KMAS9AA1 – Algebraic Topology

Exercise Sheet 5

1. Degree of a map $S^n \to S^n$ See [Hatcher, Beginning of Section 2.2]

Given a continuous map $f\colon S^n \to S^n$, we consider the induced map $H_n(f)\colon \mathbb{Z} \to \mathbb{Z}$. The *degree* of f, $\deg f$ is defined to be $H_n(f)(1)$. In other words, $H_n(f)$ is multiplication by $\deg f$. In this exercise we will prove some properties of the degree of a map.

1) Show that if f is not surjective, then $\deg f = 0$.
2) Given $f'\colon S^n \to S^n$, show that $\deg f \circ f' = \deg f \cdot \deg f'$. Conclude that if f is a homotopy equivalence, then $\deg f = \pm 1$.
3) Let $r^1\colon S^1 \to S^1$ be the reflection along the vertical axis, i.e. $r^1(x_0, x_1) = (-x_0, x_1)$. Show that $\deg r^1 = -1$.
4) Let $r\colon S^n \to S^n$ be a reflection along some hyperplane. Show that $\deg r = -1$.

   Hint: By change of coordinates we can suppose $r = r^n(x_0, \ldots, x_n) = (-x_0, \ldots, x_n)$. One can use Mayer–Vietoris to show that $\deg r^n = \deg r^{n-1}$.
5) Show that the degree of the antipodal map $x \mapsto -x$ is $(-1)^{n+1}$.
6) Suppose that f has no fixed points. Construct a homotopy between f and the antipodal map and conclude that $\deg f = (-1)^{n+1}$.

   Hint: A formula such as $(1-t)f(x) - tx$ almost does the trick, but this does not land in the S^n...

2. Actions on spheres

1) Let n be an even number. Suppose that a group G acts freely on the sphere S^n. Use the previous exercise to deduce that either $G = \{e\}$ or $G = \mathbb{Z}/2\mathbb{Z}$.

   See [Hatcher, 2.29]
2) Find one infinite group that acts freely on all spheres of odd dimension.

Identify $S^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | |(z_1, \ldots, z_n)| = 1\}$. The circle acts freely by rotating every coordinate: $e^{i\pi\theta} \cdot (z_1, \ldots, z_n) = (e^{i\pi\theta} z_1, \ldots, e^{i\pi\theta} z_n)$.

3. Homology of projective space

Recall that $\mathbb{RP}^n$ has a cellular structure with one cell in each dimension up to n and its k skeleton is $\mathbb{RP}^k$.

1) Show that there exists a commutative diagram as follows

$$\begin{tikzcd}
H_n(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \arrow[r, "\delta"] & H_{n-1}(\mathbb{RP}^{n-1}) \arrow[r] \arrow[dr] & H_{n-1}(\mathbb{RP}^{n-1}, \mathbb{RP}^{n-2}) \arrow[d, "\cong"] \\
H_n(D^n, S^{n-1}) \arrow[u, "\cong"] \arrow[r, "\cong"] & H_{n-1}(S^{n-1}) \arrow[u] \arrow[r, "f_*"] & H_{n-1}(D^{n-1}/S^{n-2}, *)
\end{tikzcd}$$

The top row corresponds to the construction of $d_{CW} = i_* \delta$ and this exercise will help us expressing this cellular differential in more explicit terms.

The left square of the diagram is horizontally given by the respective long exact sequences of pairs. We know from class (contractibility of D^n) that the bottom connecting morphism is an isomorphism. The leftmost vertical map is given by attaching the n-cell of $\mathbb{RP}^n$ and the middle vertical map is induced by the quotient by the $\mathbb{Z}/2\mathbb{Z}$ action. This square commutes because of the naturality of the LES.

The rightmost vertical map is the usual one we get from excision. This forces the remaining two maps (the diagonal one and f_*) if we require commutativity of the triangles.

2) Check that f_* is the map induced by the composite of the commuting diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\text{quotient}"] \arrow[dr] & \mathbb{RP}^{n-1} \arrow[r, "\text{pinch } \mathbb{RP}^2"] & D^{n-1}/S^{n-2} \cong S^{n-1} \\
& S^{n-1}/S^{n-2} \cong S^{n-1} \vee S^{n-1} \arrow[ur] &
\end{tikzcd}$$

where one of the maps from the wedge is the identity and the other is the antipodal map.

It is clear from the construction of f_* in the previous exercise that it is induced by the map $f = \text{pinch } \mathbb{RP}^2 \circ \text{quotient}$. To see that the square commutes, one needs to realise that when we write the isomorphism $S^{n-1}/S^{n-2} \cong S^{n-1} \vee S^{n-1}$, the wedge product is a bit misleading because the glueing isn't being done at the same basepoint. Concretely, for the first S^{n-1} the basepoint is the south pole and fort the second S^{n-1} the basepoint is the northpole, so when we're mapping both of them to a single S^{n-1} one of them is upside-down, so we need to reverse it, hence the antipodal map.

3) Conclude that the cellular complex of $\mathbb{RP}^n$ is

$$
\begin{array}{ccccccc}
\text{deg} & -1 & 0 & 1 & \cdots & n & n+1\cdots \\
& 0 \longleftarrow & R \xleftarrow[0]{} & R \xleftarrow[\times 2]{} & \cdots \xleftarrow[2 \text{ or } 0]{} & R \longleftarrow & 0 \quad \cdots
\end{array}
$$

It follows that the cellular boundary is multiplication by $d_{CW} = 1 + \deg(\text{antipode})$. From exercise 1.5 in this sheet, this is therefore equal to 0 or 2, alternating.

4) Compute explicitly (for both parities of n) and check that the homology gives very different results for
   a. $R = \mathbb{Z}$.
   b. A field of characteristic 2 (or in fact any ring in which $2 = 0$).
   c. A field of characteristic $\neq 2$ (or in fact any ring in which the endomorphism $\times 2$ is invertible.)
   d. $R = \mathbb{Z}/8\mathbb{Z}$.

   See `https://topospaces.subwiki.org/wiki/Homology_of_real_projective_space`

4. Leftovers from class

1) Construct the CW structure on $\mathbb{CP}^n$ that has one cell of each even dimension up to $2n$. This is a recursive construction in which we get $\mathbb{CP}^n$ from a single $2n$ cell attachment on $\mathbb{CP}^{n-1}$

   This can be done in the following way: Recall that elements of the complex projective space can be expressed in homogeneous coordinates as $[z_1 : \cdots : z_{n+1}]$, where $z_i \in \mathbb{C}$, there is at least one non-zero z_i and we have the identification $[z_1 : \cdots : z_{n+1}] = [\lambda z_1 : \cdots : \lambda z_{n+1}]$. Under this identification, what is $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$?

   Under this identification, $\mathbb{CP}^{n-1}$ is given by setting $z_{n+1} = 0$. Therefore we can identify its complement with $\mathbb{C}^n$ which is homeomorphic to the interior of D^{2n}.

   See also `https://en.wikipedia.org/wiki/Complex_projective_space`

2) Recall that a short exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

   is said to be *split* if it satisfies any (and hence all) of the equivalent conditions below.

a) There exists a homomorphism $s : C \to B$ such that $p \circ s = \mathrm{id}_C$ (a *section* of p).
b) There exists a homomorphism $r : B \to A$ such that $r \circ i = \mathrm{id}_A$ (a *retraction* of i).
c) The module B is isomorphic to the direct sum $A \oplus C$, and the maps i, p correspond to the canonical inclusion and projection.

Prove that the three formulations above are equivalent.

(1) $\Rightarrow$ (3): Given a section $s : C \to B$, define

$$\phi : A \oplus C \longrightarrow B, \qquad \phi(a, c) = i(a) + s(c).$$

Then $p \circ \phi(a, c) = p(i(a)) + p(s(c)) = 0 + c = c$, and ϕ is an isomorphism with inverse

$$b \longmapsto (r(b), p(b)),$$

where $r(b)$ is defined as the unique $a \in A$ such that $b - i(a) \in \mathrm{im}(s)$. Hence $B \cong A \oplus C$.

(3) $\Rightarrow$ (1): If $B \cong A \oplus C$, take p to be projection onto the second factor and s the inclusion of C. Then $p \circ s = \mathrm{id}_C$, so the sequence splits.

(1) $\Leftrightarrow$ (2): If s is a section of p, define $r = \mathrm{id}_B - s \circ p : B \to B$. Then $r(B) \subseteq i(A)$ and $r \circ i = \mathrm{id}_A$, giving a retraction. Conversely, given a retraction r, set $s(b) = b - i(r(b))$; this satisfies $p \circ s = \mathrm{id}_C$. Thus the two formulations are equivalent.

`https://en.wikipedia.org/wiki/Split_exact_sequence`

3) Check carefully that the fundamental theorem of homological algebra implies that for any two resolutions of the R-module M, $F_\bullet \to M$ and $F'_\bullet \to M$, we have that $H_n(F_\bullet \otimes B)$ is canonically isomorphic to $H_n(F'_\bullet \otimes B)$.