KMAS9AA1 – Algebraic Topology

Exercise Sheet 1

Topology

1. Quotient topology, spheres, and discs

Let X be a topological space and $A \subset X$. We denote by X/A the quotient of X by the equivalence relation

$$x \sim y \iff x = y \text{ or } x, y \in A$$

equipped with the quotient topology.

1. Let $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the closed n-dimensional disc, and $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\} \subset D^n$ the $(n-1)$-dimensional sphere. Show that D^n/S^{n-1} is homeomorphic to S^n.

   This answer can come with a varied amount of details depending on how many formulas we want to write down (also other definitions of the sphere are reasonable..). A minimalistic acceptable argument would be to claim that S^n minus a point is homeomorphic to R^n and so is the interior of D^n. By the universal property of the quotient, this homeomorphism extends to a continuous bijection $D^n/S^{n-1} \to S^n$. Since both the source and target are compact Hausdorff, this is a homeomorphism.

   Rule of thumb: An argument with low levels of rigour is acceptable if it is relatively clear that it could be given complete rigour if the writer has enough patience.

2. Given a topological space X, the cone of X is the quotient topological space $C(X) := X \times [0,1]/X \times \{0\}$. Show that $C(S^{n-1})$ is homeomorphic to D^n.

   Using the presentation above, can construct the map $S^{n-1} \times [0,1] \to D^n$ sending (x,t) to tx. Check that this map passes to the quotient and is a homeomorphism.

3. Show that the cone of a non-empty topological space is contractible.

   An explicit deformation retract into the vertex of the cone is given by $h_t(x,s) = (x,ts)$.

2. Path-connected components

1. Reprove that, up to isomorphism, the fundamental group $\pi_1(X, x)$ only depends on the path-connected component of $x \in X$. More precisely, if γ is a path between x and $y \in X$, then

$$\begin{aligned} \phi_\gamma \colon \pi_1(X, x) &\to \pi_1(X, y) \\ [\alpha] &\mapsto [\overline{\gamma} \cdot \alpha \cdot \gamma] \end{aligned}$$

is a group isomorphism.

2. If δ is another path joining x to y, then the isomorphisms ϕ_γ and ϕ_δ are conjugate. Deduce that if $\pi_1(X, x)$ is abelian, then this isomorphism is canonical.

3. Homotopy

1. Show that the homotopy equivalence relation is indeed an equivalence relation. Convince yourself that the same is not true for deformation retracts.
2. Show that $f \sim f'$ and $g \sim g'$ imply $f \circ g \sim f' \circ g'$.
3. Show with explicit formulas that any convex subset of $\mathbb{R}^n$ is contractible.
   Pick an arbitrary point $x \in X$. The homotopy $h_t(y) = x + t(y - x)$, sending y linearly to x is continuous and is fully contained in X due to convexity.

4. Fundamental Group

1. **Simple connectedness**: Let X be a path-connected topological space. Show that the following assertions are equivalent:
   a. $\pi_1(X, x)$ is trivial for any $x \in X$.
   b. There exists $x_0 \in X$ such that $\pi_1(X, x_0)$ is trivial.
   c. Any map $f \colon S^1 \to X$ extends to a map $D^2 \to X$.
   f is given equivalently by a map $I \to X$ sending the endpoints to the same point. If f is a contractible loop, there is a homotopy starting from $I \times I$... The key observation is that $I \times I \cong D^2$.
   d. There exist $x_0, x_1 \in X$ such that all paths from x_0 to x_1 are homotopic.
   e. For any $x_0, x_1 \in X$, all paths from x_0 to x_1 are homotopic.
2. **An abelian π_1**: Let x, y be two points in a path-connected topological space. Show that the following assertions are equivalent:
   a. $\pi_1(X, x)$ is abelian.
   b. For any paths α, β from x to y, the induced homomorphisms given by Exercise 2 from $\pi_1(X, x)$ to $\pi_1(X, y)$ are the same.

Homological algebra

5. Chain Complexes

1) Show that homotopy of morphisms of chain complexes is an equivalence relation. Also show that homotopy equivalence between two chain complexes (i.e., there exist morphisms in both directions such that their compositions are homotopic to the identity) is an equivalence relation. Show that $f \sim f'$ and $g \sim g'$, then if the composites are defined $f \circ f' \sim g \circ g'$.

2) Show that any short exact sequence is isomorphic to a short exact sequence of the form

$$0 \to A \to B \to B/A \to 0$$

where A is an R-module and B is a submodule of A.

If we have a SES $0 \to B \to A \xrightarrow{p} C \to 0$, we can define a map $f : C \to A/B$ by defining $f(c) = [a]$,where $p(a) = c$. We check that this is a well defined linear map. We can conclude by showing that there is a well defined map $A/B \to C$ which is a both sided inverse to f.

Alternatively, we can use the Five Lemma to argue that f is an isomorphism.

6. Deformation Retraction

Let C and A be two chain complexes over R. A *deformation retraction* of C onto A is a triple (r, i, h) with $r\colon C \to A$ and $i\colon A \to C$ chain complex morphisms satisfying $r \circ i = \mathrm{id}_A$, and $h\colon C_\bullet \to C_{\bullet+1}$ is a homotopy between $i \circ r$ and id_C (i.e., $ir - \mathrm{id}_C = h\partial + \partial h$).

1) Show that over a field, any chain complex retracts by deformation onto its homology.

Hint: Over a field you can write $C_i = \ker \partial_i \oplus I_i$, for some I_i.

Let us write $C_i = \ker \partial_i \oplus I_i$, where I_i is an arbitrary choice of complement of $\ker \partial_i$ (which exists since we are working over a field). Let us further decompose $\ker \partial_i = \mathrm{Im}(\partial_{i+1}) \oplus H_i$, where H_i is a chosen complement. Notice that that $H_i \cong H_i(C)$. Under this decomposition, we can write for all degrees $C = H \oplus \mathrm{Im}(\partial) \oplus I$, where on the right hand side the only non-trivial part of the differential is $\partial I \to \mathrm{Im}(\partial)$ and this restriction is an isomorphism.

Now, one can check that there is a deformation rectraction given by projecting (resp. including) C to H (resp. H in C) and the homotopy can be taken to be the inverse of $\partial I \to \mathrm{Im}(\partial)$.

2) Show that this is not true in general. A counterexample can be found for $R = \mathbb{Z}$ with $C_0 = C_1 = \mathbb{Z}$ and $C_{i \neq 0,1} = 0$.

Taking $C = [\mathbb{Z} \xrightarrow{2} \mathbb{Z}]$, there is no map from $H(C) = \mathbb{Z}/2\mathbb{Z}$ to C besides the zero map.

3) Show that a morphism of chain complexes that admits a left inverse (a "retract") is injective and induces an injective morphism in homology. Show that the converse is true over a field.

For one implication just take the homology of $r \circ i = \mathrm{id}_A$.

For the other implication, take $f\colon C \to D$. We do the decomposition for $C = H^C \oplus \mathrm{Im}(\partial^C) \oplus I^C$ of the previous exercise and then similarly for D, but in a way that is compatible with f, namely we pick H^D such that $H(f)(H^C) \subset H^D$. Now we can define $g\colon D \to C$ component by component. Using injectivity in homology we define g in H^D and the injectivity of f helps us defining g in the other components.

7. Euler Characteristic

Let C be a chain complex over a field such that for all i, C_i is finite-dimensional, and for $i \gg 0$ and $i \ll 0$, $C_i = 0$. The *Euler characteristic* of C is

$$\chi(C) = \sum_{i \in \mathbb{Z}} (-1)^i \dim(C_i).$$

Show that the Euler characteristic depends only on the homology of C.

This follows from the rank nullity theorem: $\dim \ker \partial_i + \dim \mathrm{Im}\partial_i = \dim C_i$ plus the fact that $\dim H_i(C) = \dim \ker \partial_i - \dim \mathrm{Im}\partial_{i+1}$

8. Five Lemma

Consider the following commutative diagram of R-modules:

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

Assume that the rows are exact at B, C, D, B', C', D', and that all vertical maps except the middle one are isomorphisms. Prove that the map $C \to C'$ is also an isomorphism.

This was done in class, but I'm recommending again the gif from the Wikipedia page.