

KMAS9AA1 – Algebraic Topology

Exercise Sheet 2

1. Simplicial Identities

Let $\sigma: \Delta^p \rightarrow X$ be a p -simplex of X and $0 \leq j < i \leq p$. Show that $\partial_j \partial_i(\sigma) = \partial_{i-1} \partial_j(\sigma)$.

2. The standard p -simplex

- 1) Show that Δ^p is homeomorphic to

$$\{(s_1, \dots, s_p) \in \mathbb{R}^p \mid 0 \leq s_1 \leq \dots \leq s_p \leq 1\}.$$

The homeomorphism can be obtained by setting $s_i = t_0 + \dots + t_{i-1}$, for $i = 1, \dots, p$. Notice that with the same formula, this extends to $s_{-1} = 0$ and $s_{p+1} = \sum t_i = 1$. We can then argue that a continuous bijection of compact Hausdorff spaces is a homeomorphism. Alternatively, we construct the inverse by setting $t_i = s_{i+1} - s_i$, for $i = 0, \dots, p$.

- 2) Express the partial boundary maps (also known as face maps) ∂_i . Following the homeomorphism above, we have

$$\partial_i(s_1, \dots, s_p) = (s_1, \dots, s_i, s_i, \dots, s_p), \quad 0 \leq i \leq p,$$

where again for convenience we set $s_0 = 0$ and $s_{p+1} = 1$.

- 3) Recall that we did a decomposition of the prism $\Delta^p \times I$ into simplices to show that homology was homotopy invariant. How does this decomposition work under this homeomorphism?

In other words, given $(s_1 \leq \dots \leq s_p, t)$ with $t \in I$, in which of the p -simplices $[0, \dots, k, k', \dots, p']$ does $(s_1 \leq \dots \leq s_p, t)$ fall on?

Since $t \in [0, 1]$, if t is different from s_i for all i , lives in an interval $[s_k, s_{k+1}]$ for a unique k . We have that $(s_1 \leq \dots \leq s_p, t)$ falls on the interior simplex $[0, \dots, k, k', \dots, p']$ for that same k .

If $t = s_k$, then it lives in two different intervals, namely $[s_k, s_{k+1}]$ and $[s_{k+1}, s_{k+2}]$ and therefore $(s_1 \leq \dots \leq s_p, t)$ lands on the boundary of the two corresponding simplices.

3. Connectedness

Let X be a topological space and $(X_\alpha)_{\alpha \in E}$ the family of its path-connected components.

Show that for all $n \in \mathbb{N}$

$$H_n(X) = \bigoplus_{\alpha \in E} H_n(X_\alpha).$$

This exercise often allows us to assume X is path-connected without loss of generality.

For complexes $H(\bigoplus_\alpha A_\alpha) = \bigoplus_\alpha H(A_\alpha)$ since both the kernel and the image decompose. Since Δ^n is connected, the image of a simplex of X must land in a single connected component. It follows that $C(X)$ decomposes as $\bigoplus_\alpha C(X_\alpha)$ as complexes.

4. Homological algebra

1. Let

$$0 \rightarrow A \rightarrow C \rightarrow F \rightarrow 0$$

be a short exact sequence and assume that F is a free R -module. Show that $C \cong A \oplus F$.

If F is free, choose a basis $\{e_j\}$ and lift each e_j to $\tilde{e}_j \in C$. These generate a submodule isomorphic to F intersecting A trivially, giving $C \cong A \oplus F$. As we learned on Lecture 11, this means that the sequence splits.

2. Find a counter example where F is non-free. This can be done over $R = \mathbb{Z}$ with $A = \mathbb{Z}$ and $F = \mathbb{Z}/2\mathbb{Z}$.

Check that the counter example is no longer a counter example if R is instead the ring $\mathbb{Z}/2\mathbb{Z}$.

$$0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

Notice that the sequence does not split seeing everything as $\mathbb{Z}$ -modules (there is no non-trivial map $\mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}$).

The second part of the question is deliberately vague, to make you think of how to fix the problem. While $\mathbb{Z}/2\mathbb{Z}$ is a free $\mathbb{Z}/2\mathbb{Z}$ -module, the same sequence would split in the world of $\mathbb{Z}/2\mathbb{Z}$, but there is no $\mathbb{Z}/2\mathbb{Z}$ -module structure on $\mathbb{Z}$.

Here is another example: As we saw in class, there are two possible ways to fill out the question mark in the category of $\mathbb{Z}$ -modules

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow ? \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

On the other hand, by the preceding question, over $\mathbb{Z}/2\mathbb{Z}$ there is only one way to fill the question mark, which is with $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Indeed, $\mathbb{Z}/4\mathbb{Z}$ is not a $\mathbb{Z}/2\mathbb{Z}$ -module

5. Relative Homology

Let (X, A) be a topological pair.

- 1) Show that $H_0(X, A) = 0$ if and only if A intersects every path-connected component of X .
- 2) Let $Z_p(X, A) = \{\sigma \in C_p(X) \mid \partial\sigma \in C_{p-1}(A)\}$. Show that there is an isomorphism of modules

$$H_p(X, A) \cong \frac{Z_p(X, A)}{B_p(X) + C_p(A)}.$$

- 3) Provide an alternative proof of 1) using 2).
- 4) Show that $H_1(X, A) = 0$ if and only if the map $H_1(A) \rightarrow H_1(X)$ is surjective and every path-connected component of X contains at most one path-connected component of A .

6. Retract

- 1) Show that if X is a topological space and $A \subset X$ is a retract of X , then for all n , the map induced by inclusion $H_n(A) \rightarrow H_n(X)$ is injective.

Does this remain true if A is just a subspace of X ?

For the first part just use that $A \hookrightarrow X \rightarrow A$ is the identity and take the homology functor.

For the second one, consider $S^1 \subset \mathbb{R}^2$.

- 2) Show that if A is a deformation retract of X , then $H_n(X, A) = 0$ for all n .

We can observe that a deformation retract gives a homotopy from (X, A) to (A, A) and conclude using that $H_n(A, A) = 0$. Alternatively, the long exact sequence of a pair also works.

7. Surjective in Homology

- 1) Show that a surjective morphism $f: A \rightarrow B$ of chain complexes is not necessarily surjective in homology, but this is the case if $\ker f$ is acyclic.

The projection $[\mathbb{Z} \rightarrow \mathbb{Z}] \rightarrow \mathbb{Z}$ is surjective but its homology is $0 \rightarrow \mathbb{Z}$. Warning: You might think there are two projections depending on which degree we choose to put the target, but notice that only one is a map of chain complexes!

The other statement follows from the long exact sequence of $0 \rightarrow \ker f \rightarrow A \rightarrow B \rightarrow 0$.

- 2) Formulate and prove that same statement for injective maps.

Injective maps do not give injective maps in homology. You can reverse the counter example above (again being careful that you have a map of complexes) to get something that in homology is $\mathbb{Z} \rightarrow 0$.

If the cokernel is acyclic the map is indeed surjective, with the same proof.