KMAS9AA1 – Algebraic Topology

Exercise Sheet 5

1. Degree of a map $S^n \to S^n$

Given a continuous map $f\colon S^n \to S^n$, we consider the induced map $H_n(f)\colon \mathbb{Z} \to \mathbb{Z}$. The *degree* of f, $\deg f$ is defined to be $H_n(f)(1)$. In other words, $H_n(f)$ is multiplication by $\deg f$. In this exercise we will prove some properties of the degree of a map.

1) Show that if f is not surjective, then $\deg f = 0$.
2) Given $f'\colon S^n \to S^n$, show that $\deg f \circ f' = \deg f \cdot \deg f'$. Conclude that if f is a homotopy equivalence, then $\deg f = \pm 1$.
3) Let $r^1\colon S^1 \to S^1$ be the reflection along the vertical axis, i.e. $r^1(x_0, x_1) = (-x_0, x_1)$. Show that $\deg r^1 = -1$.
4) Let $r\colon S^n \to S^n$ be a reflection along some hyperplane. Show that $\deg r = -1$.

   Hint: By change of coordinates we can suppose $r = r^n(x_0, \ldots, x_n) = (-x_0, \ldots, x_n)$. One can use Mayer–Vietoris to show that $\deg r^n = \deg r^{n-1}$.
5) Show that the degree of the antipodal map $x \mapsto -x$ is $(-1)^{n+1}$.
6) Suppose that f has no fixed points. Construct a homotopy between f and the antipodal map and conclude that $\deg f = (-1)^{n+1}$.

   Hint: A formula such as $(1-t)f(x) - tx$ almost does the trick, but this does not land in the S^n...

2. Actions on spheres

1) Let n be an even number. Suppose that a group G acts freely on the sphere S^n. Use the previous exercise to deduce that either $G = \{e\}$ or $G = \mathbb{Z}/2\mathbb{Z}$.
2) Find one infinite group that acts freely on all spheres of odd dimension.

3. Homology of projective space

Recall that $\mathbb{RP}^n$ has a cellular structure with one cell in each dimension up to n and its k skeleton is $\mathbb{RP}^k$.

1) Show that there exists a commutative diagram as follows

$$\begin{tikzcd}
H_n(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \arrow[r, "\delta"] & H_{n-1}(\mathbb{RP}^{n-1}) \arrow[r] \arrow[dr] & H_{n-1}(\mathbb{RP}^{n-1}, \mathbb{RP}^{n-2}) \arrow[d, "\cong"] \\
H_n(D^n, S^{n-1}) \arrow[u, "\cong"] \arrow[r, "\cong"] & H_{n-1}(S^{n-1}) \arrow[u] \arrow[r, "f_*"] & H_{n-1}(D^{n-1}/S^{n-2}, *)
\end{tikzcd}$$

2) Check that f_* is the map induced by the composite of the commuting diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\text{quotient}"] \arrow[dr] & \mathbb{RP}^{n-1} \arrow[r, "\text{pinch } \mathbb{RP}^2"] & D^{n-1}/S^{n-2} \cong S^{n-1} \\
& S^{n-1}/S^{n-2} \cong S^{n-1} \vee S^{n-1} \arrow[ur] &
\end{tikzcd}$$

where one of the maps from the wedge is the identity and the other is the antipodal map.

3) Conclude that the cellular complex of $\mathbb{RP}^n$ is

$$\begin{array}{lcccccccccccc}
\text{deg} & & -1 & & 0 & & 1 & & \cdots & & n & & n+1\cdots \\
 & & 0 & \longleftarrow & R & \xleftarrow[0]{} & R & \xleftarrow[\times 2]{} & \cdots & \xleftarrow[2 \text{ or } 0]{} & R & \longleftarrow & 0 \quad \cdots
\end{array}$$

4) Compute explicitly (for both parities of n) and check that the homology gives very different results for
   a. $R = \mathbb{Z}$.
   b. A field of characteristic 2 (or in fact any ring in which $2 = 0$).
   c. A field of characteristic $\neq 2$ (or in fact any ring in which the endomorphism $\times 2$ is invertible.)
   d. $R = \mathbb{Z}/8\mathbb{Z}$.

4. Leftovers from class

1) Construct the CW structure on $\mathbb{CP}^n$ that has one cell of each even dimension up to $2n$. This is a recursive construction in which we get $\mathbb{CP}^n$ from a single $2n$ cell attachment on $\mathbb{CP}^{n-1}$

This can be done in the following way: Recall that elements of the complex projective space can be expressed in homogeneous coordinates as $[z_1 : \cdots : z_{n+1}]$, where $z_i \in \mathbb{C}$, there is at least one non-zero z_i and we have the identification $[z_1 : \cdots : z_{n+1}] = [\lambda z_1 : \cdots : \lambda z_{n+1}]$. Under this identification, what is $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$?

2) Recall that a short exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

is said to be *split* if it satisfies any (and hence all) of the equivalent conditions below.

a) There exists a homomorphism $s : C \to B$ such that $p \circ s = \mathrm{id}_C$ (a *section* of p).
b) There exists a homomorphism $r : B \to A$ such that $r \circ i = \mathrm{id}_A$ (a *retraction* of i).
c) The module B is isomorphic to the direct sum $A \oplus C$, and the maps i, p correspond to the canonical inclusion and projection.

Prove that the three formulations above are equivalent.

3) Check carefully that the fundamental theorem of homological algebra implies that for any two resolutions of the R-module M, $F_\bullet \to M$ and $F'_\bullet \to M$, we have that $H_n(F_\bullet \otimes B)$ is canonically isomorphic to $H_n(F'_\bullet \otimes B)$.