On the circle, \( \text{GMC}^\gamma = \lim \ C^\beta E_n \) if \( \gamma = \sqrt{\frac{2}{\beta}} \leq 1 \)

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Sommaire
A puzzling identity in law

Consider \((\mathcal{N}_1^C, \mathcal{N}_2^C, \ldots)\) to be a sequence of i.i.d standard complex Gaussians i.e:

\[
P(\mathcal{N}_i^C \in dxdy) = \frac{1}{\pi} e^{-x^2-y^2} dxdy,
\]

so that:

\[
E \mathcal{N}_k^C = 0, \quad E|\mathcal{N}_k^C|^2 = 1.
\]

Let \((\alpha_j)_{j \geq 0}\) be independent random variables with uniform phases and modulii as follows:

\[
|\alpha_j|^2 \overset{\mathcal{L}}{=} \text{Beta}(1, \beta_j := \frac{\beta}{2}(j + 1))
\]

As a shadow of a more global correspondence between GMC and RMT:

**Proposition (Verblunsky expansion of Gaussians)**

The following equality in law holds, while the RHS converges almost surely (!):

\[
\sqrt{\frac{2}{\beta}} \mathcal{N}_1^C \overset{\mathcal{L}}{=} \sum_{j=0}^{\infty} \alpha_j \overline{\alpha}_{j-1}.
\]
A puzzling identity in law (II)

“Numerical proof:” Histogram of \( \Re \left( \sigma \sum_{j=0}^{\infty} \alpha_j \bar{\alpha}_{j-1} \right) , |\sigma| = 1. \)
A puzzling identity in law (III)

“Numerical proof:”
The main player of this talk will be the random Gaussian distribution on $S^1$:

$$G(e^{i\theta}) := 2\Re \sum_{k=1}^{\infty} \frac{\mathcal{N}_k^\mathbb{C}}{\sqrt{k}} e^{ik\theta}.$$ 

**Remark**

*Given the decay of Fourier coefficients, this is a Schwartz distribution in negative Sobolev spaces $\cap_{\varepsilon > 0} H^{-\varepsilon}(S^1)$ where:*

$$H^s(S^1) := \left\{ f \mid \sum_{k \in \mathbb{Z}} |k|^s |\hat{f}(k)|^2 \right\}.$$
Harmonic extension of $G$

Consider the harmonic extension of $G$ to the disc:

$$G(re^{i\theta}) := 2\Re \sum_{k=1}^{\infty} \frac{N_k^C}{\sqrt{k}} r^k e^{ik\theta} = P_r \ast G_{|S^1}(e^{i\theta}),$$

where $P_r$ is the Poisson kernel.
Sommaire
Modern motivations: “Liouville Conformal Field Theory” in 2D

Brownian Map (© Bettinelli)

\[
\text{Uniformization } \xrightarrow{\sim} \left( \vphantom{A}\right), \text{GMC}^\gamma(dz) \right) \\
\text{for } \gamma = \sqrt{\frac{8}{3}}.
\]

(Theorem by Miller-Sheffield)

**Message**

*The GMC\(^\gamma\) is the natural Riemannian measure on random surfaces which model LCFT.*

But please, ask someone else to tell you about this... E.g. Rhodes-Vargas, Miller-Sheffield and/or their students.
Our construction: On the circle, in 1d

A natural object (for Kahane and the LCFT crowd) is:

\[ GMC_{r}^{\gamma}(d\theta) := e^{\gamma G(re^{i\theta})} - \frac{1}{2} \text{Var}[G(re^{i\theta})] \frac{d\theta}{2\pi} = e^{\gamma G(re^{i\theta})} (1 - r^2)^{\gamma^2} \frac{d\theta}{2\pi}. \]

We have:

**Theorem (Kahane, Rhodes-Vargas, Berestycki)**

Define for every \( f : S^1 = \partial \mathbb{D} \to \mathbb{R}_+ \), and \( \gamma < 1 \):

\[ GMC_{r}^{\gamma}(f) := \int_{0}^{2\pi} f(e^{i\theta}) GMC_{r}^{\gamma}(d\theta). \]

Then the following convergence holds in \( L^1(\Omega) \):

\[ GMC_{r}^{\gamma}(f) \overset{r \to 1}{\longrightarrow} GMC^{\gamma}(f). \]

The limiting measure \( GMC^{\gamma} \) is called Kahane’s Gaussian Multiplicative Chaos.
Sommaire
The model

- Consider the distribution of $n$ points on the circle:

$$
(C\beta E_n) \frac{1}{Z_{n,\beta}} \prod_{1 \leq k < l \leq n} |e^{i\theta_k} - e^{i\theta_l}|^\beta \, d\theta = \frac{1}{Z_{n,\beta}} |\Delta(\theta)|^\beta \, d\theta
$$

- For $\beta = 2$, one recognizes the Weyl integration formula for central functions on the compact group $U(n)$. Therefore, this nothing but the distribution of a Haar distributed matrix on the group $U(n)$. The study of this case is very rich in the representation theory of $U_n$ (Bump-Gamburd, Borodin-Okounkov, ...)

- For general $\beta$, not as nice but still an integrable system: Jack polynomials in $n$ variables are orthogonal for the $C\beta E_n$, Eigenvectors for the trigonometric Calogero-Sutherland system ($n$ variables), “Higher” representation theory (Rational Cherednik algebras).

- The characteristic polynomial:

$$X_n(z) := \det (\text{id} - zU_n^*) = \prod_{1 \leq j \leq n} (1 - ze^{-i\theta_j})$$
CBE as regularization of Gaussian Fock space

The $C\beta E_n$ is the regularization of a Gaussian space by $n$ points at the level of symmetric functions. In fact:

$$\text{tr} \left( U_n^k \right) \xrightarrow{n \to \infty} \sqrt{\frac{2k}{\beta}} N_k^C,$$

(Strong Szegö - $\beta = 2$, Diaconis-Shahshahani - $\beta = 2$, Matsumoto-Jiang)

Short proof: Open the bible of symmetric functions
CBE as regularization of Gaussian Fock space: Proof

- Power sum polynomials: \( p_k := p_k(U_n) = \text{tr} \left( U_n^k \right) \) and \( p_\lambda := \prod_i p_{\lambda_i} \).
- Scalar product for functions in \( n \) variables: \( \langle f, g \rangle_n := \mathbb{E}_{C\beta E_n} \left( f(z_i)g(z_i) \right) \).
- Fact 1: This scalar product approximates the Hall-Macdonald scalar product in infinitely many variables \( \langle \cdot, \cdot \rangle_n \rightarrow \langle \cdot, \cdot \rangle \), where
  \[
  \langle p_\lambda, p_\mu \rangle = z_\lambda \left( \frac{2}{\beta} \right)^{\ell(\lambda)} \delta_{\lambda,\mu} = \delta_{\lambda,\mu} \text{Cste}(\lambda) .
  \]
- Fact 2: The Macdonald scalar product has a Gaussian space lurking behind as
  \[
  \delta_{\lambda,\mu} \text{Cste}(\lambda) = \mathbb{E} \left( \prod_k \left( \sqrt{\frac{2k}{\beta} \mathcal{N}_k^C} \right)^{m_k(\lambda)} \left( \sqrt{\frac{2k}{\beta} \mathcal{N}_k^C} \right)^{m_k(\mu)} \right) ,
  \]
  where \( m_k(\lambda) \) multiplicity of \( k \) in partition \( \lambda \).

\( \Rightarrow \) the \( C\beta E \) is the regularization of a Gaussian Fock space by restricting the symmetric functions to \( n \) variables.
Since:

$$\log X_n(z) = \sum_{k \geq 1} \frac{\text{tr} \left( U_n^k \right)}{k} z^k,$$

it is conceivable that:

Proposition (O’C-H-K for $\beta = 2$, C-N for $\beta > 0$)

We have the convergence in law to the log-correlated field:

$$(\log |X_n(z)|)_{z \in \mathbb{D}} \xrightarrow{n \to \infty} \left( \sqrt{\frac{2}{\beta}} G(z) \right)_{z \in \mathbb{D}}$$

- uniformly in $z \in K \subset \mathbb{D}$, $K$ compact.
- for $z \in \partial \mathbb{D}$, in the Sobolev space $H^{-\varepsilon}(\partial \mathbb{D})$. 
A step further, it is natural to construct a measure from the characteristic polynomial

$$(\log |X_n(z)|)_{z \in \mathbb{D}} \xrightarrow{n \to \infty} \left(\sqrt{\frac{2}{\beta}} G(z)\right)_{z \in \mathbb{D}}$$

and compare it to the GMC.

Here is a result whose content is very different from ours but easily confused with it.
GMC from RMT: A convergence in law (I)

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Here is a result whose content is very different from ours but easily confused with it. Building on ideas of Berestycki, then Lambert-Ostrovsky-Simm (2016):

**Proposition (Nikula, Saksman and Webb (2018))**

For \( \beta = 2 \) and for every \( \alpha \in [0, 2) \), consider \( X_n(z) = \det(I_n - zU_n^*) \) to be the characteristic polynomial of the CUE = C2E. Then, for all continuous \( f : \partial \mathbb{D} \to \mathbb{R} \), we have the convergence in law as \( n \to \infty \):

\[
\int_{[0,2\pi]} \frac{d\theta}{2\pi} f(e^{i\theta}) \frac{\left| \frac{X_n(e^{i\theta})}{\mathbb{E}\left| X_n(e^{i\theta}) \right|^\alpha} \right|^{\alpha}}{\left| X_n(e^{i\theta}) \right|^\alpha} \xrightarrow{\mathcal{L}} \text{GMC}^{\alpha/2}(f) .
\]
A few remarks are in order:

- In fact, for $\beta = 2$, there is an extremely fast convergence of traces of Haar matrices to Gaussians. For $f$ polynomial on the circle, we have:

$$d_{TV} \left( \text{Tr} f(U_n), \sum_k c_k(f) \sqrt{k} N_k^C \right) \xrightarrow{n \to \infty} C_f n^{-cn/\deg f}.$$  

- Nikula, Saksman and Webb (NSW) leverage the (notoriously technical) Riemann-Hilbert problem, which packages neatly this convergence for traces of high powers in order to compare to GMC.

- Probably hopeless for general $\beta$, where convergence to Gaussians is known to be slower and finding a machinery that replaces the RH problem, while being just as precise, is an open question of its own.

Message (Take home message)

Our statement $\text{GMC}^\gamma = \lim C \beta E_n$ is non-asymptotic and an almost sure equality for all $\beta > 0$ and $n \in \mathbb{N}$, via a non-trivial coupling. We are saying for $\gamma < 1$:

"$\text{GMC}^\gamma$ is the object whose finite $n$ approximations are given by $C \beta E_n$'s."
Sommaire
OPUC and Szegö recurrence

- OPUC: “Orthogonal Polynomials on the Unit Circle”
- Consider a probability measure \( \mu \) on the circle and apply the Gram-Schmidt procedure:
  \[
  \{1, z, z^2, \ldots \} \leadsto \{\Phi_0(z), \Phi_1(z), \Phi_2(z), \ldots \}
  \]
- Szegö recurrence:
  \[
  \begin{align*}
  \Phi_{j+1}(z) &= z\Phi_j(z) - \alpha_j \Phi_j^*(z) \\
  \Phi_{j+1}^*(z) &= -\alpha_j z\Phi_j(z) + \Phi_j^*(z).
  \end{align*}
  \]
  Here:
  \[
  \Phi_j^*(z) := z^j \Phi_j(1/\overline{z})
  \]
  is the polynomial with reversed and conjugated coefficients. The \( \alpha_j \)'s are inside the unit disc, known as Verblunsky coefficients.
The work of Killip, Nenciu

Killip and Nenciu have discovered an explicit distribution for Verblunsky coefficients so that $X_n$, the characteristic polynomial of $C_\beta E_n$, is a $\Phi_n^*$!

**Theorem (Killip, Nenciu)**

1. Let $(\alpha_j)_{j \geq 0}$, as before and $\eta$ uniform on the circle.
2. Let $(\Phi_j, \Phi_j^*)_{j \geq 0}$ be a sequence of OPUC obtained from the coefficients $(\alpha_j)_{j \geq 0}$ and the Szegö recurrence.

Then we have the equality in law between random polynomials:

$$X_n(z) = \Phi_{n-1}^*(z) - z\eta\Phi_{n-1}(z).$$

**Proof.**

Essentially computation of a Jacobian - with two important subtleties!

**IMPORTANT:** Projective family. Notice the consistency. A priori, a realization of $CBE_n$ has no reason to share the first Verblunsky coefficients with $CBE_{n+1}$.
A puzzling question

If a measure defines Verblunsky coefficients, the converse is also true:

**Theorem (Verblunsky 1930)**

Let $\mathcal{M}_1(\partial \mathbb{D})$ be the simplex of probability measures on the circle, endowed with the weak topology. The set $\mathbb{D}^\mathbb{N}$ is endowed with the topology of point-wise convergence. The Verblunsky map

$$\nabla : \mathcal{M}_1(\partial \mathbb{D}) \rightarrow \mathbb{D}^\mathbb{N} \sqcup (\bigsqcup_{n \in \mathbb{N}} \mathbb{D}^n \times \partial \mathbb{D})$$

$$\mu \mapsto (\alpha_j(\mu); j \in \mathbb{N})$$

is an homeomorphism. Atomic measures with $n$ atoms have $n$ Verblunsky coefficients, the last one being of modulus one.

This begs the question:

**Question**

The Verblunsky coefficients are consistent. Since the obvious coupling respects the Verblunksy map, we define a limiting measure $\lim \leftarrow CBE_n$, whose $n$-point approximation/projection is the $CBE_n$. What is this measure?
Sommaire
Statement

Theorem (C-Najnudel, arXiv:1904.00578)

For $\gamma = \sqrt{\frac{2}{\beta}} \leq 1$, we have equality between

- the measure $\mu^\beta$ whose Verblunsky coefficients are the $(\alpha_n; n \in \mathbb{N})$ from $C_\beta E$.
- Kahane’s $GMC^\gamma$, renormalized into a probability measure.

\[
\mu^\beta(d\theta) = \frac{1}{GMC^\gamma(\partial \mathbb{D})} GMC^\gamma(d\theta) .
\]

\[\Rightarrow\] One can theoretically sample the $GMC^\gamma$. Then upon considering the best approximating measure on $n$ points, the quadrature points are nothing but the RMT ensembles $CBE_n$.

\[\Rightarrow\] One could write a projective limit:

\[
GMC^\gamma = C_\beta E_\infty := \lim_{n \to \infty} C_\beta E_n .
\]
Ideas of proof

Finitely many Verblunsky coeff
- RMT regularization of Gaussians

Bernstein-Szegö approx.
\[ \mu^\beta_n(d\theta) = \frac{\prod_{j=0}^{n-1}(1-|\alpha_j|^2)}{|\Phi^*_n(e^{i\theta})|^2} d\theta \]

Gaussian fields
\[ \mu^\beta_r(d\theta) = \frac{e^{\omega_r(\theta)}}{C_0} GMC^{\gamma}(d\theta) \]

On the circle
\[ n \rightarrow \infty \]
\[ r \rightarrow 1 \]

Difficult points:
- Filtrations by Gaussians and Verblunsky coefficients (\( \mathbb{F} \)) have bad overlap. Top \( n \rightarrow \infty \) limit is built to be a martingale limit, with parameter \( r \).
- Doob decomposition w.r.t \( \mathbb{F} \): \( \omega_r = \sum_{k=0}^{\infty} (1 - r^2) \frac{Y_k^r}{k+1} \). \( Y^k \) has a non-trivial limiting SDE as \( r \rightarrow 1 \). SDE is ill-behaved at time 0.
- SDE = Crossing mechanism, which quickly forgets initial Verblunsky coefficients, thanks to non-trivial entrance law. Crucial for \( r \rightarrow 1 \) limit.
Consequences

- \((C\beta E_n ; \beta \geq 2, n \in \mathbb{N}^*)\) can all be coupled upon constructing \((GMC^\gamma ; 0 \leq \gamma < 1)\).
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- (Fyodoroff-Bouchaud) Another proof of G. Rémy’s identity:

\[
GMC^\gamma(\partial \mathbb{D}) = K_\beta \prod_{j=0}^{\infty} (1 - |\alpha_j|^2)^{-1} e^{-\frac{2}{\beta(j+1)}} \overset{\mathcal{L}}{=} K'_\beta \ e^{-\frac{2}{\beta}}.
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  \]
- (Beyond Fyodoroff-Bouchaud) One can also describe all moments
  \[
  c_k = \frac{1}{GMC^\gamma(\partial \mathbb{D})} \int_0^{2\pi} e^{ik\theta} GMC^\gamma(d\theta).
  \]
  via universal expressions in terms of the Verblunsky coefficients. For example:

  \[
  \begin{align*}
  c_1 &= \alpha_0 , \\
  c_2 &= \alpha_0^2 + \alpha_1(1 - |\alpha_0|^2) , \\
  c_3 &= (\alpha_0 - \alpha_1 \bar{\alpha}_0)[\alpha_0^2 + \alpha_1(1 - |\alpha_0|^2)] \\
  &\quad + \alpha_1 \alpha_0 + \alpha_2(1 - |\alpha_0|^2)(1 - |\alpha_1|^2).
  \end{align*}
  \]
Open questions

Our result brings forth other questions:

- What happens in the supercritical phase $\beta < 2 \leftrightarrow \gamma > 1$? Our intuition suggests no freezing. Conjectural answer: the KPZ dual measure.
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- $C\beta E$ has an intimate relationship to algebraic structures: Jack polynomials, the integrable Calogero-Sutherland system ($\sim$ Wick-rotated circular Dyson dynamics), Vertex algebras... Bridge between the Liouville CFT/GMC and the algebra?
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- Question to physicists: Role of $\beta_{\text{critical}} = \beta = 2$? This is where the geometry and rep. theory of unitary groups lies.
- Linking dynamics in RMT and dynamics in conformal growth: Hastings-Levitov (hint in work of Norris-Turner-Silvestri), Loewner-(Kufarev) Evolutions...
Acknowledgements

Thank you for your attention!