KMAS9AA1 – Algebraic Topology

Exercise Sheet 6

1. Tor

1) Assume that $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$ are finitely generated. Show that for any prime p, $H_n(X;\mathbb{Z}/p\mathbb{Z})$ consists of:
   i. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$,
   ii. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$,
   iii. A $\mathbb{Z}/p\mathbb{Z}$ summand for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_{n-1}(X;\mathbb{Z})$,
2) Use the universal coefficient theorem to show that if $H_*(X;\mathbb{Z})$ is finitely generated, so the Euler characteristic
$$\chi(X) = \sum_n (-1)^n \text{rank } H_n(X;\mathbb{Z})$$
is defined, then for any coefficient field $\mathbb{F}$ we have $\chi(X) = \sum_n (-1)^n \dim H_n(X;\mathbb{F})$.

2. Torsion-free I claimed in class that while $\mathbb{Q}$ is not free, it is torsion-free and therefore $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, C) = 0, \forall C \in R-\mathrm{Mod}$. Let us show this.

Let G be an abelian group.

1) Show that any element of $G \otimes \mathbb{Q}$ is of the form $g \otimes \frac{1}{n}$.
2) Show that if G is a torsion group, then $\mathbb{Q} \otimes G = 0$.
3) Show that if G is torsion free, then $g \otimes \frac{1}{n} = g' \otimes \frac{1}{n'}$ is equivalent to $gn' = ng'$.
4) Take a free resolution $F_1 \to F_0$ of G. Show that $F_1 \otimes \mathbb{Q} \to F_0 \otimes \mathbb{Q}$ is injective and conclude that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, G) = 0$.

3. Ext

1) Show if $A \to B \to C \to 0$ is exact, then $\mathrm{Hom}(A,N) \leftarrow \mathrm{Hom}(B,N) \leftarrow \mathrm{Hom}(C,N) \leftarrow 0$ is exact.
   [This is what is used to conclude that $\mathrm{Ext}_R^0(M,N) = \mathrm{Hom}_R(M,N)$!]

2) Show that $\mathrm{Ext}^i_R(A \oplus B, N) = \mathrm{Ext}^i_R(A, N) \oplus \mathrm{Ext}^i_R(B, N)$ and $\mathrm{Ext}^i_R(R^7, N) = 0$.
3) Show that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N) = N/nN$.
4) Show that $\mathrm{Ext}^1_R(M, -)$ is a covariant functor and that $\mathrm{Ext}^1(-, N)$ is a contravariant functor.
5) Show that for $n \geq 0$, $\mathrm{Ext}^n_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

4. Cup product

1) Show that $H^\bullet(X \sqcup Y; R)$ and $H^\bullet(X; R) \oplus H^\bullet(Y; R)$ are isomorphic as graded commutative R-algebras. Deduce a similar statement for the wedge product (assuming that the basepoints are deformation retracts of open neighbourhoods).
2) Let X be a CW complex with one 0-cell, one 5-cell, one 7-cell and one 10-cell. What is the cohomology ring structure of X with coefficients in $\mathbb{Q}$?

   The cohomology of the torus with coefficients in $\mathbb{F}_2$ is spanned by **degree 0:** 1, **degree 1:** α, β and **degree 2:** γ.
3) Use the same strategy that we used in class for $\mathbb{RP}^2$ to show that $\alpha \cup \beta = \gamma$.
4) Show that $\alpha \cup \alpha = 0$.

5. Eckmann–Hilton argument

The way I presented the group structure on higher homotopy groups, it seems that the first coordinate plays a privileged role, when compared to the other ones. In fact, with an argument not so different from the proof of commutativity, one can show that the product defined similarly but with other coordinates ends up giving the same result. Here, we present a purely algebraic proof of a much more general result.

1) Let $\times$ and $\bullet$ be two unital binary operations on a set X. Suppose
$$(a \times b) \bullet (c \times d) = (a \bullet c) \times (b \bullet d)$$
for all $a, b, c, d \in X$. Then $\times$ and $\bullet$ are in fact the same operation, and are commutative and associative.
2) Consider the usual product on higher homotopy groups
$$(f \times g)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & t_1 \in [0, 1/2] \\ g(2t_1 - 1, t_2, \ldots, t_n) & t_1 \in [1/2, 1]. \end{cases}$$
and define as well

$$(f \bullet g)(t_1, \ldots, t_n) = \begin{cases} f(t_1, 2t_2, \ldots, t_n) & t_1 \in [0, 1/2] \\ g(t_1, 2t_2 - 1, \ldots, t_n) & t_1 \in [1/2, 1]. \end{cases}$$

Show that these operations satisfy the conditions from the previous exercise.