# KMAS9AA1 – Algebraic Topology

# Exercise Sheet 3

## 1) Long Exact Sequence

1. Use the long exact sequence of a short exact sequence of complexes to give new proofs of exercises 5.1, 5.4, and 7 of Exercise Sheet 2.
2. Let $R = \mathbb{Z}$. Consider the short exact sequence of chain complexes complexes
$$0 \to A_\bullet \xrightarrow{f} B_\bullet \to C_\bullet \to 0,$$
where all complexes are concentrated in degrees 0 or 1. The boundary maps are: $A_1 = \mathbb{Z} \xrightarrow{\times 5} \mathbb{Z} = A_0$, $B_1 = \mathbb{Z} \oplus \mathbb{Z} \to B_0 = \mathbb{Z}$ sends $(1,0) \mapsto 2$ and $(0,1) \mapsto 3$, $C_1 = \mathbb{Z}$ and $C_0 = 0$.
The map $f$ is the diagonal in degree 1, $f_1(x) = (x,x)$ and the identity in degree 0, $f_0 = \mathrm{id}_\mathbb{Z}$.
Give a possible formula for the remaining map, show that this is indeed a short exact sequence and describe the long exact sequence associated, including a description of the connecting morphisms $\delta\colon H_1(C_\bullet) \to H_0(A_\bullet)$.
We are given complexes concentrated in degrees 1 and 0 with differentials
$$d_A : A_1 = \mathbb{Z} \xrightarrow{\times 5} A_0 = \mathbb{Z}, \qquad d_B : B_1 = \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{(2\ 3)} B_0 = \mathbb{Z}, \qquad d_C : C_1 = \mathbb{Z} \xrightarrow{0} C_0 = 0.$$
The map $f$ is specified by $f_1(x) = (x,x)$ and $f_0 = \mathrm{id}_\mathbb{Z}$. We must give the remaining map $g_1 : B_1 \to C_1$, since $g_0 : B_0 \to C_0 = 0$ must be 0.
Choose
$$g_1\colon \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}, \qquad g_1(a,b) = a - b,$$
Let us check exactness degreewise:
   - In degree 1: $\ker g_1 = \{(a,b) \mid a - b = 0\} = \{(x,x) \mid x \in \mathbb{Z}\} = \operatorname{im} f_1$, so $0 \to A_1 \xrightarrow{f_1} B_1 \xrightarrow{g_1} C_1 \to 0$ is exact.
   - In degree 0: $f_0 = \mathrm{id} : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism, so exactness holds there as well.

We must check compatibility with differentials (so we indeed have a short exact sequence of chain complexes). Compute

$$f_0 \circ d_A(x) = \mathrm{id}(5x) = 5x, \qquad d_B \circ f_1(x) = d_B(x, x) = 2x + 3x = 5x,$$

so $f_0 d_A = d_B f_1$. Also $d_C \circ g_1 = 0$ and $g_0 \circ d_B = 0$, so the squares commute. Thus we have a short exact sequence of chain complexes.

**Homology of each complex.**
Compute $H_*$ degreewise.
*For $A$:* $d_A = \times 5$, so

$$H_1(A) = \ker(d_A) = \ker(\times 5 : \mathbb{Z} \to \mathbb{Z}) = 0, \qquad H_0(A) = \mathrm{coker}(d_A) = \mathbb{Z}/5\mathbb{Z}.$$

*For $B$:* $d_B(a, b) = 2a + 3b$. So

$$\ker d_B = \{(a, b) \in \mathbb{Z}^2 \mid 2a + 3b = 0\}.$$

A $\mathbb{Z}$-basis for $\ker d_B$ is spanned by $(3, -2)$ (since $2 \cdot 3 + 3 \cdot (-2) = 0$). In particular $\ker d_B \cong \mathbb{Z}$ and

$$H_1(B) = \ker d_B \cong \mathbb{Z}.$$

For $H_0(B) = \mathrm{coker} d_B = \mathbb{Z}/\operatorname{im}(d_B)$ and $\operatorname{im}(d_B) = (2, 3) \cdot \mathbb{Z}^2 \cong \gcd(2, 3)\mathbb{Z} = \mathbb{Z}$, so $H_0(B) = 0$. If you are not used to this sort of computation just notice that $1 = 3 - 2$ is in the image of $d_B$ and therefore $d_B$ is surjective.
*For $C$:* $d_C = 0$ and $C_0 = 0$, so

$$H_1(C) = \mathbb{Z}, \qquad H_0(C) = 0.$$

**Long exact sequence and the connecting map $\delta$.**
The LES of homology for $0 \to A \to B \to C \to 0$ in the relevant range is

$$H_1(A) \xrightarrow{f_*} H_1(B) \xrightarrow{g_*} H_1(C) \xrightarrow{\delta} H_0(A) \xrightarrow{f_*} H_0(B) \xrightarrow{g_*} H_0(C).$$

Plugging in the computed groups:

$$0 \longrightarrow \langle(-3, 2)\rangle \xrightarrow{g_*} \mathbb{Z} \xrightarrow{\delta} \mathbb{Z}/5\mathbb{Z} \longrightarrow 0 \longrightarrow 0.$$

Let us give an explicit formula for $\delta$. The usual definition: for $[c] \in H_1(C)$ take a cycle representative $c \in C_1$, lift to $b \in B_1$ with $g_1(b) = c$, then $d_B(b) \in B_0$ maps to zero in $C_0$ so $d_B(b) = f_0(a)$ for a unique $a \in A_0$, and $\delta([c]) = [a] \in H_0(A)$.

Apply to the generator $1 \in H_1(C) = \mathbb{Z}$. Lift $1 \in C_1$ to $b = (1, 0) \in B_1$. Then

$$d_B(1, 0) = 2 \in B_0 = \mathbb{Z}.$$

Since $f_0 = \mathrm{id}$, $a = 2 \in A_0 = \mathbb{Z}$, hence

$$\delta(1) = [2] \in H_0(A) = \mathbb{Z}/5\mathbb{Z}.$$

Thus the connecting homomorphism $\delta\colon \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ is the reduction map $k \mapsto 2k \pmod 5$.

Finally, one checks that the map $g_*\colon H_1(B) \to H_1(C)$ sends the generator $(3, -2)$ to $g_1(3, -2) = 5 \in \mathbb{Z}$, and in particular $\operatorname{im} g_* = 5\mathbb{Z} = \ker \delta$, as required by exactness.

## 2) Leftovers from Lecture 6

1) Find $A \subset \mathbb{R}^n - 0$ such that

$$H_\bullet(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \neq H_\bullet(\mathbb{R}^n - A, (\mathbb{R}^n - \{0\}) - A).$$

Take A=$\mathbb{R}^n - \{0\}$.

2) Show that $(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ does not deformation retract into $(\mathbb{R}^n, S^{n-1})$.
Suppose that was the case. Restricting to the subspaces, in particular this would give a retract of $D^n - \{0\}$ into $S^{n-1}$ which would extend to a map $D^n \to S^{n-1} \cup \{f(0)\}$. But $S^{n-1}$ is closed, so $f(0)$ must necessarily be sent to $S^{n-1}$, which gives a retract of $D^n$ into $S^{n-1}$ which we know to be impossible (by exercise 4 of this exercise sheet).

## 3) Cone and Suspension of a Topological Space

Let $X$ be a topological space. The *suspension* of $X$ is the topological space

$$SX = X \times [-1, 1]/(x, -1) \sim (x', -1);\ (x, 1) \sim (x', 1)\ \forall x, y \in X.$$

The *cone* of $X$ is the subspace of $SX$

$$CX = X \times [0, 1]/(x, 1) \sim (x', 1).$$

1) Describe an explicit homeomorphism $SX \cong CX/X \times 0$.
Choose a monotonic homeomorphism $\phi : [-1, 1] \to [0, 1]$. For instance $\phi(t) = (t + 1)/2$.
Then, define $\Phi : SX \longrightarrow CX/(X \times \{0\})$ defined by $Phi(x, t) = (x, \phi(t))$. This is well defined on the quotients and it is indeed a homemorphism whose inverse is $\Phi^{-1}(x, t) = (x, \phi^{-1}(t))$.

2) Compute $H_\bullet(CX)$.

As we saw on the first exercise sheet, $CX$ is contractible. It follows that $H_\bullet(CX) = H_0(CX) = R$.

3) Show that $H_{n+1}(SX) \cong H_n(X)$ for $n \geq 1$ and that if $X$ is path-connected, $H_1(SX) = 0$.

Consider the long exact sequence of the pair $CX, X \times 0$. Notice that $X \times 0$ is a deformation retract of a neighbourhood $X \times [0, \epsilon)$, so the relative homology is a homology of the quotient which is $SX$ by the first exercise.

## 4) Brouwer Fixed-Point Theorem

1) Show that the boundary of the disk $\partial D^n$ is not a deformation retract of $D^n$.

2) Use the previous point to show that every continuous map $D^n \to D^n$ has a fixed point.

## 5) Local Homology

Let $X$ be a topological space and $x \in X$. Recall from class that the *local homology* of $X$ at $x$ is defined as

$$H_n(X, X - \{x\}) \quad \text{for } n \geq 0.$$

We will assume all points are closed in $X$.

1) Show that if $V \subset X$ is an open set containing $x$, then

$$H_n(X, X - \{x\}) \cong H_n(V, V - \{x\}).$$

2) Show that if $f\colon X \to Y$ is a local homeomorphism, then it induces isomorphisms at the level of local homology for all $x \in X$.

Let $f : X \to Y$ be a local homeomorphism and fix $x \in X$. There exists an open $U \ni x$ such that $f|_U : U \xrightarrow{\cong} f(U)$ is a homeomorphism. Using part 1) we have

$$H_n(X, X - \{x\}) \cong H_n(U, U - \{x\}) \xrightarrow{\cong} H_n(f(U), f(U) - \{f(x)\}) \cong H_n(Y, Y - \{f(x)\}),$$

where the middle isomorphism is induced by the homeomorphism $f|_U$. Hence $f$ induces isomorphisms on local homology at corresponding points.

3) Show that the $f$ above does not induce necessarily an isomorphism $H_\bullet(X) \to H_\bullet(Y)$.

Consider the projection $f\colon \mathbb{R} \to S^1$, seeing $S^1 = \mathbb{R}/\mathbb{Z}$.