KMAS9AA1 – Algebraic Topology

Exercise Sheet 4

1. Computation of Homologies

1) Let (M, m) and (N, n) be two pointed spaces with open neighbourhoods deformation retracting to the respective points. Show that $H_d(M \vee N, *) = H_d(M, n) \oplus H_d(N, n)$.

   Remark: This condition is satisfied by manifolds, but also by CW complexes.

   Pick $U \subset M \vee N$ the union of M with the small open neighbourhood (this is open in $M \vee N$) and similarly for V. The Mayer–Vietoris LES simplifies to

   $$0 \to \tilde{H}_k(M) \oplus \tilde{H}_k(N) \to \tilde{H}_k(M \vee N) \to 0$$

2) The *connected sum* $M\#N$ of two connected manifolds M and N of the same dimension n is obtained by removing a small neighborhood of a point[1] formed by an open disc from each, and gluing the resulting manifolds along the two spheres S^{n-1} that appear. For example, $\Sigma_g\#\Sigma_{g'} = \Sigma_{g+g'}$, where Σ_g is the oriented surface of genus g.

   Show that for $i \neq n-1, n$, we have $\tilde{H}_i(M\#N) = \tilde{H}_i(M) \oplus \tilde{H}_i(N)$.

   Show that if R is a field, the dimension of $\tilde{H}_i(M\#N)$ and $\tilde{H}_i(M) \oplus \tilde{H}_i(N)$ differ by at most 1. The same strategy as before applies, but now the intersection $U \cap V \sim S^{n-1}$, so the isomorphism can't be obtained in the relevant dimensions.

   Another way to get to this result is to realise that $M \vee N$ is a quotient of $M\#N$ and use the previous exercise.

   Taking $N = S^n$ we have $M\#N = M$, which provides a counter example for additivity of homology in degree n.

   See also: `https://math.stackexchange.com/questions/187413/computing-the-homology-`

[1] Up to homeomorphism it doesn't matter which point we choose.

3) Compute the homology of the torus, seeing it as square identified opposite edges. This can be done by taking the Mayer–Vietoris sequence on U being a small disc inside the square and V being a thickening of the complement of U, in such a way that $U \cap V \sim S^1$.

4) Compute the homology of Σ_2. This can be done using the strategy of the last exercise, but also seeing $\Sigma_2 = \Sigma_1 \# \Sigma_1$.

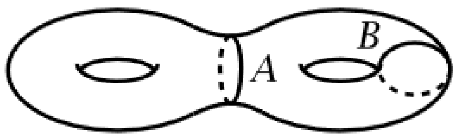
   Can use Mayer–Vietoris splitting Σ_2 into two opens that are each homotopy equivalent to the torus minus one point, which in turn is homotopy equivalent to a wedge of two circles. Alternatively, cellular homology works.

5) Let $R = \mathbb{Z}$. Suppose that a topological space is written as a union of two open subspaces $X = U \cup V$ and consider the associated Mayer–Vietoris long exact sequence. Assuming that $U \cap V$ is path connected, use the explicit construction of the connecting morphism $\delta \colon H_1(X) \to H_0(U \cap V)$ to show that is the zero map.

6) Compute the homology of $S^n \times S^m$.

   Use Mayer?Vietoris removing one point from each factor.

7) Compute the homology $H(\Sigma_2, A)$ and $H(\Sigma_2, B)$, where A and B are the following circles[2]:



   See the solutions from [H, Exercise 2.1.17]

2. Products, coproducts, pullbacks and pushouts

Let $\mathcal{C}$ be a category.

1) Recall the notions in the title of this exercise and show that if they exist they are unique up to unique isomorphism.

2) Show that the category of fields does not have all coproducts.

   There are no maps between fields of different characteristic so a coproduct of two such fields cannot exist.

3) Show that in the category of unital commutative rings, the coproduct of R and S is given by $R \otimes S$ with the maps $R \to R \otimes S, r \mapsto r \otimes 1$ and $S \to R \otimes S, s \mapsto 1 \otimes s$.

4) We say that $I \in \mathcal{C}$ is an *initial* object if for any object $X \in \mathcal{C}$, there is a unique morphism $I \to X$. Similarly, $T \in \mathcal{C}$ is *terminal* if there is a unique morphism from any $X \to T$.

[2]Drawing from [H, Exercise 2.1.17]

a. Show that if such objects exist, they are unique.
b. Show that if an initial object exists, any coproduct can be written as a pushout. Similarly, if a terminal object exists, a product is a pullback.
The pushout of $X \leftarrow I \rightarrow Y$ is the same as $X \sqcup Y$. Similarly, the pullback of $X \rightarrow T \leftarrow Y$ is the product $X \times Y$.
c. Determine the initial and terminal object in the categories **Top**, $\mathbf{Top}_*$, **Groups**, $R-\mathbf{Mod}$.
$(\emptyset, *), (*, *), (\{e\}, \{e\}), (0, 0)$

5) In **Set**, show that the pullback of $f\colon X \rightarrow Z$ and $g\colon Y \rightarrow Z$ is given by the set of pairs $X \times_Z Y = \{(x,y)|x \in X, y \in Y, f(x) = g(y)\}$.

3. CW Complexes

1) Show that a single cell attachment is a pushout along the attachment map.
2) Let X, Y be CW complexes containing finitely many cells. Show that $X \times Y$ is a CW complex with d-cells given by products of k and $d-k$ cells.
We use that $D^n \times D^m$ is homeomorphic to D^{n+m} and under this identification the boundary is given by $\partial D^n \times D^m \cup D^n \times \partial D^m$. Then, the "inclusion" of the cell in the product $X \times Y$ is given by the product of the two maps $\phi_1 \times \phi_2 : D^n \times D^m \rightarrow X \times Y$. See [H, Theorem A.6] for a solution with no finiteness assumptions.
3) Let X, Y be CW complexes, A a subcomplex of X and $f : A \rightarrow Y$ a cellular map. Show that the pushout of f along the inclusion $A \hookrightarrow X$ is a CW complex with the cells of Y and of $X - A$ as cells. You can assume for convenience that all CW complexes have finitely many cells (and therefore it is enough to consider the case of a single cell attachment)
See Theorem 4.14 of `https://www.mat.univie.ac.at/~kriegl/Skripten/2011WS.pdf` for a solution with no finiteness assumptions
4) We constructed in class a cellular structure on $\mathbb{RP}^n$ as the quotient of a cellular structure on S^n. Describe with precise formulas for the attaching maps the cellular structure of $\mathbb{RP}^2$, without using quotients at any point.
5) Let Γ be a finite graph with e edges, v vertices and c connected components.
Compute $H_\bullet(\Gamma; \mathbb{Z})$ in terms of e, v and c.
Hint: Use the fact that given a CW-complex X and A a contractible subcomplex $X \rightarrow X/A$ is a homotopy equivalence.

Suppose the graph is connected. We can contract every edge whose endpoints are distinct and end up with a single vertex and a bouquet of circles. How many circles? For every contraction we reduce the vertices and edges by one $(e, v) \mapsto (e-1, v-1)$, each and we stop at $v = 1$, i.e. at $(e - v + 1, 1)$. It follows that $H_1(\Gamma) = \mathbb{Z}^{e-v+1}$.

The non-connected case follows from $H_\bullet(X \sqcup Y) = H_\bullet(X) \oplus H_\bullet(Y)$. In principle the result should depend on $v_1, \ldots v_c$, the number of vertices of each connected component, rather than just their sum $v = \sum v_i$, but writing everything out we see that this just depends on the total number of vertices.

Instead of doing that, let us give an alternative proof using the Euler characteristic:

By contracting every edge with different end points, we already know that this is a dijoint union of c bouquets of circles. So in particular the only non-zero homologies are in degrees 0 and 1 and these are free $\mathbb{Z}$-modules: $H_0(\Gamma) = \mathbb{Z}^x, H_1(\Gamma) = \mathbb{Z}^y$. We know that $x = c$, the number of connected components, we just need to determine y. We can tensor with $\mathbb{Q}$ to be able to talk about the Euler characteristic of the complex, which is the same as the Euler characteristic of its homology. It follows

$$x - y = \chi(\Gamma) = v - e,$$

and therefore $y = c - v + e$.

4. **Chain Complexes** [This exercise was supposed to appear on an earlier exercise sheet, but didn't by mistake]

1) Show that the kernel, image, and cokernel of a chain complex morphism $f\colon C \to D$ are chain complexes. Show that if $H(\ker f) = 0 = H(\mathrm{coker} f)$, then f induces an isomorphism in homology.
2) Show that the direct sum of chain complexes is a chain complex, and compare $H(A \oplus B)$ with $H(A) \oplus H(B)$.
3) Compute the homology of the complex

$$0 \to R \xrightarrow{\times 2} R \to 0$$

for $R = \mathbb{Z}$ and R a field (pay attention to the characteristic). [Important to retain from this: The ring we are working with changes a lot the result. Hatcher only works with $\mathbb{Z}$ which does not allow us to see such differences later on.]
4) Show that the dual of a chain complex $C_\bullet^\vee = \{\mathrm{Hom}_R(C_{-i}, R)\}_{i\in\mathbb{Z}}$ is a chain complex with boundary map given by the dual of the original one. Prove that over $\mathbb{Z}$, the dual of the homology of C is not isomorphic to the homology of the dual of C.

Hint: The subexercise just above.