$ matrices driven by $W$:

$$dB_t(W^{(\mu)}) = B_t(W^{(\mu)}) \circ \begin{pmatrix} dW_t^{(\mu)} & 0 \\ 2dt & -dW_t^{(\mu)} \end{pmatrix}$$

where $\circ$ stands for the Stratononich integral.

Its solution is:

$$B_t(W^{(\mu)}) = \begin{pmatrix} e^{W_t^{(\mu)}} & 0 \\ e^{W_t^{(\mu)}} & 2e^{2\int_0^t -2W_s^{(\mu)} \, ds} e^{-W_t^{(\mu)}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{W_t^{(\mu)}} & 0 \\ 2e^{\Lambda_t^{(\mu)}} & e^{-W_t^{(\mu)}} \end{pmatrix}$$
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Analytic construction (1)

Consider the Weyl chamber $C = \{ x \in \mathbb{R}^n | x_1 > x_2 > \cdots > x_n \}$ and let $W^{(\mu)}$ be a BM with drift $\mu \in C$ killed upon touching $\partial C$.

- Infinitesimal generator with Dirichlet boundary conditions:
  \[ \mathcal{L}^{(\mu)} = \frac{1}{2} \Delta + \langle \mu, \nabla \rangle \]

- Special harmonic function for $\mathcal{L}^{(\mu)}$:
  \[ \mathbb{P}_x \left( W^{(\mu)} \text{survives} \right) \]
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- Special harmonic function for $\frac{1}{2} \Delta - \frac{1}{2} \| \mu \|^2$:
  \[ h_\mu(x) = \frac{e^{\langle \mu, x \rangle}}{\prod_{i<j}(\mu_i - \mu_j)} \mathbb{P}_x \left( W^{(\mu)} \text{ survives} \right) = \frac{\det (e^{\mu_i x_j})_{i,j=1}^n}{\prod_{i<j}(\mu_i - \mu_j)} \]
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  Notice that we have analytic extension to $\mu \in \mathbb{C}^n$ and symmetry in the variables $(\mu_1, \ldots, \mu_n)$.

- Process conditioned to survive has generator:
  $$G^{(\mu)} = h_\mu(x)^{-1} \left( \frac{1}{2} \Delta - \frac{1}{2} ||\mu||^2 \right) h_\mu(x) = \frac{1}{2} \Delta + \langle \nabla \log h_\mu, \nabla \rangle$$

As $\mu \to 0$, $h_\mu(x) \to \Delta(x) := \prod_{i<j} (x_i - x_j)$ (Not obvious!). And $G^{(\mu=0)}$ is the generator of Dyson’s Brownian motion.
Geometric construction via RMT

Consider the $n \times n$ Hermitian Brownian motion - marginally distributed as $\sqrt{t}GUE$:

$$GUE_t := \begin{pmatrix}
B_{11}^t & B_{12}^t + i\tilde{B}_{12}^t & \cdots & B_{1n}^t + i\tilde{B}_{1n}^t \\
B_{12}^t - i\tilde{B}_{12}^t & B_{22}^t & \cdots & B_{2n}^t + i\tilde{B}_{2n}^t \\
\cdots & \cdots & \cdots & \cdots \\
B_{1n}^t - i\tilde{B}_{1n}^t & B_{2n}^t + i\tilde{B}_{2n}^t & \cdots & B_{nn}^t
\end{pmatrix}$$

Its spectrum $\{\Lambda_1^t > \Lambda_2^t > \cdots > \Lambda_n^t\}$ is a Markovian diffusion called Dyson’s Brownian motion with generator:

$$\mathcal{G} = \frac{1}{2}\Delta + \langle \nabla \log \Delta, \nabla \rangle = \frac{1}{2}\Delta + \sum_{i<j} \frac{\partial \Lambda_i}{\Lambda_i - \Lambda_j}$$

Moreover (GUE density):

$$\mathbb{P}(\Lambda_t \in dx) = \frac{1}{Z_n} \Delta(x)^2 e^{-\|x\|^2_{2t}}$$
Via a Pitman-type construction

Partially due to O’Connell-Yor for type $A$, and to Biane, Bougerol and O’Connell for general Lie type. In type $A$:

- There is a (deterministic) Pitman transform that folds paths in $\mathbb{R}^n$ into the cone $C$:
  \[ \mathcal{P}_{w_0} : C_0 ([0, t], \mathbb{R}^n) \to C_0 ([0, t], C) \]
  Coincides with the original Pitman transform for $n = 2$. 

\[\rightarrow\] 

"Algebraic" construction of non-intersecting particles.
Via a Pitman-type construction

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- Such transform is of representation-theoretic significance: It is the highest weight transform in the continuous Littelmann path model.
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- Such transform is of representation-theoretic significance: It is the highest weight transform in the continuous Littelmann path model.

- If $W^{(\mu)}$ is a Brownian motion in $\mathbb{R}^n$ with drift $\mu$ then:

  $\left( \mathcal{P}_{w_0} \left( W^{(\mu)} \right) (t); t \geq 0 \right)$

  is the Markovian diffusion given by Brownian motion conditioned to remain in $C$ - as in the “Analytic” construction.

  $\rightsquigarrow$ “Algebraic” construction of non-intersecting particles.
Application to Last Passage Percolation

(Blackboard explanation of LPP in an $n \times M$ box)
Application to Last Passage Percolation

(Blackboard explanation of LPP in an $n \times M$ box)

When expliciting the first coordinate of the Pitman transform on $\mathbb{R}^n$:

$$\left( \mathcal{P}_{w_0} W \right)_1 (t) = \sup_{0 = t_0 < t_1 < \cdots < t_n = t} \sum_{i=1}^{n} W^i (t_i, t_{i-1})$$

which is interpreted as a semi-discrete LPP obtained from the diffusive rescaling as $M \to \infty$.

Since $\mathcal{P}_{w_0} W$ is distributed as Dyson's Brownian motion, the above quantity is distributed as $\sqrt{t}$ times the largest eigenvalue of a GUE matrix (Baryshnikov and Tracy-Widom).
Application to Last Passage Percolation

(Blackboard explanation of LPP in an \( n \times M \) box)

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Since \( P_{w_0}W \) is distributed as Dyson’s Brownian motion, the above quantity is distributed as \( \sqrt{t} \) times the largest eigenvalue of a GUE matrix (Baryshnikov and Tracy-Widom).

Random matrix theory gives the weak convergence:

\[
\frac{(P_{w_0}W)_1(t) - 2\sqrt{tn}}{\sqrt{tn^{\frac{1}{6}}}} \xrightarrow{n \to \infty} TW_2
\]
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Analytic construction (1)

Following naively the one dimensional logic, we will add the exponential potential for each wall in the Weyl chamber

\[ C = \{ x \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_n \} \]

hence the Toda potential

\[ V(x) = \sum_{i=1}^{n-1} e^{-(x_{i+1} - x_i)} \]

- Infinitesimal generator of \( W^{(\mu)} \) “slowly killed” BM:

\[ \mathcal{L}^{(\mu)} = \frac{1}{2} \Delta + \langle \mu, \nabla \rangle - V(x) \]

- Special harmonic function for \( \mathcal{L}^{(\mu)} \):

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\[ V(x) := \sum_{i=1}^{n-1} e^{-(x_{i+1} - x_i)} \]

- Infinitesimal generator of \( W^{(\mu)} \) “slowly killed” BM:

\[ L^{(\mu)} = \frac{1}{2} \Delta + \langle \mu, \nabla \rangle - V(x) = e^{-\langle \mu, x \rangle} \left( \frac{1}{2} \Delta - V(x) - \frac{1}{2} \| \mu \|^2 \right) e^{\langle \mu, x \rangle} \]

- Special harmonic function for \( \frac{1}{2} \Delta - V(x) - \frac{1}{2} \| \mu \|^2 \):

\[ \psi_{\mu}(x) = \prod_{i<j} \Gamma(\mu_i - \mu_j) e^{\langle \mu, x \rangle} \mathbb{P}_x \left( W^{(\mu)} \text{survives} \right) \]

Following Jacquet, this is the Archimedean Whittaker function.

- Process conditioned to survive is the Whittaker process. Generator:

\[ G^{(\mu)} = \psi_{\mu}(x)^{-1} \left( \frac{1}{2} \Delta - \frac{1}{2} \| \mu \|^2 \right) \psi_{\mu}(x) = \frac{1}{2} \Delta + \langle \nabla \log \psi_{\mu}, \nabla \rangle \]
Geometric construction - “Hypoelliptic BM on a lower triangular matrices”

Let $W^{(\mu)}$ be a Brownian motion with drift $\mu$ on $\mathbb{R}^n$. For notational reasons, we drop the superscript $(\mu)$ and put indices as exponents. Consider the SDE on lower triangular matrices:

$$dB_t(W^{(\mu)}) = B_t(W^{(\mu)}) \circ \begin{pmatrix}
    dW_t^1 & 0 & 0 & \cdots & 0 \\
    dt & dW_t^2 & 0 & \cdots & \cdots \\
    0 & dt & \cdots & \cdots & 0 \\
    \vdots & \vdots & \ddots & dW_t^{n-1} & 0 \\
    0 & \cdots & 0 & dt & dW_t^n
\end{pmatrix}$$

and its solution $B_t(W^{(\mu)})$ is given by:

$$
\begin{pmatrix}
    e^{W_t^1} & 0 & 0 & \cdots \\
    e^{W_t^1 \int_0^t e^{W_s^2-W_s^1} ds} & e^{W_t^2} & 0 & \cdots \\
    e^{W_t^1 \int_0^t e^{W_s^2-W_s^1} ds_1 \int_0^{s_1} e^{W_s^2-W_s^1} ds_2} & e^{W_t^2 \int_0^t e^{W_s^3-W_s^2} ds} & e^{W_t^3} & \cdots \\
    \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
$$
Via a Pitman-type construction

- There is a geometric Pitman transform:
  \[ \mathcal{T}_{w_0} : C_0 ([0, t], \mathbb{R}^n) \rightarrow C_0 ([0, t], \mathbb{R}^n) \]
  which degenerates to \( \mathcal{P}_{w_0} = \lim_{h \rightarrow 0} h \mathcal{T}_{w_0} h^{-1} \). In fact:
  \[ (\mathcal{T}_{w_0} W)_{k} (t) = \log \det \left( B_t (W)_{i=n,\ldots,n-k+1}^{j=1,\ldots,k} \right) \]

- Such transform is of representation-theoretic significance: It is the highest weight transform in the geometric Littelmann path model (constructed in chapter 4 of thesis).
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- (Givental in type A; chapter 5 of thesis for general Lie type) The Whittaker function \( \psi_\mu (x) \) is a symmetric and entire function in \( \mu = (\mu_1, \ldots, \mu_n) \).
Via a Pitman-type construction

- There is a geometric Pitman transform:
  \[ T_{w_0} : C_0 ([0, t], \mathbb{R}^n) \to C_0 ([0, t], \mathbb{R}^n) \]
  which degenerates to \( P_{w_0} = \lim_{h \to 0} h T_{w_0} h^{-1} \). In fact:
  \[
  (T_{w_0} W)_k (t) = \log \det \left( B_t (W)^{j=1,\ldots,k}_{i=n,\ldots,n-k+1} \right)
  \]

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- (O'Connell 2009 in type A; chapter 6 of thesis for general Lie type) If \( W^{(\mu)} \) is a Brownian motion in \( \mathbb{R}^n \) with drift \( \mu \) then:
  \[
  \left( T_{w_0} \left( W^{(\mu)} \right) (t); t \geq 0 \right)
  \]
  is the Whittaker process - as in the “Analytic” construction.

\( \rightsquigarrow \) “Algebraic” construction of weakly non-intersecting particles.
Application to directed polymers

(Blackboard explanation of $1 + 1$ directed polymer in an $n \times M$ box)
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(Blackboard explanation of $1 + 1$ directed polymer in an $n \times M$ box)

When expliciting the first coordinate of the geometric Pitman transform on $\mathbb{R}^n$:

$$(T_{w_0} W)_1 (t) = \log \int_{0=0}^{t_0 < t_1 < \ldots < t_n = t} e^{\sum_{i=1}^{n} W^i(t_i, t_{i-1})}$$

which is interpreted as a semi-discrete partition function obtained from the diffusive rescaling as $M \to \infty$.

Since $T_{w_0} W$ is distributed as the Whittaker process, the relevance to mathematical physics is clear.
Application to directed polymers

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Since $\mathcal{T}_{w_0} W$ is distributed as the Whittaker process, the relevance to mathematical physics is clear.

**Theorem (Borodin-Corwin-Ferrari)**

There is a $\beta^* > 0$ and constants $a_t$ $b_t$ such that for all $0 \leq \beta < \beta^*$:

$$\frac{\beta}{n} \left( \mathcal{T}_{w_0} \frac{n W}{\beta} \right)_1 (t) - a_t \sqrt{n} \quad \xrightarrow{n \to \infty} \quad TW_2$$
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Conclusion

What we did:

- We presented the Whittaker process as $n$ particles slowly killed with an exponential potential and conditioned to survive.

- It degenerates via rescaling to Dyson’s Brownian motion, hence expected random-matrix behaviors.

- It has a representation-theoretic (or Pitman-type) construction.