KMAS9AA1 – Algebraic Topology

Exercise Sheet 2

1. Simplicial Identities

Let $\sigma\colon \Delta^p \to X$ be a p-simplex of X and $0 \leq j < i \leq p$. Show that $\partial_j \partial_i(\sigma) = \partial_{i-1}\partial_j(\sigma)$.

2. The standard p-simplex

1) Show that Δ^p is homeomorphic to

$$\{(s_1, \dots, s_p) \in \mathbb{R}^p \mid 0 \leq s_1 \leq \cdots \leq s_p \leq 1\}.$$

2) Express the partial boundary maps (also known as face maps) ∂_i.
3) Recall that we did a decomposition of the prism $\Delta^p \times I$ into simplices to show that homology was homotopy invariant. How does this decomposition work under this homeomorphism?
   In other words, given $(s_1 \leq \cdots \leq s_p, t)$ with $t \in I$, in which of the p-simplices $[0, \dots, k, k', \dots, p']$ does $(s_1 \leq \cdots \leq s_p, t)$ fall on?

3. Connectedness

Let X be a topological space and $(X_\alpha)_{\alpha \in E}$ the family of its path-connected components.

Show that for all $n \in \mathbb{N}$

$$H_n(X) = \bigoplus_{\alpha \in E} H_n(X_\alpha).$$

This exercise often allows us to assume X is path-connected without loss of generality.

4. Homological algebra

1. Let

$$0 \to A \to C \to F \to 0$$

be a short exact sequence and assume that F is a free R-module. Show that $C \cong A \oplus F$.

2. Find a counter example where F is non-free. This can be done over $R = \mathbb{Z}$ with $A = \mathbb{Z}$ and $F = \mathbb{Z}/2\mathbb{Z}$.

   Check that the counter example is no longer a counter example if R is instead the ring $\mathbb{Z}/2\mathbb{Z}$.

5. Relative Homology

Let (X, A) be a topological pair.

1) Show that $H_0(X, A) = 0$ if and only if A intersects every path-connected component of X.
2) Let $Z_p(X, A) = \{\sigma \in C_p(X) \mid \partial\sigma \in C_{p-1}(A)\}$. Show that there is an isomorphism of modules

$$H_p(X, A) \cong \frac{Z_p(X, A)}{B_p(X) + C_p(A)}.$$

3) Provide an alternative proof of 1) using 2).
4) Show that $H_1(X, A) = 0$ if and only if the map $H_1(A) \to H_1(X)$ is surjective and every path-connected component of X contains at most one path-connected component of A.

6. Retract

1) Show that if X is a topological space and $A \subset X$ is a retract of X, then for all n, the map induced by inclusion $H_n(A) \to H_n(X)$ is injective.

   Does this remain true if A is just a subspace of X?
2) Show that if A is a deformation retract of X, then $H_n(X, A) = 0$ for all n.

7. Surjective in Homology

1) Show that a surjective morphism $f\colon A \to B$ of chain complexes is not necessarily surjective in homology, but this is the case if $\ker f$ is acyclic.
2) Formulate and prove that same statement for injective maps.