KMAS9AA1 – Algebraic Topology

Exercise Sheet 1

Topology

1. Quotient topology, spheres, and discs

Let X be a topological space and $A \subset X$. We denote by X/A the quotient of X by the equivalence relation

$$x \sim y \iff x = y \text{ or } x, y \in A$$

equipped with the quotient topology.

1. Let $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the closed n-dimensional disc, and $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\} \subset D^n$ the $(n-1)$-dimensional sphere. Show that D^n/S^{n-1} is homeomorphic to S^n.
2. Given a topological space X, the cone of X is the quotient topological space $C(X) := X \times [0,1]/X \times \{0\}$. Show that $C(S^{n-1})$ is homeomorphic to D^n.
3. Show that the cone of a non-empty topological space is contractible.

2. Path-connected components

1. Reprove that, up to isomorphism, the fundamental group $\pi_1(X, x)$ only depends on the path-connected component of $x \in X$. More precisely, if γ is a path between x and $y \in X$, then

$$\begin{aligned} \phi_\gamma \colon \pi_1(X, x) &\to \pi_1(X, y) \\ [\alpha] &\mapsto [\overline{\gamma} \cdot \alpha \cdot \gamma] \end{aligned}$$

   is a group isomorphism.
2. If δ is another path joining x to y, then the isomorphisms ϕ_γ and ϕ_δ are conjugate. Deduce that if $\pi_1(X, x)$ is abelian, then this isomorphism is canonical.

3. Homotopy

1. Show that the homotopy equivalence relation is indeed an equivalence relation. Convince yourself that the same is not true for deformation retracts.
2. Show that $f \sim f'$ and $g \sim g'$ imply $f \circ g \sim f' \circ g'$.
3. Show with explicit formulas that any convex subset of $\mathbb{R}^n$ is contractible.

4. Fundamental Group

1. **Simple connectedness**: Let X be a path-connected topological space. Show that the following assertions are equivalent:
   a. $\pi_1(X, x)$ is trivial for any $x \in X$.
   b. There exists $x_0 \in X$ such that $\pi_1(X, x_0)$ is trivial.
   c. Any map $f\colon S^1 \to X$ extends to a map $D^2 \to X$.
   d. There exist $x_0, x_1 \in X$ such that all paths from x_0 to x_1 are homotopic.
   e. For any $x_0, x_1 \in X$, all paths from x_0 to x_1 are homotopic.
2. **An abelian** π_1: Let x, y be two points in a path-connected topological space. Show that the following assertions are equivalent:
   a. $\pi_1(X, x)$ is abelian.
   b. For any paths α, β from x to y, the induced homomorphisms given by Exercise 2 from $\pi_1(X, x)$ to $\pi_1(X, y)$ are the same.

Homological algebra

5. Chain Complexes

1) Show that homotopy of morphisms of chain complexes is an equivalence relation. Also show that homotopy equivalence between two chain complexes (i.e., there exist morphisms in both directions such that their compositions are homotopic to the identity) is an equivalence relation. Show that $f \sim f'$ and $g \sim g'$, then if the composites are defined $f \circ f' \sim g \circ g'$.
2) Show that any short exact sequence is isomorphic to a short exact sequence of the form

$$0 \to A \to B \to B/A \to 0$$

where A is an R-module and B is a submodule of A.

6. Deformation Retraction

Let C and A be two chain complexes over R. A *deformation retraction* of C onto A is a triple (r, i, h) with $r\colon C \to A$ and $i\colon A \to C$ chain complex morphisms satisfying $r \circ i = \mathrm{id}_A$, and $h\colon C_\bullet \to C_{\bullet+1}$ is a homotopy between $i \circ r$ and id_C (i.e., $ir - \mathrm{id}_C = h\partial + \partial h$).

1) Show that over a field, any chain complex retracts by deformation onto its homology.

   Hint: Over a field you can write $C_i = \ker \partial_i \oplus I_i$, for some I_i.

2) Show that this is not true in general. A counterexample can be found for $R = \mathbb{Z}$ with $C_0 = C_1 = \mathbb{Z}$ and $C_{i \neq 0,1} = 0$.

3) Show that a morphism of chain complexes that admits a left inverse (a "retract") is injective and induces an injective morphism in homology. Show that the converse is true over a field.

7. Euler Characteristic

Let C be a chain complex over a field such that for all i, C_i is finite-dimensional, and for $i \gg 0$ and $i \ll 0$, $C_i = 0$. The *Euler characteristic* of C is

$$\chi(C) = \sum_{i \in \mathbb{Z}} (-1)^i \dim(C_i).$$

Show that the Euler characteristic depends only on the homology of C.

8. Five Lemma

Consider the following commutative diagram of R-modules:

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Assume that the rows are exact at B, C, D, B', C', D', and that all vertical maps except the middle one are isomorphisms. Prove that the map $C \to C'$ is also an isomorphism.