# KMAS9AA1 – Algebraic Topology

## Exercise Sheet 6

### 1. Tor

1) Assume that $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$ are finitely generated. Show that for any prime $p$, $H_n(X;\mathbb{Z}/p\mathbb{Z})$ consists of:

   i. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$,
   ii. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$,
   iii. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_{n-1}(X;\mathbb{Z})$,

   This problem is a direct application of the Universal Coefficient Theorem (UCT) for homology. The theorem states that for a topological space $X$ and an abelian group $G$, there exists a natural short exact sequence 1:

$$0 \to H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} G \to H_n(X;G) \to \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), G) \to 0$$

   Furthermore, this sequence splits (though not naturally), yielding an isomorphism:

$$H_n(X;G) \cong (H_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} G) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), G)$$

   We are given $G = \mathbb{Z}/p\mathbb{Z}$ for a prime $p$, and that $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$ are finitely generated. By the classification of finitely generated abelian groups, we can decompose them into a free part and a torsion part. Specifically, the torsion part can be decomposed into its $q$-primary components for all primes $q$.

$$H_k(X;\mathbb{Z}) \cong \mathbb{Z}^{\beta_k} \oplus \left( \bigoplus_j \mathbb{Z}/p^{e_j}\mathbb{Z} \right) \oplus \left( \bigoplus_{q \neq p} \bigoplus_i \mathbb{Z}/q^{k_i}\mathbb{Z} \right)$$

   Here, $\beta_k$ is the rank (the number of $\mathbb{Z}$ summands), and the sums represent the $p$-torsion and non-$p$-torsion components. Let $t_k(p)$ be the number of $\mathbb{Z}/p^k\mathbb{Z}$ summands in $H_k(X;\mathbb{Z})$.We analyze the two terms of the UCT isomorphism separately.

Term 1: The Tensor Product $H_n(X;\mathbb{Z}) \otimes \mathbb{Z}/p\mathbb{Z}$.
Since the tensor product distributes over direct sums, we analyze each type of summand:
$\mathbb{Z}$ summands: $\mathbb{Z} \otimes \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$. Each of the $\beta_n$ summands of $\mathbb{Z}$ in $H_n$ contributes one $\mathbb{Z}/p\mathbb{Z}$ summand. This accounts for item i.
$\mathbb{Z}/p^k\mathbb{Z}$ summands: $\mathbb{Z}/p^k\mathbb{Z} \otimes \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/\gcd(p^k,p)\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$. Each $\mathbb{Z}/p^k\mathbb{Z}$ summand in $H_n$ contributes one $\mathbb{Z}/p\mathbb{Z}$ summand. This accounts for item ii.
$\mathbb{Z}/q^k\mathbb{Z}$ summands ($q \neq p$): $\mathbb{Z}/q^k\mathbb{Z} \otimes \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/\gcd(q^k,p)\mathbb{Z} \cong \mathbb{Z}/1\mathbb{Z} \cong 0$. Summands with torsion prime $q$ different from $p$ are eliminated.
Thus, $(H_n(X;\mathbb{Z}) \otimes \mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^{\beta_n + t_n(p)}$.

Term 2: The Tor Functor $\mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), \mathbb{Z}/p\mathbb{Z})$. $\mathrm{Tor}_1$ also distributes over direct sums.
$\mathbb{Z}$ summands: $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$, because $\mathbb{Z}$ is a free module.
$\mathbb{Z}/p^k\mathbb{Z}$ summands:
As we can deduce from the Tor formula we saw in class,

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p^k\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/\gcd(p^k,p)\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$$

Each $\mathbb{Z}/p^k\mathbb{Z}$ summand in $H_{n-1}$ contributes one $\mathbb{Z}/p\mathbb{Z}$ summand to $H_n(X;\mathbb{Z}/p\mathbb{Z})$. This accounts for item iii.
$\mathbb{Z}/q^k\mathbb{Z}$ summands ($q \neq p$):

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/q^k\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/\gcd(q^k,p)\mathbb{Z} \cong 0.$$

Thus, $\mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^{t_{n-1}(p)}$.

2) Use the universal coefficient theorem to show that if $H_*(X;\mathbb{Z})$ is finitely generated, so the Euler characteristic

$$\chi(X) = \sum_n (-1)^n \mathrm{rank}\ H_n(X;\mathbb{Z})$$

is defined, then for any coefficient field $\mathbb{F}$ we have $\chi(X) = \sum_n (-1)^n \dim H_n(X;\mathbb{F})$.
This is [Hatcher,3A.1.1.]
The Euler characteristic with integer coefficients is $\chi(X) = \sum_n (-1)^n \beta_n$, where $\beta_n = \mathrm{rank}(H_n(X;\mathbb{Z}))$. We wish to show this equals

$$\chi_{\mathbb{F}}(X) = \sum_n (-1)^n \dim_{\mathbb{F}} H_n(X;\mathbb{F})$$

for any field $\mathbb{F}$. From the UCT, we have

$$\dim_{\mathbb{F}} H_n(X;\mathbb{F}) = \dim_{\mathbb{F}}(H_n \otimes \mathbb{F}) + \dim_{\mathbb{F}}(\mathrm{Tor}_1(H_{n-1},\mathbb{F})).$$

Let $H_n(X;\mathbb{Z}) \cong \mathbb{Z}^{\beta_n} \oplus T_n$, where $T_n$ is the torsion subgroup.

$$\dim(H_n(X;\mathbb{F})) = \dim(\mathbb{Z}^{\beta_n}\otimes\mathbb{F})+\dim(T_n\otimes\mathbb{F})+\dim(\mathrm{Tor}_1(\mathbb{Z}^{\beta_{n-1}},\mathbb{F}))+\dim(\mathrm{Tor}_1(T_{n-1},\mathbb{F}))$$

We analyze this based on the characteristic of $\mathbb{F}$.
Case 1: $\mathrm{char}(\mathbb{F}) = 0$. Then $\mathbb{F}$ contains $\mathbb{Q}$ and therefore we can assume $\mathbb{F} = \mathbb{Q}$, since for bigger fields extensions the homology corresponds to just tensoring over $\mathbb{Q}$ with $\mathbb{F}$.
We know that $\dim(\mathbb{Z}^{\beta_n} \otimes \mathbb{Q}) = \dim(\mathbb{Q}^{\beta_n}) = \beta_n$, that $\dim(T_n \otimes \mathbb{Q}) = 0$ and I claimed in class (next exercise) that $\mathbb{Q}$ is flat, and therefore $\dim(\mathrm{Tor}_1(T_{n-1}, \mathbb{Q})) = 0$.
Thus, $\dim(H_n(X;\mathbb{Q})) = \beta_n$.

$$\chi_{\mathbb{Q}}(X) = \sum_n (-1)^n \dim(H_n(X;\mathbb{Q})) = \sum_n (-1)^n \beta_n = \chi(X)$$

Case 2: $\mathrm{char}(\mathbb{F}) = p > 0$ In this case, $\mathbb{F}$ is a $\mathbb{Z}/p\mathbb{Z}$-vector space. As $\otimes$ and Tor commute with direct sums, it suffices to compute for $\mathbb{F} = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.
From the analysis in 1.1), let $t_k(p)$ be the number of $p$-torsion summands in $H_k(X;\mathbb{Z})$. $\dim(\mathbb{Z}^{\beta_n} \otimes \mathbb{F}_p) = \dim(\mathbb{F}_p^{\beta_n}) = \beta_n$. $\dim(T_n \otimes \mathbb{F}_p) = t_n(p)$. $\dim(\mathrm{Tor}_1(\mathbb{Z}^{\beta_{n-1}}, \mathbb{F}_p)) = 0$. $\dim(\mathrm{Tor}_1(T_{n-1}, \mathbb{F}_p)) = t_{n-1}(p)$. So, $\dim_{\mathbb{F}_p}(H_n(X;\mathbb{F}_p)) = \beta_n + t_n(p) + t_{n-1}(p)$. Now we compute the alternating sum:

$$\chi_{\mathbb{F}p}(X) = \sum n(-1)^n \dim(H_n(X;\mathbb{F}p)) = \sum n(-1)^n(\beta_n+t_n(p)+t_{n-1}(p))$$

$$\chi_{\mathbb{F}p}(X) = \sum n(-1)^n\beta_n + \sum_n (-1)^n t_n(p) + \sum_n (-1)^n t_{n-1}(p)$$

The first term is $\chi(X)$. The second two sums cancel each other out in a telescoping fashion. Therefore,

$$\chi_{\mathbb{F}p}(X) = \chi(X) + \left(\sum n(-1)^n t_n(p)\right) - \left(\sum_n (-1)^n t_n(p)\right) = \chi(X)$$

**2. Torsion-free** I claimed in class that while $\mathbb{Q}$ is not free, it is torsion-free and therefore $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, C) = 0, \forall C \in R-\mathrm{Mod}$. Let us show this.

Let $G$ be an abelian group.

1) Show that any element of $G \otimes \mathbb{Q}$ is of the form $g \otimes \frac{1}{n}$.
Any element of the tensor product is a finite sum of pure tensors, so it suffices to show that any sum of two pure tensors can be written a single pure tensor. Indeed

$$g\otimes\frac{a}{b}+g'\otimes\frac{a'}{b'}=ag\otimes\frac{1}{b}+a'g'\otimes\frac{1}{b'}=ab'g\otimes\frac{1}{bb'}+a'bg'\otimes\frac{1}{bb'}=(ab'g+a'bg')\otimes\frac{1}{bb'}.$$

2) Show that if $G$ is a torsion group, then $\mathbb{Q}\otimes G=0$.
Suppose $mg=0$, for $m\in\mathbb{Z}$ and $g\in G$. Then

$$b\otimes g=m\frac{b}{m}\otimes g=\frac{b}{m}\otimes mg=0.$$

3) Show that if $G$ is torsion free, then $g\otimes\frac{1}{n}=g'\otimes\frac{1}{n'}$ is equivalent to $gn'=ng'$.
$g\otimes\frac{1}{n}=g'\otimes\frac{1}{n'}\Rightarrow(n'g-g'n)\otimes\frac{1}{nn'}=0$. We're done if we show that $x\otimes 1/k=0\Rightarrow x=0$.
Let us more generally give a complete characterisation of $G\otimes\mathbb{Q}$. Let us define the *rationalization* of $G$ to be $E$, the set of formal symbols $g/n$, where $g\in G$ and $n\in\mathbb{Z}-0$. We define the equivalence relation $g/n=g'/n'$ if $n'g=ng'$ (this is only an equivalence relation since $G$ is torsion free!). There is an obvious addition that can be defined making $E$ into an abelian group. There is a bilinear map $G\times\mathbb{Q}\to E$, sending $(g,a/b)$ to $ag/b$, which thus induces a map $f\colon G\otimes\mathbb{Q}\to E$. We already see from this that if $x\neq 0$, then $f(x\otimes 1/k)\neq 0$, which concludes the proof. But furthermore, we can see that $f$ is an isomorphism, by defining the inverse $f^{-1}(g/n)=g\otimes\frac{1}{n}$ and checking that it is indeed well defined and an inverse.

4) Take a free resolution $F_1\to F_0$ of $G$. Show that $F_1\otimes\mathbb{Q}\to F_0\otimes\mathbb{Q}$ is injective and conclude that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q},G)=0$.
The question as stated might seem mildly incorrect. We need to take a free resolution such that the map $F_1\to F_0$ is injective, which we know to exist[1]. We know this exists since we can take $F_0$ to be generated by $G$ as a set, and $F_1$ to be the kernel of the morphism $F_0\to G$. Then, by the question above $F_1\otimes\mathbb{Q}\to F_0\otimes\mathbb{Q}$ has trivial kernel. But in fact this kernel is by definition $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q},G)$, using $F_\bullet$ as a resolution.

## 3. Ext

1) Show if $A\to B\to C\to 0$ is exact, then $\mathrm{Hom}(A,N)\leftarrow\mathrm{Hom}(B,N)\leftarrow\mathrm{Hom}(C,N)\leftarrow 0$ is exact.
[This is what is used to conclude that $\mathrm{Ext}^0_R(M,N)=\mathrm{Hom}_R(M,N)$!]

[1] In fact, it is not incorrect. Even if we weren't over a PID, if there is a two step resolution, then we know that $\partial_1\colon F_1\to F_0$ must be injective. This is the case since $0=H_1(F_\bullet)=\ker\partial_1/\mathrm{Im}\partial_2=\ker\partial_1$. But being over a PID guarantees that a two step resolution always exists.

2) Show that $\mathrm{Ext}^i_R(A \oplus B, N) = \mathrm{Ext}^i_R(A, N) \oplus \mathrm{Ext}^i_R(B, N)$ and $\mathrm{Ext}^i_R(R^7, N) = 0$.
Not a typo. It's really just that $R^7$ is free.

3) Show that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N) = N/nN$.

4) Show that $\mathrm{Ext}^1_R(M, -)$ is a covariant functor and that $\mathrm{Ext}^1(-, N)$ is a contravariant functor.

5) Show that for $n \geq 0$, $\mathrm{Ext}^n_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.
There is a $(\mathbb{Z}/4\mathbb{Z})$-free resolution of $\mathbb{Z}/2\mathbb{Z}$ given by

$$\ldots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z}$$

which we can use to compute the Ext functors. Notice that in particular we deduce that there is no finite free resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}/4\mathbb{Z}$, otherwise $\mathrm{Ext}^n_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$ for $n \gg 0$.

## 4. Cup product

1) Show that $H^\bullet(X \sqcup Y; R)$ and $H^\bullet(X; R) \oplus H^\bullet(Y; R)$ are isomorphic as graded commutative $R$-algebras. Deduce a similar statement for the wedge product (assuming that the basepoints are deformation retracts of open neighbourhoods).

2) Let $X$ be a CW complex with one 0-cell, one 5-cell, one 7-cell and one 10-cell. What is the cohomology ring structure of $X$ with coefficients in $\mathbb{Q}$?

   The cohomology of the torus with coefficients in $\mathbb{F}_2$ is spanned by **degree 0:** 1, **degree 1:** $\alpha, \beta$ and **degree 2:** $\gamma$.

3) Use the same strategy that we used in class for $\mathbb{RP}^2$ to show that $\alpha \cup \beta = \gamma$.

4) Show that $\alpha \cup \alpha = 0$.

## 5. Eckmann–Hilton argument

The way I presented the group structure on higher homotopy groups, it seems that the first coordinate plays a privileged role, when compared to the other ones. In fact, with an argument not so different from the proof of commutativity, one can show that the product defined similarly but with other coordinates ends up giving the same result. Here, we present a purely algebraic proof of a much more general result.

1) Let $\times$ and $\bullet$ be two unital binary operations on a set $X$. Suppose

$$(a \times b) \bullet (c \times d) = (a \bullet c) \times (b \bullet d)$$

for all $a, b, c, d \in X$. Then $\times$ and $\bullet$ are in fact the same operation, and are commutative and associative.

2) Consider the usual product on higher homotopy groups

$$(f \times g)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & t_1 \in [0, 1/2] \\ g(2t_1 - 1, t_2, \ldots, t_n) & t_1 \in [1/2, 1]. \end{cases}$$

and define as well

$$(f \bullet g)(t_1, \ldots, t_n) = \begin{cases} f(t_1, 2t_2, \ldots, t_n) & t_1 \in [0, 1/2] \\ g(t_1, 2t_2 - 1, \ldots, t_n) & t_1 \in [1/2, 1]. \end{cases}$$

Show that these operations satisfy the conditions from the previous exercise.

https://en.wikipedia.org/wiki/Eckmann%E2%80%93Hilton_argument