KMAS9AA1 – Algebraic Topology

Exercise Sheet 3

1) Long Exact Sequence

1. Use the long exact sequence of a short exact sequence of complexes to give new proofs of exercises 5.1, 5.4, and 7 of Exercise Sheet 2.
2. Let $R = \mathbb{Z}$. Consider the short exact sequence of chain complexes complexes
$$0 \to A_\bullet \stackrel{f}{\to} B_\bullet \to C_\bullet \to 0,$$
where all complexes are concentrated in degrees 0 or 1. The boundary maps are: $A_1 = \mathbb{Z} \stackrel{\times 5}{\to} \mathbb{Z} = A_0$, $B_1 = \mathbb{Z} \oplus \mathbb{Z} \to B_0 = \mathbb{Z}$ sends $(1,0) \mapsto 2$ and $(0,1) \mapsto 3$, $C_1 = \mathbb{Z}$ and $C_0 = 0$.
The map f is the diagonal in degree 1, $f_1 x = (x,x)$ and the identity in degree 0, $f_0 = \mathrm{id}_{\mathbb{Z}}$.
Give a possible formula for the remaining map, show that this is indeed a short exact sequence and describe the long exact sequence associated, including a description of the connecting morphisms $\delta\colon H_1(C_\bullet) \to H_0(A_\bullet)$.

2) Leftovers from Lecture 6

1) Find $A \subset \mathbb{R}^n - 0$ such that
$$H_\bullet(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \neq H_\bullet(\mathbb{R}^n - A, (\mathbb{R}^n - \{0\}) - A).$$

2) Show that $(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ does not deformation retract into $(\mathbb{R}^n, S^{n-1})$.

3) Cone and Suspension of a Topological Space

Let X be a topological space. The *suspension* of X is the topological space

$$SX = X \times [-1,1]/(x,-1) \sim (x',-1);\ (x,1) \sim (x',1)\ \forall x, y \in X.$$

The *cone* of X is the subspace of SX

$$CX = X \times [0,1]/(x,1) \sim (x',1).$$

1) Describe an explicit homeomorphism $SX \simeq CX/X \times 0$.
2) Compute $H_\bullet(CX)$.
3) Show that $H_{n+1}(SX) \simeq H_n(X)$ for $n \geq 1$ and that if X is path-connected, $H_1(SX) = 0$.

4) Brouwer Fixed-Point Theorem

1) Show that the boundary of the disk ∂D^n is not a deformation retract of D^n.
2) Use the previous point to show that every continuous map $D^n \to D^n$ has a fixed point.

5) Local Homology

Let X be a topological space and $x \in X$. Recall from class that the *local homology* of X at x is defined as

$$H_n(X, X - \{x\}) \quad \text{for } n \geq 0.$$

We will assume all points are closed in X.

1) Show that if $V \subset X$ is an open set containing x, then

$$H_n(X, X - \{x\}) \cong H_n(V, V - \{x\}).$$

2) Show that if $f\colon X \to Y$ is a local homeomorphism, then it induces isomorphisms at the level of local homology for all $x \in X$.
3) Show that the f above does not induce necessarily an isomorphism $H_\bullet(X) \to H_\bullet(Y)$.