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Gravity formality

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ABSTRACT

We show that Willwacher's cyclic formality theorem can be extended to preserve natural Gravity operations on cyclic multivector fields and cyclic multidifferential operators. We express this in terms of a homotopy Gravity quasi-isomorphism with explicit local formulas. For this, we develop operadic tools related to mixed complexes and cyclic homology and prove that the operad $\mathsf{M}_{\circlearrowleft}$ of natural operations on cyclic operators is formal and hence quasi-isomorphic to the Gravity operad.

Contents

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0. Introduction

The original deformation quantization problem aims to obtain a formal deformation of the associative product of functions of a Poisson manifold M, called a *star product*. The space governing such deformations is essentially the Lie algebra of multidifferential operators D_{poly}, the smooth version of the Hochschild complex of the algebra $C^\infty(M)$. In his celebrated paper [21], Kontsevich showed that the Lie algebra D_{poly} is formal i.e., it is quasi-isomorphic to its homology, the Lie algebra of multivector fields T_{poly}. His proof involves the construction of the "formality morphism", a homotopy quasi-isomorphism of Lie algebras

$$\mathcal{U} \colon T_{\text{poly}} \to D_{\text{poly}},$$

with explicit local formulas depending on integrals over configurations of points and expressed in terms of graphs. This results solves the deformation quantization problem by establishing a correspondence between formal Poisson structures and star products (bijective up to gauge equivalence).

Kontsevich's result, however, ignores the richer structures existent on T_{poly} and D_{poly}. Let now M be an oriented D-dimensional manifold with a fixed volume form ω. The pull back of the de Rham differential via contraction with ω endows the space T_{poly} with the structure of a BV algebra. On the other hand, there is a natural action of the cyclic group of order $n+1$ on D^n_{poly} given by "integration by parts" which, after the cyclic Deligne's conjecture (see [18]), induces a natural BV_∞ algebra structure on D_{poly}. The natural question to ask is whether Kontsevich's formality morphism can be extended to a BV_∞ quasi-isomorphism. Tamarkin [26,16] constructed a non-explicit Ger_∞ (homotopy Gerstenhaber) quasi-isomorphism $T_{\text{poly}} \to D_{\text{poly}}$ depending on a solution of Deligne's conjecture whose underlying Lie_∞ morphism was later shown by Willwacher [31] to be homotopy equivalent to Kontsevich's map if one uses the Alekseev–Torossian associator to construct a solution to Deligne's conjecture. Furthermore, Willwacher shows that the original formality morphism can be strictly extended to a Ger_∞ morphism. The full extension to the BV setting was given by the first author [2] who constructed a BV_∞ quasi-isomorphism $T_{\text{poly}} \to D_{\text{poly}}$ with explicit local formulas depending on integrals over configurations of framed points. One advantage of incorporating these richer structures into the discussion is that we may now view the algebraic operations as being parametrized by geometric objects, namely by the moduli spaces of genus zero surfaces with parametrized boundary components.

The subspace of cyclic invariants of D_{poly}, denoted by $D^\sigma_{\text{poly}} := \bigoplus_{n\geq 0}(D^n_{\text{poly}})^{\mathbb{Z}_{n+1}}$, is preserved by the Lie bracket and the Hochschild differential. The differential graded Lie algebra D^σ_{poly} is associated to a different deformation problem, namely the construction of *closed star products*. This led to the conjecture of an analogous formality statement, the "cyclic formality conjecture" [25]. Let $\operatorname{div}_\omega \colon T^\bullet_{\text{poly}} \to T^{\bullet-1}_{\text{poly}}$ be the divergence operator on the space of multivector fields. In [32] Willwacher gave an affirmative answer to the cyclic formality conjecture by constructing a homotopy Lie quasi-isomorphism

$$\mathcal{U}^{cyc} \colon (T_{\mathrm{poly}}[u], u\,\mathrm{div}_\omega) \to (D^\sigma_{\mathrm{poly}}, d_{Hoch}).$$

As in the non-cyclic case, both of these objects have structures richer than just Lie algebras. Namely, viewing these objects as models for cyclic invariants associated to the non-cyclic case above, it will be possible to show that they each have operations parametrized by models of the moduli spaces $\mathcal{M}_*$ of genus zero surfaces with *unparametrized* boundary components.

To make this precise we use the presentation of the Gravity operad, introduced by Getzler in [11]. The graded vector spaces $\Sigma H_*(\mathcal{M}_{n+1})$ form an operad Grav which injects into Ger, which is generated operadically by the classes of points in $H_0(\mathcal{M}_{n+1})$ (ranging over $n \geq 2$), and whose sub-operad of top degree homology $\Sigma H_{n-2}(\mathcal{M}_{n+1})$ is isomorphic to the suspension of the Lie operad $\mathfrak{s}\mathsf{Lie}$. In particular every gravity algebra is a (shifted) Lie algebra.

Both spaces $(T_{\mathrm{poly}}[u], u\,\mathrm{div}_\omega)$ and $H(D^\sigma_{\mathrm{poly}})$ are naturally gravity algebras with first bracket equal to the usual Lie bracket. The natural question to ask is then whether Willwacher's homotopy Lie quasi-isomorphism can be extended to the Gravity setting, as conjectured in [29]. However, before attempting to answer this question one must find a Grav_∞ structure on D_{poly} inducing the Gravity structure in homology, which is in some sense a dual version of the cyclic Deligne's conjecture.

In [29] the second author constructed the operad $\mathsf{M}_\circlearrowright$, a cyclic variation of the braces/minimal operad M that acts naturally on spaces of cyclic invariants such as D^σ_{poly}, and whose homology is Grav. Our first result shows that the dg operad $\mathsf{M}_\circlearrowright$ is formal.

Theorem A. *The operad* $\mathsf{M}_\circlearrowright$ *is quasi-isomorphic to* Grav.

The proof of this theorem combines three ingredients: formality of the framed little disks after [14] and [24], the homology calculations of [29], and the theory of cyclic homology of operads valued in mixed complexes. This final ingredient is developed in section 1 and should be of independent interest.

From Theorem A we obtain a Grav_∞ structure on D_{poly} after picking a homotopy lift $\mathsf{Grav}_\infty \xrightarrow{\sim} \mathsf{M}_\circlearrowright$. Having this Grav_∞ structure on D_{poly} we can formulate the main result of this paper.

Theorem B. *Let M be an oriented smooth manifold with a fixed volume form ω. There is a* Grav_∞ *quasi-isomorphism $(T_{\mathrm{poly}}[u], u\,\mathrm{div}_\omega) \to (D^\sigma_{\mathrm{poly}}, d_{Hoch})$ extending Willwacher's* Lie_∞ *quasi-isomorphism.*

In particular, the first component is the (cyclic [25]) HKR map. In the $M = \mathbb{R}^D$ case this formula admits an explicit expression in terms of integrals over configuration spaces in the upper half plane, parametrized by graphs, similar to the original paper from Kontsevich.

We emphasize the paradigm when considering formality-like theorems, that the natural structure on $D^{\sigma}_{\mathrm{poly}}$ is not that of a Grav_{∞} algebra but rather that of a $\mathsf{M}_{\circlearrowleft}$ algebra, the same way that the natural structure on D_{poly} is not the one of a $\mathsf{Ger}_{\infty}/\mathsf{BV}_{\infty}$ algebra but rather the Braces/Cyclic Braces structure. For this reason, operadic tools and concretely the language of operadic bimodules are a neat way to work simultaneously with the $\mathsf{M}_{\circlearrowleft}$ algebra structure on $D^{\sigma}_{\mathrm{poly}}$ and the Gravity algebra structure on T_{poly}.

0.1. Organization

This paper is organized as follows. We begin in section 1 by studying the interaction of operads, mixed complexes, and cyclic homology. We then apply this theory in section 2 to prove Theorem A and to define the Grav_{∞} structures that will be the subject of Theorem B. In section 3 we apply our constructions from section 1 to categories of colored operads and operadic bimodules. The resulting structures are then used in section 4 to prove Theorem B in the case $M = \mathbb{R}^D$ using the theory of operadic torsors. Finally in section 5 we globalize the results using a suitable modification of the usual formal geometry techniques developed in [21].

0.2. Notation and conventions

We work in the category of differential graded (dg) vector spaces over a field k of characteristic 0. We use the notation Σ to denote the suspension of vector spaces and $\mathfrak{s}$ to denote operadic suspension, such that for a vector space V, $(\Sigma V)_d = V_{d-1}$ and ΣV is an $\mathcal{O}$ algebra if and only if V is an $\mathfrak{s}\mathcal{O}$ algebra, for any operad $\mathcal{O}$.

We assume familiarity with operads, operadic twisting, and graph complexes. A table of the graph complex operads appearing in this paper and relevant references follows:

Notation	Graphs	Differential	cf.
B	planar rooted trees	none	e.g. [10]
M	stable planar rooted trees w/ internal and external vertices	via Tw of B	[22]
$\mathsf{B}_{\circlearrowleft}$	planar connected and genus 0	none	[29]
$\mathsf{M}_{\circlearrowleft}$	planar, connected, stable, genus 0, with internal and external vertices	via Tw of $\mathsf{B}_{\circlearrowleft}$	[29]
Gra	graphs without tadpoles	none	[31]
Graphs	internal and external vertices and no tadpoles	via Tw of Gra	[31]
vKGra	boundary and bulk vertices with tadpoles and powers of v	$\partial(v) =$ 	Section 3.2

0.3. Acknowledgments

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1. Operads in mixed complexes and S^1-operads

In this section we consider the interaction of mixed complexes, operads, and cyclic homology.

Definition 1.1. A mixed complex is a triple (V, d, Δ) such that (V, d) is a cochain complex and $\Delta\colon V \to V$ is a degree -1 operator such that $\Delta^2 = 0$ and $d\Delta + \Delta d = 0$.

The category of mixed complexes is naturally a symmetric monoidal category with monoidal unit $(k, 0, 0)$. The monoidal product is

$$(A, d_A, \Delta_A) \otimes (B, d_B, \Delta_B) = (A \otimes_k B, d_A \otimes 1_B + 1_A \otimes d_B, \Delta_A \otimes 1_B + 1_A \otimes \Delta_B)$$

where we follow the Koszul rule for evaluation over a tensor product. Explicitly $d(a \otimes b) = d(a) \otimes b + (-1)^{|a|} a \otimes d(b)$.

Since mixed complexes form a symmetric monoidal category, one can talk about operads valued in mixed complexes. The category of such will be denoted $\mathcal{O}ps^{\mathsf{MxCpx}}$. An object in $\mathcal{O}ps^{\mathsf{MxCpx}}$ is given by a triple $(\mathcal{O}, d, \Delta)$; where $\mathcal{O}$ is a graded operad and where d and Δ are maps of $\mathbb{S}$-modules which anti-commute and which are compatible with the operad structure.

If (A, d, Δ) is a mixed complex, the operad End_A can be viewed as an operad in $\mathcal{O}ps^{\mathsf{MxCpx}}$, but it has more structure. Thus we introduce the following definition:

Definition 1.2. An S^1-operad is an operad under the operad $H_*(S^1)$. The category of such is denoted S^1-$\mathcal{O}ps$.

We denote the fundamental class of S^1 by δ and by abuse of notation we often use δ to denote its image in an S^1-operad.

Construction 1.3. Let $H_*(S^1) \to \mathcal{Q}$ be a morphism of dg operads. Define $\Delta := \{\delta, -\}$ where $\{-, -\}$ is the external Lie bracket associated to $\mathcal{O}$ (see e.g. [29] Lemma 1.9). Explicitly for $a \in \mathcal{O}(n)$ of degree d we define:

$$\Delta(a) := \delta \circ_1 a - (-1)^d \sum_{i=1}^{n} a \circ_i \delta$$

Then $(\mathcal{Q}, d_{\mathcal{Q}}, \Delta)$ is an operad in mixed complexes. This gives a functor from S^1-$\mathcal{O}ps \to \mathcal{O}ps^{\mathsf{MxCpx}}$, which we call X for eXternal.

Example 1.4. Viewing the operad Ger as a suboperad of the S^1-operad BV, we define Δ as above, and then show it restricts to these subspaces. Hence, $(\mathsf{Ger}, 0, \{\delta, -\})$ is an operad in mixed complexes. Since the operator $\{\delta, -\}$ captures the rotation of a configuration of little disks, we will also write $(\mathsf{Ger}, 0, R)$ for this object in $\mathcal{O}ps^{\mathsf{MxCpx}}$.

Example 1.5. More generally, we define $\mathsf{Gra}(n)$ to be the S_n-module spanned by graphs with n numbered vertices having no tadpoles. (Recall a tadpole is an edge which is incident to the same vertex at both ends.) Insertion of graphs makes Gra an operad; in particular it is a suboperad of the S^1-operad of all graphs in which δ is the tadpole graph (one edge and one vertex). One may then form an operad in mixed complexes $(\mathsf{Gra}, 0, \{\delta, -\})$. There is an inclusion of $(\mathsf{Ger}, 0, \{\delta, -\}) \hookrightarrow (\mathsf{Gra}, 0, \{\delta, -\})$ in $\mathcal{O}ps^{\mathsf{MxCpx}}$ given by sending the commutative product to the graph with two vertices and no edges and sending the bracket to the graph with two vertices connected by an edge. We will revisit this example in greater detail in Section 3.2.

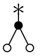
Example 1.6. Let $\mathcal{X}$ be an operad in the category of S^1-spaces and let $S\colon S^1\text{-spaces} \to \text{cdga}$ be a strict symmetric monoidal functor.[1] Then, $(S_*(\mathcal{X}), d, \Delta)$ form naturally an operad in mixed complexes. We will often consider the case $\mathcal{X} = \mathcal{D}_2$, the little disks operad.

Example 1.7. If (A, d, Δ) is a mixed complex then by default we consider $\mathrm{End}_A \in S^1$-$\mathcal{O}ps$ by $\delta \mapsto \Delta$. We may also view End as internal to $\mathcal{O}ps^{\mathsf{MxCpx}}$ by defining $\mathrm{End}_A^{\mathsf{mxd}} := X(\mathrm{End}_A)$. The terminology "an algebra over" either an object in S^1-$\mathcal{O}ps$ or $\mathcal{O}ps^{\mathsf{MxCpx}}$ is understood as a morphism to the respective End.

We would now like to observe the following **non-example:** the minimal operad (M, d, R) of [22] is not an object in $\mathcal{O}ps^{\mathsf{MxCpx}}$. Let us first recall that the minimal operad (M, d) is a dg operad whose arity n space is spanned by planar rooted trees with n distinguished vertices labeled by $\{1, \ldots, n\}$. These labeled vertices are often drawn as white, the remaining vertices are drawn as black, and the root will be denoted by an asterisk. The operad structure is given by graph insertion (summing over possible reconnections) at the white vertices. The differential d can be described combinatorially, or it may equivalently be encoded via operadic twisting and the Maurer–Cartan formalism (see [6]). The

[1] In this paper since our topological spaces are typically piecewise algebraic spaces we can consider the functor of semi-algebraic chains, see Section 2. We may also consider the model of [7] Section 2.2.

operator R is as defined in [29]; it moves the root from black to white in all ways and from white to zero.

The operator R is a square zero operator which commutes with d, but it does not distribute over the compositions maps. To see this let $\mu =$  represent the product (with white vertex labeling suppressed). One easily checks that $R(\mu \circ_1 \mu) \neq R(\mu) \circ_1 R(\mu)$ for the left hand side has three summands while the right hand side has eight summands. In order to accommodate this non-example, we introduce the following weaker notion:

Definition 1.8. Let $\mathcal{P}$ be a dg operad and ρ a degree -1, square zero operator on the underlying dg $\mathbb{S}$-module $\rho_n \colon \mathcal{P}(n) \to \mathcal{P}(n)$ (so $d\rho + \rho d = 0$). A pair $(\mathcal{P}, \rho)$ is called a *rotational operad* if $\rho(a \circ_i \rho(b)) = \rho(a) \circ_i \rho(b)$. We denote the category of rotational operads as $\mathcal{O}ps^{\mathsf{Rot}}$.

Example 1.9. Every operad in mixed complexes can be viewed as a rotational operad, via $\rho = \Delta$, (but not vice versa as per the following example). The induced functor will be denoted $\iota \colon \mathcal{O}ps^{\mathsf{MxCpx}} \to \mathcal{O}ps^{\mathsf{Rot}}$.

Example 1.10. (M, d, R) is a rotational operad. This follows from Lemma 2.6 of [29].

Example 1.11. Every S^1-operad may be viewed as a rotational operad by defining $\rho := \delta \circ_1 -$.

To introduce the next example we must first recall the topological operad of spineless cacti, denoted $\mathcal{C}act$. First note that the arity n component of the minimal operad $(\mathsf{M}(n), d)$ may be realized as the cellular chains of a CW complex by assigning weights to each white vertex arc which add to 1 [19]. These spaces do not form a topological operad because the analogous composition operation is only associative up-to homotopy due to the necessity to rescale weights. On the other hand if we drop the normalization requirement, i.e. the requirement that the weights add to 1 at each vertex, we get an honest topological operad (although no longer CW complexes). This topological operad is (up to contraction of associahedra) equal to the topological operad of spineless cacti $\mathcal{C}act$. This topological operad was introduced by Voronov [28] and studied in detail by Kaufmann [17]. In particular:

Example 1.12. The topological operad $\mathcal{C}act$ has a level-wise S^1 action given by moving the base point, but is not an operad in the category of S^1 spaces. Consequently the induced structure on chains $(S_*(\mathcal{C}act), d, R)$ is not an operad in mixed complexes. However it is a rotational operad.

Proposition 1.13. *There is a weak equivalence of rotational operads* $(\mathsf{M}, d, R) \sim (S_*(\mathcal{D}_2), d, \Delta)$.

Proof. By weak equivalence of rotational operads we mean a zig-zag of quasi-isomorphisms of dg operads which preserves the ρ operator at each stage.

From [30, Lemma 7.8] we know there exists a zig-zag of weak equivalences of topological operads connecting $\mathcal{C}act \xleftarrow{\sim} W(\mathcal{D}_2) \xrightarrow{\sim} \mathcal{D}_2$ which preserve the S^1 actions level-wise. Taking chains we have an equivalence of rotational operads $(S_*(\mathcal{C}act), d, R) \sim (S_*(\mathcal{D}_2), d, \Delta)$.

We now consider the inclusion of the normalization $\mathcal{C}act^1 \overset{\sim}{\hookrightarrow} \mathcal{C}act$ after [17], where the spaces $\mathcal{C}act^1$ are CW complexes forming an operad up to homotopy as described above. Taking chains we find the following sequence of homotopy operads:

$$CC_*(\mathcal{C}act^1) \xrightarrow{\sim} S_*(\mathcal{C}act^1) \overset{\sim}{\rightsquigarrow} S_*(\mathcal{C}act)$$

where $\rightsquigarrow$ denotes an ∞-quasi-isomorphism whose first component is induced by the inclusion of spaces. We emphasize that this sequence respects the underlying mixed complex structure at each arity.

From [17] we know that the cellular chains $CC_*(\mathcal{C}act^1)$ form an honest dg operad. Hence the composite $CC_*(\mathcal{C}act^1) \overset{\sim}{\rightsquigarrow} S_*(\mathcal{C}act)$, may be realized as a map of honest dg co-operads $\mathsf{B}(CC_*(\mathcal{C}act^1)) \xrightarrow{\sim} \mathsf{B}(S_*(\mathcal{C}act))$ and this morphism allows us to construct a zig-zag of dg operads:

$$CC_*(\mathcal{C}act^1) \xleftarrow{\sim} \Omega(\mathsf{B}(CC_*(\mathcal{C}act^1))) \xrightarrow{\sim} \Omega(\mathsf{B}(S_*(\mathcal{C}act^1))) \xrightarrow{\sim} S_*(\mathcal{C}act) \tag{1.1}$$

The ends of this sequence are rotational operads with operator R induced by the S^1-action on the underlying spaces. If $\mathcal{P}$ is a rotational operad then each $\Omega(\mathsf{B}(\mathcal{P}))(n)$ inherits the structure of an operad in mixed complexes. Since the original ∞-quasi-isomorphism was compatible with the underlying mixed-complex structure, it follows that the diagram in line (1.1) constitutes a weak equivalence of rotational operads between $(CC_*(\mathcal{C}act^1), d, R)$ and $(S_*(\mathcal{C}act), d, R)$.

To finish the proof we recall (see [29] Lemma 4.6) that contracting associahedra in the minimal operad commutes with the operator R and gives us a weak equivalences of rotational operads $(\mathsf{M}, d, R) \sim (CC_*(\mathcal{C}act^1), d, R)$. □

Construction 1.14. Define a functor $\theta\colon \mathcal{O}ps^{\mathsf{Rot}} \to \mathcal{O}ps^{\mathsf{MxCpx}}$ by taking a dg rotational operad $\mathcal{O}$ to $(\theta_\rho(\mathcal{O}), d, \rho) \in \mathcal{O}ps^{\mathsf{MxCpx}}$ where $\theta_\rho(\mathcal{O})(n) := \Sigma^{-1}\mathcal{O}(n)$, with "twist gluings" $a\tilde{\circ}_i b := a \circ_i \rho(b)$. (It is easy to check that the twist gluings satisfy associativity and are compatible with d and ρ.) For every such $\mathcal{O}$ there is a morphism of rotational operads $\iota(\theta_\rho(\mathcal{O})) \to \mathcal{O}$ given by $a \mapsto \rho(a)$. We denote the induced natural transformation $\iota \circ \theta \Rightarrow id_{\mathcal{O}ps^{\mathsf{Rot}}}$ by θ^{-1}.

Remark 1.15. The operad $\theta(\mathcal{O})$ does not come with a unit for the composition in $\theta(\mathcal{O})(1) = \Sigma\mathcal{O}(1)$. Thus here we are considering non-unital or "pseudo-operads" in the parlance of some authors.

Lemma 1.16. *Given $(\mathcal{P}, d, \rho) \in \mathcal{O}ps^{\mathsf{Rot}}$, the natural transformation θ^{-1} above factors as:*

$$\theta(\mathcal{P}) \xrightarrow{\rho} \mathrm{im}(\rho) \hookrightarrow \ker(\rho) \to \mathcal{P}$$

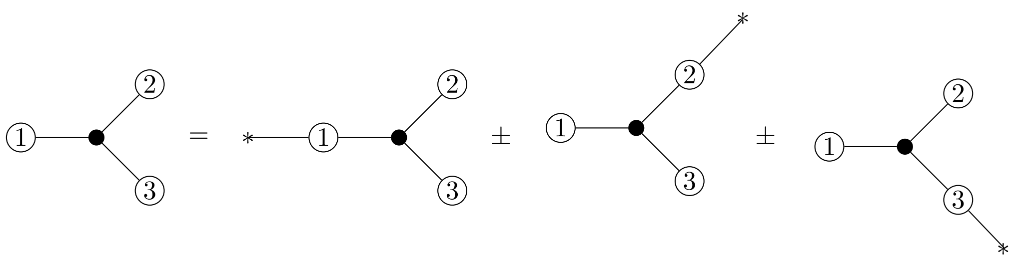
We conclude this subsection by recalling, in the present terminology, the definition of the gravity operad due to [11], and a chain model for this operad due to [29].

Definition 1.17. Consider $(\mathsf{Ger}, 0, R)$ as a rotational operad. The gravity operad Grav is defined to be the suboperad $\mathrm{im}(R)$.

Let us recall that Getzler shows $\mathsf{Grav}(n)_d \cong H_{d-1}(\mathcal{M}_{0,n+1})$, the homology of the moduli space of punctured Riemann spheres, he shows this operad is generated in homological degree 0 with one generator in each arity, and he gives an explicit description of the relations in this operad. See [11] for details.

Definition 1.18. Consider the rotational operad (M, d, R). We define the dg operad $(\mathsf{M}_{\circlearrowleft}, d)$ to be $(\mathrm{im}(R), d)$. In particular there exists an inclusion of dg operads $\mathsf{M}_{\circlearrowleft} \hookrightarrow \mathsf{M}$.

Recall that the minimal operad M may be described via black and white planar rooted trees. Therefore the dg operad $\mathsf{M}_{\circlearrowleft}$ may be described via unrooted black and white planar trees and the map between them sums over all ways to attach a root to a white vertex. Using this description, the homology of $\mathsf{M}_{\circlearrowleft}$ is generated operadically by graphs with n white vertices connected to a common black vertex over all $n \geq 2$. For example the homology class of the graph pictured here is an operadic generator in arity 3. We refer to [29] for complete details on the dg operad $\mathsf{M}_{\circlearrowleft}$.



1.1. Adjoints and algebras

We have seen that if (A, d, Δ) is a mixed complex then End_A is an S^1-operad. As such, algebras over operads in $\mathcal{O}ps^{\mathsf{MxCpx}}$ are controlled by morphisms to $X(\mathrm{End}_A)$; this prompts us to construct the left adjoint to X.

Construction 1.19. Define a functor $W\colon \mathcal{O}ps^{\mathsf{MxCpx}} \to S^1\text{-}\mathcal{O}ps$ by

$$W(\mathcal{Q}, d_{\mathcal{Q}}, R) = (\mathcal{Q} \star k[\delta])/\langle R - \{\delta, -\}\rangle,$$

where $\star$ denotes the categorical coproduct of dg operads.

In words: take the free S^1-operad on the underlying operad and identify the two candidates for rotation; the original R and the external bracket with the newly added δ. This S^1 operad is given the differential induced by $d_{\mathcal{Q}}$ and the relation $d(\delta) = 0$.

Lemma 1.20. *(W, X) are an adjoint pair.*

Proof. Given $\phi \in Hom_{\mathcal{O}ps^{\mathsf{MxCpx}}}(A, X(B))$, we may forget the mixed complex structures and take the adjoint to forgetting the morphism from $H_*(S^1)$ to get a map $Free_{S^1}(A) \to B$, which we call $\tilde{\phi}$. We then calculate

$$\tilde{\phi}(R(a) - \{\Delta_{W(A)}, a\}) = \phi(R(a)) - \tilde{\phi}(\{\Delta_{W(A)}, a\}) = \phi(R(a)) - \{\Delta_B, \phi(a)\}$$

but we now remember that ϕ was a map of mixed complexes so this last expression equals 0. Thus $\tilde{\phi}$ lifts over the quotient of such expressions, that is $\tilde{\phi} \in Hom_{S^1\text{-}\mathcal{O}ps}(W(A), B)$; and conversely. □

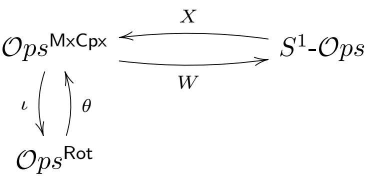
Recall that for $\mathcal{O} \in \mathcal{O}ps^{\mathsf{MxCpx}}$, the structure of an $\mathcal{O}$-algebra on a mixed complex (A, d, Δ) is a morphism $\mathcal{O} \to \mathrm{End}_A^{\mathsf{mxd}} := X(\mathrm{End}_A)$. Thus we immediately see:

Corollary 1.21. *Let $\mathcal{O} \in \mathcal{O}ps^{\mathsf{MxCpx}}$. The $\mathcal{O}$-algebra structures on a mixed complex (A, d, Δ_A) are in bijective correspondence with morphisms $W(\mathcal{O}) \to \mathrm{End}_A$ in S^1-$\mathcal{O}ps$.*

Example 1.22. $W(\mathsf{Ger}) = \mathsf{BV}$. To see this, notice $R(\mu) = \{\Delta, \mu\} = b$ (the bracket) and $R(b) = \{\Delta, b\} = 0$. In otherwords, a $W(\mathsf{Ger})$ algebra is a Gerstenhaber algebra and a mixed complex such that Δ is a derivation of the bracket, and the failure to be a derivation of the product is the bracket. In particular a mixed complex is a (dg) BV algebra iff and only if it is a Gerstenhaber algebra for which the two inherent notions of rotation coincide.

Remark 1.23. The functor $S^1\text{-}\mathcal{O}ps \to \mathcal{O}ps^{\mathsf{Rot}}$ defined in Example 1.11 also has a left adjoint by a similar construction, and hence we may also encode algebras over rotational operads in the category of S^1-operads. However, this will not be needed for our present purposes.

Let us now gather together the relevant constructions of this subsection:



$$\begin{cases} \iota \text{ via inclusion} \\ X \text{ via } \Delta := \{\delta, -\} \\ \theta \text{ via } \Sigma \text{ and twist gluings } a \tilde{\circ}_i b := a \circ_i \rho(b) \\ (W, X) \text{ an adjoint pair.} \end{cases}$$

1.2. Levelwise cyclic homology

Given an operad $(\mathcal{O}, d, \Delta) \in \mathcal{O}ps^{\mathsf{MxCpx}}$ we may take the cyclic homology of each level/arity. These spaces still form a dg operad. We will also need to consider negative and periodic variants. Having fixed cohomological conventions for our mixed complexes, we have $|d| = 1$, $|\Delta| = -1$, $|u| = 2$. We also define $v := u^{-1}$ so that $|v| = -2$, and abuse notation by writing u for the linear map $k[v] \to k[v]$ of degree 2 sending $v^r \mapsto v^{r-1}$ with the convention that $v^0 = 1$ and $v^{-1} = 0$.

Construction 1.24. Define functors $\mathsf{CC}, \mathsf{CC}^-, \mathsf{CC}^{\mathrm{per}} \colon \mathcal{O}ps^{\mathsf{MxCpx}} \to \mathcal{O}ps^{\mathsf{MxCpx}}$ by:

$$\mathsf{CC}(\mathcal{O}, d, \Delta)(n) = (\mathcal{O}(n) \otimes k[v], d + \Delta u, \Delta)$$
$$\mathsf{CC}^-(\mathcal{O}, d, \Delta)(n) = (\mathcal{O}(n) \otimes k[u], d + \Delta u, \Delta)$$
$$\mathsf{CC}^{\mathrm{per}}(\mathcal{O}, d, \Delta)(n) = (\mathcal{O}(n) \otimes k[u, v], d + \Delta u, \Delta)$$

with the operad structure:

$$(a \otimes v^r) \circ_i (b \otimes v^s) := (a \circ_i b) \otimes v^{r+s}$$

It is then straight forward to check associativity and compatibility of the differential and the operad structure.

The functor CC will be called the level-wise cyclic chain functor and its homology is called the level-wise cyclic homology, denoted $\mathsf{HC}(\mathcal{O})$. We similarly refer to the negative CC^- and $\mathsf{CC}^{\mathrm{per}}$ periodic variants. Notice that we call this constructions cyclic *homology* regardless of the degree conventions of our mixed complexes. This is because we are considering the mixed complexes themselves and not functions on them. We also observe that there is a useful modification of this construction which takes the completed tensor product, but since we will be considering $\mathcal{O}$ which are bounded and of finite type, we are not concerned with this distinction.

A weak equivalence in the category $\mathcal{O}ps^{\mathsf{MxCpx}}$ is a zig-zag of morphisms each of which are level-wise quasi-isomorphisms. Note that since $\mathsf{CC}, \mathsf{CC}^-, \mathsf{CC}^{\mathrm{per}}$ preserve level-wise quasi-isomorphisms, they preserve weak equivalences.

Definition 1.25. We define the functor $\mathsf{CC}^\theta\colon \mathcal{O}ps^{\mathsf{Rot}} \to \mathcal{O}ps^{\mathsf{MxCpx}}$ by $\mathsf{CC}^\theta := \mathsf{CC} \circ \theta$. If $\mathcal{O} \in \mathcal{O}ps^{\mathsf{MxCpx}}$ we write $\mathsf{CC}^\theta(\mathcal{O})$ in place of $\mathsf{CC}^\theta(\iota(\mathcal{O}))$ without further ado. We also define $\mathsf{HC}^\theta(-) := H^*(\mathsf{CC}^\theta(-))$.

Spelling out the definition of the functor CC^θ, we see that as an $\mathbb{S}$-module we can identify $\mathsf{CC}^\theta(\mathcal{O}) = \Sigma^{-1}\mathcal{O}[v]$ and under this identification the composition maps are given by "twisted gluings"

$$(p \otimes v^r)\tilde{\circ}_i(q \otimes v^s) = (p \circ_i \rho(q)) \otimes v^{r+s}, \quad \text{for } p, q \in \mathcal{O}.$$

Given $\mathcal{O} \in \mathcal{O}ps^{\mathsf{MxCpx}}$, there is a short exact sequence in dg-$\mathbb{S}$-Mod

$$0 \to \mathsf{CC}^-(\mathcal{O}) \hookrightarrow \mathsf{CC}^{\mathrm{per}}(\mathcal{O}) \xrightarrow{u} \Sigma^{-2}\mathsf{CC}(\mathcal{O}) \to 0 \tag{1.2}$$

the map labeled by u is "multiplication by u" and sends v to 1, 1 to 0, etc.

The connecting homomorphism in the associated long exact sequence can be described via θ (Construction 1.14). First observe that there is an isomorphism of $\mathbb{S}$-modules $\mathsf{HC}^\theta(\mathcal{O}) \cong \Sigma^{-1}\mathsf{HC}(\mathcal{O})$ and this endows the right hand side with an operad structure.

Lemma 1.26. *The boundary map in the long exact sequence associated to equation (1.2) is a morphism of operads* $\mathsf{HC}^\theta(\mathcal{O}) \cong \Sigma^{-1}\mathsf{HC}(\mathcal{O}) \to \mathsf{HC}^-(\mathcal{O})$.

Proof. This follows from the fact that if $c_0 + c_1u^{-1} + \dots$ is a $d + u\Delta$ cycle in $\mathsf{CC}(\mathcal{O})(n)$ then the image of its homology class under the connecting homomorphism is $[\Delta(c_0)]$. □

We also remark that if $d_{\mathcal{O}} = 0$, the connecting homomorphism coincides with the homology of θ^{-1}, else it is a combination of θ^{-1} and projection $u \mapsto 0$.

Lemma 1.27. *Let $(\mathcal{O}, d, \Delta) \in \mathcal{O}ps^{\mathsf{MxCpx}}$ and suppose that $[\Delta]$ is exact on $H(\mathcal{O}, d)$ and that each $(\mathcal{O}(n), d)$ is bounded above. Then the morphism of operads in Lemma 1.26 is an isomorphism.*

Proof. It is enough to show that $\mathsf{HC}^{\mathrm{per}}(\mathcal{O})$ vanishes. Consider a filtration of $\mathcal{O}(r)[u, v]$ by the powers of u. The exactness of $[\Delta]$ will result in the E^2 page of the associated spectral sequence being exactly 0. Since $\mathcal{O}(r)[u, v]$ is bounded in each filtration degree, this spectral sequences converges to $\mathsf{HC}^{\mathrm{per}}(\mathcal{O})(r)$, hence the claim. □

Corollary 1.28. *Let $(\mathcal{O}, d, \Delta) \in \mathcal{O}ps^{\mathsf{MxCpx}}$. There are maps of dg operads:*

$$\mathsf{CC}^\theta(\mathcal{O}) \longrightarrow (\ker(\Delta), d) \longrightarrow \mathsf{CC}^-(\mathcal{O}) \tag{1.3}$$

which are both weak equivalences if the conditions of Lemma 1.27 are satisfied.

Proof. Define the left hand map by $c_0 + c_1u^{-1} + \cdots + c_nu^{-n} \mapsto \Delta(c_0)$ in each arity. Define the right hand map by inclusion at u^0 in each arity. It is straight forward to check that these are dg operad maps.

Now we assume the conditions of Lemma 1.26 which implies that the composition of these two maps is a weak equivalence. We then claim the left hand map is surjective on homology at each level. For if $[a]$ is a class in $H(\ker(\Delta), d)$ then $[\Delta(a)] = 0$ implies $[a] \in Im([\Delta])$ and hence there exists $b \in \mathcal{O}(r)$ with $db = 0$ such that $[\Delta(b)] = [a] \in H(\mathcal{O}(r))$. Since $\Delta(b)$ is in the image of the left hand map, the claim follows.

So if we consider the sequence on line (1.3), the composite being a level-wise isomorphism on homology forces the left hand map to be a level-wise injection on homology. Since it is also a level-wise surjection on homology, the left hand map is a weak equivalence. Hence the right hand map is an weak equivalence by the 2-out-of-3 property. □

By a *truncated operad* we refer to the truncation of an operad to its arity ≥ 2 terms.

Example 1.29. Consider $(\mathsf{Ger}, 0, R)$ as a truncated operad in mixed complexes. Then R is exact on Ger, see [11]. In arity ≥ 2, $\mathsf{CC}^-(\mathsf{Ger})$ has cycles and boundaries:

$$Z(\mathsf{CC}^-(\mathsf{Ger})) = \ker(R) \otimes k[u] \qquad \text{and} \qquad B(\mathsf{CC}^-(\mathsf{Ger})) = Im(R) \otimes uk[u]$$

so $\mathsf{HC}^-(\mathsf{Ger}) \cong Im(R) \cong \ker(R)$. On the other hand $\mathsf{CC}(\mathsf{Ger})$ has cycles and boundaries:

$$Z(\mathsf{CC}(\mathsf{Ger})) = \mathsf{Ger} \oplus (\ker(R) \otimes vk[v]) \qquad \text{and} \qquad B(\mathsf{CC}(\mathsf{Ger})) = Im(R) \otimes k[v]$$

so $\Sigma\mathsf{HC}(\mathsf{Ger}) \cong \Sigma\mathsf{Ger}/Im(R)$. In particular the generators are the n-fold commutative products. The corollary gives us weak equivalences of dg operads:

$$\mathsf{CC}^\theta(\mathsf{Ger}) \xrightarrow{\sim} \mathsf{Grav} \xrightarrow{\sim} \mathsf{CC}^-(\mathsf{Ger})$$

where Grav is (by Definition 1.17 above) the graded operad $(\mathrm{im}(R), 0) = (\ker(R), 0)$. This weak equivalence $\mathsf{CC}^\theta(\mathsf{Ger}) \xrightarrow{\sim} \mathsf{CC}^-(\mathsf{Ger})$ can be interpreted as a dg version of [30, Corollary 2.8].

Recall that two objects in $\mathcal{O}ps^{\mathsf{M{\times}Cpx}}$ (resp. $\mathcal{O}ps^{\mathsf{Rot}}$) are said to be weakly equivalent (denoted $\sim$) if they are connected by a zig-zag of levelwise quasi-isomorphisms of dg operads which preserve the rotation operator.

From the level-wise homotopy invariance of CC^θ and CC^- we immediately see:

Corollary 1.30. *If $(\mathcal{O}, d, \Delta)$ in $\mathcal{O}ps^{\mathsf{M{\times}Cpx}}$ is weakly equivalent to $(\mathsf{Ger}, 0, R)$, then $\mathsf{CC}^\theta(\mathcal{O}) \sim \mathsf{Grav} \sim \mathsf{CC}^-(\mathcal{O})$ are weakly equivalent dg operads.*

If $(\mathcal{O}, d, \Delta)$ in $\mathcal{O}ps^{\mathsf{Rot}}$ is weakly equivalent to $(\mathsf{Ger}, 0, R)$ (viewed as a rotational operad) then $\mathsf{CC}^\theta(\mathcal{O}) \sim \mathsf{Grav}$ are weakly equivalent dg operads.

1.3. Operations on cyclic homology

In this section we fix a mixed complex (A, d_A, δ_A) and consider its cyclic homology as well as negative and periodic variants. This is the same construction as Construction 1.24 above, except the input and output is just a mixed complex (as opposed to an operad in mixed complexes). Let us use the same notation to denote these constructions for both algebras and operads; so explicitly we consider chain complexes $\mathsf{CC}(A) := (A \otimes k[v], d + \delta u)$, $\mathsf{CC}^{-}(A) := (A \otimes k[u], d + \delta u)$, and $\mathsf{CC}^{\mathrm{per}}(A) := (A \otimes k[u, v], d + \delta u)$.

Recall that for our mixed complex A we may consider End_A as an S^1-operad or as an operad in mixed complexes End_A^{mxd}, via $\Delta = \{\delta, -\}$. In this section we take the latter consideration as the default. The following lemma will allow us to study operations on cyclic cohomology:

Lemma 1.31. *Let $A = (A, d, \Delta)$ be a mixed complex. There is an inclusion $\mathsf{CC}^{-}(End_A) \hookrightarrow End_{\mathsf{CC}^{-}(A)}$ in $\mathcal{O}ps^{\mathsf{MxCpx}}$.*

Proof. Define a map ψ_n:

$$Hom(A^{\otimes n}, A) \otimes k[u] \xrightarrow{\psi_n} Hom(A[u]^{\otimes n}, A[u])$$

as the k-linear extension of the assignment:

$$f \otimes u^r \mapsto \Big[(a_1 u^{i_1} \otimes \cdots \otimes a_n u^{i_n}) \mapsto f(a_1, \ldots, a_n) u^{r + \sum_j u_{i_j}}\Big]$$

This map is clearly injective. In particular, a multi-linear operation on $A[u]$ is in the image of this map if and only if it is u-linear and has bounded support in the codomain. We remark that the extension of this map to $Hom(A^{\otimes n}, A)\hat{\otimes} k[u]$ would encompass all multi-linear operations in its image, but this intermediary operad is not needed for our purposes. We now claim that the ψ_n constitute a map of dg operads.

Let us first check the differential. The operad End_A has an internal differential, call it ∂, induced by $d_A \in End_A(1)$. Notice that it can be described via the operadic Lie bracket as $\partial(f) = \{d_A, f\}$. Therefore the total differential on $\mathsf{CC}^{-}(End_A)$ which is *a priori* of the form $\partial + \Delta u$, can be rewritten as $\{d_A, -\} + \{\delta, -\}u = \{d_A + u\delta, -\}$. On the other hand, the operad $End_{\mathsf{CC}^{-}(A)}$ has differential induced from the complex $(\mathsf{CC}^{-}(A), d_A + u\delta)$ via the operadic Lie bracket. Thus we again find $\{d_A + u\delta, -\}$, and so the differentials agree.

It is then an easy exercise to see that ψ respects the operad compositions. In particular let f and g be multi-linear operations on A of arities n and m. Then we see that both $\psi(f \otimes u^r) \circ_l \psi(g \otimes u^s)$ and $\psi(f \otimes u^r \circ_l g \otimes u^s) := \psi(f \circ_l g \otimes u^{r+s})$ are evaluated at a pure tensor $\otimes_{j=1}^{n+m-1} a_j u^{i_j}$ by evaluating $f \circ_l g$ at $\otimes_j a_j$ and multiplying by u to the power $(s + \sum_{j=l}^{l+m-1} i_j) + (r + \sum_{j=1}^{l-1} i_j + \sum_{j=l+m}^{n+m-1} i_j)$ in the former case and $r + s + \sum_j i_j$ in the latter; and these two expressions are equal. □

Remark 1.32. We have given the statement of the Lemma using the negative variant of cyclic cohomology because it will be the result we need subsequently. However the same result can be proven for the other variants.

Corollary 1.33. *If (A, d, Δ) is an algebra over the S^1-operad $W(\mathcal{O})$ then $\mathsf{CC}^-(A)$ inherits the structure of an algebra over $\mathsf{CC}^-(\mathcal{O})$.*

Proof. Associated to the map of S^1-operads $W(\mathcal{O}) \to End_A$ is the adjoint map $\mathcal{O} \to End_A$ in $\mathcal{O}ps^{\mathsf{M{\times}Cpx}}$ (suppressing the notation X used above). Taking CC^- of this map and applying the lemma we have $\mathsf{CC}^-(\mathcal{O}) \to \mathsf{CC}^-(End_A) \hookrightarrow End_{\mathsf{CC}^-(A)}$. □

Example 1.34. If A is a BV-algebra, then $\mathsf{CC}^-(A)$ inherits the structure of a gravity algebra via the sequence

$$\mathsf{Grav} \xrightarrow{\sim} \mathsf{CC}^-(\mathsf{Ger}) \hookrightarrow End_{\mathsf{CC}^-(A)} \tag{1.4}$$

after Example 1.22.

More generally, combining this example with Example 1.29 above we see that if A is a BV-algebra, there is a sequence of (truncated) dg operads:

$$\mathsf{Grav}_\infty \xrightarrow{\sim} \mathsf{CC}^\theta(\mathsf{Ger}) \xrightarrow{\sim} \mathsf{CC}^-(\mathsf{Ger}) \hookrightarrow End_{\mathsf{CC}^-(A)} \tag{1.5}$$

We will use this construction in the following section to associate a gravity algebra to the poly-vector fields of an oriented manifold.

Remark 1.35. In this section we have studied the interaction of operads, mixed complexes, and cyclic homology. Although it is not needed for the purposes of this paper, it should be possible to generalize these results and examples to apply to operads valued in multi-complexes. In particular, the suitably derived version of Example 1.22 is an interesting question.

2. Formality, cyclic formality, and gravity structures

In this section we recall the statement of Kontsevich's formality theorem [20,21] and the cyclic variant of the theorem due to Willwacher [32]. We also apply our work from Section 1 to establish the Grav_∞ structures on the respective sides of the cyclic formality theorem that will be the subject of our results in subsequent sections.

In this section we fix an oriented manifold M of dimension d and equip it with a fixed volume form ω. In this section and beyond we take our ground field to be the real numbers.

2.1. Multivector fields

The graded vector space $T_{\text{poly}}(M)$, or just T_{poly}, of multivector fields on M is

$$T^{\bullet}_{\text{poly}} = \Gamma(M, \bigwedge^{\bullet} T_M),$$

where T_M is the tangent bundle of M. This space is naturally a $\mathfrak{s}^{-1}\mathsf{Lie}$ algebra with Lie bracket given by the Schouten–Nijenhuis bracket. It is also a graded commutative algebra under the exterior product and these operations combine to make T_{poly} a Gerstenhaber algebra.

We now define a map $f\colon T^{\bullet}_{\text{poly}}(M) \to \Omega^{d-\bullet}_{dR}(M)$ that sends a multivector field to its contraction with the volume form of M. This map is easily checked to be an isomorphism of vector spaces. We define the divergence operator div_ω to be the pullback of the de Rham differential via f, i.e. $\text{div}_\omega := f^{-1} \circ d_{dR} \circ f$. The square zero operator div_ω combines with the Gerstenhaber structure to make T_{poly} a BV algebra.

Our work in Section 1, namely Example 1.34, assigns to the complex $(T_{\text{poly}}[u], u\,\text{div}_\omega)$ the structure of a dg gravity algebra. Explicit formulas for this structure can be given as the u-linear extension of those given in [11, Lemma 4.4]. In particular this complex is a dg $\mathfrak{s}^{-1}\mathsf{Lie}$ algebra whose bracket is the u-linear extension, $[Xu^k, Yu^l] := [X,Y]u^{k+l}$.

2.2. Multidifferential operators

In this section we describe the differential graded Lie algebra of multidifferential operators of M, denoted by $D_{\text{poly}}(M)$ or just D_{poly}. We do an operadic construction which is less standard but allows us to introduce notation that suits our needs better.

Consider the endomorphism operad $\text{End}(C^\infty_c(M)) = \text{Hom}(C^\infty_c(M)^{\otimes \bullet}, C^\infty_c(M))$ on the algebra of compactly supported smooth functions on M, concentrated in degree zero. We define $\mathcal{D}_{\text{poly}} \subset \text{End}(C^\infty_c(M))$ to be the suboperad given by endomorphisms that vanish on constant functions[2] and that can be locally expressed in the form

$$\sum f \frac{\partial}{\partial x_{I_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{I_n}}$$

where the I_j are finite sequences of indices between 1 and $dim(M)$ and $\partial/\partial x_{I_j}$ is the multi-index notation representing the composition of partial derivatives.

Associated to $\mathcal{D}_{\text{poly}}$ is the graded vector space $\tilde{D}_{\text{poly}} = \bigoplus_n \Sigma\mathfrak{s}\mathcal{D}_{\text{poly}}(n)$ (graded internally so that arity n operators are of concentrated in degree n) which inherits a natural graded $\mathfrak{s}^{-1}\mathsf{Lie}$ algebra structure from the symmetrization of the total composition maps

[2] Some authors call this space the “normalized cochains” or “normalized multidifferential operators” due to this condition.

$$D \circ D' = \sum_{i=1}^{|D|} (-1)^{(i-1)(|D'|-1)} D \circ_i D'.$$

The product $\mu \in \mathcal{D}_{\mathrm{poly}}(2)$ of compactly supported functions is associative. This can be rewritten as $[\mu, \mu] = 0$ which amounts to saying that μ is a Maurer–Cartan element of the shifted Lie algebra $\tilde{D}_{\mathrm{poly}}$. The differential graded Lie algebra of multidifferential operators D_{poly} is defined to be $\tilde{D}^{\mu}_{\mathrm{poly}} := (\tilde{D}_{\mathrm{poly}}, [\mu, -])$, the twist of $\tilde{D}_{\mathrm{poly}}$ by the Maurer–Cartan element μ.

The Hochschild–Kostant–Rosenberg map is a quasi-isomorphism of cochain complexes $T_{\mathrm{poly}} \to D_{\mathrm{poly}}$ that is not compatible with the Lie algebra structures. Kontsevich's result states that the obstructions of the Hochschild–Kostant–Rosenberg map to commute with the Lie algebra structure are homotopically trivial.

Theorem 2.1 *(Kontsevich formality). There exists a homotopy $\mathfrak{s}^{-1}\mathsf{Lie}_\infty$ quasi-isomorphism*

$$\mathcal{U} \colon T_{\mathrm{poly}} \to D_{\mathrm{poly}}$$

extending the Hochschild–Kostant–Rosenberg map.

We will now describe an action of the group $\mathbb{Z}_{n+1} = \langle \sigma_n | \sigma_n^{n+1} = e \rangle$ on $\mathcal{D}_{\mathrm{poly}}(n)$. Consider the natural map

$$\mathrm{Hom}(C_c^\infty(M)^{\otimes n}, C_c^\infty(M)) \to \mathrm{Hom}(C_c^\infty(M)^{\otimes n+1}, \mathbb{R})$$

induced from the pairing $\int_M \colon C_c^\infty(M) \otimes C_c^\infty(M) \to \mathbb{R}$. The restriction of this map to $\mathcal{D}_{\mathrm{poly}}(n)$ is injective and therefore $\mathcal{D}_{\mathrm{poly}}(n)$ inherits the $\mathbb{Z}_{n+1}$ action of $\mathrm{Hom}(C_c^\infty(M)^{\otimes n+1}, \mathbb{R})$ coming from cyclic permutation of the inputs.

We define the cyclic multi-differential operators $D^{\sigma}_{\mathrm{poly}}$ to be those multi-differential operators which are left invariant under the cyclic action.

Proposition 2.2. *The Lie algebra structure and differential of D_{poly} restrict to this subspace making $D^{\sigma}_{\mathrm{poly}}$ a differential graded Lie algebra.*

Similarly to the non-cyclic setting, there is a cyclic Hochschild–Kostant–Rosenberg map [25] $T_{\mathrm{poly}}[u] \to D^{\sigma}_{\mathrm{poly}}$ which is a quasi-isomorphism of chain complexes. Willwacher's result shows that also in the cyclic setting this map can be made compatible with the Lie bracket up to homotopy.

Theorem 2.3 *(Willwacher's cyclic formality). There exists a homotopy $\mathfrak{s}^{-1}\mathsf{Lie}_\infty$ quasi-isomorphism*

$$\mathcal{U}^{\mathrm{cyc}}\colon T_{\mathrm{poly}}[u] \to D^{\sigma}_{\mathrm{poly}}$$

extending the cyclic Hochschild–Kostant–Rosenberg map.

Remark 2.4. The cyclic group actions combine with the operad structure to endow $\mathcal{D}_{\mathrm{poly}}$ with the structure of a cyclic operad (cf. [13]). Proposition 2.2 actually holds for any cyclic operad (see e.g. [29]) and so we term the associated dg Lie algebra $(\mathcal{O}^{\sigma}, d_{\mu})$ the cyclic deformation complex of $(\mathcal{O}, \mu)$. In particular this complex controls equivalence classes of deformations of our algebra which are inner-product preserving, see [4].

The fact that $\mathcal{D}_{\mathrm{poly}}$ is a cyclic operad equipped with the invariant Maurer–Cartan element μ allows us to apply [29, Theorem C] to show that the dg Lie algebra structure on $D^{\sigma}_{\mathrm{poly}}$ lifts to an action of the dg operad $\mathsf{M}_{\circlearrowleft}$ of Definition 1.18. This will be used below to endow $D^{\sigma}_{\mathrm{poly}}$ with the structure of a Grav_{∞}-algebra.

2.3. FM_2 *and formality of* $\mathsf{M}_{\circlearrowleft}$

Let

$$Conf_n(\mathbb{C}) = \{(x_1, \ldots, x_n) \in (\mathbb{C})^n | x_i \neq x_j \text{ for } i \neq j\}/\mathbb{R}_+ \ltimes \mathbb{C}$$

be the configuration space of n labeled points in $\mathbb{C}$ modulo the action of the Lie group $\mathbb{R}_+ \ltimes \mathbb{C}$ acting by scaling and translations. Notice that $Conf_n(\mathbb{C})$ is a $2n-3$ dimensional smooth manifold.

The Fulton–MacPherson topological operad FM_2, introduced by Getzler and Jones [12] after [9] is constructed in such a way that the n-ary space $\mathsf{FM}_2(n)$ is a compactification of the $Conf_n(\mathbb{C})$. The spaces $\mathsf{FM}_2(n)$ are manifolds with corners with each boundary stratum representing a set of points that got infinitely close.

Formally, the compactification is done by considering the closure of $Conf_n(\mathbb{C})$ under the embedding $Conf_n(\mathbb{C}) \hookrightarrow (S^1)^{n(n-1)} \times [0, +\infty]^{n^2(n-1)^2}$ that maps every pair of points to their angle and every triple of points to their relative distances.

The first few terms are

- $\mathsf{FM}_2(0) = \emptyset$,
- $\mathsf{FM}_2(1) = \{*\}$,
- $\mathsf{FM}_2(2) = S^1$.

The operadic composition $\circ_i$ is given by inserting a configuration at the boundary stratum at the point labeled by i. For details on this construction see also [9, Part IV] or [20].

The homology of the Fulton–MacPherson operad is the Gerstenhaber operad Ger [1]. The formality of this operad was established by Kontsevich with the exhibition of the following explicit zig-zag of quasi-isomorphisms.

Theorem 2.5 *([20,23]). There is a zig-zag of quasi-isomorphisms of operads*

$$\mathsf{Chains}_*(\mathsf{FM}_2) \to \mathsf{Graphs} \leftarrow \mathsf{Ger}.$$

The operad Graphs is an operad of graphs, constructed by considering the twisted operad of Gra (see section 3.2 for the definitions). Concretely, it is the suboperad of $\mathsf{Tw}\,\mathsf{Gra}$ consisting of graphs containing no connected components without external vertices and all internal vertices have valence at least 3. The construction of this operad using operadic twisting was first done in [31].

We mention the technical point that the various projection maps $\mathsf{FM}_2(n+k) \to \mathsf{FM}_2(n)$ obtained by forgetting k points of the configuration are not smooth fiber bundles. Since Kontsevich's construction requires integration of forms along fibers, one has to work in a semi-algebraic setting. In particular, the functor Chains_*, used by Kontsevich is the functor of semi-algebraic chains (see [15] for an extensive study of this functor) and the morphism $\mathsf{Chains}_*(\mathsf{FM}_2) \to \mathsf{Graphs}$ is best constructed in the dual setting, as a map of cooperads

$$\omega_\bullet \colon \mathsf{Graphs}^* \to \Omega(\mathsf{FM}_2),$$

where Ω represents the functor of PA (piecewise-algebraic) forms.

Remark 2.6. The functor Ω is not comonoidal since the canonical map $\Omega(A) \otimes \Omega(B) \to \Omega(A \times B)$ goes "in the wrong direction", therefore $\Omega(\mathsf{FM}_2)$ is not a cooperad but still satisfies cooperad-like relations (see [23]). Nevertheless, by abuse of language throughout this paper we will refer to these spaces as cooperads and refer to maps such as $\mathsf{Gra}^* \to \Omega(\mathsf{FM}_2)$ as maps of (colored) cooperads if they satisfy a compatibility relation such as commutativity of the following diagram:

$$\begin{array}{ccc}
\mathsf{Graphs}^*(n) & \longrightarrow & \Omega(\mathsf{FM}_2(n)) \\
\downarrow & & \downarrow \\
 & & \Omega(\mathsf{FM}_2(n-k+1) \times \mathsf{FM}_2(k)) \\
 & & \uparrow \\
\mathsf{Graphs}^*(n-k+1) \otimes \mathsf{Graphs}^*(k) & \longrightarrow & \Omega(\mathsf{FM}_2(n-k+1)) \otimes \Omega(\mathsf{FM}_2(k)).
\end{array}$$

To describe the map $\omega_\bullet$, first let us take Γ, a graph in $\mathsf{Graphs}^*(n)$ with no internal vertices. We define

$$\omega_\Gamma := \bigwedge_{(i,j) \text{ edge of } \Gamma} d\phi_{i,j} \in \Omega(\mathsf{FM}_2(n)),$$

where $d\phi_{i,j} = p_{i,j}^*(\mathrm{vol}_{S^1})$[3] is the pullback of the volume form of the circle via the projection map $p_{i,j} \colon \Omega(\mathsf{FM}_2(n)) \to \Omega(\mathsf{FM}_2(2)) = \Omega(S^1)$.

If the graph $\Gamma \in \mathsf{Graphs}^*(n)$ contains k internal vertices, one can construct a graph $\Gamma' \in \mathsf{Graphs}^*(n+k)$ by replacing all internal vertices by external vertices labeled in some way from $n+1$ to $n+k$. The map $\omega_\bullet \colon \mathsf{Graphs}^*(n) \to \Omega(\mathsf{FM}_2(n))$ is defined by sending Γ to $\int_{n+1,\dots,n+k} \omega_{\Gamma'}$, where the integral runs over all possible configuration of the points that correspond to the internal vertices.

Notice that the induced composition map $\mathsf{Gra}^* \to \mathsf{Graphs}^* \to \Omega(\mathsf{FM}_2)$ is just the map of commutative algebras defined by sending the edge connecting vertices i and j to $\phi_{i,j}$.

Remark 2.7. The operad FM_2 can be directly related to a shifted version of the homotopy Lie operad via the operad morphism

$$\mathfrak{s}^{-1}\mathsf{Lie}_\infty \to \mathsf{Chains}_*(\mathsf{FM}_2),$$

given by sending the generator $\mu_n \in \mathfrak{s}^{-1}\mathsf{Lie}_\infty$ to the fundamental chain of $\mathsf{FM}_2(n)$.[4] This is essentially lifting the formality zig-zag for $\mathfrak{s}^{-1}\mathsf{Lie}_\infty \subset \mathsf{Ger}_\infty$.

The action of S^1 on FM_2 allows us to consider the operad $(\mathsf{Chains}_*(\mathsf{FM}_2), d, \Delta) \in \mathcal{O}ps^{\mathsf{MxCpx}}$.

Theorem 2.8. *There exists a zig-zag of quasi-isomorphisms in $\mathcal{O}ps^{\mathsf{MxCpx}}$ connecting*

$$(S_*(\mathcal{D}_2; \mathbb{R}), d, \Delta) \sim (S_*(\mathsf{FM}_2; \mathbb{R}), d, \Delta) \sim (\mathsf{Ger}, 0, R).$$

Proof. Recall that the usual proof of the homotopy equivalence of $\mathcal{D}_2$ and FM_2 [8, Chapter 4] makes use of the Boardman–Vogt W-construction to construct the following zig-zag of homotopy equivalences

$$\mathcal{D}_2 \xleftarrow{\sim} W(\mathsf{FM}_2) \xrightarrow{\sim} \mathsf{FM}_2.$$

One readily notices that for a fixed arity both maps preserve the natural S^1 actions on the three topological spaces, from which it follows that $\mathcal{D}_2$ and FM_2 are homotopy equivalent as S^1 operads. From the functoriality of the semi-direct product of a topological group with a topological operad it also follows [14] that the framed versions $\mathcal{D}_2^{fr}$ and FM_2^{fr} are homotopy equivalent topological operads.

At the algebraic level we obtain that $(S_*(\mathcal{D}_2), d, \Delta) \sim (S_*(\mathsf{FM}_2), d, \Delta)$, since both Δ operators are given by the composition with the unary framed element.

[3] Notice that $d\phi_{i,j}$ is not an exact form, since the angle $\phi_{i,j}$ is only well defined up to a constant.

[4] Note that due to our cohomological conventions the generator $\mu_n \in \mathfrak{s}^{-1}\mathsf{Lie}_\infty$ has degree $(1-n)+(2-n) = 3-2n$ as desired.

Recall from [15], that the equivalence between the functor of singular chains and the one of semi-algebraic chains is given by a zig-zag of natural quasi-isomorphisms

$$\mathsf{Chains}_*(-) \xleftarrow{\sim} S_*^{PA}(-) \xrightarrow{\sim} S_*(-),$$

where $S_p^{PA}(X) = \{\sigma\colon \Delta^p \to X \mid \sigma \text{ is a semi-algebraic map}\}$. Both maps are easily seen to be compatible with the mixed complex structure, and therefore $(S_*(\mathcal{D}_2), d, \Delta) \sim (\mathsf{Chains}_*(\mathsf{FM}_2), d, \Delta)$.

Kontsevich's quasi-isomorphism of operads $\mathsf{Chains}_*(\mathsf{FM}_2) \to \mathsf{Graphs}$ is compatible with the mixed complex structure, as shown in [14, Lemma 3.1]. It remains to see that the map $\mathsf{Ger} \to \mathsf{Graphs}$ is also compatible with the mixed complex structure. It suffices to check this statement on generators, where it is clear since Δ sends the graph with no edges in $\mathsf{Graphs}(2)$ to the graph containing only two external vertices and an edge connecting them. □

As a corollary to this theorem we can relate the rotational operads discussed above in Examples 1.4 and 1.10. The proof follows immediately from the Theorem and Proposition 1.13.

Corollary 2.9. *There is an equivalence of rotational operads* $(\mathsf{M}, d, R) \sim (\mathsf{Ger}, 0, R)$.

Combining this corollary with our work in Section 1 yields the following result:

Theorem 2.10. *The operad* $\mathsf{M}_{\circlearrowright}$ *of [29] is weakly equivalent to the gravity operad.*

Proof. Applying CC^θ to the above result, and using Corollary 1.30, we find $\mathsf{CC}^\theta(\mathsf{M}, R) \sim \mathsf{Grav}$ as truncated dg operads, where the commutative products generate the gravity operations. We then define a map of truncated operads $\mathsf{CC}^\theta(\mathsf{M}, R) \xrightarrow{\sim} \mathsf{M}_{\circlearrowright} \subset \mathsf{M}$ by R (with $v \mapsto 0$). This is a morphism of operads with the same homologies, and on homology it takes generators to generators so it's a quasi-isomorphism. □

Remark 2.11. This proof works over $\mathbb{R}$. For other fields of characteristic 0 we can prove this result by appealing to the formality of the gravity operad along with Proposition 1.13. The analog of Theorem 2.8 in the non real case is an open problem.

We recall from [29, Theorem C] that the dg operad $\mathsf{M}_{\circlearrowright}$ acts on the cyclic deformation complex of any cyclic operad. This action extends the Lie algebra structure discussed above (Remark 2.4), is compatible with the action of M on the (non-cyclic) deformation complex, and recovers the expected gravity structure on the homology of this complex. We choose a weak equivalence $\mathsf{Grav}_\infty \xrightarrow{\sim} \mathsf{M}_{\circlearrowright}$, whose existence is guaranteed by the Theorem and then define:

Definition 2.12. If $\mathcal{O}$ is a cyclic operad with associated MC element μ, we define a Grav_∞ structure on cyclic deformation complex $(\mathcal{O}^\sigma, d_\mu)$ via $\mathsf{Grav}_\infty \xrightarrow{\sim} \mathsf{M}_\circlearrowleft \to End_{\mathcal{O}^\sigma}$. In particular, in the case $\mathcal{O} = \mathcal{D}_{\text{poly}}$ this defines a Grav_∞ structure on D^σ_{poly}.

3. Cyclic Swiss Cheese type operads

In this Section we introduce the 2-colored operads that we will work with throughout the paper. They all have a compatible cyclic structure encoded by the following definition.

Definition 3.1. Let $\mathcal{P}$ be a 2-colored operad that is non-symmetric in color 2. We say that $\mathcal{P}$ is of Swiss Cheese type if $\mathcal{P}^1(m,n) = 0$ if $n > 0$.

A Swiss Cheese type operad $\mathcal{P}$ endowed with a right action of the cyclic group $\mathbb{Z}_{n+1}$ on each $\mathcal{P}^2(m,n)$ is said to be of Cyclic Swiss Cheese type (abbreviated CSC) if:

- The cyclic action is $\mathcal{P}^1$ equivariant,
- The cyclic action and the color 2 compositions satisfy the same compatibility as in a cyclic operad.

In particular, this last axiom implies that if $\mathcal{P}$ is of CSC type then the partial compositions and the cyclic action combine to endow $\prod_m \mathcal{P}^2(m,n)$ with the structure of a cyclic operad.

A morphism of CSC type operads is a map of colored operads that is moreover equivariant with respect to the cyclic action.

3.1. Configuration spaces of points

The (original) Swiss Cheese operad is a colored operad introduced by Voronov [27] whose operations in color 1 are given by rectilinear embeddings of discs in a big disc, while operations in color 2 consist of rectilinear embeddings of discs and semi-disc in a big semi-disc. In [27], Voronov considers also a homotopy equivalent operad $(\mathsf{FM}_2, \mathbb{H})$ made out of configuration spaces of points on the plane or upper half-plane. This second construction has some advantages over the first one, one of them being that there is a natural Cyclic Swiss Cheese structure on $(\mathsf{FM}_2, \mathbb{H})$ as we describe in this subsection.

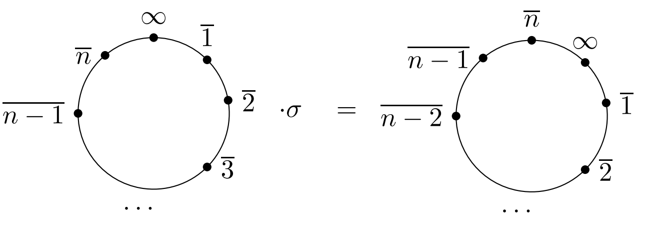
Analogously to section 2.3, one can consider the configuration space of m points on the upper half-plane and n points at the boundary, modulo scaling and horizontal translations

$$Conf_{m,n}(\mathbb{C}) = \{(x_1, \dots, x_m; y_1, \dots, y_n) \in \mathbb{C}^{n+m} | \Im(x_i) > 0, \Im(y_i) = 0, \\ \text{no points overlap}\}/\mathbb{R}_+ \ltimes \mathbb{R}.$$

There is an embedding $Conf_{m,n}(\mathbb{C}) \hookrightarrow Conf_{2m+n}(\mathbb{C})$ by mirroring the bulk points along the real axis. Compactifying as in section 2.3, we obtain the space $\mathbb{H}_{m,n}$.

The spaces FM_2 and $\mathbb{H}_{\bullet,\bullet}$ assemble into a Swiss Cheese type operad, with the two color compositions into $\mathbb{H}_{\bullet,\bullet}$ being still done by insertion into boundary strata. There is in fact a cyclic action extending the Swiss Cheese structure to a Cyclic Swiss Cheese type operad structure as follows:

The open upper half plane is isomorphic to the Poincaré disc via a conformal map. This isomorphism sends the boundary of the plane to the boundary of the disc except one point that we label by ∞. We define the cyclic action of $\mathbb{Z}_{n+1}$ in $\mathbb{H}_{m,n}$ by cyclic permutation of the point labeled by infinity with the other points at the boundary.



3.2. Graphs

For $m, n \geq 0$, let $\mathsf{vKGra}(m, n)$ be the free differential graded commutative algebra generated by "edges" $\Gamma^{i,j}$, $1 \leq i, j \leq m$; "edges" $\Gamma^{i}_{\overline{j}}$, $1 \leq i \leq m$; $1 \leq \overline{j} \leq n$ in degree -1 and symbols v_i, $1 \leq i \leq m$ of degree -2.

The differential sends v_i to $\Gamma^{i,i}$ and vanishes on every other generator. The reason for the notation is that this cdga can be considered a variation of Kontsevich's graphs, used in [21].

We interpret $\mathsf{vKGra}(m, n)$ as the space spanned by directed graphs with m vertices of *type I* labeled with the numbers $\{1, \dots, m\}$ that can be additionally decorated with a power of v, n vertices labeled with the numbers $\{\overline{1}, \dots, \overline{n}\}$ of *type II* and edges that can not start on a vertex of type II.

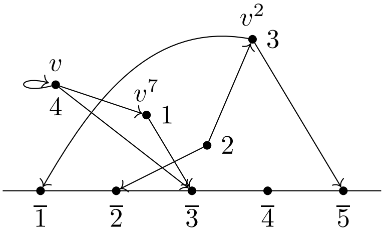
Let us consider a different free cdga, $\mathsf{Gra}(n)$ (cf. [31]), to be generated by symbols $\Gamma^{i,j}$, $1 \leq i \neq j \leq n$, that is to say, $\mathsf{Gra}(n)$ is the subspace of graphs of $\mathsf{vKGra}(n, 0)$ containing no tadpoles or positive powers of v.

We define a symmetric operad structure on Gra by setting the symmetric action to permute the labels on vertices and the operadic composition $\Gamma \circ_i \Gamma'$ to be the insertion of Γ' in the i-th vertex of Γ and taking a signed sum over all possible ways of connecting the edges incident to i to Γ.

Remark 3.2 *(Sign rules).* To obtain the appropriate signs one has to consider the full data of graphs with an ordering on the set of edges. In this situation the orientation of the edges of Γ is preserved and one uses the symmetry relations on Γ in such a way that the labels of the edges of the subgraph Γ come before the labels of the edges of the subgraphs Γ'. The operad axioms are a straightforward verification.

We can form a Swiss Cheese type operad by setting Gra to be the operations in color 1 and vKGra to be the operations in color 2, considering the symmetric action permuting the labels of type I vertices and ignoring the symmetric action of type II vertices. The partial compositions are given as in Gra, i.e., by insertion on the corresponding vertex, connecting in all possible ways and distributing corresponding the powers of v also in all possible ways.

Following Kontsevich's conventions, since type II vertices in vKGra will be seen as boundary vertices, we draw them with a line passing by the type II vertices.



$\in \mathsf{vKGra}(4,5)$.

We define a cyclic $\mathbb{Z}_{n+1} = \langle \sigma | \sigma^{n+1} = e \rangle$ action on $\mathsf{vKGra}(m,n)$ on generators as follows: For all $1 \leq i \leq m$ and $2 \leq j \leq n$, we have $\sigma(\Gamma^i_{\bar{j}}) = \Gamma^i_{\overline{j-1}}$ and $\sigma(\Gamma^i_{\bar{1}}) = -\sum_{k=1}^n \Gamma^i_{\bar{k}} - \sum_{k=1}^m \Gamma^{i,k}$. The action is trivial on other generators, namely for $1 \leq i,j \leq m$, $\sigma(\Gamma^{i,j}) = \Gamma^{i,j}$ and $\sigma(v_i) = v_i$.

The cyclic action is extended to the whole $\mathsf{vKGra}(m,n)$ by requiring it to be compatible with the product in the sense that $\sigma(ab) = \sigma(a)\sigma(b), \forall a,b \in \mathsf{vKGra}(m,n)$. Since $\sigma^2(\Gamma^i_{\bar{1}}) = \Gamma^i_{\bar{m}}$, we have that σ^{n+1} acts as the identity in every one-edge graph, and therefore the action of $\mathbb{Z}_{n+1}$ is well defined.

3.3. Representation of a morphism

Let $\mathcal{P}$ be a cyclic operad and V a chain complex. Notice that there is an obvious operad of Cyclic Swiss Cheese type

$$\left(\mathrm{End}_V, \mathrm{Hom}\left(V^{\otimes \bullet}, \mathcal{P}\right)\right)$$

given by insertion of functions in to tensor powers of V and the cyclic operadic compositions in $\mathcal{P}$. If V is an algebra over the operad $\mathcal{O}$, then this induces the structure of an operad of CSC type on $(\mathcal{O}, \mathrm{Hom}\,(V^{\otimes \bullet}, \mathcal{P}))$ in which a cross color composition is given by pushing forward along the given morphism $\mathcal{O} \to End_V$ and then composing as in the first example. In particular there is an induced map:

$$\left(\mathcal{O}, \mathrm{Hom}\left(V^{\otimes \bullet}, \mathcal{P}\right)\right) \to \left(\mathrm{End}_V, \mathrm{Hom}\left(V^{\otimes \bullet}, \mathcal{P}\right)\right)$$

Now suppose (A, d, Δ) is an algebra over $\mathcal{O} \in \mathcal{O}ps^{\mathsf{M{\times}Cpx}}$. Combining this example with the sequence of dg operads $\mathsf{CC}^{\theta}(\mathcal{O}) \to \mathsf{CC}^{-}(\mathcal{O}) \to \mathsf{CC}^{-}(\mathrm{End}_A) \to End_{\mathsf{CC}^{-}(A)}$ constructed in Section 1 (via Corollaries 1.28 and 1.33) we have a morphism of CSC type operads:

$$\left(\mathsf{CC}^{\theta}(\mathcal{O}), \mathrm{Hom}\left(\mathsf{CC}^{-}(A)^{\otimes \bullet}, \mathcal{P}\right)\right) \to \left(\mathrm{End}_{\mathsf{CC}^{-}(A)}, \mathrm{Hom}\left(\mathsf{CC}^{-}(A)^{\otimes \bullet}, \mathcal{P}\right)\right)$$

Example 3.3. We will make subsequent use of the following example of such an operad of CSC type. Let A be the mixed complex $(T_{\mathrm{poly}}, 0, u\,\mathrm{div}_{\omega})$, $\mathcal{O} = End_{T_{poly}} \in \mathcal{O}ps^{\mathsf{M{\times}Cpx}}$, and $\mathcal{P} = \mathcal{D}_{\mathrm{poly}}$. Then we may consider the consequent morphism of CSC type operads:

$$\left(\mathsf{CC}^{\theta}(\mathrm{End}_{T_{\mathrm{poly}}}), \mathrm{Hom}(\mathsf{CC}^{-}(T_{\mathrm{poly}})^{\otimes \bullet}, \mathcal{D}_{\mathrm{poly}})\right)$$
$$\to \left(\mathrm{End}_{\mathsf{CC}^{-}(T_{\mathrm{poly}})}, \mathrm{Hom}(\mathsf{CC}^{-}(T_{\mathrm{poly}})^{\otimes \bullet}, \mathcal{D}_{\mathrm{poly}})\right)$$

3.4. The functor CC^{θ} on operads of Cyclic Swiss Cheese type

Proposition 3.4. *If $\mathcal{P} = (\mathcal{P}^1, \mathcal{P}^2)$ is an operad of CSC type and if the operad $\mathcal{P}^1$ is a rotational operad, then $\mathsf{CC}^{\theta}(\mathcal{P}) = (\mathsf{CC}^{\theta}(\mathcal{P}^1), \mathcal{P}^2)$ is still an operad of CSC type, with compositions given by*

$$p_2 \ \tilde{\circ}_l \ p_1 v^k = \begin{cases} p_2 \circ_l \rho(p_1) & \textit{if } k = 0 \\ 0 & \textit{if } k > 0, \end{cases}$$

for $p_i \in \mathcal{P}^i$.

Proof. Let $p_i, p_i' \in \mathcal{P}^i$. We start by showing the associativity of the composition, which is clear if we take three elements of $\mathcal{P}^2$ or three elements of $\mathcal{P}^1$. Otherwise, if a positive power of v appears in an element of $\mathsf{CC}^{\theta}(\mathcal{P}^1)$, both double compositions will be zero and associativity holds trivially. If $p_2 \ \tilde{\circ}_l \ (p_1 v^0 \ \tilde{\circ}_j \ p_1' v^0)$ compose sequentially (as opposed to parallely), we have $p_2 \ \tilde{\circ}_l \ (p_1 v^0 \ \tilde{\circ}_j \ p_1' v^0) = p_2 \circ_l \rho(p_1 \circ_j \rho(p_1')) = p_2 \circ_l (\rho(p_1) \circ_j \rho(p_1')) = (p_2 \circ_l \rho(p_1)) \circ_{l+j-1} \rho(p_1')) = (p_2 \tilde{\circ}_l p_1) \ \tilde{\circ}_{l+j-1} \ p_1'$. The other associativity verifications are straightforward.

For the compatibility with the differential, consider that $d(p_2 \ \tilde{\circ}_i \ p_1 v^k) = 0$ if $k > 0$. In that case, $dp_2 \ \tilde{\circ}_l \ p_1 v^k \pm p_2 \ \tilde{\circ}_l \ dp_1 v^k \pm p_2 \ \tilde{\circ}_l \ \rho(p_1) v^{k-1} = 0$, owing to the compatibility of d with $\circ_i$ and the fact that $\rho^2 = 0$. If $k = 0$, then $dp_2 \ \tilde{\circ}_l \ p_1 \pm p_2 \ \tilde{\circ}_l \ (d + u\rho) p_1 = dp_2 \circ_l \rho(p_1) \pm p_2 \ \tilde{\circ}_l \ dp_1 = d(p_2 \circ_l \rho(p_1)) \mp p_2 \circ_l d\rho(p_1) \pm p_2 \circ_l \rho(dp_1) = d(p_2 \ \tilde{\circ}_l \ p_1)$.

The cyclic action on $\mathcal{P}^2$ is still $\mathsf{CC}^{\theta}(\mathcal{P}^1)$ equivariant since

$$p_2^{\sigma} \ \tilde{\circ}_l \ p_1 = p_2^{\sigma} \circ_l \rho(p_1) = (p_2 \circ_l \rho(p_1))^{\sigma} = (p_2 \ \tilde{\circ}_i \ p_1)^{\sigma}. \quad \square$$

Remark 3.5. Notice that this construction defines an endofunctor on the category whose objects are operads of CSC type that are operads on mixed complexes in color 1 and whose morphisms are maps of colored operads that are equivariant with respect to the cyclic action and commute with the rotational structure in color 1.

Recall from [29] the operad $\mathsf{B}_{\circlearrowleft}$ constructed as the image of the rotational operator R on the operad of rooted planar trees. It is the untwisted version of the operad $\mathsf{M}_{\circlearrowleft}$.

Proposition 3.6. *Let $\mathcal{P} = (\mathcal{P}^1, \mathcal{P}^2)$ be an operad of CSC type. The totalized space of cyclic invariants $\prod_n \Sigma^n \mathcal{P}^2(\bullet, n)^{\mathbb{Z}_{n+1}}$ is a $\mathsf{B}_{\circlearrowleft} - \mathcal{P}^1$ bimodule.*

Proof. The left module structure follows from [29, Corollary 2.11]. The colored operad structure defines a right $\mathcal{P}^1$-module structure on $\prod_n \Sigma^n \mathcal{P}^2(\bullet, n)$ and the fact that this right module structure restricts to the space of invariants is a consequence of the equivariance of $\mathcal{P}^1$ with respect to the cyclic structure.

The compatibility of the left and right actions follows from the associativity for parallel composition on operads, as the left action only involves insertions of color 2 and the right action only involves insertions of color 1. □

Proposition 3.7. *This construction is functorial.*

Proof. The equivariance of the morphism with respect to the cyclic action guarantees that cyclic invariants are mapped to cyclic invariants. Since a morphism of CSC type operads is in particular a morphism of colored operads, the induced map on the total space is a morphism of right bimodules. As for the right $\mathsf{B}_{\circlearrowleft}$ action, the compatibility follows from the compatibility of the compositions with the cyclic structure, given in the axioms of a cyclic operad. □

4. Bimodule maps

A homotopy Grav_∞ morphism from a Grav_∞ algebra A to a Grav_∞ algebra B can be expressed as a representation of the canonical Grav_∞ operadic bimodule on the colored vector space $A \oplus B$.

The strategy to find such a representation for our case $A = T_{\text{poly}}[u], B = D^\sigma_{\text{poly}}$ is to construct a certain bimodule $\mathsf{M}_{\circlearrowleft} \circlearrowright \ker \Delta_{\mathbb{H}} \circlearrowleft \mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2))$ which is homotopy equivalent to the Grav_∞ canonical bimodule and construct a map of bimodules into $\mathrm{End}_{D^\sigma_{\text{poly}}} \circlearrowright \mathrm{End}^{T_{\text{poly}}[u]}_{D^\sigma_{\text{poly}}} \circlearrowleft \mathrm{End}_{T_{\text{poly}}[u]}$, where $\mathrm{End}^{T_{\text{poly}}[u]}_{D^\sigma_{\text{poly}}}(n) = \mathrm{Hom}(T_{\text{poly}}[u]^{\otimes n}, D^\sigma_{\text{poly}})$.

More concretely, the goal of this Section is to define several operadic bimodules and construct the following series of bimodule maps, whose composition determines our desired Grav_∞ formality map.

$$
\begin{array}{ccccc}
\mathsf{Grav}_\infty & \circlearrowright & \mathsf{Grav}_\infty^{\mathrm{bimod}} & \circlearrowleft & \mathsf{Grav}_\infty \\
\downarrow & & \downarrow{\scriptstyle 4.3} & & \downarrow \\
\mathsf{M}_\circlearrowright & \circlearrowright & \ker \Delta_{\mathbb{H}} & \circlearrowleft & \mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2)) \\
\downarrow & & \downarrow{\scriptstyle 4.3} & & \downarrow \\
\mathsf{M}_\circlearrowright & \circlearrowright & \mathsf{Chains}_*(\mathbb{H}_{\bullet,0}) & \circlearrowleft & \mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2)) \\
\downarrow & & \downarrow{\scriptstyle 4.3} & & \downarrow \\
\mathsf{M}_\circlearrowright & \circlearrowright & \left(\prod_n \Sigma^n \mathsf{Chains}_*(\mathbb{H}_{\bullet,n})^{\mathbb{Z}_{n+1}}\right)^\mu & \circlearrowleft & \mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2)) \\
\downarrow & & \downarrow{\scriptstyle 4.2} & & \downarrow \\
\mathsf{M}_\circlearrowright & \circlearrowright & \left(\prod_n \Sigma^n \mathsf{vKGra}(\bullet, n)^{\mathbb{Z}_{n+1}}\right)^\mu & \circlearrowleft & \mathsf{CC}^\theta(\mathsf{Gra}) \\
\downarrow & & \downarrow{\scriptstyle 4.4} & & \downarrow \\
\mathsf{M}_\circlearrowright & \circlearrowright & \mathrm{End}^{T_{\mathrm{poly}}[u]}_{D^\sigma_{\mathrm{poly}}} & \circlearrowleft & \mathsf{CC}^\theta(\mathrm{End}_{T_{\mathrm{poly}}}) \\
\downarrow & & \downarrow{\scriptstyle 4.4} & & \downarrow \\
\mathrm{End}_{D^\sigma_{\mathrm{poly}}} & \circlearrowright & \mathrm{End}^{T_{\mathrm{poly}}[u]}_{D^\sigma_{\mathrm{poly}}} & \circlearrowleft & \mathrm{End}_{T_{\mathrm{poly}}[u]}
\end{array}
\tag{4.1}
$$

Most of these maps follow from the application of Propositions 3.6 and 3.7, sometimes after using Proposition 3.4. The labels on the arrows represent the section in which the respective map is constructed.

The top-most map in the diagram is due to the theory of quasi-torsors that was developed in [3] and which we now briefly recall.

Definition 4.1. Let $\mathcal{P}$ and $\mathcal{Q}$ be two differential graded operads and let $\mathcal{M}$ be a $\mathcal{P} - \mathcal{Q}$ operadic differential graded bimodule, i.e., there are compatible actions

$$\mathcal{P} \circlearrowright \mathcal{M} \circlearrowleft \mathcal{Q}.$$

We say that $\mathcal{M}$ is a $\mathcal{P}$-$\mathcal{Q}$ quasi-torsor if there is an element $\mathbf{1} \in M^0(1)$ such that the canonical maps

$$
\begin{aligned}
l \colon \mathcal{P} &\to \mathcal{M} & \qquad r \colon \mathcal{Q} &\to \mathcal{M} \\
p &\mapsto p \circ (\mathbf{1}, \dots, \mathbf{1}) & q &\mapsto \mathbf{1} \circ q
\end{aligned}
\tag{4.2}
$$

are quasi-isomorphisms.

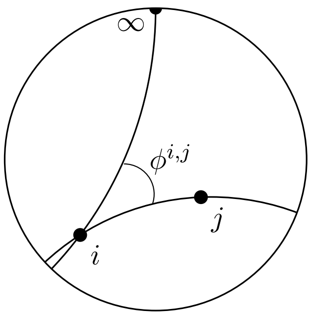


Fig. 1. The hyperbolic angle $\phi^{i,j}$.

The main Theorem of [3] states that if the $\mathcal{P}-\mathcal{Q}$-bimodule $\mathcal{M}$ is an operadic quasi-torsor, then there is a zig-zag of quasi-isomorphisms connecting $\mathcal{P} \circlearrowright \mathcal{M} \circlearrowleft \mathcal{Q}$ to the canonical bimodule $\mathcal{P} \circlearrowright \mathcal{P} \circlearrowleft \mathcal{P}$. Under good conditions, one can then homotopy lift the zig-zag to a cofibrant resolution $\mathcal{P}_\infty \circlearrowright \mathcal{P}_\infty^{\text{bimod}} \circlearrowleft \mathcal{P}_\infty \to \mathcal{P} \circlearrowright \mathcal{M} \circlearrowleft \mathcal{Q}$.

From this discussion it follows that to obtain Grav_∞ morphism $A \to B$ it suffices to construct a representation of a Grav quasi-torsor. We will show that the second row of diagram (4.1) is such a quasi-torsor, from which Theorem B (for $M = \mathbb{R}^d$) follows.

4.1. From topology to graphs

Recall from Section 2.3 the map of cooperads $\omega_\bullet \colon \mathsf{Gra}^* \to \Omega(\mathsf{FM}_2)$. We wish to define a similar map $\omega_\bullet \colon \mathsf{vKGra}^* \to \Omega(\mathbb{H}_{\bullet,\bullet})$ denoted by the same symbol by abuse of notation.

Let us consider the (multivalued) angle function θ on $\mathbb{H}_{m,n}$ such that

$$\theta(z, w, x) = \frac{1}{2\pi} \arg\left(\frac{(w-z)(1-\bar{z}x)}{(1-\bar{z}w)(x-z)} \right)$$

giving the angle between the geodesics $[w, z]$ and $[z, x]$. Since all values differ by an integer, the differential $d\theta$ is a well-defined 1-form.

The map $\omega_\bullet \colon \mathsf{vKGra}^* \to \Omega(\mathbb{H}_{\bullet,\bullet})$ is defined to be a map of commutative algebras as follows:

- The one-edge graph $\Gamma^{i,j} \in \mathsf{vKGra}^*$[5] for $i \neq j$ is sent to $d\phi^{i,j} := d\theta(z_i, z_j, z_\infty)$. Here, $\phi^{i,j}$ can be pictured as the hyperbolic geodesic passing through i and j and the vertical line passing by i or alternatively, on the hyperbolic disc, this angle can be pictured as the angle between the lines $[\infty, i]$ and $[i, j]$ (see Fig. 1).
- Similarly, the one-edge graphs $\Gamma^{i}_{\bar{j}} \in \mathsf{vKGra}^*$ are sent to $d\phi^{i}_{\bar{j}} := d\theta(z_i, z_{\bar{j}}, z_\infty)$.

[5] We identify the basis of vKGra with its dual basis in vKGra^*, except the dual of the elements v that we denote by u.

- The tadpole graphs $\Gamma^{k,k}$ are sent to Willwacher's form η_{z_k}, where

$$\eta_z = \sum_{i=0}^{n} \theta(z, z_{\overline{i+1}}, z_{\bar{i}}) d\theta(z, z_{\bar{i}}, z_\infty).$$

  Notice that $\theta(z, z_{\overline{i+1}}, z_{\bar{i}})$ is a well defined smooth function since the points $z_{\bar{i}}$ and $z_{\overline{i+1}}$ are on the boundary of the disc.
- To define the image of a graph Γ with no edges and with a vertex decorated with $u = v^* = v^{-1}$, in order for f_2 to commute with the differential, we have no choice but to define $f_2(v_k) = d\eta_{z_k}$.

Remark 4.2. Recall from Theorem 2.8 that the map $\mathsf{Chains}_*(\mathsf{FM}_2) \to \mathsf{Gra}$ is compatible with the mixed complex structure and therefore induces a map $\mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2)) \to \mathsf{CC}^\theta(\mathsf{Gra})$. One would like to have a map of CSC type operads

$$(\mathsf{Chains}_*(\mathsf{FM}_2), \mathsf{Chains}_*(\mathbb{H}_{\bullet,\bullet})) \to (\mathsf{Gra}, \mathsf{vKGra}),$$

however, due to the existence of tadpoles in vKGra this map is not compatible with the operadic composition. The next proposition states that we obtain nevertheless a map of CSC operads after taking the functor CC^θ.

Proposition 4.3. *The map*

$$\omega_\bullet^* \colon \left(\mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2)), \mathsf{Chains}_*(\mathbb{H}_{\bullet,\bullet})\right) \to (\mathsf{CC}^\theta(\mathsf{Gra}), \mathsf{vKGra})$$

is a morphism of operads of Cyclic Swiss Cheese type.

Proof. Let us start by showing that the map $\mathsf{vKGra}^* \to \Omega(\mathbb{H}_{\bullet,\bullet})$ is compatible with the cyclic structure. It suffices to check this on the generators of vKGra^*. For this, we start by observing that the cyclic structure on vKGra^*, being dual to the one of vKGra is the following:

- $\left(\Gamma^{i,j}\right)^\sigma = \Gamma^{i,j} - \Gamma^i_{\bar{1}}$,
- $\left(\Gamma^i_{\bar{j}}\right)^\sigma = \Gamma^i_{\overline{j+1}} - \Gamma^i_{\bar{1}}$ with the convention that $\Gamma^i_{\overline{n+1}} = 0$,
- $(v_i)^\sigma = v_i$.

The cyclic structure on $\Omega(\mathbb{H}_{m,n})$ is given by the pullback of the cyclic structure on $\mathbb{H}_{m,n}$. It follows that $\theta(z, z_{\bar{i}}, z_{\bar{j}}) = \theta(z, z_{\overline{i+1}}, z_{\overline{j+1}})$, from which it follows that

$$d\phi^{i\ \sigma}_{\bar{j}} = d\theta(z_i, z_{\overline{j+1}}, z_{\bar{1}}) = d\theta(z_i, z_{\overline{j+1}}, z_\infty) - d\theta(z_i, z_{\bar{1}}, z_\infty) = d\phi^i_{\overline{j+1}} - d\phi^i_{\bar{1}}$$

Similarly, it follows that $d\phi^{i,j\,\sigma} = d\phi^{i,j} - d\phi^i_{\bar{1}}$.

To show that $\eta^\sigma_{z_i} = \eta_{z_i} - d\phi^i_{\overline{1}}$, we make use of the fact that $\sum_{i=0}^n \theta(z, z_{\overline{i+1}}, z_{\overline{i}}) = 1$.

$$\begin{aligned}\eta^\sigma_z &= \sum_{i=0}^n \theta(z, z_{\overline{i+2}}, z_{\overline{i+1}}) d\theta(z, z_{\overline{i+2}}, z_{\overline{1}}) \\ &= \sum_{i=0}^n \theta(z, z_{\overline{i+1}}, z_{\overline{i}}) d\theta(z, z_{\overline{i+1}}, z_{\overline{1}}) \\ &= \sum_{i=0}^n \theta(z, z_{\overline{i+1}}, z_{\overline{i}}) (d\theta(z, z_{\overline{i+1}}, z_\infty) - d\theta(z, z_{\overline{1}}, z_\infty)) \\ &= \eta_z - d\phi^z_{\overline{1}}.\end{aligned}$$

Notice that the same computation, but with $d\theta$ instead of θ shows that $d\eta_z$ is invariant by the cyclic action.

To show the compatibility with the cooperadic structure, let us start by noticing that the pullback of forms of the type $\eta_{z_k}, d\phi^{i,j}$ and $d\eta_{z_k}$ under the composition map $\circ_i \colon \mathbb{H}_{m,n} \times \mathsf{FM}_2(k) \to \mathbb{H}_{m+k-1,n}$ is expressible with forms of the same type. For instance, the image of $d\phi^{1,2} \in \Omega(\mathbb{H}_{m,n})$ inside $\Omega(\mathbb{H}_{m-2,n} \times \mathsf{FM}_2(3))$ under the map $\circ_1^*$ is the form $1 \otimes d\phi_{1,2} \in \Omega(\mathbb{H}_{m-2,n}) \otimes \Omega(\mathsf{FM}_2(3)) \subset \Omega(\mathbb{H}_{m-2,n} \times \mathsf{FM}_2(3))$, while $\circ_1^*(\eta_{z_1}) = \eta_{z_1} \otimes 1$.

Let $X \in \mathsf{Chains}_l(\mathbb{H}_{m,n})$ and $Y \in \mathsf{Chains}_r(\mathsf{FM}_2)$. The operadic compatibility in mixed colors amounts to showing that[6]

$$\sum_{\Gamma \in \mathsf{vKGra}_{l+r}(m+k-1,n)} \left(\int_{X \circ_i \Delta(Y)} \omega_\Gamma \right) \cdot \Gamma$$
$$= \sum_{\tilde{\Gamma} \in \mathsf{vKGra}_l(m,n)} \left(\int_X \omega_{\tilde{\Gamma}} \right) \cdot \tilde{\Gamma} \circ_i \sum_{\Gamma' \in \mathsf{Gra}_r(k)} \left(\int_Y \omega_{\Gamma'} \right) \cdot \Delta(\Gamma').$$

We need therefore to show that for every graph Γ in $\mathsf{vKGra}_{l+r}(m+k-1,n)$, we have

$$\int_{X \circ_i \Delta(Y)} \omega_\Gamma = \sum_{\substack{\Gamma' \in \mathsf{Gra}_r(k) \\ \tilde{\Gamma} \in \mathsf{vKGra}_l(m,n) \\ \Gamma \in \tilde{\Gamma} \circ_i \Delta(\Gamma')}} \left(\int_X \omega_{\tilde{\Gamma}} \right) \left(\int_Y \omega_{\Gamma'} \right). \tag{4.3}$$

Consider those vertices in Γ labeled with numbers $i, i+1, \dots, i+k-1$ and let γ be the subgraph of Γ induced by these vertices. Furthermore, let us consider $\gamma^{\not{\circlearrowleft}}$ the subgraph of γ where we disregard tadpoles and powers of v.

[6] The sums are meant to be taken over the basis of graphs.

Notice that $\int_{X\circ_i\Delta(Y)}\omega_\Gamma = \int_{X\times\Delta(Y)} \circ_i^*\omega_\Gamma$ and that we can decompose this integral into $\int_X \omega_{rest}\cdot\int_{\Delta(Y)}\omega_{\gamma^{\not\circlearrowleft}}$. Notice that if γ has at least two tadpoles, there will be two copies of η_{z_i} in ω_{rest} and therefore $\omega_{rest}=0$. If that is the case, then also the right hand side of (4.3) must vanish as $\tilde\Gamma$ would need to have two tadpoles at the vertex i. Here we have used the fact that Γ' and hence $\Delta(\Gamma')$ are tadpole free (after Example 1.5).

Suppose now that γ has one tadpole. Then, a decomposition $\Gamma\in\tilde\Gamma\circ_i\Delta(\Gamma')$ allows just one choice of $\tilde\Gamma$ (that requires $\tilde\Gamma$ to have a tadpole at i and a power of v equal to the total amount of powers of v in γ).

It suffices to check that $\int_{\Delta(Y)}\omega_{\gamma^{\not\circlearrowleft}} = \sum_{\Gamma'}\int_Y\omega_{\Gamma'}$, where the sum is being taken over the admissible Γ' such that we find Γ as a summand in $\tilde\Gamma\circ_i\Delta(\Gamma')$. Since $\Delta(\Gamma')$ adds an edge in every possible way, those admissible graphs correspond precisely to all possible graphs that one obtains by removing one edge from $\gamma^{\not\circlearrowleft}$. Notice however that $\int_{\Delta(Y)}\omega_{\gamma^{\not\circlearrowleft}} = \int_Y\Delta^*(\omega_{\gamma^{\not\circlearrowleft}})$ and Δ^* is a derivation that sends every $d\phi_{i,j}$ to the constant function 1. This is because the projection maps $p_{ij}\colon \mathsf{FM}_2(n)\to\mathsf{FM}_2(2)$ are S^1 equivariant and therefore $\Delta^*(d\phi_{ij}) = p^*(\Delta^*(d\phi_{12})) = p^*(1) = 1$. It follows that expanding $\Delta^*(\omega_{\gamma^{\not\circlearrowleft}})$ one obtains exactly the same admissible graphs Γ'.

Suppose now that γ has no tadpoles. In general there are two possibilities for the choice of $\tilde\Gamma$, one containing a tadpole at i and other not containing a tadpole at i. Suppose we consider $\tilde\Gamma$ with a tadpole at i. Then, for every admissible choice of Γ', when we compute $\tilde\Gamma\circ_i\Delta(\Gamma')$ two copies of Γ appear with opposite signs, since, like Δ, inserting at a tadpole vertex produces every possible edge. Therefore there is no contribution on the right hand side of (4.3) if we take $\tilde\Gamma$ containing a tadpole at i. If $\tilde\Gamma$ contains no tadpole at i, then, as before, $\omega_{rest}=\omega_{\tilde\Gamma}$ and $\Delta^*(\omega_{\gamma^{\not\circlearrowleft}}) = \sum_{\text{admissible }\Gamma'}\omega_{\Gamma'}$.

The compatibility of $\omega_\bullet$ with the composition in color 1 is clear. $\square$

By applying Propositions 3.6 and 3.7 we obtain the following Corollary.

Corollary 4.4. *There exist bimodules and a bimodule morphism*

$$\begin{array}{ccccc}
\mathsf{B}_{\circlearrowleft} & \circlearrowleft & \prod_n \Sigma^n\mathsf{Chains}_*(\mathbb{H}_{\bullet,n})^{\mathbb{Z}_{n+1}} & \circlearrowright & \mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2)) \\
\Big\downarrow{\scriptstyle id} & & \Big\downarrow & & \Big\downarrow \\
\mathsf{B}_{\circlearrowleft} & \circlearrowleft & \prod_n \Sigma^n\mathsf{vKGra}(\bullet,n)^{\mathbb{Z}_{n+1}} & \circlearrowright & \mathsf{CC}^\theta(\mathsf{Gra})
\end{array}$$

4.2. *Twisting left modules*

Let us consider μ, the $n-2$ dimensional chain of $\mathbb{H}_{0,n}$ in which the points at the boundary are free, the “fundamental chain of the boundary”. This chain is invariant under the $\mathbb{Z}_{n+1}$ action and therefore defines a degree 2 element of $\prod_n\Sigma^n\mathsf{Chains}_*(\mathbb{H}_{\bullet,n})^{\mathbb{Z}_{n+1}}$. This is a Maurer–Cartan element with respect to the $\mathfrak{s}^{-1}\mathsf{Lie}$ action induced by $\mathfrak{s}^{-1}\mathsf{Lie}\to$

$\mathsf{B}_{\circlearrowright}$. We can therefore twist the left modules of the diagram from Corollary 4.4 by μ and its image μ' to obtain the following bimodule map

$$\begin{array}{ccc} \mathsf{Tw}\,\mathsf{B}_{\circlearrowright} & \circlearrowright & \left(\prod_n \Sigma^n \mathsf{Chains}_*(\mathbb{H}_{\bullet,n})^{\mathbb{Z}_{n+1}}\right)^{\mu} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ \mathsf{Tw}\,\mathsf{B}_{\circlearrowright} & \circlearrowright & \left(\prod_n \Sigma^n \mathsf{vKGra}(\bullet, n)^{\mathbb{Z}_{n+1}}\right)^{\mu} \end{array}$$

Since the left action concerns boundary points and the right action concerns bulk points, the two actions are compatible, giving us the bimodule map

$$\begin{array}{ccccc} \mathsf{Tw}\,\mathsf{B}_{\circlearrowright} & \circlearrowright & \left(\prod_n \Sigma^n \mathsf{Chains}_*(\mathbb{H}_{\bullet,n})^{\mathbb{Z}_{n+1}}\right)^{\mu} & \circlearrowleft & \mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2)) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow \\ \mathsf{Tw}\,\mathsf{B}_{\circlearrowright} & \circlearrowright & \left(\prod_n \Sigma^n \mathsf{vKGra}(\bullet, n)^{\mathbb{Z}_{n+1}}\right)^{\mu} & \circlearrowleft & \mathsf{CC}^{\theta}(\mathsf{Gra}) \end{array} \tag{4.4}$$

We can also consider the restriction of the left actions to $\mathsf{M}_{\circlearrowright}$, giving us the fourth map in (4.1).

4.3. Topological maps

The projection map $p\colon \mathbb{H}_{m,n} \to \mathbb{H}_{m,0}$ that forgets the points at the boundary induces a strongly continuous chain [15] $p^{-1}_{m,n}\colon \mathbb{H}_{m,0} \to \mathsf{Chains}_*(\mathbb{H}_{m,n})$. The image of a configuration of points in $\mathbb{H}_{m,0}$ can be interpreted as the same configuration of points but with n points at the real line that are freely allowed to move. If we consider the total space $\mathsf{Chains}_*(\mathbb{H}_{\bullet,0}) = \bigoplus_{m\geq 1} \mathsf{Chains}_*(\mathbb{H}_{m,0})$, this induces a degree preserving map

$$p^{-1}\colon \mathsf{Chains}_*(\mathbb{H}_{\bullet,0}) \to \prod_{n\geq 0} \Sigma^n \mathsf{Chains}^{\mu}_*(\mathbb{H}_{\bullet,n}).$$

Notice that this map actually lands in the cyclic invariant space

$$\left(\prod_{n\geq 0} \Sigma^n \mathsf{Chains}^{\mathbb{Z}_{n+1}}_*(\mathbb{H}_{\bullet,n})\right)^{\mu}.$$

Proposition 4.5. *The map p^{-1} is a morphism of right $\mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2))$ modules and its image is stable under the action of $\mathsf{M}_{\circlearrowright}$.*

The proof of this result is essentially in [31, Appendix A.2] where the reader can find further details.

Proof. The morphism clearly commutes with the right action. Let $c \in \mathsf{Chains}_*(\mathbb{H}_{m,0})$.

The boundary term $\partial p^{-1}_{m,n}(c)$ has two kind of components. When at least two points at the upper half plane get infinitely close, giving us the term $p^{-1}_{m,n}(\partial c)$, and when points at the real line get infinitely close, giving us $\pm p^{f\partial}_{m,n}(c)$, where the $f\partial$ superscript represents that we are considering the boundary at every fiber.

Then, we have $p^{-1}(\partial c) = \prod_{n\geq 0} p^{-1}_{m,n}(\partial c) = \prod_{n\geq 0} \partial p^{-1}_{m,n}(c) \pm p^{f\partial}_{m,n}(c)$. The first summand corresponds to the normal differential in $\mathsf{Chains}_*(\mathbb{H}_{m,n})$ and the second summand is precisely the extra piece of the differential induced by the twisting.

It remains to check the stability under the left $\mathsf{M}_{\circlearrowright} \subset \mathsf{M}$ action. In fact, the stronger statement that the image is stable by the M holds. To show this, it is enough to check the stability under the action of the generators T_n and T'_n.

Let $c_0, \dots, c_n$ be chains on $\mathsf{Chains}_*(\mathbb{H}_{m,0})$ and consider the action of generators of the form $T_n \in \mathsf{M}(n+1)$ on their images, i.e. consider $T_n(p^{-1}(c_0), p^{-1}(c_1), \dots, p^{-1}(c_n))$. The result follows from computing that

$$T_n(p^{-1}(c_0), p^{-1}(c_1), \dots, p^{-1}(c_n)) = p^{-1}(p(T_n(p^{-1}(c_0), p^{-1}(c_1), \dots, p^{-1}(c_n))))$$

and a similar equality for T'_n. □

Since p^{-1} is right inverse to the projection map, from this proposition it follows that $\mathsf{Chains}_*(\mathbb{H}_{\bullet,0})$ has a natural left $\mathsf{M}_{\circlearrowright}$ module structure. This gives us the third map of bimodules from diagram (4.1)

$$\begin{array}{ccccc}
\mathsf{M}_{\circlearrowright} & \circlearrowright & \mathsf{Chains}_*(\mathbb{H}_{\bullet,0}) & \circlearrowleft & \mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2)) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow \\
\mathsf{M}_{\circlearrowright} & \circlearrowright & \left(\prod_n \Sigma^n \mathsf{Chains}_*(\mathbb{H}_{\bullet,n})^{\mathbb{Z}_{n+1}}\right)^{\mu'} & \circlearrowleft & \mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2))
\end{array}$$

We want now to make the first row a quasi-torsor. The left and right operads have the correct homology Grav however, as a symmetric sequence $H(\mathbb{H}_{\bullet,0}) = \mathsf{Ger}$.

Notice that there is no analog of the S^1 action of FM_2 on $\mathbb{H}_{\bullet,0}$. We can nevertheless define a mixed complex structure at the chain level in the following way. Let $i\colon \mathsf{FM}_2 \to \mathbb{H}_{\bullet,0}$ be the map resulting from collapsing a configuration into one point, or alternatively, composing a configuration in FM_2 with the single element $\mathbb{1} \in \mathbb{H}_{1,0}$. This is a homotopy equivalence and admits a retract $r\colon \mathbb{H}_{\bullet,0} \to \mathsf{FM}_2$ by forgetting the boundary line. In particular $ri = \mathrm{id}$.

Denoting the induced maps on chains also by i and r, we see $\mathsf{Chains}_*(\mathbb{H}_{\bullet,0})$ has a mixed complex structure by defining the degree 1 map $\Delta_{\mathbb{H}}\colon \mathsf{Chains}_*(\mathbb{H}_{\bullet,0}) \to \mathsf{Chains}_*(\mathbb{H}_{\bullet,0})$ to be $\Delta_{\mathbb{H}} = i\Delta_{\mathsf{FM}_2} r$. From $ri = \mathrm{id}$ it follows $\Delta^2_{\mathbb{H}} = 0$.

Proposition 4.6. *The subspace* $\ker \Delta_{\mathbb{H}} \subset \mathsf{Chains}_*(\mathbb{H}_{\bullet,0})$ *is a* $\mathsf{M}_{\circlearrowright} - \mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2))$ *sub-bimodule.*

Proof. Let $h \in \mathsf{Chains}_*(\mathbb{H}_{\bullet,0})$ and let $c \in \mathsf{Chains}_*(\mathsf{FM}_2)$ so that $v^k c \in \mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2))$.

We have $\Delta_{\mathbb{H}}(h \circ_i c) = i\Delta r(h \circ_i c) = i\Delta(r(h) \circ_i c) = i(\Delta r(h) \circ_i c) + ir(h) \circ_i \Delta(c) = \Delta_{\mathbb{H}}(h) \circ_i c + ir(h) \circ_i \Delta(c)$.

Therefore, $\Delta_{\mathbb{H}}(h \;\tilde{\circ}_i\; c) = \Delta_{\mathbb{H}}(h \circ_i \Delta c) = \Delta_{\mathbb{H}} h \circ_i \Delta c$, so if $h \in \ker \Delta$, also $h \;\tilde{\circ}_i\; c \in \ker \Delta$. For higher powers of k we have $\Delta(h \;\tilde{\circ}_i\; v^k c) = 0$, therefore $\ker \Delta_{\mathbb{H}}$ is trivially stable by the right action.

On the other hand, $\ker \Delta_{\mathbb{H}}$ is stable by the left action since it only involves cyclic operations and compositions, all of which are compatible with $\Delta_{\mathbb{H}}$. □

Proposition 4.7.

$$\mathsf{M}_{\circlearrowright} \qquad \circlearrowright \qquad \ker \Delta_{\mathbb{H}} \qquad \circlearrowleft \qquad \mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2))$$

is a quasi-torsor.

Proof. We may apply Corollary 1.28 to the underlying $\mathbb{S}$-modules to find

$$H(\ker \Delta_{\mathbb{H}}) = \mathsf{HC}^{-}(\mathsf{Chains}_*(\mathsf{FM}_2)) = \mathsf{Grav} = H(\mathsf{M}_{\circlearrowright}),$$

and we just need to check that the maps $p\colon \mathsf{M}_{\circlearrowright} \to \ker \Delta_{\mathbb{H}}$ and $q\colon \mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2)) \to \ker \Delta_{\mathbb{H}}$ induce quasi-isomorphisms.

The map q fits in the following commutative diagram

$$\begin{array}{ccc} \mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2)) & \xrightarrow{\quad q \quad} & \ker \Delta_{\mathbb{H}} \\ \downarrow{\scriptstyle \theta^{-1}} & & \downarrow \\ \mathsf{CC}^{-}(\mathsf{Chains}_*(\mathsf{FM}_2)) & \xrightarrow{\mathsf{CC}^{-}(i)} & \mathsf{CC}^{-}(\mathsf{Chains}_*(\mathbb{H}_{\bullet,0})) \end{array}$$

Since all other maps are quasi-isomorphisms, so is q.

To see that $p\colon \mathsf{M}_{\circlearrowright} \to \ker \Delta_{\mathbb{H}}$ is a quasi-isomorphism, notice that since we already know the homologies are isomorphic, it suffices to show that the generators $g_n \in \mathsf{M}^{\pm 1}_{\circlearrowright}(n)$ are sent to generators of the homology of $\ker \Delta_{\mathbb{H}}(n)$.

Notice that since $\mathsf{Grav}_1(n)$ is 1-dimensional, in fact in suffices to show that $p(g_n)$ is non-zero in homology.

For this, notice that if we denote by $\angle_{12}\colon \mathbb{H}_{n,0} \to S^1$ the map remembering only the angle between points 1 and 2, then, the image of $p(g_n)$ under the composition $\ker \Delta_{\mathbb{H}} \hookrightarrow$

$C(\mathbb{H}_{n,0}) \stackrel{\angle_{12}}{\to} C(S^1)$ is (homologous to) the fundamental chain of the circle and is therefore non-zero. □

Remark 4.8. This is another way to show the formality of $\mathsf{M}_{\circlearrowright}$, as it would be now quasi-isomorphic to $\mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2))$ and therefore to $\mathsf{CC}^\theta(\mathsf{Graphs})$ and $\mathsf{Grav} \to \mathsf{CC}^\theta(\mathsf{Graphs})$ is a quasi-isomorphism.

4.4. *Action of graphs on $T_{\mathrm{poly}}[u]$ and D^σ_{poly}*

In this Section we construct the action on $T_{\mathrm{poly}}[u]$ and D^σ_{poly}. We express this in the form of an operadic bimodule morphism

$$\begin{array}{ccccc}
\mathsf{M}_{\circlearrowright} & \circlearrowright & \left(\prod_n \Sigma^n \mathsf{vKGra}(\bullet, n)^{\mathbb{Z}_{n+1}}\right)^{\mu'} & \circlearrowleft & \mathsf{CC}^\theta(\mathsf{Gra}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{End}_{D^\sigma_{\mathrm{poly}}} & \circlearrowright & \mathrm{End}^{T_{\mathrm{poly}}[u]}_{D^\sigma_{\mathrm{poly}}} & \circlearrowleft & \mathrm{End}_{T_{\mathrm{poly}}[u]}
\end{array}$$

Remark 4.9. If one tries to replicate the arguments of the previous section, the starting place would be to construct a map of CSC type operads

$$(\mathsf{Gra}, \mathsf{vKGra}) \to \left(\mathrm{End}_{T_{\mathrm{poly}}}, \mathrm{Hom}\left(T_{\mathrm{poly}}[u]^{\otimes \bullet}, D_{\mathrm{poly}}\right)\right),$$

and the argument would continue with the application of Proposition 3.4. Unfortunately, on the right hand side we don't have a colored operad due to the non-compatibility of the differential with the operadic composition.

We will rectify this problem by using the operad of CSC type

$$\left(\mathsf{CC}^\theta(\mathrm{End}_{T_{\mathrm{poly}}}), \mathrm{Hom}\left(T_{\mathrm{poly}}[u]^{\otimes \bullet}, \mathcal{D}_{\mathrm{poly}}\right)\right)$$

(see Example 3.3) as an intermediary.

Recall Kontsevich's action of the operad Gra on T_{poly} [21] given for every graph $\Gamma \in \mathsf{Gra}(k)$ and vector fields $X_1, \ldots, X_k \in T_{\mathrm{poly}}(\mathbb{R}^d)$ by

$$\Gamma(X_1, \ldots, X_k) = \left(\prod_{(i,j)\in\Gamma} \sum_{l=1}^{d} \frac{\partial}{\partial x_l^{(j)}} \wedge \frac{\partial}{\partial \xi_l^{(i)}}\right)(X_1 \wedge \cdots \wedge X_k),$$

where $x_1, \ldots, x_d$ are the coordinates in $\mathbb{R}^d$ and $\xi_1, \ldots, \xi_d$ be the corresponding basis of vector fields. Notice that this map is compatible with the mixed complex structure on both sides.

Proposition 4.10. *There is a map of Cyclic Swiss Cheese type operads*

$$\Big(\mathsf{CC}^\theta(\mathsf{Gra}), \mathsf{vKGra}\Big) \to \Big(\mathsf{CC}^\theta(\mathrm{End}_{T_{\mathrm{poly}}}), \mathrm{Hom}\left(T_{\mathrm{poly}}[u]^{\otimes\bullet}, \mathcal{D}_{\mathrm{poly}}\right)\Big).$$

Proof. The map $\mathsf{CC}^\theta(\mathsf{Gra}) \to \mathsf{CC}^\theta(\mathrm{End}_{T_{\mathrm{poly}}})$ is obtained by taking the functor CC^θ to Kontsevich's map above and is therefore a map of dg operads. The map $\mathsf{vKGra} \to \mathrm{Hom}\,(T_{\mathrm{poly}}[u]^{\otimes\bullet}, \mathcal{D}_{\mathrm{poly}})$ is essentially[7] defined as described in [32, Section 4.2]. For $X_1u^{i_1}, \dots, X_m u^{i_m} \in T_{\mathrm{poly}}[u]$ the action of $\Gamma \in \mathsf{vKGra}(m,n)$ on $X_1u^{i_1}, \dots, X_m u^{i_m}$ is zero if there exists a vertex l of type I in Γ such that the power of v at the vertex l does not match i_l. Otherwise, for $f_1, \dots, f_n \in C_c^\infty(\mathbb{R}^d)$, the action is given by

$$\begin{aligned}&\Gamma(X_1u^{i_1}, \dots, X_m u^{i_m})(f_1, \dots, f_n)\\ &\quad= \left(\prod_{(i,j)\in\Gamma} \sum_{r=1}^{d} \frac{\partial}{\partial x_r^{(j)}} \wedge \frac{\partial}{\partial \xi_r^{(i)}}\right)(X_1, \dots, X_m; f_1, \dots, f_n),\end{aligned} \tag{4.5}$$

where the product runs over all edges of Γ in the order given by the numbering of edges and the superscripts (i) and (j) mean that the partial derivative is being taken on the i-th and j-th component of $X_1, \dots, X_m$ (or f_j, if j corresponds to a type II vertex).

We need to check compatibility with the differentials. For simplicity of notation, let us focus on the piece of the differential acting on the vertex 1 and suppose this is decorated by v^k and let us denote by d_1 the piece of the differential only acting on the first vertex, i.e., the piece that lowers k by 1 and adds a tadpole. Since the differential on $\mathcal{D}_{\mathrm{poly}}$ is zero, we need to show that $0 = d_1\Gamma(X_1u^{i_1}, \dots) - \Gamma(d(X_1u^{i_1}), \dots)$. Both summands are zero if $k \neq i_1 - 1$ and if $k = i_1 - 1$ they cancel since the action of a tadpole on a multivector field produces its divergence.

The compatibility of the map with the mixed color composition is clear, as the map $\mathsf{Gra} \to \mathrm{End}(T_{\mathrm{poly}})$ is given by essentially the same formula (4.5).

To check the compatibility with the cyclic action in color 2 we notice that the cyclic action on $\mathrm{Hom}\,(T_{\mathrm{poly}}[u]^{\otimes\bullet}, \mathcal{D}_{\mathrm{poly}})$ is given by the cyclic action on $\mathcal{D}_{\mathrm{poly}}$ and integration by parts produces exactly the kind of graphs given by the cyclic action on vKGra. An explicit computation can be found in [2, Lemma 20]. □

Combining this result with Example 3.3 we find:

Corollary 4.11. *There is a map of Cyclic Swiss Cheese type operads*

$$\Big(\mathsf{CC}^\theta(\mathsf{Gra}), \mathsf{vKGra}\Big) \longrightarrow \left(\mathrm{End}_{T_{\mathrm{poly}}[u]}, \mathrm{Hom}\left(T_{\mathrm{poly}}[u]^{\otimes\bullet}, \mathcal{D}_{\mathrm{poly}}\right)\right)$$

[7] Notice that Willwacher's graphs do not involve powers of v.

Applying Propositions 3.6 and 3.7 to this result, we get the following bimodule maps:

$$
\begin{array}{ccccc}
\mathsf{B}_{\circlearrowright} & \circlearrowright & \left(\prod_n \Sigma^n \mathsf{vKGra}(\bullet, n)^{\mathbb{Z}_{n+1}}\right) & \circlearrowleft & \mathsf{CC}^{\theta}(\mathsf{Gra}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{B}_{\circlearrowright} & \circlearrowright & \mathrm{End}^{T_{\text{poly}}[u]}_{\tilde{D}^{\sigma}_{\text{poly}}} & \circlearrowleft & \mathsf{CC}^{\theta}(\mathrm{End}_{T_{\text{poly}}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{B}_{\circlearrowright} & \circlearrowright & \mathrm{End}^{T_{\text{poly}}[u]}_{\tilde{D}^{\sigma}_{\text{poly}}} & \circlearrowleft & \mathrm{End}_{T_{\text{poly}}[u]}
\end{array}
$$

By twisting we get the last 3 rows from diagram (4.1). Notice that we can replace the last row by $\mathrm{End}_{D^{\sigma}_{\text{poly}}}$ as the action on the bimodule is, by definition, the action on D^{σ}_{poly}.

4.5. An extension of Willwacher's morphism

In this section we remark that the restriction of the Grav_∞ morphism to $\mathfrak{s}^{-1}\mathsf{Lie}_\infty$ is Willwacher's morphism [32]. This follows essentially from the rigidity of bimodule maps from $\mathfrak{s}^{-1}\mathsf{Lie}_\infty^{\text{bimod}}$ to $\mathsf{Chains}_*(\mathbb{H}_{\bullet,0})$. Concretely, suppose we take two maps $f, g\colon \mathfrak{s}^{-1}\mathsf{Lie}_\infty^{\text{bimod}} \to \mathsf{Chains}_*(\mathbb{H}_{\bullet,0})$ that agree in arity 1 (notice that $\mathbb{H}_{1,0}) = \{pt\}$). An inductive argument shows that then f and g must be the same map.

Let us consider the family $(\mu_n)_{n\geq 1}$ of generators of $\mathfrak{s}^{-1}\mathsf{Lie}_\infty^{\text{bimod}}$. The element $\mu_n \in \mathfrak{s}^{-1}\mathsf{Lie}_\infty^{\text{bimod}}(n)$ has degree $2-2n$. Assume by induction that $f(\mu_k) = g(\mu_k)$ for all $k < n$. Then $d(f(\mu_n) - g(\mu_0)) = f(d\mu_n) - g(d\mu_n) = 0$, since the differential of μ_n only involves elements μ_k for $k < n$. Therefore $f(\mu_n) - g(\mu_n)$ represent a homology class in $H_{2-2n}(\mathbb{H}_{n,0}) = \mathfrak{s}^{-1}\mathsf{Lie}_{2-2n}^{\text{bimod}}(n) = 0$.

It follows that there exists some chain $c \in \mathsf{Chains}_{1-2n}(\mathbb{H}_{n,0})$ such that $dc = f(\mu_n) - g(\mu_n)$, but since $\dim(\mathbb{H}_{n,0}) = 2n-2$ there can be no such (non-zero) chain c from which our conclusion follows.

5. Globalization

Let M be a d-dimensional oriented manifold. In this section we show that the Grav_∞ quasi-isomorphism $T_{\text{poly}}[u](\mathbb{R}^d) \to D^{\sigma}_{\text{poly}}(\mathbb{R}^d)$ constructed in the previous sections can be globalized to a quasi-isomorphism $T_{\text{poly}}[u](M) \to D^{\sigma}_{\text{poly}}(M)$. All work is essentially already done as the globalized version follows from formal geometry techniques as in the original Kontsevich map [21, Section 7] and its cyclic version [32, Appendix].

Before reading this section, we recommend the non-expert reader to read [32, Appendix] that contains all the crucial arguments. We also recommend [5, Section 4] for a detailed introduction to the Fedosov resolutions that we use. Let us nevertheless sketch the general argument.

We start by remarking that the entire construction of the Grav_∞ quasi-isomorphism $T_{\mathrm{poly}}[u](\mathbb{R}^d) \to D^\sigma_{\mathrm{poly}}(\mathbb{R}^d)$ still holds if we replace $\mathbb{R}^d$ by $\mathbb{R}^d_{\mathrm{formal}}$, its formal completion at the origin.

One considers $\mathcal{T}^{\mathrm{formal}}_{\mathrm{poly}}$ (resp. $\mathcal{D}^{\mathrm{formal}}_{\mathrm{poly}}$), the vector bundle on M of fiberwise formal multivector fields (resp. multidifferential operators) tangent to the fibers. As in the flat case, one can also consider their cyclic versions $\mathcal{T}^{\mathrm{formal}}_{\mathrm{poly}}[u]$ (with appropriate differential) and $(\mathcal{D}^{\mathrm{formal}}_{\mathrm{poly}})^\sigma$.

We can then construct the vector bundles $\Omega(\mathcal{T}^{\mathrm{formal}}_{\mathrm{poly}}[u], M)$ of forms valued in $\mathcal{T}_{\mathrm{poly}}[u]$ and $\Omega((\mathcal{D}^{\mathrm{formal}}_{\mathrm{poly}})^\sigma, M)$ of forms valued in $(\mathcal{D}^{\mathrm{formal}}_{\mathrm{poly}})^\sigma$ with appropriate differentials.

The fibers of the bundles $\mathcal{T}^{\mathrm{formal}}_{\mathrm{poly}}[u]$ and $(\mathcal{D}^{\mathrm{formal}}_{\mathrm{poly}})^\sigma$ are isomorphic to $T_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ and $D_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$, respectively. Therefore, the formal version of the formality map can be used to find a vector bundle Grav_∞ quasi-isomorphism[8]

$$U^f\colon \Omega(\mathcal{T}^{\mathrm{formal}}_{\mathrm{poly}}[u], M) \to \Omega((\mathcal{D}^{\mathrm{formal}}_{\mathrm{poly}})^\sigma, M). \tag{5.1}$$

These two vector bundles can be related with $T_{\mathrm{poly}}[u](M)$ and $D^\sigma_{\mathrm{poly}}(M)$. In fact, with an appropriate change of differential that comes from a choice of a flat connection, $\Omega(\mathcal{T}^{\mathrm{formal}}_{\mathrm{poly}}[u], M)$ becomes a resolution of $T_{\mathrm{poly}}[u](M)$ and $\Omega((\mathcal{D}^{\mathrm{formal}}_{\mathrm{poly}})^\sigma, M)$ becomes a resolution of $D^\sigma_{\mathrm{poly}}(M)$. Both changes of differential can be seen locally as a twist via a Maurer–Cartan element B[9] sitting inside $\Omega^1(\mathcal{T}^{\mathrm{formal},1}_{\mathrm{poly}}[u], U)$ or $\Omega^1((\mathcal{D}^{\mathrm{formal}}_{\mathrm{poly}})^{\sigma,1}, U)$.

However, the linear part of B (in the fiber coordinates) is not globally well defined. It follows that to show that the globalization of the Grav_∞ map is possible, it suffices to see that its construction is compatible with twisting by Maurer–Cartan elements in a way that is not using the linear part of B.

There are three main components in the globalization procedure:

(1) The Grav_∞ formality morphism needs to be made compatible with twisting,
(2) The $\mathfrak{s}^{-1}\mathsf{Lie}_\infty$ piece of the Grav_∞ map must send B to itself,
(3) The twisting procedure must not use the linear part of B.

We remark that the second condition is automatically satisfied since the $\mathfrak{s}^{-1}\mathsf{Lie}_\infty$ piece of the Grav_∞ map is precisely Willwacher's formality map which satisfies this property.

The first component is essentially done by operadic twisting together with the verification of a condition of native twistability at the level of $\mathsf{Chains}_*(\mathbb{H}_{\bullet,0})$. The third component consists of checking that after the twisting procedure, the obtained Grav_∞ morphism factors through graphs whose action does not use the linear part of B. As we will see later, this would occur whenever there exist internal vertices with exactly one

8 Using the fact that the formality morphism is invariant by linear transformation of coordinates.
9 This B is the same one that one uses in the non-cyclic setting. The fact that B is still a Maurer–Cartan element in $\Omega(\mathcal{T}^{\mathrm{formal}}_{\mathrm{poly}}[u], U)$ follows from it being divergence free [32, Proposition 27].

outgoing edge and at most one incoming edge (since more incoming edges would kill the linear part).

5.1. The approach using operadic twisting

Let us recall the formalism of operadic twisting, developed extensively in [6]. Most of it adapts in a straightforward manner to the operadic bimodule setting, as explained in the Appendix of [2]. Let $\mathcal{P}$ be an operad under $\mathfrak{s}^{-1}\mathsf{Lie}_\infty$. If one twists a $\mathcal{P}$-algebra A (in particular a $\mathfrak{s}^{-1}\mathsf{Lie}_\infty$-algebra) by a Maurer–Cartan element $\mu \in A$, the resulting twisted algebra A^μ is not an algebra over $\mathcal{P}$ but rather over the twisted $\mathsf{Tw}\,\mathcal{P}$.

However, if $\mathcal{P}$ is *natively twistable*, i.e., there exists an operad morphism $\mathcal{P} \to \mathsf{Tw}\,\mathcal{P}$ such that $\mathcal{P} \to \mathsf{Tw}\,\mathcal{P} \to \mathcal{P}$ is the identity, then $\mathcal{P}$ still acts on A.

Recall that the action of Grav_∞ on $T_{\text{poly}}[u]$ can be expressed as a map

$$\mathsf{Grav}_\infty \to \mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2)) \to \mathsf{CC}^\theta(\mathsf{Gra}) \to \mathrm{End}_{T_{\text{poly}}[u]} \tag{5.2}$$

inducing a similar action on $\Omega(\mathcal{T}_{\text{poly}}^{\text{formal}}[u], M)$. Unfortunately, the functor CC^θ does not behave well with respect to operadic twisting. For instance, given a map $\mathfrak{s}^{-1}\mathsf{Lie} \to \mathcal{P}$, there is no natural map $\mathfrak{s}^{-1}\mathsf{Lie} \to \mathsf{CC}^\theta(\mathcal{P})$. On the other hand, as the following lemma shows, we can circumvent this issue by considering the functor CC^- instead.

Lemma 5.1. *Let $\mu\colon (\mathfrak{s}^{-1}\mathsf{Lie}, 0, 0) \to (\mathcal{P}, d, \Delta)$ be a morphism in $\mathcal{O}ps^{\mathsf{MxCpx}}$. (So in particular $\mu(l_2) \in ker(\Delta)$.) Then there is a morphism $\hat{\mu}\colon \mathfrak{s}^{-1}\mathsf{Lie} \to \mathsf{CC}^-(\mathcal{P})$ for which*

$$\mathsf{CC}^-(\mathsf{Tw}^\mu(\mathcal{P})) \hookrightarrow \mathsf{Tw}^{\hat{\mu}}(\mathsf{CC}^-(\mathcal{P}))$$

Proof. The morphism $\hat{\mu}\colon \mathfrak{s}^{-1}\mathsf{Lie} \to \mathsf{CC}^-(\mathcal{P})$ is given by $f(-) \otimes u^0$, which is a dg map since f lands in the kernel of d and of Δ.

Now on the level of graded vector spaces we can include

$$(\prod_{r\geq 0} \mathcal{P}(n+r)) \otimes k[u] \hookrightarrow \prod_{r\geq 0} (\mathcal{P}(n+r) \otimes k[u])$$

as the subset of lists whose powers of u match. Here we view $\mathsf{Tw}(\mathcal{P})$ as having a mixed complex structure via the product over r of $\Delta_{n+r}\colon \mathcal{P}(n+r) \to \mathcal{P}(n+r)$. The differential on the left hand side is $(d_\mathcal{P} + d_\mu^{\mathsf{Tw}}) + u\Delta$. The differential on the right hand side is $(d_\mathcal{P} + u\Delta) + d_{\hat{\mu}}^{\mathsf{Tw}}$. So since the inclusion takes $\{\mu(l_2), -\} \otimes u^0$ to $\{\mu(l_2) \otimes u^0, -\}$ it turns d_μ^{Tw} into $d_{\hat{\mu}}^{\mathsf{Tw}}$, whence the claim. □

We can then reexpress the action (5.2) as

$$\mathsf{Grav}_\infty \to \mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2)) \to \mathsf{CC}^\theta(\mathsf{Gra}) \to \mathsf{CC}^-(\mathsf{Gra}) \to \mathrm{End}_{T_{\text{poly}}[u]}\,.$$

If we factor the map $\mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2)) \to \mathsf{CC}^-(\mathsf{Gra})$ through the canonical projection $\mathsf{Tw}\,\mathsf{CC}^-(\mathsf{Gra}) \to \mathsf{CC}^-(\mathsf{Gra})$, we will obtain a Grav_∞ structure on $\Omega(\mathcal{T}_{\text{poly}}^{\text{formal}}[u], M)^\mu$ for every Maurer–Cartan element μ given by the following maps

$$\mathsf{Grav}_\infty \to \mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2)) \to \mathsf{Tw}\,\mathsf{CC}^-(\mathsf{Gra}) \to \mathrm{End}_{\Omega(\mathcal{T}_{\text{poly}}^{\text{formal}}[u])^\mu}\,.$$

In fact, looking at diagram (4.1) using operadic bimodule twisting,[10] we see that the same argument can be used to twist the Grav_∞ morphism, as long as we can find a factorization of the following form:

$$\begin{array}{ccccc}
\mathsf{M}_{\circlearrowright} & \circlearrowright & \mathsf{Chains}_*(\mathbb{H}_{\bullet,0}) & \circlearrowleft & \mathsf{CC}^\theta(\mathsf{Chains}_*(\mathsf{FM}_2)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{Tw}\,\mathsf{M}_{\circlearrowright} & \circlearrowright & \mathsf{Tw}\left(\prod_n \Sigma^n \mathsf{vKGra}(\bullet, n)^{\mathbb{Z}_{n+1}}\right)^\mu & \circlearrowleft & \mathsf{Tw}\,\mathsf{CC}^-(\mathsf{Gra}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{M}_{\circlearrowright} & \circlearrowright & \left(\prod_n \Sigma^n \mathsf{vKGra}(\bullet, n)^{\mathbb{Z}_{n+1}}\right)^\mu & \circlearrowleft & \mathsf{CC}^-(\mathsf{Gra})
\end{array} \tag{5.3}$$

In fact, due to the ill-definedness of the linear part of the Maurer–Cartan element B that we consider, we must in fact factor the morphism through a smaller bimodule which we construct in the next section.

5.2. Twisting of graphs

The construction of this section is essentially a formal adaptation of the globalization section in [2], so we will only sketch it and refer to [2] for the missing proofs. We first need the following proposition whose proof is immediate.

Proposition 5.2. *If $\mathcal{P} = (\mathcal{P}^1, \mathcal{P}^2)$ is an operad of CSC type and if the operad $\mathcal{P}^1$ is a rotational operad, then $\mathsf{CC}^-(\mathcal{P}) = (\mathsf{CC}^-(\mathcal{P}^1), \mathcal{P}^2)$ is still an operad of CSC type, with compositions given by*

$$p_2 \;\tilde{\circ}_l\; p_1 u^k = \begin{cases} p_2 \circ_l p_1 & \text{if } k = 0 \\ 0 & \text{if } k > 0, \end{cases}$$

for $p_i \in \mathcal{P}^i$. Moreover, the map from Corollary 1.28 induces a morphism of CSC type operads $\mathsf{CC}^\theta(\mathcal{P}) \to \mathsf{CC}^-(\mathcal{P})$.

[10] Cf. [2, Appendix] regarding twisting of operadic bimodules.

The $\mathsf{M}_{\circlearrowleft} - \mathsf{CC}^{-}(\mathsf{Gra})$-bimodule $\left(\prod_n \Sigma^n \mathsf{vKGra}(\bullet, n)^{\mathbb{Z}_{n+1}}\right)^{\mu}$ constructed in section 4.4 (together with Proposition 5.2) can be twisted to obtain the $\mathsf{Tw}\,\mathsf{M}_{\circlearrowleft} - \mathsf{Tw}\,\mathsf{CC}^{-}(\mathsf{Gra})$-bimodule $\mathsf{Tw}\left(\prod_n \Sigma^n \mathsf{vKGra}(\bullet, n)^{\mathbb{Z}_{n+1}}\right)^{\mu}$. Notice that $\mathsf{M}_{\circlearrowleft}$ arises itself from operadic twisting and we can therefore restrict the left action of $\mathsf{Tw}\,\mathsf{M}_{\circlearrowleft}$ to $\mathsf{M}_{\circlearrowleft}$ using the map $\mathsf{M}_{\circlearrowleft} \to \mathsf{Tw}\,\mathsf{M}_{\circlearrowleft}$.

Recall from section 2.3 the operad Graphs, defined as the suboperad of Tw Gra spanned by graphs such that all internal vertices have ≥ 3 valence and every connected component contains at least an external vertex.

We can restrict the bimodule right action to $\mathsf{CC}^{\theta}(\mathsf{Graphs})$ via the chain of inclusions $\mathsf{CC}^{-}(\mathsf{Graphs}) \subset \mathsf{CC}^{-}(\mathsf{Tw}\,\mathsf{Gra}) \subset \mathsf{Tw}\,\mathsf{CC}^{-}(\mathsf{Gra})$.

Definition/Proposition 5.3. *The* $\mathsf{M}_{\circlearrowleft} - \mathsf{CC}^{-}(\mathsf{Graphs})$ *bimodule*

$$\mathsf{Tw}\left(\prod_n \Sigma^n \mathsf{vKGra}(\bullet, n)^{\mathbb{Z}_{n+1}}\right)^{\mu}$$

has a sub-quotient denoted by $\mathsf{vKGraphs}^{\sigma}$ *constructed in the following way:*

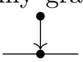
We first consider the quotient Q of $\mathsf{Tw}\left(\prod_n \Sigma^n \mathsf{vKGra}(\bullet, n)^{\mathbb{Z}_{n+1}}\right)^{\mu}$ *by the subspace consisting of graphs with tadpoles or powers of v on type I internal vertices and then the subspace of Q spanned by the graphs with the following properties:*

(1) There is at least one type I external vertex,
(2) There are no 0-valent type I internal vertices
(3) There are no 1-valent type I internal vertices with an outgoing edge,
(4) There are no 2-valent type I internal vertices with one incoming and one outgoing edge.

Proof. This result is essentially [2, Def./Prop. 24], where it was done for BVKGraphs, since BVKGra can be interpreted as the quotient of vKGra by graphs containing non-zero powers of v. We sketch the proof pointing out the adaptations to our case.

The right $\mathsf{CC}^{-}(\mathsf{Graphs})$ action cannot destroy tadpoles on internal vertices hence it descends to Q. $\mathsf{vKGraphs}^{\sigma}$ is clearly stable by the right action.

To verify the stability by the left action and by the differential one uses two properties of the Maurer–Cartan element m (the image of the generators of $\mathfrak{s}^{-1}\mathsf{Lie}_{\infty}^{\mathrm{bimod}}$) by which we twist:

(a) The only graph in m containing a 1-valent type I internal vertex is the 2 vertex graph  , with coefficient 1.
(b) There are no graphs with vertices like the ones in property *(4)*.

The proof of these properties is the same as for the original Kontsevich vanishing lemmas.

Using these properties it is a straightforward (but lengthy) combinatorial verification that non-cyclic invariant graphs $\mathsf{vKGraphs} \supset \mathsf{vKGraphs}^{\sigma}$ are preserved by the left M action. It follows that the cyclic invariant $\mathsf{vKGraphs}^{\sigma}$ are preserved by the $\mathsf{M}_{\circlearrowleft}$.

Similarly, one can check that $\mathsf{vKGraphs}$ are stable by the differential and to see that the cyclic invariant $\mathsf{vKGraphs}^{\sigma}$ are preserved by the differential it is enough to notice that the image of the generators of $\mathfrak{s}^{-1}\mathsf{Lie}_{\infty}^{\mathrm{bimod}}$ is cyclic invariant itself. □

5.3. Factorization of the bimodule morphism

To conclude the globalization procedure it is enough to construct the first bimodule morphism of the following diagram:

$$\begin{array}{ccccc}
\mathsf{M}_{\circlearrowleft} & \circlearrowright & \mathsf{Chains}_*(\mathbb{H}_{\bullet,0}) & \circlearrowleft & \mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2)) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
\mathsf{M}_{\circlearrowleft} & \circlearrowright & \mathsf{vKGraphs}^{\sigma} & \circlearrowleft & \mathsf{CC}^{-}(\mathsf{Graphs}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{End}_{\Omega((\mathcal{D}_{\mathrm{poly}}^{\mathrm{formal}})^{\sigma},M)^B} & \circlearrowright & \mathrm{End}_{\Omega((\mathcal{D}_{\mathrm{poly}}^{\mathrm{formal}})^{\sigma},M)^B}^{\Omega(\mathcal{T}_{\mathrm{poly}}^{\mathrm{formal}}[u],M)^B} & \circlearrowleft & \mathrm{End}_{\Omega(\mathcal{T}_{\mathrm{poly}}^{\mathrm{formal}}[u],M)^B}
\end{array} \tag{5.4}$$

The map g is defined to be the composition

$$\begin{aligned}
\mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2)) \to \mathsf{CC}^{-}(\mathsf{Chains}_*(\mathsf{FM}_2)) &\overset{\mathsf{CC}^{-}(\pi^{-1})}{\to} \mathsf{CC}^{-}(\mathsf{Tw}\,\mathsf{Chains}_*(\mathsf{FM}_2)) \\
&\to \mathsf{CC}^{-}(\mathsf{Tw}\,\mathsf{Gra}).
\end{aligned}$$

Here we consider the maps

$$\begin{aligned}
\pi_n^{-1} = \prod_k \pi_{n,k}^{-1} \colon \mathsf{Chains}_*(\mathsf{FM}_2(n)) &\to \mathsf{Tw}\,\mathsf{Chains}_*(\mathsf{FM}_2(n)) \\
&= \prod_k \Sigma^{2k}\mathsf{Chains}_*(\mathsf{FM}_2(n+k))^{\mathbb{S}_k},
\end{aligned}$$

obtained as the strongly continuous chain associated to the SA bundle corresponding to the map $\pi_{n,k} \colon \mathsf{FM}_2(n+k) \to \mathsf{FM}_2(n)$ that forgets the last k points. Informally, the map $\pi_{n,k}^{-1}$ is obtained by creating k points that move freely.

The maps π_n^{-1} are clearly compatible with the cyclic action and therefore induce the desired

$$\mathsf{CC}^{-}(\mathsf{Chains}_*(\mathsf{FM}_2)) \to \mathsf{CC}^{-}(\mathsf{Tw}\,\mathsf{Chains}_*(\mathsf{FM}_2)).$$

Notice that fact that the composition $\mathsf{Chains}_*(\mathsf{FM}_2) \to \mathsf{Tw}\,\mathsf{Chains}_*(\mathsf{FM}_2) \to \mathsf{Tw}\,\mathsf{Gra}$ actually lands inside Graphs uses Kontsevich's vanishing lemmas [21].

The map f is given by the composition: $\mathsf{Chains}_*(\mathbb{H}_{m,0}) \overset{\pi^{-1}}{\to} \prod_k \Sigma^{2k}\mathsf{Chains}_*(\mathbb{H}_{m+k,0}) \to$

$$\left(\prod_{n,k\geq 0} \Sigma^{n+2k}\mathsf{Chains}_*(\mathbb{H}_{m+k,n})^{\mathbb{Z}_{n+1}}\right)^{\mu} \to \mathsf{Tw}\left(\prod_n \Sigma^n \mathsf{vKGra}(m,n)^{\mathbb{Z}_{n+1}}\right)^{\mu}.$$

Here, π^{-1} is defined, as above, as the strongly continuous chain associated to the projection $\mathbb{H}_{m+k,n} \to \mathbb{H}_{m,n}$.

To finish the globalization argument, one needs to check the following two properties:

(i) f is a map of bimodules,

(ii) f lands in $\mathsf{vKGraphs}^{\sigma}(m)$ seen as a subquotient of $\mathsf{Tw}\left(\prod_n \Sigma^n \mathsf{vKGra}(m,n)^{\mathbb{Z}_{n+1}}\right)^{\mu}$.

5.3.1. Proof of (i)

We start by noticing that the compatibility with the left $\mathsf{M}_{\circlearrowleft}$ is immediate. As for the right action, notice that f as a right module map can be decomposed as

$$\begin{array}{ccc} \mathsf{Chains}_*(\mathbb{H}_{\bullet,0}) & \circlearrowleft & \mathsf{CC}^{\theta}(\mathsf{Chains}_*(\mathsf{FM}_2)) \\ \downarrow & & \downarrow g' \\ \left(\prod_{n,k\geq 0} \Sigma^{n+2k}\mathsf{Chains}_*(\mathbb{H}_{\bullet+k,n})^{\mathbb{Z}_{n+1}}\right)^{\mu} & \circlearrowleft & \mathsf{CC}^{-}(\mathsf{Tw}\,\mathsf{Chains}_*(\mathsf{FM}_2)) \\ \downarrow & & \downarrow \\ \mathsf{Tw}\left(\prod_n \Sigma^n \mathsf{vKGra}(m,n)^{\mathbb{Z}_{n+1}}\right)^{\mu} & \circlearrowleft & \mathsf{CC}^{-}(\mathsf{Graphs}). \end{array} \tag{5.5}$$

The upper map is easily checked to be a morphism of right modules. However, due to Remark 4.2 the bottom map is not a morphism of right modules. However, it is so if we restrict it to the image of g', essentially by Proposition 4.3. This guarantees that f itself is a morphism of right modules.

The compatibility of f with the differential follows from the same arguments as the functoriality of bimodule twisting.

5.3.2. Proof of (ii)

One has to show that every graph not satisfying at least one of properties *(1), (2), (3)* or *(4)* appears in the image of f with coefficient zero. This is clear for the first property.

As for property *(2)*, if a graph contains an isolated type I internal vertex, its coefficient will involve the integration of a 0-form over a two dimensional space, which is zero.

Similarly, if a graph contains a 1-valent internal vertex, its coefficient will involve an integral of a 1-form over a two dimensional space and is therefore 0.

Finally, if a graph has an internal vertex i connected to vertices a and b as in property *(4)*, in the computation of its coefficient we find the factor

$$\int_{X_{z_a,z_b}} d\phi^{ai} d\phi^{ib}$$

where X_{z_a,z_b} is the space of configurations in which the points labeled by a and b are in positions z_a and z_b, and the point labeled by i moves freely. Here the notation assumes that both a and b are type I vertices but the argument also holds if they are type II vertices.

By Stokes' theorem for SA bundles, we have

$$\underbrace{d \int_{Y_{z_a,z_b}} d\phi^{ai} d\phi^{ij} d\phi^{jb}}_{0} = \int_{Y_{z_a,z_b}} \underbrace{d(d\phi^{ai} d\phi^{ij} d\phi^{jb})}_{0} \pm \int_{\partial Y_{z_a,z_b}} d\phi^{ai} d\phi^{ij} d\phi^{jb},$$

where Y_{z_a,z_b} is the configuration space of four points (i, j, a and b) where a and b are fixed at z_a and z_b and the points labeled by i and j are free. The integral on the left hand side vanishes by degree reasons. The boundary terms on the right hand side vanish except on the following cases:

- The boundary stratum in which a and i are infinitely close,
- The boundary stratum in which i and j are infinitely close,
- The boundary stratum in which j and b are infinitely close.

In each of these cases, the result is an integral of the form $\int_{X_{z_a,z_b}} d\phi^{ai} d\phi^{ib}$, therefore it is zero.

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