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Homotopy Equivalence of Shifted Cotangent Bundles

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Abstract. Given a bundle of chain complexes, the algebra of functions on its shifted cotangent bundle has a natural structure of a shifted Poisson algebra. We show that if two such bundles are homotopy equivalent, the corresponding Poisson algebras are homotopy equivalent.
We apply this result to L_∞-algebroids to show that two homotopy equivalent bundles have the same L_∞-algebroid structures and explore some consequences about the theory of shifted Poisson structures.

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Key Words: Differential graded geometry, infinity algebroids, shifted Poisson structures.

1. Introduction

A Lie algebroid consists of a vector bundle A over a manifold M together with a compatible Lie algebra structure on the space of sections $\Gamma(A)$ of A. More recently, due to the application of homotopy theoretical tools to theoretical physics [19, 20] and to differential geometry (resolution of singular foliations) [23, 24], as well as the study of derived Poisson structures [8, 26, 27], there has been much interest in a derived version of Lie algebroids.

In the early 90's, T. Lada and J. Stasheff [22] introduced the notion of L_∞ algebras in the context of mathematical physics as a natural extension of differential graded Lie algebras. In an L_∞ algebra, the Jacobi identity is only satisfied up to higher coherent homotopies given by multilinear brackets. The same approach of intertwining L_∞ algebras and manifolds gives rise to the homotopical version of Lie algebroids, the so-called L_∞ algebroids [33, 34].

It is often convenient to work in the dual setting of differential graded (dg) manifolds which are generalizations of smooth manifolds to higher geometry, in which spaces are locally modeled by chain complexes. We recall that in [38] Voronov shows that given a graded vector bundle E, L_∞ algebroids over E are in one-to-one correspondence with non-positive dg manifold structures on E. Given this correspondence, we call E a *split graded manifold*.

Assume now that E dg vector bundle i.e., E is a sequence of vector bundles $(E_i)_{i\in\mathbb{Z}}$ endowed with a global differential $d\colon E_i \to E_{i+1}$ squaring to zero. One of the goals of this manuscript is to understand the behavior of the space of L_∞ algebroid structures

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on E when we replace E by a homotopy equivalent split dg manifold F over the same base manifold M.

One of our results states that two homotopy equivalent split dg manifolds have essentially the same L_∞ algebroid structures, which can be seen a a version of the Homotopy Transfer Theorem for Lie algebroids, see [28, Theorem 2.5].

Theorem 4.1. *Let E and F be homotopy equivalent split dg manifolds concentrated in non-positive degrees. Then, there is a bijection*

$$\left\{\begin{matrix} L_\infty \text{ algebroid} \\ \text{structures on } E \end{matrix}\right\} /\text{gauge eq.} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix} L_\infty \text{ algebroid} \\ \text{structures on } F \end{matrix}\right\} /\text{gauge eq.}$$

This correspondence can be obtained by explicit formulas that are given by sums of trees in the spirit of the homotopy transfer theorem [25]. The setting to prove this result is the shifted cotangent bundle $T^*[1]E$ [29]. The commutative algebra of functions of this space extends to a shifted Poisson algebra via Kosmann-Schwarzbach's *big bracket* [15, 16, 38].

There is an of analog of Voronov's result stating that the space of L_∞ algebroid structures over E can be identified with the set of Maurer-Cartan elements of the algebra of functions on $T^*[1]E$, the shifted cotangent bundle of E.

This prompts us to understand how the shifted cotangent bundle behaves under homotopy equivalence. Our main result, in the form of Theorem 3.2, states that if E and F are two homotopy equivalent dg vector bundles, their algebras of functions are homotopy equivalent as Poisson algebras.

Theorem. *Let E and F be two homotopy equivalent split dg manifolds. Then, there exist $C^\infty(M)$-linear ∞-quasi-isomorphisms $\mathcal{O}_{T^*[1]E} \rightsquigarrow \mathcal{O}_{T^*[1]F}$ and $\mathcal{O}_{T^*[1]F} \rightsquigarrow \mathcal{O}_{T^*[1]E}$ of shifted Poisson algebras. Furthermore, this homotopy equivalence of shifted Poisson algebras respects a natural notion of weight.*

When E is concentrated in degree 0, L_∞ algebroids are precisely Lie algebroids, and 1-shifted Poisson structures [8, 27] are seen to be what is refereed to in the literature as quasi-Lie bialgebroids [4]. In section 4 we see that under certain conditions our result allows us to conclude that two homotopy equivalent L_∞ algebroids have equivalent spaces of shifted Poisson structures. This matches recent advances by [3] and [32].

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Notation and conventions. Throughout this manuscript the phrase *differential graded* or *dg* should be implicit everywhere. Concretely, unless otherwise explicit, a vector space V is a dg vector space (i.e. a cochain complex), Lie algebras are differential graded Lie algebras, locally ringed spaces are dg $\mathbb{R}$-algebras etc. We use cohomological conventions, i.e. all differentials have degree $+1$. In particular this means that taking linear duals negates degrees, that is to say $V_i^* = (V_{-i})^*$.

All vector spaces (such as the ones arising from dg manifolds) considered are assumed to be finite dimensional in every degree and but not necessarily of bounded degree. Given two differential graded vector spaces A and B, the induced differential on the space $\mathrm{Hom}(A,B)$ is the commutator, denoted by $[d,-]$, satisfying the equation $[d,f] = f\circ d_A + (-1)^k d_B \circ f$, for $f\in \mathrm{Hom}(A,B)$ of degree k.

The notation $A \rightsquigarrow B$ will be reserved for ∞-morphisms of Lie or Poisson algebras, while $A\to B$ will always denote a single map.
Finally, we consider the ground field to be $\mathbb{R}$ for concreteness but the reader will notice that all algebraic proofs hold over any field.

Remark about degree shifts. Given a vector space V, the notation $[k]$ denotes a shift of degree by k units, i.e. $(V[k])_i = V_{k+i}$. Throughout the text we will encounter algebraic structures whose operations are not in degree zero. Concretely, the functions on the shifted cotangent bundle form a 2-shifted Lie algebra or a $\mathsf{Lie}\{2\}$ algebra, a Lie algebra whose Lie bracket has degree -2. When it is unambiguous, we might omit the shifts for simplicity.
As a precise definition one defines a $\mathsf{Lie}\{k\}$ algebra structure on V to be a Lie algebra structure on $V[-k]$. One should notice that this means that for odd k, a $\mathsf{Lie}\{k\}$ algebra has symmetric brackets, but when shifts are even, the defining axioms of (including signs) stay the same.
We remark that one of the consequences of the degree shifts and the cohomological conventions is that on a $\mathsf{Lie}\{k\}$ algebra, a Maurer-Cartan element has degree $k+1$.

2. Differential graded manifolds and the shifted cotangent bundle

In this section, we intend to recall in detail the constructions and results associated to the shifted cotangent bundle of a split dg manifold. We recommend [2, 6, 12] for a more thorough introduction to the topics of this section.

2.1. Dg manifolds

The origins of graded geometry and dg (differential graded) geometry can be traced back to physics, where ($\mathbb{Z}/2\mathbb{Z}$ graded) manifolds give for instance a proper treatment of ghosts in BRST deformation. Graded (resp. dg) manifolds [18] are locally modeled by a graded (resp. dg) vector space V in the sense that a function on such a manifold is locally given by a function on the base manifold and a polynomial function on V.

Definition 2.1. A *graded manifold* is a locally ringed space $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$, where the base M is a smooth manifold and around every point $x\in M$ there is an open set U containing x such that the structure sheaf can be expressed as $\mathcal{O}_\mathcal{M}(U) = C^\infty(U)\otimes S(V^*)$ for some some graded vector space V.
A *dg manifold* (also called a Q-manifold) is a graded manifold equipped with a degree $+1$ cohomological vector field Q, i.e., a derivation of the algebra of functions such that $Q^2=0$.

In the present article we will be mostly interested in a subclass of dg manifolds that originate from vector bundles.

Example 2.2 (Dg vector bundles). Given a differential graded vector bundle E over M, i.e., a sequence of vector bundles $(E_i)_{i\in\mathbb{Z}}$ with differentials d

$$\begin{array}{ccccc} \dots\ E_{i-1} & \xrightarrow{d} & E_i & \xrightarrow{d} & E_{i+1}\dots \\ & \searrow & \downarrow & \swarrow & \\ & & M & & \end{array}$$

such that $d^2 = 0$, one has a naturally associated dg manifold also denoted by E, given by its sheaf of sections $E = (M, \mathcal{O}_E = \Gamma(S(E^*)))$. Notice that $d\colon E \to E$ induces a degree $+1$ map $Q\colon E^* \to E^* \subset S(E^*)$ that extends to a square zero $C^\infty(M)$-linear derivation on $\Gamma(S(E^*))$. Such dg manifolds are called *split dg manifolds*.

In fact, Batchelor's theorem [5] (or rather, it's $\mathbb{N}$-graded version) states that every non-negatively graded manifold originates from such a construction, even though the vector bundle E is non-canonically determined.

2.2. Shifted cotangent bundle and the big bracket

Given a graded vector bundle $E \to M$, one can consider its shifted cotangent bundle $T^*[1]E = (M, \mathcal{O}_{T^*[1]E})$ (see [30, 2] for the constructions in the ungraded setting)[1] Locally this space has coordinates

$$x^i \in M, \xi^a \in E, \underbrace{p_i \in TM, \theta_a \in E^*}_{\text{momentum coordinates}}.$$

In these coordinates, the cohomological degree in the algebra $\mathcal{O}_{T^*[1]E}$ is given by $\deg(x^i) = 0$, $\deg(p_i) = 2$, $\deg(\xi^a) = d+1$ for $\xi^a \in E_d$ and $\deg(\theta_a) = -d+1$ for $\theta_a \in (E_d)^*$. We will also consider a biweight w on $\mathcal{O}_{T^*[1]E}$ compatible with the product[2], where $w(x^i) = (0,0)$, $w(p_i) = (1,1)$, $w(\xi^a) = (0,1)$ and $w(\theta_a) = (1,0)$.

Notice that there are natural inclusions

$$C^\infty|_M \hookrightarrow \mathcal{O}_{T^*[1]E}, \text{ and}$$

$$\begin{array}{ccc} \Gamma(E[-1]) \searrow & & \\ & \Gamma(S(E[-1] \oplus E^*[-1])) \hookrightarrow \mathcal{O}_{T^*[1]E}. & \\ \Gamma(E^*[-1]) \nearrow & & \end{array}$$

Remark 2.3. Let us choose connections ∇_i on E_i for all i, and let us consider the corresponding dual connections ∇_i^* on E_i^*. This defines a (non-canonical) inclusion $\Gamma(TM) \hookrightarrow \mathcal{O}_{T^*[1]E}$. With this choice one has an isomorphism of algebras

$$\mathcal{O}_{T^*[1]E} \cong_\nabla S(TM[-2] \oplus E^*[-1] \oplus E[-1]).$$

Besides the commutative product, the space $\mathcal{O}_{T^*[1]E}$ has a natural Lie bracket $\{-,-\}$, the so-called *big bracket* [16, 30] extending the natural pairing of E^* and E.

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[1] A more accurate notation for this object from the graded geometry point of view could be $T^*(E[-1])[2]$. In [2, 29] the notation $T^*\Pi E$ is used.

[2] In the sense that the product is additive with respect to the biweights.

More concretely, the bracket has degree -2, biweight $(-1,-1)$ and it satisfies the following identities on generators

$$\{X, f\} = X \cdot f, \text{ for } X \in \Gamma(TM), f \in C^\infty|_M,$$
$$\{\epsilon, e\} = \langle \epsilon, e\rangle, \text{ for } e \in \Gamma(E), \epsilon \in \Gamma(E^*),$$

Even though the bracket is intrinsically defined, with the choice of a connection ∇ as in Remark 2.3 we also have $\{X, e\} = \nabla_X(e)$ and $\{X, \epsilon\} = \nabla_X(\epsilon)$.

The bracket is extended to the full algebra $\mathcal{O}_{T^*[1]E}$ by the Leibniz rule with respect to the product of functions, making $\mathcal{O}_{T^*[1]E}$ a shifted version of a Poisson algebra, also called a Pois_3 or e_3 algebra in the literature.

Remark 2.4. Since the differential has weight zero and the bracket has weight $(-1,-1)$, the (shifted) Poisson algebra $\mathcal{O}_{T^*[1]E}$ can be decomposed into a direct sum of (shifted) Lie algebras

$$\mathcal{O}_{T^*[1]E} = \bigoplus_{k\geq 0} W_k,$$

where the Lie algebra $W_k = \bigoplus_{n\geq 0} W_{(n,n+k)}$ is spanned by all the elements whose biweights components have a common difference, i.e, elements of biweight $(0,k)$, $(1,k+1)$, $(2,k+2)$ and so on.

Suppose now that E was a dg vector bundle with differential d_E. It is easy to see that these constructions are compatible with the differential and that in this case $\mathcal{O}_{T^*[1]E}$ is a dg Poisson algebra.

Remark 2.5. Another way to see this is by noting that d_E is a Maurer-Cartan element of $\mathcal{O}_{T^*[1]E}$ (seen as a non-differential Poisson algebra), i.e. $\{d_E, d_E\} = 0$. Indeed, it follows from $d_E^2 = 0$ that $\{\{d_E, d_E\}, x\} = 0$ for every x element of E or E^*. Therefore, $\{d_E, d_E\}$ is central in $S(E \oplus E^*)$ but the center of this Lie algebra is $\mathbb{R}$ and therefore $\{d_E, d_E\} = 0$.
By twisting the (Lie part of the) Poisson algebra $\mathcal{O}_{T^*[1]E}$ by this Maurer-Cartan element, we recover a dg Poisson algebra structure on $\mathcal{O}^{d_E}_{T^*[1]E}$ that we will denote by $\mathcal{O}_{T^*[1]E}$ only.

2.3. (Infinity) Algebroids

The constructions from the previous section allow us to encode neatly some classical notions. For example, a Lie algebroid structure over M i.e., a Lie algebra bundle E concentrated in degree zero, with a compatible anchor map $\rho\colon E \to TM$, can be expressed as a solution of the Maurer-Cartan equation on $T^*[1]E$:

Proposition 2.6 ([29, 35])**.** *Let M be a manifold and $E \to M$ a vector bundle concentrated in degree zero. A Lie algebroid structure on E is equivalent to an element $\mu \in \mathcal{O}_{T^*[1]E}(M)$ of biweight $(1,2)$ such that $\{\mu, \mu\} = 0$.
The correspondence is given by $\rho(X) \cdot f = \{\{X, \mu\}, f\}$ and $[X, Y] = \{\{X, \mu\}, Y\}$, for $X, Y \in \Gamma(TM, M)$ and $f \in C^\infty(M)$.*

The same way the *homotopically correct* version of a Lie algebra is an L_∞ algebra, the notion of a Lie algebroid over a manifold M can be homotopically relaxed leading to the concept of an L_∞ algebroid. In what follows we will suppose that all objects are non-positively graded.

Definition 2.7. Let M be a smooth manifold and let $(E = (E_i)_{i\leq 0}, d)$ be a dg vector bundle over M concentrated in non-positive degree. An L_∞ *algebroid structure* on E is:

1. A dg bundle map $\rho\colon E \to TM$ called *the anchor* and
2. A sequence of antisymmetric brackets $l_k = [\dots]_k\colon \Gamma(E^{\otimes k}) \to \Gamma(E)$ of degree $2-k$, for $k \geq 2$.

such that

1. All brackets are $C^\infty(M)$ linear except the binary bracket if one of the entries is in degree 0. If that is the case, then it behaves as a vector field in the sense that if $X \in \Gamma(E_0)$ and $e \in \Gamma(E)$,
$$[X, fe]_2 = f[X, e]_2 + (\rho(X)\cdot f)e.$$
2. The anchor intertwines l_2 and the bracket of vector fields
$$[\rho(x), \rho(y)] = \rho([x, y]), \forall x, y \in \Gamma(E_0).$$
3. These brackets satisfy the structural axioms of an L_∞ algebra (5).

Remark 2.8. Some authors such as [13] consider all brackets to be symmetric and of degree 1 (from an operadic perspective one would call these $L_\infty\{-1\}$ algebroids) while we follow conventions such as the ones of [6]. These are equivalent up to a degree shift of E.

Analogous to Proposition 2.6 one can show that L_∞ algebroids are also given as solutions of the Maurer-Cartan equation.

Proposition 2.9 (Folklore). *Let $E \to M$ be a split dg manifold concentrated in non-positive degrees, finite dimensional in every degree. The set of L_∞ algebroid structures over E is in biunivocal correspondence with the space of solutions of the Maurer-Cartan equation in $\mathcal{O}_{T^*[1]E}$ of biweight $(*, 1)$ such that the term in $E^*\otimes E = \mathrm{Hom}(E, E)$ is the differential $d\colon E \to E$.*

Sketch of proof. Due to the assumption of finite dimension, a map of bundles $E \to TM$ is equivalent to a section of $E^* \otimes TM$ and the data of the brackets corresponds to a section of $S(E^*) \otimes E$. The degree conditions imply that these correspond to elements of degree 3 in $\mathcal{O}_{T^*[1]E}$.

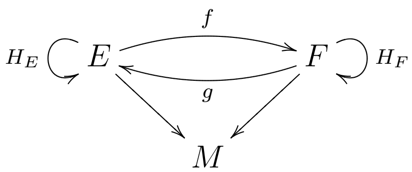
The bracket condition its easy to verify: The Maurer-Cartan equation can be split by left weight. On left weight 2 the terms with the differential do not exist due to our degree restraints on E. On higher weight we find the L_∞ structure equations and so the Maurer-Cartan equation gives us the same compatibility with the anchor as in the Lie algebroid case. ∎

Remark 2.10. Some authors suppose that E is a graded manifold from the start and the L_∞ algebroid structure includes the datum of the differential d as a unary bracket l_1 (see [24, Definition 1.1.6] for instance). The natural analog of the previous proposition holds, with the differential is recovered from the $E^* \otimes E$ component. Recall that the differential d_E is itself a Maurer-Cartan element of $\mathcal{O}_{T^*[1]E}$, the two results are related from the general fact that if $\mathfrak{g}$ is a Lie algebra and $\mu \in \mathrm{MC}(\mathfrak{g})$, then $\nu \in \mathrm{MC}(\mathfrak{g}^\mu) \Leftrightarrow \nu + \mu \in \mathrm{MC}(\mathfrak{g})$.

3. Proof of the main result

The natural notion of homotopy equivalences on cochain complexes generalize naturally to the setting of dg vector bundles.

Definition 3.1. Two dg vector bundles E and F are said to be *homotopy equivalent* if there exist bundle maps $f\colon E \to F$ and $g\colon F \to E$ and *homotopies* $H_E\colon E^\bullet \to E^{\bullet+1}$ and $H_F\colon F^\bullet \to F^{\bullet+1}$ such that $id_E - g \circ f = H_E d_E + d_E H_E$ and $id_F - f \circ g = H_F d_F + d_F H_F$



(1)

One can also consider the weaker notion of a *quasi-isomorphism* of dg vector bundles, i.e., a dg vector bundle map $f\colon E \to F$ that induces a quasi-isomorphism on sections. Bear in mind that in general the homology of a dg vector bundle is not a graded vector bundle as it can shift dimensions.
Our main result states that if we take two homotopy equivalent dg vector bundles and consider their shifted cotangent bundles, the respective algebras of functions are homotopy equivalent as Poisson algebras.

Theorem 3.2. *Let E and F be two dg vector bundles over M that are homotopy equivalent as in the previous definition. Then, there exists a $C^\infty(M)$ linear $L_\infty\{2\}$ quasi-isomorphism $\mathcal{U}\colon \mathcal{O}_{T^*[1]E} \rightsquigarrow \mathcal{O}_{T^*[1]F}$. Furthermore, this map:*

(1) *is compatible with the symmetric algebra product,*

(2) *is compatible with the biweight in the sense that it preserves each component W_k from Remark 2.4 (for all $n \geq 1$, $\mathcal{U}_n$ has biweight $(-n+1, -n+1)$),*

(3) *its first component $\mathcal{U}_1$ is the natural extension of $f \oplus g^*\colon E \oplus E^* \to F \oplus F^*$ to a graded commutative algebra morphism.*

To be more precise, by compatibility with the symmetric algebra product we mean that every $(\mathcal{U}_n)_{n\geq 2}$ acts as a derivation with respect to the map $\mathcal{U}_1$. In particular, this means that $\mathcal{U}$ actually defines a weak equivalence of shifted Poisson algebras (an $\infty - \mathsf{Pois}_3$ algebra quasi-isomorphism). This is the notion of morphism considered in [3].

3.1. The case $M = *$

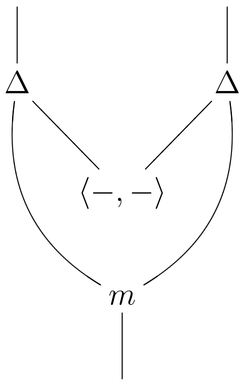
In this section we prove the main Theorem 3.2 over $M = *$ a point, which reduces to a problem in homotopical algebra. As we will see, this is the main part of the proof, as the formulas we will obtain over a point readily extend to a more general base. In this case, E and F are just two dg vector spaces that are quasi-isomorphic with a prescribed homotopy. The functions on the shifted cotangent bundle $T^*[1]E$ are given by the symmetric algebra $S(E[-1] \oplus E^*[-1])$. We define a map $\mathcal{U}_1 \colon S(E[-1] \oplus E^*[-1]) \to S(F[-1] \oplus F^*[-1])$ by extending $f \colon E \to F$ and $g^* \colon E^* \to F^*$ to a map of commutative algebras.

Recall that given a dg vector space V, the space $S(V)$ admits a bialgebra structure given by the canonical coproduct $\Delta \colon S(V) \to S(V) \otimes S(V)$ by

$$\Delta(v_1 \dots v_n) = \sum_{p \leq n,\ \sigma \in \mathbb{S}_n} \pm v_{\sigma^{-1}(1)} \dots v_{\sigma^{-1}(p)} \otimes v_{\sigma^{-1}(p+1)} \dots v_{\sigma^{-1}(n)}.$$

Notice that under this description, the Poisson bracket on $S(E[-1] \oplus E^*[-1]) = \mathcal{O}_{T^*[1]E}$ has the following nice form

$$l_{T^*[1]E} \coloneqq \{-,-\} =$$



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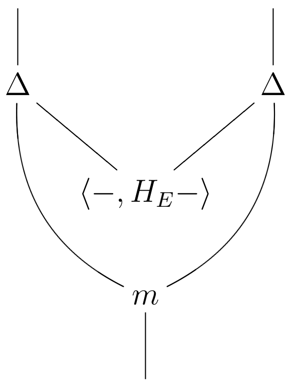
where m stands for the multiplication (in the symmetric algebra) and $\langle -,- \rangle$ denotes the pairing between E^* and E being zero otherwise. By convention, elements of E^* will be placed on the first entry of $\langle -,- \rangle$ and elements of E will be placed on the second entry. We define the operator $\mathcal{R}_2 \coloneqq \mathcal{O}_{T^*[1]E} \otimes \mathcal{O}_{T^*[1]E} \to \mathcal{O}_{T^*[1]E}$



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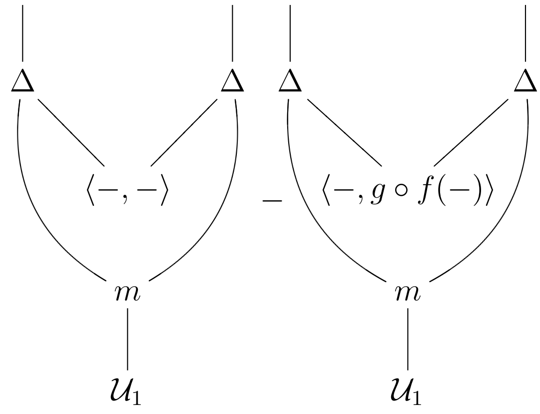
Finally, we define $\mathcal{U}_2 \colon \mathcal{O}_{T^*[1]E} \to \mathcal{O}_{T^*[1]F}$ to be $\mathcal{U}_2 \coloneqq \mathcal{U}_1 \circ \mathcal{R}_2$.

Notice that besides the homotopy, all the operations involved in $\mathcal{U}_2$ commute with the differentials, from which it follows that

$$[d,\mathcal{U}_2] =$$



$$=: \mathcal{U}_1 \circ l_{T^*[1]E} - \mathcal{U}_1 \circ \tilde{l}_{T^*[1]E} \tag{2}$$

and a similar formula without the terms $\mathcal{U}_1$ holds if we replace $\mathcal{U}_2$ by $\mathcal{R}_2$. One can easily check on generators that the second term of the equation, $\mathcal{U}_1 \circ \tilde{l}_{T^*[1]E}$ is equal to $\{\mathcal{U}_1(-),\mathcal{U}_1(-)\}_{T^*[1]E}$ from where the case $n = 2$ from equation (6) follows.

Defining the higher components of the L_∞ morphism requires some set-up. Let Tree_n be the set of trees with n labeled vertices. To an element $T \in \mathrm{Tree}_n$ one can associate a map $T\colon \mathcal{O}_{T^*[1]E}^{\otimes n} \to \mathcal{O}_{T^*[1]E}$ of degree $n-1$. The value of $T(x_1, x_2, \dots, x_n)$ is obtained in the following way:

Let $e_1, \dots, e_{n-1}$ be the set of edges of T and consider a choice of $2n-2$ elements $\alpha_1, \beta_1, \dots, \alpha_{n-1}, \beta_{n-1}$ each one of them from either E or E^* such that:

(1) For every k, if e_k connects vertices i and j, α_k is a factor of x_i and β_k is a factor of x_j,
(2) There is no repetition of choices.

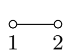
Given such a choice one can consider the product

$$\langle \alpha_1, H(\beta_1)\rangle \dots \langle \alpha_{n-1}, H(\beta_{n-1})\rangle\, x_1\widehat{x_2 \dots x}_n,$$

where $x_1\widehat{x_2 \dots x}_n$ denotes the product of all x_i's but with our choice of α's and β's removed[3] together with the appropriate Koszul sign corresponding to the elements removed. Finally, the value of

$$T(x_1, x_2, \dots, x_n) = \sum_{\substack{\text{choices of}\\ \alpha_1,\dots,\beta_{n-1}}} \langle \alpha_1, H(\beta_1)\rangle \dots \langle \alpha_{n-1}, H(\beta_{n-1})\rangle\, x_1\widehat{x_2 \dots x}_n$$

is obtained by summing over all possible choices the products described.

Heuristically, to every edge of a tree we associate an application of the operator $\mathcal{R}_2$ to its vertices. In particular, $\mathcal{R}_2 =$  $\in \mathrm{Tree}_2$.

Remark 3.3. Notice that some choices regarding the ordering and orientation of edges of T has to be done to compute $T(x_1, x_2, \dots, x_n)$. Since the target $\mathcal{O}_{T^*[1]E}$ is commutative, all choices lead to the same result up to a sign.

We fix the convention that edges are oriented from the smaller vertex to the bigger vertex and the ordering of edges is done by comparing the smaller label and then the bigger label. In particular it follows that the natural action of $\mathbb{S}_n$ permuting the labels of the vertices produces signs.

[3] Keep in mind that $\langle a, b\rangle$ is zero unless one of a, b is in E and the other one in E^*.

For all $n \geq 1$ we define the operators $\mathcal{R}_n \colon \mathcal{O}_{T^*[1]E}^{\otimes n} \to \mathcal{O}_{T^*[1]E}$ of degree $n-1$ as $\mathcal{R}_n = \sum_{T \in \mathrm{Tree}_n} T$. Notice that this definition gives $\mathcal{R}_1 = \mathrm{id}_{\mathcal{O}_{T^*[1]E}}$. We also define $\mathcal{U}_n \coloneqq \mathcal{U}_1 \circ \mathcal{R}_n \circ \mathcal{O}_{T^*[1]E}^{\otimes n} \to \mathcal{O}_{T^*[1]F}$.

Proposition 3.4. *The maps $\mathcal{R}_n$ satisfy the following equations, for all $n \geq 2$:*

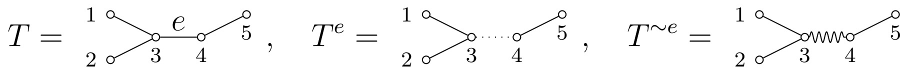
$$[d, \mathcal{R}_n] = \sum_{\sigma \in \mathrm{Sh}_{2,n-2}^{-1}} \mathrm{sgn}(\sigma)(\mathcal{R}_{n-1} \circ_1 l_{T^*[1]E})^\sigma - \sum_{\substack{p+q=n \\ \sigma \in \mathrm{Sh}_{p,q}^{-1}}} \mathrm{sgn}(\sigma)(-1)^{p-1} \tilde{l}_{T^*[1]E} \circ (\mathcal{R}_p, \mathcal{R}_q)^\sigma \quad (3)$$

where $\circ_1$ represents insertion in the first slot and $\tilde{l}_{T^*[1]E}$ is the twisted bracket defined in equation (2)*.*

Before proving this proposition notice that by composing the equations above with $\mathcal{U}_1$ and using the observation that $\mathcal{U}_1 \circ \tilde{l}_{T^*[1]E} = l_{T^*[1]F} \circ (\mathcal{U}_1, \mathcal{U}_1)$ we recover exactly the equations (6) defining an L_∞ morphism.

Corollary 3.5. *The maps $\mathcal{U}_n$ defined above form an L_∞ algebra morphism.*

Proof of Proposition 3.4. Given a tree $T \in \mathrm{Tree}_n$ and e an edge of T, we denote by T^e the same tree T but with the edge e replaced by a dashed edge. Similarly, we denote by $T^{\sim e}$ the same edge e replaced by a wavy edge instead.



For $T \in \mathrm{Tree}_n$ we define an action of these modified trees, $T^e, T^{\sim e} \colon \mathcal{O}_{T^*[1]E}^{\otimes n} \to \mathcal{O}_{T^*[1]E}$ of degree $n-2$ by the same formula as T, except that on the action of the edge corresponding to e connecting vertices i and j, with T^e we perform the pairing $\langle \alpha_i, \beta_j \rangle$ and with $T^{\sim e}$ we perform the twisted pairing $\langle \alpha_i, g \circ f(\beta_j) \rangle$.

Notice that since the commutator with the differential $[d, -]$ acts by derivations, the computation of $[d, \mathcal{R}_n]$ produces the same kind of composition, except that it replaces one instance of $\langle -, H- \rangle$ by $\langle -, - \rangle - \langle -, g \circ f- \rangle$, just as in equation (2). In terms of trees, we have that $[d, T] = \sum_{e \text{ edge}} T^e - T^{\sim e}$, so we can also interpret $[d, \mathcal{R}_n]$ as a sum of all possible trees of n vertices with a dotted edge, minus a sum of all trees with n vertices with a wavy edge.

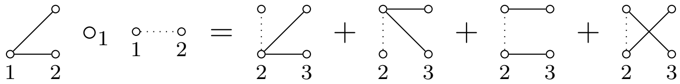


Figure 1: An example of an insertion of the graph corresponding to $l_{T^*[1]E}$.

We claim that the summands corresponding to the terms T^e correspond to the terms

$$\sum_{\sigma \in \mathrm{Sh}_{2,n-2}^{-1}} \mathrm{sgn}(\sigma)(\mathcal{R}_{n-1} \circ_1 l_{T^*[1]E})^\sigma.$$

This follows from the observation that, given a tree $\Gamma \in \mathrm{Tree}_{n-1}$, the operation $T(\{x_1, x_2\}, x_3, \dots, x_n)$ can be expressed as a sum of trees with a dotted edge. Concretely, as a quick inspection shows, $\Gamma \circ_1 l_{T^*[1]E}$ is obtained by inserting a graph $\underset{1}{\circ}\cdots\underset{2}{\circ}$ on the vertex labeled by 1 of Γ and summing over all possible ways (there exist $2^{\text{valence of } 1}$) of reconnecting the incident edges, followed by a shift by 1 of all other labels. This allows us to conclude that all terms of $(\mathcal{R}_{n-1} \circ_1 l_{T^*[1]E})^\sigma$ are dotted trees. We just need to show that every dotted tree appears exactly once on the sum over all $(2, n-2)$ unshuffles.

Let us consider an arbitrary dotted tree T^e, where $T \in \mathrm{Tree}_n$. Suppose that e connects vertices $i < j$. There is a unique unshuffle $\tau \in Sh^{-1}_{2,n-2}$ sending i to 1 and j to 2. It is then clear that only $(\mathcal{R}_{n-1} \circ_1 l_{T^*[1]E})^\tau$ produces trees with a dotted edge connecting vertices i and j. Conversely, if we denote by $T/e \in \mathrm{Tree}_{n-1}$ be the graph obtained by the contraction of the edge e one sees that we recover T^e from the insertion of $l_{T^*[1]E}$ in T/e.[4] To finish the proof it remains to show that

$$\sum_{\substack{p+q=n \\ \sigma \in \mathrm{Sh}^{-1}_{p,q}}} \mathrm{sgn}(\sigma)(-1)^{p-1} \tilde{l}_{T^*[1]E} \circ (\mathcal{R}_p, \mathcal{R}_q)^\sigma = \sum_{T \in \mathrm{Tree}_n} \sum_{e \text{ edge of } T} T^{\sim e}.$$

The proof is analogous to the other case. We start by noting that for $T_p \in \mathrm{Tree}_p$ and $T_q \in \mathrm{Tree}_q$, $\tilde{l}_{T^*[1]E} \circ (T_i, T_q)$ is obtained summing over all possible ways $(p \times q)$ of connecting T_p and T_q with a wavy edge, and shifting the labels of T_q up by p units. It follows that $\tilde{l}_{T^*[1]E} \circ (\mathcal{R}_p, \mathcal{R}_q)^\sigma$ is a sum of elements of the form $T^{\sim e}$, where $T \in \mathrm{Tree}_n$. To see that every tree appears exactly once, one notices that given a tree with a wavy edge, removing the wavy edge results in a disconnected graph composed of two trees, one in Tree_p and the other one in Tree_{n-p} whose labels are uniquely retained by an element of $\mathrm{Sh}^{-1}_{p,q}$. ∎

3.2. The global case

Suppose now that E and F are split dg manifolds over an arbitrary manifold M with maps f, g and homotopies H_E and H_F as in equation (1).
Suppose for the moment being that we can choose connections ∇^E and ∇^F that are compatible with f and g, i.e. for all $X \in \Gamma(TM), e \in \Gamma(E)$ and $s \in \Gamma(F)$ we have

$$f(\nabla^E_X(e)) = \nabla^F_X(f(e)) \text{ and } g(\nabla^F_X(s)) = \nabla^E_X(g(s)). \tag{4}$$

Under the identifications induced by ∇^E and ∇^F,

$$\mathcal{O}_{T^*[1]E} \cong S(TM[-2] \oplus E^*[-1] \oplus E[-1]) \text{ and } \mathcal{O}_{T^*[1]F} \cong S(TM[-2] \oplus F^*[-1] \oplus F[-1]),$$

the two maps f and g induce a map $\mathcal{U}_1 \colon \mathcal{O}_{T^*[1]E} \to \mathcal{O}_{T^*[1]F}$ of commutative algebras by extending the maps $f \colon E \to F$, $g^* \colon E^* \to F^*$ and $\mathrm{id} \colon TM \to TM$.

Equations (4) imply that $\mathcal{U}_1$ intertwines the Lie brackets on $\mathcal{O}_{T^*[1]E}$ and $\mathcal{O}_{T^*[1]F}$ whenever at least one of the elements being bracketed is a vector field.

We note that the formulas used for $\mathcal{U}_n, n \geq 2$ in the previous section can be defined over any manifold, since the homotopy H_E is globaly defined on the bundle E. It

[4] Notice that there is an appearance of a sign factor $\mathrm{sgn}(\tau)$ due to the considerations from Remark 3.3.

follows that the natural extensions of the maps $\mathcal{U}_n$ give a well defined L_∞ quasi-isomorphism $\mathcal{O}_{T^*[1]E} \rightsquigarrow \mathcal{O}_{T^*[1]F}$:

$$\mathcal{U}_n(x_1, \dots, x_n) := \begin{cases} \text{same formula as before if all } x_i \in S(E \oplus E^*) \\ 0 \text{ otherwise.} \end{cases}$$

Even though is not true in general that we can choose connections ∇^E and ∇^F that are compatible with f and g, one situation where such choice can be made is if $g \circ f = \mathrm{id}_E$. Indeed, the condition $g \circ f = \mathrm{id}_E$ implies that both f and g are maps of constant rank, so their images and kernels are bundles. Identifying $E = \operatorname{Im} f$, we can decompose $F = E \oplus \ker g$.

We can now take an arbitrary connection on E, an arbitrary connection on $\ker g$ and define the sum of the two connections as the connection on F. This makes the maps $f\colon E \to F$ and $g^*\colon E^* \to F^*$ compatible with the respective connections. Therefore, the global version of Theorem 3.2 follows from the following proposition:

Proposition 3.6. *Given a homotopy equivalence of vector bundles over M*

$$H_E \circlearrowright E \underset{g}{\overset{f}{\rightleftarrows}} F \circlearrowleft H_F$$

there is a dg vector bundle C and homotopy equivalences

$$E \underset{p_E}{\overset{i_E}{\rightleftarrows}} C \circlearrowleft H_1 \quad \textit{and} \quad H_2 \circlearrowright C \underset{i_F}{\overset{p_F}{\rightleftarrows}} F,$$

such that $p_E \circ i_E = \mathrm{id}_E$ and $p_F \circ i_F = \mathrm{id}_F$.

Proof. We mimic the standard mapping cylinder construction from homological algebra, see for example [14]. We define $C = E \oplus E[1] \oplus F$ with differential $d(e, e', y) = (de - e', -de', dy + f(e'))$.

The second homotopy equivalence depends only on f and is given by the maps $i_F(y) = (0, 0, y)$, $p_F(e, e', y) = f(e) + y$ and $H_2(e, e', y) = (0, e, 0)$.

The other homotopy equivalence is given by the maps $i_E(e) = (e, 0, 0)$, $p_E(e, e', y) = e + H_E(e') + g(y)$ and $H_1(e, e', y) =$

$$\begin{pmatrix} -gH_F(y + H_F f(e') - gfH_E(e')) + H_E g(y + H_F f(e') - fH_E(e')) + H_E H_E(e') \\ -g(y + H_F f(e') - fH_E(e')) - H_E(e') \\ H_F(y + H_F f(e') + fH_E(e'))) \end{pmatrix}. \blacksquare$$

3.3. Remarks about the hypothesis of homotopy equivalence

The reader might be surprised that it seems that we almost did not use F (in particular H_F) at all in the proofs in section 3, by reducing the problem to work with $\mathcal{R}_n$ instead of $\mathcal{U}_n$. The reason for this is that one can consider a "twisted shifted cotangent bundle" $\widetilde{\mathcal{O}_{T^*[1]E}}$ given by the same base space space but with the twisted Lie bracket $\tilde{l}_{T^*[1]E}$.

What we have shown is that $(\mathcal{R}_n)_{n>1}$ realise an L_∞ isomorphism $\mathcal{O}_{T^*[1]E} \rightsquigarrow \widetilde{\mathcal{O}_{T^*[1]E}}$ extending the identity map. The result follows from the fact that $\mathcal{U}_1$ defines a strict Lie quasi-isomorphism $\widetilde{\mathcal{O}_{T^*[1]E}} \to \mathcal{O}_{T^*[1]F}$.

However, over manifolds M different from a point, to say that $T^*[1]E$ and $T^*[1]F$ are homotopy equivalent as locally ringed spaces we need the full homotopy data. The reason for this is while ∞-quasi-isomorphisms are quasi-invertible over $\mathbb{R}$, that is not necessarily the case over $C^\infty(M)$.

4. Applications

4.1. Equivalence of L_∞ algebroid structures

Let E and F be non-positively graded split dg manifolds that are homotopy equivalent via maps f, g, H_E and H_F as in the conditions of the main Theorem 3.2. Recall from Proposition 2.9 that L_∞ algebroid structures over E are the same as Maurer-Cartan elements of $\mathcal{O}_{T^*[1]E}(M)$ of biweight $(*,1)$. Since $\mathcal{U}_n$ has biweight $(-n+1,-n+1)$, it sends n elements of biweight $(*,1)$ to an element of biweight $(*,1)$. It follows that $\mathcal{U}\colon \mathcal{O}_{T^*[1]E} \rightsquigarrow \mathcal{O}_{T^*[1]F}$ maps Maurer-Cartan elements of biweight $(*,1)$ to Maurer-Cartan elements of biweight $(*,1)$. It follows from the Goldman-Millson Theorem A.7 and the main Theorem 3.2 that E and F have the same L_∞ algebroid structures.

Theorem 4.1. *Let E and F be split dg manifolds concentrated in non-positive degrees that are homotopy equivalent. Then, there is a set bijection*

$$\left\{ \begin{matrix} L_\infty \textit{ algebroid} \\ \textit{structures on } E \end{matrix} \right\} / \textit{gauge eq.} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} L_\infty \textit{ algebroid} \\ \textit{structures on } F \end{matrix} \right\} / \textit{gauge eq.}$$

This result can be compared to the similar result of Pym and Safronov [28, Theorem 2.5]. Their approach follows the classical proof of the Homotopy Transfer Theorem [25, Theorem 10.3.9] while ours is closer to its interpretation in terms of Maurer-Cartan elements and gauge actions as in [11].

4.2. Isotropy of L_∞ algebroids

Let us consider $E = (E, d_E, \{l_n^E\}_{n\geq 2})$, an L_∞ algebroid of bounded degree over a manifold M and let us fix a point $m \in M$. It is well known that on a neighborhood $U \subset M$ of m, there exists a dg vector bundle (F, d_F) which is homotopy equivalent to $E|_U$, such that the restriction of the differential d_F to the point m is trivial $d_F|_m = 0$ (see the proof of [23, Proposition 1.3.5]).

Using Theorem 4.1 we can transfer the L_∞ algebroid structure from $E|_U$ to one in $F = (F, d_F, \{l_n^F\}_{n\geq 2})$. This structure restricts to an L_∞ algebra on the point m, where we have the identification $F_m = H^\bullet(E_m, d_E)$. Notice that when considering the cohomology of an L_∞ algebroid, authors typically consider the anchor $\rho\colon E_0 \to TM$ as part of the cochain complex therefore changing the cohomology in degree zero[5]. In that case, the identification becomes $F_m \supset \ker \rho|_m = H^\bullet(E_m, d_E + \rho)$.

[5] Keep in mind that the anchor is in principle not of constant rank, which means that $\ker \rho \subset F$ is not a vector bundle. This is an important point in the study of singular foliations of [23].

Moreover, since the restricted L_∞ algebra structure on F_m has zero differential, (F_m, l_2^F) is a strict (graded) Lie algebra. The higher brackets $\{l_n^F\}_{n\geq 3}$ can be seen as Lie analogs of the Massey products [36].

Concretely, the higher brackets $\{l_n^F\}_{n\geq 3}$ correspond to a Maurer-Cartan element in the Chevalley-Eilenberg complex of (F_m, l_2^F), denoted by $\mathrm{CE}(F_m, l_2^F)$. Since the differential is trivial, they actually correspond to an obstruction class living in the Chevalley-Eilenberg cohomology $[l_n^F] \in H^n_{\mathrm{CE}}(H(E_m, d_E), H(E_m, d_E))$. In fact, since the differential is zero, for every $N \geq 3$, $\{l_n^F\}_{3\leq n\leq N}$ gives an L_∞ structure on F_m and therefore a Maurer-Cartan element in $\mathrm{CE}(F_m, l_2)$. In particular, we obtain

Proposition 4.2. *Given an L_∞ algebroid E over M and a point $m \in M$, there is a canonically associated class $[l_3] \in H^3_{\mathrm{CE}}(H(E_m, d_E), H(E_m, d_E))$ that vanishes if the L_∞ algebroid structure is homotopically trivial.*

The NMRLA (No Minimal Rank Lie Algebroid) class [23] is an example of this class.

4.3. Shifted Poisson structures

The results that we present here are certainly connected to the theory of shifted Poisson structures [8, 27], see also [3, 7] and [32].

Let E be a split dg manifold and (E, ϕ^E) an L_∞ algebroid structure on E, i.e., ϕ^E is a Maurer-Cartan element of $\mathcal{O}_{T^*[1]E}$, as in the last section. One can then twist the Poisson algebra $\mathcal{O}_{T^*[1]E}$ by ϕ^E to obtain the Lie algebra

$$\mathcal{O}^{\phi^E}_{T^*[1]E} = (\mathcal{O}_{T^*[1]E}, d^E + \{\phi^E, -\},\ \{-,-\}).$$

We propose the following definition of a 1-shifted Poisson structure.

Definition 4.3. A 1-shifted Poisson structure over the L_∞ algebroid (E, ϕ^E) is a Maurer-Cartan element in $\mathcal{O}^{\phi^E}_{T^*[1]E}$ of biweight $(*, \geq 2)$.

This is notion is very close to what in [21, 17, 4] is referred to as an L_∞ quasi-bialgebra(oid). From Theorem 3.2 and Lemma A.5, we obtain the following result.

Corollary 4.4. *Let E and F be homotopy equivalent split dg manifolds. Suppose that ϕ^E and ϕ^F are L_∞ algebroid structures on E and F respectively, such that the map $\mathcal{U}$ constructed in Theorem 3.2 satisfies $\mathcal{U}(\phi^E) = \phi^F$. Then, the 1-shifted Poisson structures over E and F are in bijection up to gauge equivalence.*

Recall from [37] "Voronov's trick" that out of a decomposition of a Lie algebra $\mathfrak{g}$ into two Lie sub-algebras $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is abelian and a Maurer-Cartan element $\pi \in \mathrm{MC}(\mathfrak{h})$ produces an L_∞ structure on $\mathfrak{a}[1]$ with higher brackets l_n given by the iterated adjoint action $l_n(a_1, \dots, a_n) = \mathrm{pr}_{\mathfrak{a}}[\dots[\pi, a_1], \dots, a_n]$, where $\mathrm{pr}_{\mathfrak{a}} \colon \mathfrak{g} \to \mathfrak{a}$ denotes the projection.

Consider an L_∞ algebroid (E, ϕ^E). Voronov's trick, applied to $\mathfrak{g} = \mathcal{O}^{\phi^E}_{T^*[1]E}$, $\mathfrak{a} = S(\Gamma(E^*[-1]))$, $\mathfrak{h}$ the natural complement (elements of biweight $(*, \geq 1)$) and π a 1-shifted Poisson structure, yields the following proposition:

Proposition 4.5. *There exists an L_∞ algebra structure on $S(\Gamma(E^*[-1]))$ whose differential is the one coming from the L_∞ algebroid structure.*

Due to the compatibility with the product of functions, we actually obtain that the L_∞ structure extends to a homotopy shifted Poisson structure [8, 27] that is strict on the product. This is what is called a *derived Poisson algebra* in [3].

Finally, for E concentrated in degree zero, we recover the classical notion of quasi-Lie bialgebroids (see [31] for a definition of those and, e.g. [1], for the description in terms of big bracket).

A. Recollections about L_∞ algebras and Maurer-Cartan elements

In this Appendix we recall some of the classical homotopy theory of Lie algebras and their Maurer-Cartan elements that are used in this paper. We assume that the Lie algebras are unshifted, i.e., the bracket has degree zero, but all statements hold for shifted Lie algebras c.f. Section 1.

Recall that an L_∞ algebra structure on the differential graded vector space (A, d) is a family of multilinear antisymmetric maps (the multibrackets) $[-, \dots, -] = l_n \colon A^{\otimes n} \to A$ of degree $|l_n| = 2 - n$ for $n \geq 2$ satisfying the higher Jacobi identities:

$$\sum_{\substack{p+q=n+1 \\ p,q>1}} \sum_{\sigma \in \mathrm{Sh}^{-1}_{q,p-1}} \mathrm{sgn}(\sigma)(-1)^{(p-1)q}(l_p \circ_1 l_q)^\sigma = [d, l_n], \tag{5}$$

where $\mathrm{Sh}^{-1}_{q,p-1} \subset \mathbb{S}_{q+p-1}$ denotes the $(q, p-1)$ unshuffles.

Most results in this section can be generalized to L_∞ algebras but since they are not necessary for us they are stated in terms of Lie algebras and L_∞ morphisms for simplicity of formulas.

Definition A.1. An L_∞ *morphism* $\mathcal{U} \colon A \rightsquigarrow B$ between two Lie algebras (A, l_A, d_A) and (B, l_B, d_B) is a sequence of maps $\mathcal{U}_n \colon S^n A \to B$, $\forall n \geq 1$ of degree $1-n$ such that $\mathcal{U}_1$ commutes with the differentials, i.e $[d, \mathcal{U}_1] = 0$ and

$$[d, \mathcal{U}_n] = \sum_{\sigma \in \mathrm{Sh}^{-1}_{2,n-2}} \mathrm{sgn}(\sigma)(\mathcal{U}_{n-1} \circ_1 l_A)^\sigma - \sum_{\substack{p+q=n \\ \sigma \in \mathrm{Sh}^{-1}_{p,q}}} \mathrm{sgn}(\sigma)(-1)^{p-1} l_B \circ (\mathcal{U}_p, \mathcal{U}_q)^\sigma \tag{6}$$

Definition A.2. Let $\mathfrak{g}$ be a differential graded Lie algebra. A *Maurer-Cartan element* is an element $\mu \in \mathfrak{g}^1$ of degree 1 that satisfies the equation

$$d\mu + \frac{1}{2}[\mu, \mu] = 0$$

The set of Maurer-Cartan elements of a Lie algebra $\mathfrak{g}$ is denoted by $\mathrm{MC}(\mathfrak{g})$.

Definition A.3. A filtered Lie algebra is a Lie algebra $\mathfrak{g}$ equipped with a complete descending filtration $\mathcal{F}^\bullet$ of Lie algebras i.e. $\mathfrak{g} = \mathcal{F}^1\mathfrak{g} \supset \mathcal{F}^2\mathfrak{g} \supset \mathcal{F}^2\mathfrak{g} \supset \dots$ satisfying $[\mathcal{F}^i\mathfrak{g}, \mathcal{F}^j\mathfrak{g}] \subset \mathcal{F}^{i+j}\mathfrak{g}$, such that $\mathfrak{g}$ is complete with respect to this filtration

$$\mathfrak{g} = \varprojlim_k \mathfrak{g}/\mathcal{F}^k\mathfrak{g}.$$

Let $\mathfrak{g}$ and $\mathfrak{h}$ be filtered Lie algebras. It is easy to check that given an L_∞ morphism $\mathcal{U} = (\mathcal{U}_k)_{k\geq 1} \colon \mathfrak{g} \rightsquigarrow \mathfrak{h}$ compatible with the filtrations [6] and a Maurer-Cartan element

[6] In the sense that $\mathcal{U}_k(\mathcal{F}^{i_1}\mathfrak{g}, \dots, F^{i_k}\mathfrak{g}) \subset \mathcal{F}^{i_1+\dots+i_k}\mathfrak{h}$.

$\mu \in \mathfrak{g}$, then, the element

$$\mathcal{U}(\mu) := \sum_{n=1}^{\infty} \frac{1}{n!}\mathcal{U}_n(\mu, \ldots, \mu) \in \varprojlim \mathfrak{h}/\mathcal{F}^k\mathfrak{h} = \mathfrak{h} \tag{7}$$

is a Maurer-Cartan element of $\mathfrak{h}$. Given a Maurer-Cartan element μ of a Lie algebra $\mathfrak{g}$ one often consider the corresponding twisted Lie algebra $\mathfrak{g}^\mu$.

Definition A.4. Let $\mathfrak{g}$ be a differential graded Lie algebra and $\mu \in \mathrm{MC}(\mathfrak{g})$. We denote by $\mathfrak{g}^\mu$ the *twist* of $\mathfrak{g}$ by μ, which is a differential graded Lie algebra that is equal to $\mathfrak{g}$ as a graded Lie algebra, with differential given by

$$d_{\mathfrak{g}^\mu} = d_{\mathfrak{g}} + [\mu, -].$$

Twisting is a homotopically stable property. The following result follows from a simple spectral sequence argument.

Proposition A.5 ([9], Proposition 1)**.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and $U\colon \mathfrak{g} \to \mathfrak{h}$ be an L_∞ morphism. If for all k, $\mathcal{U}_1\colon \mathcal{F}^k\mathfrak{g} \to \mathcal{F}^k\mathfrak{h}$ is a quasi-isomorphism, then for any $\mu \in \mathrm{MC}(\mathfrak{g})$, the induced map $\mathcal{U}^\mu\colon \mathfrak{g}^\mu \rightsquigarrow \mathfrak{h}^{\mathcal{U}(\mu)}$ is an L_∞ quasi-isomorphism.*

Given a Lie algebra $\mathfrak{g}$ and a commutative algebra A, the space $\mathfrak{g} \otimes A$ inherits a natural Lie algebra structure by declaring the bracket to be A-bilinear, that is $[X \otimes a, X' \otimes a'] = [X, X'] \otimes aa'$. In the case of the polynomial forms $A = \Omega_{\mathrm{poly}}([0,1]) = \mathbb{K}[t, dt]$, we get a natural Lie algebra structure on $\mathfrak{g}[t, dt]$.

Definition A.6. Let $\mathfrak{g}$ be a Lie algebra and $\mu_0, \mu_1 \in \mathrm{MC}(\mathfrak{g})$ two Maurer-Cartan elements. They are said to be *gauge equivalent* if there is a Maurer-Cartan element $\mu_t \in \mathfrak{g}[t, dt]$ interpolating μ_0 and μ_1.

This definition amounts to say that μ_t can be written for all $t \in [0,1]$ as

$$\mu_t = m_t + h_t dt$$

where m_t can be understood as a family of Maurer-Cartan elements in $\mathfrak{g}$, connected by a family of infinitesimal homotopies (gauge transformations) $h_t \in \mathfrak{g}^0$. The Maurer-Cartan equation for μ_t translates into the two equations

$$dm_t + \frac{1}{2}[m_t, m_t] = 0, \qquad \dot{m}_t + dh_t + [h_t, m_t] = 0.$$

Remarkably, the Goldman-Millson theorem states that under appropriate conditions one can identify the Maurer-Cartan spaces of quasi-isomorphic Lie algebras.

Theorem A.7 (Goldman-Millson [10])**.** *Let $U\colon \mathfrak{g} \to \mathfrak{h}$ be an L_∞ morphism of filtered Lie algebras. Suppose furthermore that on the associated graded level the map $\mathrm{gr}\, U\colon \mathrm{gr}\, \mathfrak{g} = \bigoplus \mathcal{F}^\bullet\mathfrak{g}/\mathcal{F}^{\bullet+1}\mathfrak{g} \to \mathrm{gr}\, \mathfrak{h}$ is a quasi-isomorphism. Then, formula* (7) *induces a bijection of sets*

$$U\colon \mathrm{MC}(\mathfrak{g})/\textit{gauge equiv.} \to \mathrm{MC}(\mathfrak{h})/\textit{gauge equiv.}$$

References

[1] P. Antunes: *Poisson quasi-Nijenhuis structures with background*, Letters Math. Physics 86(1) (2008) 33–45.

[2] P. Antunes: *Crochets de Poisson gradués et applications: structures compatibles et généralisations des structures hyperkählériennes*, PhD thesis, Ecole Polytechnique X (2010).

[3] R. Bandiera, Z. Chen, M. Stiénon, P. Xu: *Shifted derived Poisson manifolds associated with Lie pairs*, arXiv: 1712.00665 (2017).

[4] D. Bashkirov, A. Voronov: *On homotopy Lie bialgebroids*, arXiv: 1612.02026 (2016).

[5] M. Batchelor: *Two approaches to supermanifolds*, Trans. Amer. Math. Soc. 258(1) (1980) 257–270.

[6] G. Bonavolontà, N. Poncin: *On the category of Lie n-algebroids*, J. Geometry Physics 73 (2013) 70–90.

[7] F. Bonechi, N. Ciccoli, C. Laurent-Gengoux, P. Xu: *Shifted Poisson structures on differentiable stacks*, arXiv: 1803.06685 (2018).

[8] D. Calaque, T. Pantev, B. Toën, M. Vaquié, G. Vezzosi: *Shifted Poisson structures and deformation quantization*, J. Topology 10(2) (2017) 483–584.

[9] V. Dolgushev: *A proof of Tsygan's formality conjecture for an arbitrary smooth manifold*, Ph. D. Thesis, Massachusetts Inst. of Technology (2005).

[10] V. Dolgushev, C. Rogers: *A version of the Goldman-Millson theorem for filtered L_∞-algebras*, J. Algebra 430 (2015) 260–302.

[11] V. Dotsenko, S. Shadrin, B. Vallette: *Pre-Lie deformation theory*, Moscow Math. J. 16 (2016) 505–543.

[12] M. Fairon: *Introduction to graded geometry*, Eur. J. Math. 3(2) (2017) 208–222.

[13] E. Getzler: *Higher derived brackets*, arXiv: 1010.5859 (2010).

[14] J. Huebschmann, T. Kadeishvili: *Small models for chain algebras*, Math. Zeitschrift 207 (1991) 245–280.

[15] Y. Kosmann-Schwarzbach: *From Poisson algebras to Gerstenhaber algebras*, Ann. de l'Institut Fourier 46 (1996) 1243–1274.

[16] Y. Kosmann-Schwarzbach: *Derived brackets*, Letters Math. Physics 69(1) (2004) 61–87.

[17] Y. Kosmann-Schwarzbach: *Quasi, twisted, and all that... in Poisson geometry and Lie algebroid theory*, in: *The Breadth of Symplectic and Poisson Geometry*, Progress in Mathematics 232, Birkhäuser, Basel (2005) 363–389.

[18] B. Kostant: *Graded manifolds, graded Lie theory, and prequantization*, in: *Differential Geometrical Methods in Mathematical Physics*, Lecture Notes in Mathematics 570, Springer (1977) 177–306.

[19] A. Kotov, T. Strobl: *Generalizing geometry-algebroids and sigma models*, in: *Handbook of Pseudo-Riemannian Geometry and Supersymmetry*, IRMA Lect. Math. Theor. Phys. 16, European Mathematical Society, Zürich (2010) 209–262.

[20] A. Kotov, T. Strobl: *Characteristic classes associated to q-bundles*, Int. J. Geometric Methods Modern Physics 12(01) (2015) 1550006.

[21] O. Kravchenko: *Strongly homotopy Lie bialgebras and Lie quasi-bialgebras*, Letters Math. Physics 81(1) (2007) 19–40.

[22] T. Lada, J. Stasheff: *Introduction to sh Lie algebras for physicists*, Int. J. Theo. Physics 32(7) (1993) 1087–1103.

[23] C. Laurent-Gengoux, S. Lavau, T. Strobl: The universal Lie infinity-algebroid of a singular foliation, arXiv: 1806.00475 (2018).

[24] S. Lavau: *Lie infini-algébroides et feuilletages singuliers*, PhD thesis, Lyon (2016).

[25] J.-L. Loday, B. Vallette: *Algebraic Operads*, Grundlehren der Mathematischen Wissenschaften 346, Springer, Berlin (2012).

[26] T. Pantev, G. Vezzosi: *Symplectic and Poisson derived geometry and deformation quantization*, in: *Algebraic Geometry: Salt Lake City 2015, Part 2*, Proc. Symposia Pure Mathematics 97.2, American Mathematical Society, Providence (2018) 405–457.

[27] J. P. Pridham: *Shifted Poisson and symplectic structures on derived N-stacks.* Journal of Topology, **10**, (2017), 178–210.

[28] B. Pym, P. Safronov: *Shifted symplectic Lie algebroids*, Int. Math. Research Notices, rny215 (2018).

[29] D. Roytenberg: *Courant algebroids, derived brackets and even symplectic supermanifolds*, arXiv: 9910078 (1999).

[30] D. Roytenberg: *On the structure of graded symplectic supermanifolds and Courant algebroids*, Contemp. Mathematics 315 (2002) 169–186.

[31] D. Roytenberg: *Quasi-Lie bialgebroids and twisted Poisson manifolds*, Letters Math. Physics 61(2) (2002) 123–137.

[32] P. Safronov: *Lectures on shifted Poisson geometry*, arXiv: 1709.07698 (2017).

[33] H. Sati, U. Schreiber, J. Stasheff: *Twisted differential string and fivebrane structures*, Comm. Math. Physics 315(1) (2012) 169–213.

[34] P. Severa: *Some title containing the words "homotopy" and "symplectic", eg this one*, Travaux Mathématiques, Fasc. XVI, Univ. Luxembourg (2005) 121–137.

[35] A. Vaintrob: *Lie algebroids and homological vector fields*, Russian Math. Surveys 52(2) (1997) 428–429.

[36] B. Vallette: *Algebra + homotopy = operad*, in: *Symplectic, Poisson, and Noncommutative Geometry*, Math. Sci. Res. Inst. Publ. 62, Cambridge University Press, New York (2014) 229–290.

[37] T. Voronov: *Higher derived brackets and homotopy algebras*, J. Pure Appl.Algebra 202(1-3) (2005) 133–153.

[38] T. Voronov: *Q-manifolds and higher analogs of Lie algebroids*, AIP Conference Proceedings 1307 (2010) 191–202.

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