THE FROBENIUS PROPERAD IS KOSZUL

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Abstract
We show the Koszulness of the properad governing involutive Lie bialgebras and also of the properads governing nonunital and unital-counital Frobenius algebras, solving a long-standing problem. This gives us minimal models for their deformation complexes, and for deformation complexes of their algebras which are discussed in detail. Using an operad of graph complexes we prove, with the help of an earlier result of one of the authors, that there is a highly nontrivial action of the Grothendieck–Teichmüller group GRT_1 *on (completed versions of) the minimal models of the properads governing Lie bialgebras and involutive Lie bialgebras by automorphisms. As a corollary, one obtains a large class of universal deformations of (involutive) Lie bialgebras and Frobenius algebras, parameterized by elements of the Grothendieck–Teichmüller Lie algebra. We also prove that for any given homotopy involutive Lie bialgebra structure on a vector space, there is an associated homotopy Batalin–Vilkovisky algebra structure on the associated Chevalley–Eilenberg complex.*

Contents

1. Introduction

The notion of a Lie bialgebra was introduced by Drinfeld in [8] in the context of the theory of Yang–Baxter equations. Later, this notion played a fundamental role in his

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theory of Hopf algebra deformations of universal enveloping algebras (see [13] and the references cited therein).

Many interesting examples of Lie bialgebras automatically satisfy an additional algebraic condition, the so-called *involutivity* or *diamond* $\diamond$ constraint. A remarkable example of such a Lie bialgebra structure was discovered by Turaev [36] on the vector space generated by all nontrivial free homotopy classes of curves on an orientable surface. Chas [4] proved that such a structure is in fact always involutive. This example was generalized to arbitrary manifolds within the framework of *string topology*: the equivariant homology of the free loop space of a compact manifold was shown by Chas and Sullivan [5] to carry the structure of a graded involutive Lie bialgebra. An involutive Lie bialgebra structure was also found by Cieliebak and Latschev [7] in the contact homology of an arbitrary exact symplectic manifold, while Schedler [34] introduced a natural involutive Lie bialgebra structure on the necklace Lie algebra associated to a quiver. It is worth pointing out that the construction of quantum A_∞-algebras given in [1] (see also [17]) stems from the fact that the vector space of cyclic words in elements of a graded vector space W equipped with a (skew-)symmetric pairing admits a canonical involutive Lie bialgebra structure. Therefore, involutive Lie bialgebras appear in many different areas of modern research.

In the study of the deformation theory of dg involutive Lie bialgebras one needs to know a minimal resolution of the associated properad. Such a minimal resolution is particularly nice and explicit if the properad happens to be *Koszul* (see [37]). Koszulness of the prop(erad) of Lie bialgebras $\mathcal{L}ie\mathcal{B}$ was established independently by Vallette [37] and Markl and Voronov [29], the latter following an idea of Kontsevich [23]. The proof of [29] made use of a new category of *small props*, which are often called $\frac{1}{2}$-*props* nowadays, and a new technical tool, the *path filtration* of a dg free properad. Attempts to settle the question of Koszulness or non-Koszulness of the properad of involutive Lie bialgebras, $\mathcal{L}ie\mathcal{B}^\diamond$, have been made since 2004. The Koszulness proof of $\mathcal{L}ie\mathcal{B}$ in [29] does not carry over to $\mathcal{L}ie^\diamond\mathcal{B}$ since the additional involutivity relation is not $\frac{1}{2}$-properadic in nature. The proof of [37] does not carry over to $\mathcal{L}ie^\diamond\mathcal{B}$ either since $\mathcal{L}ie^\diamond\mathcal{B}$ is not of distributive law type. Motivated by some computer calculations, the authors of [11] conjectured in 2009 that the properad of involutive Lie bialgebras, $\mathcal{L}ie\mathcal{B}^\diamond$, *is* Koszul. In Section 2 of this paper, we settle this long-standing problem. Our result in particular justifies some ad hoc definitions of "homotopy Lie bialgebras" which have appeared in the literature (see, e.g., [6]).

There are at least two not very straightforward steps in our solution. First, we extend Kontsevich's exact functor from small props to props by twisting it with the relative simplicial cohomologies of graphs involved. This step allows us to incorporate operations in arities $(1,1)$, $(1,0)$, and $(0,1)$ into the story, which were strictly prohibited in the Kontsevich construction, as they destroy the exactness of his functor. Second, we reduce the cohomology computation of some important auxiliary dg properad to a computation checking Koszulness of some ordinary quadratic algebra, which might be of independent interest.

By the Koszul duality theory of properads (see [37]), our result implies immediately that the properad of nonunital Frobenius algebras is Koszul. By the curved Koszul duality theory (see [18]), the latter result implies, after some extra work, the Koszulness of the prop of unital-counital Frobenius algebras. These Frobenius properads also admit many applications in various areas of mathematics and mathematical physics, for example, in representation theory, algebraic geometry, combinatorics, and, recently, in two-dimensional topological quantum field theory.

Another main result of this paper is a construction of a highly nontrivial action of the Grothendieck–Teichmüller group GRT_1 (see [9]) on minimal models of the properads of involutive Lie bialgebras/Frobenius algebras, and hence on the sets of homotopy involutive Lie bialgebras/Frobenius structures on an arbitrary dg vector space $\mathfrak{g}$. The Grothendieck–Teichmüller group GRT_1 has recently been shown to include a prounipotent subgroup freely generated by an infinite number of generators (see [2]); hence our construction provides a rich class of universal symmetries of the aforementioned objects.

In Section 5 of this paper, we furthermore show that the Chevalley–Eilenberg complex of an involutive Lie bialgebra carries a Batalin–Vilkovisky (BV) algebra structure, that is, an action of the homology operad of the framed little disks operad. This statement remains true (up to homotopy) for homotopy involutive Lie bialgebras.

Some notation

In this paper, $\mathbb{K}$ denotes a field of characteristic 0. The set $\{1, 2, \ldots, n\}$ is abbreviated to $[n]$. Its group of automorphisms is denoted by $\mathbb{S}_n$. The sign representation of $\mathbb{S}_n$ is denoted by sgn_n. The cardinality of a finite set A is denoted by $\#A$. If $V = \bigoplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ stands for the graded vector space with $V[k]^i := V^{i+k}$. For $v \in V^i$, we set $|v| := i$. The phrase *differential graded* is abbreviated by dg. In some situations we will work with complete topological vector spaces. For our purposes, the following "poor man's" definition suffices: A complete topological graded vector space for us is a graded vector space V together with a family of graded subspaces V_p, $p = 0, 1, \ldots$, such that $V = \prod_{p=0}^{\infty} V_p$. If a graded vector space U comes with a direct sum decomposition $U = \bigoplus_{p=0}^{\infty} U_p$, then we call $\prod_{p=0}^{\infty} U_p$ the *completion* of U (along the given decomposition). We define the completed tensor product of two complete graded vector spaces $V = \prod_{p=0}^{\infty} V_p$, $W = \prod_{q=0}^{\infty} W_q$ as

$$V \hat{\otimes} W = \prod_{r=0}^{\infty} \bigoplus_{p=0}^{r} V_p \otimes W_{r-p}.$$

The n-fold symmetric product of a (dg) vector space V is denoted by $\odot^n V$, the full symmetric product space by $\odot^\bullet V$ or just $\odot V$, and the completed (along $\bullet$) symmetric product by $\hat{\odot}^\bullet V$. For a finite group G acting on a vector space V, we denote via V^G the space of invariants with respect to the action of G, and by V_G the space of coinvariants $V_G = V/\{gv - v | v \in V, g \in G\}$. We always work over a field $\mathbb{K}$ of characteristic 0 so that, for finite G, we have a canonical isomorphism $V_G \cong V^G$.

We use freely the language of operads and properads and their Koszul duality theory. For a background on operads, we refer to [22], while the Koszul duality theory of properads has been developed in [37]; note, however, that we always work with differentials of degree $+1$ rather than -1 as in the aforementioned texts. For a properad $\mathcal{P}$, we denote by $\mathcal{P}\{k\}$ the unique properad which has the following property: for any graded vector space V there is a one-to-one correspondence between representations of $\mathcal{P}\{k\}$ in V and representations of $\mathcal{P}$ in $V[-k]$; in particular, $\mathcal{E}nd_V\{k\} = \mathcal{E}nd_{V[-k]}$. For $\mathcal{C}$ a coaugmented co(pr)operad, we will denote by $\Omega(\mathcal{C})$ its cobar construction. Concretely, $\Omega(\mathcal{C}) = \mathcal{F}ree\langle\overline{\mathcal{C}}[-1]\rangle$ is a graded (pr)operad, where $\overline{\mathcal{C}}$ is the cokernel of the coaugmetation and $\mathcal{F}ree\langle\dots\rangle$ denotes the free (pr)operad generated by an $\mathbb{S}$-(bi)module. We will often use complexes of derivations of (pr)operads and deformation complexes of (pr)operad maps. For a map of properads $f : \Omega(\mathcal{C}) \to \mathcal{P}$, we will denote by

$$\mathsf{Def}\big(\Omega(\mathcal{C}) \xrightarrow{f} \mathcal{P}\big) \cong \prod_{m,n} \mathrm{Hom}_{\mathbb{S}_m \times \mathbb{S}_n}\big(\mathcal{C}(m,n), \mathcal{P}(m,n)\big) \tag{1}$$

the associated convolution complex. (It is naturally a dg Lie algebra; see [33].) We will also consider derivations of the properad $\mathcal{P}$. However, we will use a minor variation of the standard definition. First, let us define a properad $\mathcal{P}^+$ generated by $\mathcal{P}$ and one other operation, say, D, of arity $(1,1)$ and cohomological degree $+1$. On $\mathcal{P}^+$ we define a differential such that whenever $\mathcal{P}^+$ acts on a dg vector space (V,d), then the action restricts to an action of $\mathcal{P}$ on the vector space with modified differential $(V, d + D)$. Clearly, any $\mathcal{P}$-algebra is a $\mathcal{P}^+$-algebra by letting D act trivially, so that we have a properad map $\mathcal{P}^+ \to \mathcal{P}$. Now, slightly abusively, we define $\mathrm{Der}(\mathcal{P})$ as the complex of derivations of $\mathcal{P}^+$ preserving the map $\mathcal{P}^+ \to \mathcal{P}$. Specifically, in all relevant cases $\mathcal{P} = \Omega(\mathcal{C})$ is the cobar construction of a coaugmented coproperad $\mathcal{C}$. The definition is then made such that $\mathrm{Der}(\mathcal{P})[-1]$ is identified with (1) as a complex. On the other hand, if we were using ordinary derivations we would have to modify (1) by replacing $\mathcal{C}$ by the cokernel of the coaugmentation $\overline{\mathcal{C}}$ on the right-hand side, thus complicating statements of several results. We assure the reader that this modification is minor and made for technical reasons in the cases we consider, and results about our $\mathrm{Der}(\mathcal{P})$ can be easily transcribed into results about the ordinary derivations if necessary.

Note, however, that $\mathrm{Der}(\mathcal{P})$ carries a natural Lie bracket through the commutator, which is not directly visible on the level of the deformation complex.

2. Koszulness of the prop of involutive Lie bialgebras

2.1. Reminder on props, $\frac{1}{2}$-props, properads, and operads

There are several ways to define these notions (see [26] for a short and clear review of different approaches and [38] for an elementary introduction to operads), but for practical computations and arguments used in our work the approach via decorated graphs is most relevant.

2.1.1. Directed graphs

Let m and n be arbitrary nonnegative integers. A *directed* (m,n)*-graph* is a triple $(\Gamma, f_{\mathrm{in}}, f_{\mathrm{out}})$, where Γ is a finite one-dimensional *CW* complex whose one-dimensional cells ("edges") are oriented ("directed"), and

$$f_{\mathrm{in}} : [n] \to \left\{ \begin{array}{c} \text{the set of all 0-cells, } v\text{, of } \Gamma \\ \text{which have precisely one} \\ \text{adjacent edge directed from } v \end{array} \right\},$$

$$f_{\mathrm{out}} : [m] \to \left\{ \begin{array}{c} \text{the set of all 0-cells, } v\text{, of } \Gamma \\ \text{which have precisely one} \\ \text{adjacent edge directed toward } v \end{array} \right\}$$

are injective maps of finite sets (called *labeling maps* or simply *labelings*) such that $\mathrm{Im} f_{\mathrm{in}} \cap \mathrm{Im} f_{\mathrm{out}} = \emptyset$. The set, $\mathfrak{G}^{\circlearrowright}(m,n)$, of all possible directed (m,n)-graphs carries an action, $(\Gamma, f_{\mathrm{in}}, f_{\mathrm{out}}) \to (\Gamma, f_{\mathrm{in}} \circ \sigma^{-1}, f_{\mathrm{out}} \circ \tau)$, of the group $\mathbb{S}_m \times \mathbb{S}_n$ (more precisely, the *right* action of $\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n$, but we declare this detail implicit from now on). We often abbreviate a triple $(\Gamma, f_{\mathrm{in}}, f_{\mathrm{out}})$ to Γ. For any $\Gamma \in \mathfrak{G}^{\circlearrowright}(m,n)$, the set

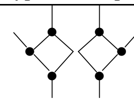
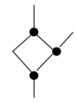
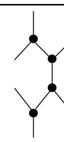
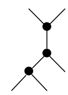
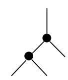
$$V(\Gamma) := \{\text{all 0-cells of } G\} \setminus \{\mathrm{Im} f_{\mathrm{in}} \cup \mathrm{Im} f_{\mathrm{out}}\}$$

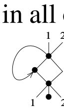
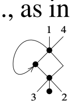
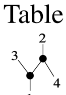
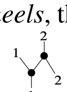
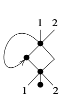
of all unlabeled 0-cells is called the set of *vertices* of Γ. The edges attached to labeled 0-cells, that is, the ones lying in $\mathrm{Im} f_{\mathrm{in}}$ or in $\mathrm{Im} f_{\mathrm{out}}$, are called *incoming* or, respectively, *outgoing legs* of the graph Γ. The set

$$E(\Gamma) := \{\text{all 1-cells of } \Gamma\} \setminus \{\text{legs}\},$$

is called the set of *(internal) edges* of Γ. Legs and edges of Γ incident to a vertex $v \in V(\Gamma)$ are often called *half-edges* of v; the set of half-edges of v splits naturally into two disjoint sets, In_v and Out_v, consisting of incoming and outgoing half-edges, respectively. In all our pictures, the vertices of a graph will be denoted by bullets,

Table 1. A list of $\mathfrak{G}$-algebras.

$\mathfrak{G}$	Definition	$\mathfrak{G}$-algebra	Typical examples
$\mathfrak{G}^{\uparrow}$	The set of all possible oriented graphs	Prop	
$\mathfrak{G}_c^{\uparrow}$	A subset $\mathfrak{G}_c^{\uparrow} \subset \mathfrak{G}^{\uparrow}$ consisting of all *connected* graphs	Properad	
$\mathfrak{G}_{c,0}^{\uparrow}$	A subset $\mathfrak{G}_{c,0}^{\uparrow} \subset \mathfrak{G}_c^{\uparrow}$ consisting of graphs of genus zero	Dioperad	
$\mathfrak{G}^{\frac{1}{2}}$	A subset $\mathfrak{G}^{\frac{1}{2}} \subset \mathfrak{G}_{c,0}^{\uparrow}$ consisting of all (m,n)-graphs with the number of directed paths from input legs to the output legs equal to mn and with at least trivalent vertices	$\frac{1}{2}$-Prop	
$\mathfrak{G}^{\curlywedge}$	A subset $\mathfrak{G}^{\curlywedge} \subset \mathfrak{G}_{c,0}^{\uparrow}$ consisting of graphs whose vertices have precisely one output leg	Operad	
$\mathfrak{G}^{\vert}$	A subset $\mathfrak{G}^{\vert} \subset \mathfrak{G}^{\curlywedge}$ consisting of graphs whose vertices have precisely one input leg	Associative algebra	

the edges by intervals (or sometimes, curves) connecting the vertices, and legs by intervals attached from one side to vertices. A choice of orientation on an edge or a leg will be visualized by the choice of a particular direction (arrow) on the associated interval/curve; unless otherwise explicitly shown, the direction of each edge in all our pictures is assumed to go *from bottom to the top*. For example, the graph  $\in \mathfrak{G}^{\circlearrowright}(2,2)$ has four vertices, four legs, and five edges; the orientation of all legs and of four internal edges is *not* shown explicitly and hence, by default, flows *upward*. Sometimes we skip showing explicitly labelings of legs (e.g., as in Table 1). Note that elements of $\mathfrak{G}^{\circlearrowright}$ are not necessarily connected, for example,   $\in \mathfrak{G}^{\circlearrowright}(4,4)$. A directed graph Γ is called *oriented* if it has no *wheels*, that is, sequences of directed edges forming a closed path; for example, the graph  is oriented, while the graph  is not. Let $\mathfrak{G}^{\uparrow}(m,n) \subset \mathfrak{G}^{\circlearrowright}(m,n)$ denote the subset of oriented (m,n)-graphs.

We will work from now on in this section with the set $\mathfrak{G}^{\uparrow} := \coprod_{m,n\geq 0} \mathfrak{G}^{\uparrow}(m,n)$ of oriented graphs, although everything said below applies to the general case as well (giving us *wheeled* versions of the classical notions of prop, properad, and operad; see [28], [31]).

2.1.2. Decorated oriented graphs

Let E be an $\mathbb{S}$-*bimodule*, that is, a family $\{E(m,n)\}_{m,n\geq 0}$ of vector spaces on which the group $\mathbb{S}_m$ acts on the left and the group $\mathbb{S}_n$ acts on the right, and both actions commute with each other. We will use elements of E to decorate vertices of an arbitrary graph $\Gamma \in \mathfrak{G}^{\uparrow}$ as follows. First, for each vertex $v \in V(\Gamma)$ with q input edges and p output edges, we construct a vector space

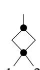
$$E(\mathrm{Out}_v, \mathrm{In}_v) := \langle \mathrm{Out}_v \rangle \otimes_{\mathbb{S}_p} E(p,q) \otimes_{\mathbb{S}_q} \langle \mathrm{In}_v \rangle,$$

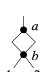
where $\langle \mathrm{Out}_v \rangle$ (resp., $\langle \mathrm{In}_v \rangle$) is the vector space spanned by all bijections $[p] \to \mathrm{Out}_v$ (resp., $\mathrm{In}_v \to [q]$). It is (noncanonically) isomorphic to $E(p,q)$ as a vector space and carries natural actions of the automorphism groups of the sets Out_v and In_v. These actions make the following *unordered tensor product* over the set $V(\Gamma)$ (of cardinality, say, k),

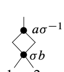
$$\bigotimes_{v\in V(\Gamma)} E(\mathrm{Out}_v, \mathrm{In}_v) := \Big(\bigoplus_{i:[k]\to V(\Gamma)} E(\mathrm{Out}_{i(1)}, \mathrm{In}_{i(1)}) \otimes \cdots \otimes E(\mathrm{Out}_{i(k)}, \mathrm{In}_{i(k)}) \Big)_{\mathbb{S}_k},$$

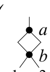
into a representation space of the automorphism group, $\mathrm{Aut}(\Gamma)$, of the graph Γ which, by definition, is the subgroup of the symmetry group of the one-dimensional *CW*-complex underlying the graph Γ which fixes its legs. Hence with an arbitrary graph $\Gamma \in \mathfrak{G}^{\uparrow}$ and an arbitrary $\mathbb{S}$-bimodule E one can associate a vector space,

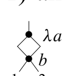
$$\Gamma\langle E\rangle := \Big(\bigotimes_{v\in V(\Gamma)} E(\mathrm{Out}_v, \mathrm{In}_v) \Big)_{\mathrm{Aut}\Gamma},$$

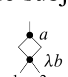
whose elements are called *decorated (by E) oriented graphs*. For example, the automorphism group of the graph $\Gamma =$ 

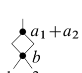
is $\mathbb{Z}_2$ so that $\Gamma\langle E\rangle \cong E(1,2) \otimes_{\mathbb{Z}_2} E(2,2)$. It is useful to think of an element in $\Gamma\langle E\rangle$ as of the graph Γ whose vertices are decorated by some elements $a \in E(1,2)$ and $b \in E(2,1)$ and are subject to the following relations: 

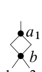
$=$ 

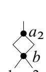
for $\sigma \in \mathbb{S}_2$, $\lambda \Big($ 

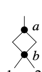
$\Big) =$ 

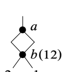
$=$ 

for any $\lambda \in \mathbb{K}$, and 

$=$ 

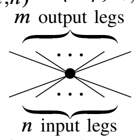
$+$ 

and similarly for b. It also follows from the definition that 

$=$ 

, $(12) \in \mathbb{S}_2$.

If $E = \{E(m,n)\}$ is a dg $\mathbb{S}$-bimodule, that is, if each vector space $E(m,n)$ is a complex equipped with an $(\mathbb{S}_m \times \mathbb{S}_n)$-equivariant differential δ, then, for any graph $\Gamma \in \mathfrak{G}^{\circlearrowright}(m,n)$, the associated graded vector space $\Gamma\langle E\rangle$ comes equipped with an induced $(\mathbb{S}_m \times \mathbb{S}_n)$-equivariant differential (which we denote by the same symbol δ) so that the collection $\{\bigoplus_{G\in\mathfrak{G}^{\circlearrowright}(m,n)} G\langle E\rangle\}_{m,n\geq 0}$ is again a dg $\mathbb{S}$-bimodule.

The 1-vertex graph $\mathfrak{C}_{m,n} :=$ 

$\in \mathfrak{G}^{\uparrow}(m,n)$ is often called the (m,n)-*corolla*. It is clear that for any $\mathbb{S}$-bimodule E, one has $\mathfrak{C}_{m,n}\langle E\rangle \cong E(m,n)$.

2.1.3. Props

A *prop* is an $\mathbb{S}$-bimodule $\mathcal{P} = \{\mathcal{P}(m,n)\}$ together with a family of linear $(\mathbb{S}_m \times \mathbb{S}_n)$-equivariant maps,

$$\left\{\mu_\Gamma : \Gamma\langle \mathcal{P}\rangle \to \mathcal{P}(m,n)\right\}_{\Gamma\in\mathfrak{G}^{\uparrow}(m,n)}, \quad m,n \geq 0,$$

which satisfy the following "associativity" condition,

$$\mu_\Gamma = \mu_{\Gamma/\gamma} \circ \mu'_\gamma, \tag{2}$$

for any subgraph $\gamma \subset \Gamma$ such that the quotient graph Γ/γ (which is obtained from Γ by shrinking all the vertices and internal edges of γ into a single internal vertex) is oriented, and $\mu'_\gamma : \Gamma\langle E\rangle \to (\Gamma/\gamma)\langle E\rangle$ stands for the map which equals μ_γ on the decorated vertices lying in γ and which is identity on all other vertices of Γ.

If the $\mathbb{S}$-bimodule $\mathcal{P}$ underlying a prop has a differential δ satisfying, for any $\Gamma \in \mathfrak{G}^{\circlearrowright}$, the condition $\delta \circ \mu_\Gamma = \mu_\Gamma \circ \delta$, then the prop $\mathcal{P}$ is called a *differential*.

As $\mathfrak{C}_{m,n}\langle E\rangle = E(m,n)$, the values of the maps μ_Γ can be identified with decorated corollas, and hence the maps themselves can be visually understood as *contraction* maps, $\mu_{\Gamma\in\mathfrak{G}^{\uparrow}(m,n)} : \Gamma\langle\mathcal{P}\rangle \to \mathfrak{C}_{m,n}\langle\mathcal{P}\rangle$, contracting all the edges and vertices of Γ into a single vertex.

Strictly speaking, the notion introduced just above should be called a prop *without unit*. A prop *with unit* can be defined as above provided one enlarges $\mathfrak{G}^{\uparrow}$ by adding a family of graphs, $\{\uparrow\uparrow\cdots\uparrow\}$, *without vertices*.

2.1.4. Props, properads, operads, and so on as $\mathfrak{G}$-algebras

Let $\mathfrak{G} = \coprod_{m,n} \mathfrak{G}(m,n)$ be a subset of the set $\mathfrak{G}^{\uparrow}$, say, one of the subsets defined in Table 1. A subgraph γ of a graph $\Gamma \in \mathfrak{G}$ is called *admissible* if $\gamma \in \mathfrak{G}$ and $\Gamma/\gamma \in \mathfrak{G}$. A $\mathfrak{G}$-*algebra* is, by definition, an $\mathbb{S}$-bimodule $\mathcal{P} = \{\mathcal{P}(m,n)\}$ together with a family of linear $(\mathbb{S}_m \times \mathbb{S}_n)$-equivariant maps, $\{\mu_\Gamma : \Gamma\langle\mathcal{P}\rangle \to \mathcal{P}(m,n)\}_{G\in\mathfrak{G}^{\circlearrowright}(m,n)}$, parameterized by elements $\Gamma \in \mathfrak{G}$, which satisfy condition (2) for any admissible subgraph

$H \subset \Gamma$. Applying this idea to the subfamilies $\mathfrak{G} \subset \mathfrak{G}^{\circlearrowright}$ from Table 1 gives us, in chronological order, the notions of *prop, operad, dioperad,* $\frac{1}{2}$*-prop*, and *properad* introduced, respectively, in [16], [23], [25], [30], and [37].

2.1.5. Basic examples of $\mathfrak{G}$-algebras

(i) For any $\mathfrak{G}$ and any vector space V, the $\mathbb{S}$-bimodule $\mathcal{E}nd_V = \{\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})\}$ is naturally a $\mathfrak{G}$-algebra with contraction maps $\mu_{G\in\mathfrak{G}}$ being ordinary compositions of linear maps; it is called the *endomorphism $\mathfrak{G}$-algebra of V*.

(ii) With any $\mathbb{S}$-bimodule, $E = \{E(m,n)\}$, there is associated another $\mathbb{S}$-bimodule, $\mathcal{F}ree^{\mathfrak{G}}\langle E\rangle = \{\mathcal{F}^{\mathfrak{G}}\langle E\rangle(m,n)\}$ with $\mathcal{F}ree^{\mathfrak{G}}\langle E\rangle(m,n) := \bigoplus_{\Gamma\in\mathfrak{G}(m,n)} \Gamma\langle E\rangle$, which has a natural $\mathfrak{G}$-algebra structure with the contraction maps μ_G being tautological. The $\mathfrak{G}$-algebra $\mathcal{F}ree^{\mathfrak{G}}\langle E\rangle$ is called the *free $\mathfrak{G}$-algebra generated by the $\mathbb{S}$-bimodule E*. We often abbreviate the notation by replacing $\mathcal{F}ree^{\mathfrak{G}}$ by $\mathcal{F}ree$.

(iii) Definitions of $\mathfrak{G}$-*sub*algebras, $\mathcal{Q} \subset \mathcal{P}$, of $\mathfrak{G}$-algebras, of their ideals, $\mathcal{I} \subset \mathcal{P}$, and the associated quotient $\mathfrak{G}$-algebras, $\mathcal{P}/\mathcal{I}$, are straightforward. We omit the details.

2.1.6. Morphisms and resolutions of $\mathfrak{G}$-algebras

A morphism of $\mathfrak{G}$-algebras, $\rho : \mathcal{P}_1 \to \mathcal{P}_2$, is a morphism of the underlying $\mathbb{S}$-bimodules such that, for any graph G, one has $\rho \circ \mu_G = \mu_G \circ (\rho^{\otimes G})$, where $\rho^{\otimes G}$ is a map, $G\langle\mathcal{P}_1\rangle \to G\langle\mathcal{P}_2\rangle$, which changes decorations of each vertex in G in accordance with ρ. A morphism of $\mathfrak{G}$-algebras, $\mathcal{P} \to \mathcal{E}nd_V$, is called a *representation* of the $\mathfrak{G}$-algebra $\mathcal{P}$ in a graded vector space V.

A *free resolution* of a dg $\mathfrak{G}$-algebra $\mathcal{P}$ is, by definition, a dg free $\mathfrak{G}$-algebra, $(\mathcal{F}^{\mathfrak{G}}\langle E\rangle, \delta)$, together with a morphism, $\pi : (\mathcal{F}\langle E\rangle, \delta) \to \mathcal{P}$, which induces a cohomology isomorphism. If the differential δ in $\mathcal{F}\langle\mathcal{E}\rangle$ is decomposable with respect to compositions μ_G, then it is called in [33, Section 5.1] a *minimal model* of $\mathcal{P}$ and is often denoted by $\mathcal{P}_\infty$. To ensure better properties, one may require in addition that there is a filtration on the space of generators

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset \bigcup_{n\geq 0} E_n = E$$

such that $\delta(E_n) \subset \mathcal{F}(E_{n-1})$. This stronger condition will also hold in the examples we consider.

2.2. Involutive Lie bialgebras

A *Lie bialgebra* is a graded vector space $\mathfrak{g}$, equipped with degree 0 linear maps,

$$\triangle\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g} \qquad \text{and} \qquad [\,,\,] : \wedge^2\mathfrak{g} \to \mathfrak{g},$$

such that

- the data $(\mathfrak{g}, \triangle)$ is a Lie coalgebra;
- the data $(\mathfrak{g}, [\,,\,])$ is a Lie algebra;
- the compatibility condition,

$$\triangle\,[a,b] = \sum a' \otimes [a'',b] + [a,b'] \otimes b'' - (-1)^{|a||b|}\big([b,a'] \otimes a'' + b' \otimes [b'',a]\big),$$

holds for any $a, b \in \mathfrak{g}$. Here $\triangle\, a =: \sum a' \otimes a''$, $\triangle\, b =: \sum b' \otimes b''$.

A Lie bialgebra $(\mathfrak{g}, [\,,\,], \triangle)$ is called *involutive* if the composition map

$$V \xrightarrow{\triangle} \Lambda^2 V \xrightarrow{[\,,\,]} V,$$
$$a \longrightarrow \sum a' \otimes a'' \longrightarrow \sum [a', a'']$$

vanishes. A dg (involutive) Lie bialgebra is a complex $(\mathfrak{g}, d)$ equipped with the structure of an (involutive) Lie bialgebra such that the maps $[\,,\,]$ and Δ are morphisms of complexes.

2.2.1. An example

Let W be a finite-dimensional graded vector space over a field $\mathbb{K}$ of characteristic 0 equipped with a degree 0 skew-symmetric pairing,

$$\omega : W \otimes W \longrightarrow \mathbb{K},$$
$$w_1 \otimes w_2 \longrightarrow \omega(w_1, w_2) = -(-1)^{|w_1||w_2|}\omega(w_2, w_1).$$

Then the associated vector space of "cyclic words in W"

$$\mathrm{Cyc}^\bullet(W) := \bigoplus_{n \geq 0} (W^{\otimes n})_{\mathbb{Z}_n}$$

admits an involutive Lie bialgebra structure given by [4]:

$$\begin{aligned}\big[(w_1 \otimes \cdots \otimes w_n)_{\mathbb{Z}_n}, (v_1 \otimes \cdots \otimes v_m)_{\mathbb{Z}_n}\big] \\ := \sum_{\substack{i \in [n] \\ j \in [m]}} \pm\omega(w_i, v_j)(w_1 \otimes \cdots \otimes w_{i-1} \otimes v_{j+1} \otimes \cdots \otimes v_m \otimes v_1 \otimes \cdots \\ \otimes v_{j-1} \otimes w_{i+1} \otimes \cdots \otimes w_n)_{\mathbb{Z}_{n+m-2}}\end{aligned}$$

and

$$\begin{aligned}&\triangle\,(w_1 \otimes \cdots \otimes w_n)_{\mathbb{Z}_n} \\ &:= \sum_{i \neq j} \pm\omega(w_i, w_j)(w_{i+1} \otimes \cdots \otimes w_{j-1})_{\mathbb{Z}_{j-i-1}} \\ &\otimes (w_{j+1} \otimes \cdots \otimes w_{i-1})_{\mathbb{Z}_{n-j+i-1}}.\end{aligned}$$

This example has many applications in various areas of modern research (see, e.g., [1], [4], [7], [17]).

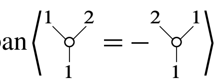
2.3. Properad of involutive Lie bialgebras

By definition, the properad of involutive Lie bialgebras $\mathcal{L}ie^{\diamond}\mathcal{B}$ is a quadratic properad given as the quotient,

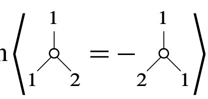
$$\mathcal{L}ie^{\diamond}\mathcal{B} := \mathcal{F}ree\langle E\rangle/\langle\mathcal{R}\rangle,$$

of the free properad generated by an $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geq 1}$ with all $E(m,n) = 0$, except

$$E(2,1) := \mathbf{1}_1 \otimes \mathrm{sgn}_2 = \mathrm{span}$$

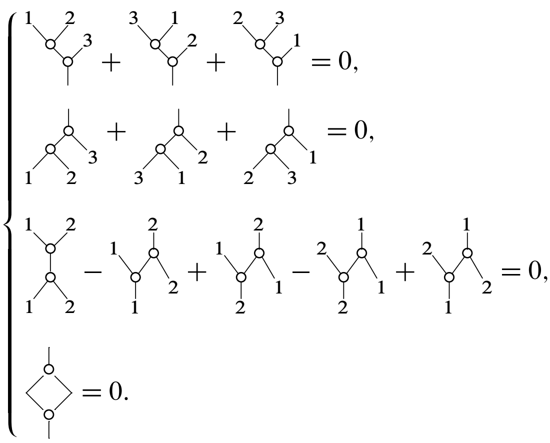


$$E(1,2) := \mathrm{sgn}_2 \otimes \mathbf{1}_1 = \mathrm{span}$$



modulo the ideal generated by the following relations:

$\mathcal{R}$:



(3)

The properad governing Lie bialgebras $\mathcal{L}ie\mathcal{B}$ is defined in the same manner, except that the last relation of (3) is omitted.

Recall (see [37]) that any quadratic properad $\mathcal{P}$ has an associated Koszul dual coproperad $\mathcal{P}^{¡}$ such that its cobar construction

$$\Omega(\mathcal{P}^{¡}) = \mathcal{F}ree\big\langle\bar{\mathcal{P}}^{¡}[-1]\big\rangle$$

comes equipped with a differential d and a canonical surjective map of dg properads

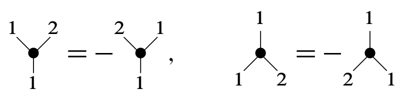
$$\big(\Omega(\mathcal{P}^{¡}), d\big) \longrightarrow (\mathcal{P}, 0).$$

This map always induces an isomorphism in cohomology in degree 0. If, additionally, the map is a quasi-isomorphism, then the properad $\mathcal{P}$ is called *Koszul*. In this case, the cobar construction $\Omega(\mathcal{P}^{¡})$ gives us a minimal resolution of $\mathcal{P}$ and is denoted by $\mathcal{P}_\infty$. It is well known, for example, that the properad governing Lie bialgebras $\mathcal{L}ie\mathcal{B}$ is Koszul (see [29]).

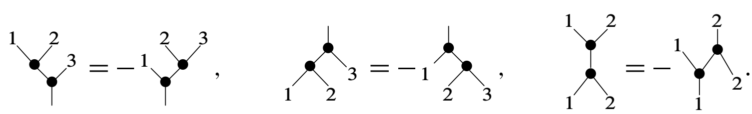
We study below the Koszul dual properad $\mathcal{L}ie^\diamond\mathcal{B}^{¡}$ and its cobar construction $\Omega(\mathcal{L}ie^\diamond\mathcal{B}^{¡})$, and prove that the natural surjection $\Omega(\mathcal{L}ie^\diamond\mathcal{B}^{¡}) \to \mathcal{L}ie^\diamond\mathcal{B}$ is a quasi-isomorphism. Anticipating this conclusion, we often use the symbol $\mathcal{L}ie^\diamond\mathcal{B}_\infty$ as a shorthand for $\Omega(\mathcal{L}ie^\diamond\mathcal{B}^{¡})$.

2.4. An explicit description of the dg properad $\mathcal{L}ie^\diamond\mathcal{B}_\infty$

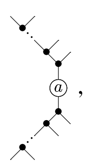
The Koszul dual of $\mathcal{L}ie^\diamond\mathcal{B}$ is a coproperad $\mathcal{L}ie^\diamond\mathcal{B}^{¡}$ whose (genus-)graded dual, $(\mathcal{L}ie^\diamond\mathcal{B}^{¡})^*$, is the properad generated by degree 1 corollas,



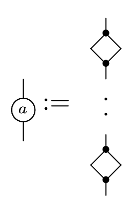
with the following relations:



Hence the following graphs

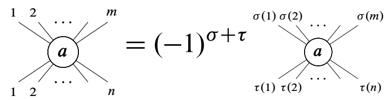
 (4)

where

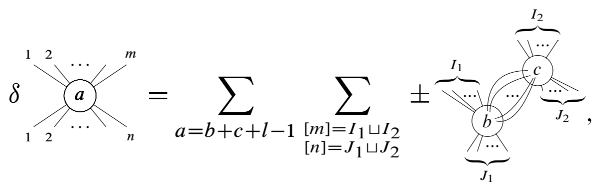
 (5)

is the composition of $a = 0, 1, 2, \ldots$ graphs of the form , form a basis of $(\mathcal{L}ie^\diamond\mathcal{B}^{¡})^*$.
If the graph (4) has n input legs and m output legs, then it has $m+n+2a-2$ vertices and its degree is equal to $m+n+2a-2$. Hence the properad $\Omega((\mathcal{L}ie^\diamond\mathcal{B})^{¡}) =$

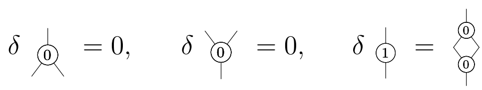
$\mathcal{F}ree\langle\overline{(\mathcal{L}ie^{\diamond}\mathcal{B})^{¡}}[-1]\rangle$ is a free properad generated by the following skew-symmetric corollas of degree $3-m-n-2a$:


$\forall\sigma\in\mathbb{S}_m, \forall\tau\in\mathbb{S}_n,$ (6)

where $m+n+a\geq 3$, $m\geq 1$, $n\geq 1$, $a\geq 0$. The nonnegative number a is called the *weight* of the generating corollas (6). The differential in $\Omega((\mathcal{L}ie^{\diamond}\mathcal{B})^{¡})$ is given by†


(7)

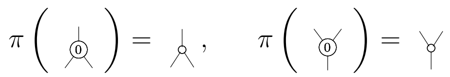
where the parameter l counts the number of internal edges connecting the two vertices on the right-hand side. We have, in particular,



so that the map

$$\pi : \mathcal{L}ie^{\diamond}\mathcal{B}_{\infty} \longrightarrow \mathcal{L}ie^{\diamond}\mathcal{B}, \tag{8}$$

which sends to zero all generators of $\mathcal{L}ie^{\diamond}\mathcal{B}_{\infty}$ except the following ones,



is a morphism of dg properads, as expected. Showing that the properad $\mathcal{L}ie^{\diamond}\mathcal{B}$ is Koszul is equivalent to showing that the map π is a quasi-isomorphism. As $\mathcal{L}ie^{\diamond}\mathcal{B}_{\infty}$ is nonpositively graded, the map π is a quasi-isomorphism if and only if the cohomology of the dg properad $\mathcal{L}ie^{\diamond}\mathcal{B}_{\infty}$ is concentrated in degree 0. We will prove this property below in Section 2.9 with the help of several auxiliary constructions, which we discuss next.

2.5. A decomposition of the complex $\mathcal{L}ie^{\diamond}\mathcal{B}_{\infty}$

As a vector space the properad $\mathcal{L}ie^{\diamond}\mathcal{B}_{\infty}=\{\mathcal{L}ie^{\diamond}\mathcal{B}_{\infty}(m,n)\}_{m,n\geq 1}$ is spanned by oriented graphs built from corollas (6). For such a graph $\Gamma\in\mathcal{L}ie^{\diamond}\mathcal{B}_{\infty}$, we set

†The precise sign factors in this formula can be determined via a usual trick: the analogous differential in the degree shifted properad $\Omega((\mathcal{L}ie^{\diamond}\mathcal{B})^{¡})\{1\}$ must be given by the same formula but with all sign factors equal to $+1$.

$$\|\Gamma\| := g(\Gamma) + w(\Gamma) \in \mathbb{N},$$

where $g(\Gamma)$ is its genus and $w(\Gamma)$ is its total weight defined as the sum of weights of its vertices (corollas). It is obvious that the differential δ in $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$ respects this total grading,

$$\|\delta\Gamma\| = \|\Gamma\|.$$

Therefore, each complex $(\mathcal{L}ie^{\diamond}\mathcal{B}_\infty(m,n), \delta)$ decomposes into a direct sum of subcomplexes

$$\mathcal{L}ie^{\diamond}\mathcal{B}_\infty(m,n) = \sum_{s \geq 0} \mathcal{L}ie^{\diamond}\mathcal{B}_\infty(m,n)^{(s)},$$

where $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty(m,n)^{(s)} \subset \mathcal{L}ie^{\diamond}\mathcal{B}_\infty(m,n)$ is spanned by graphs Γ with $\|\Gamma\| = s$.

2.5.1 LEMMA
For any fixed $m, n \geq 1$ and $s \geq 0$, the subcomplex $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty(m,n)^{(s)}$ is finite-dimensional.

Proof
The number of bivalent vertices in every graph Γ with $\|\Gamma\| = s$ is finite. As the genus of the graph Γ is also finite, it must have a finite number of vertices of valence ≥ 3 as well. □

This lemma guarantees convergence of all spectral sequences which we consider below in the context of computing the cohomology of $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$ and which, for general dg free properads, can be ill-behaved.

2.6. *An auxiliary graph complex*

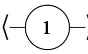
Let us consider a graph complex,

$$C = \bigoplus_{n \geq 1} C^n,$$

where C^n is spanned by graphs of the form $-\!\!\textcircled{\scriptsize a_1}\!\!-\!\!\textcircled{\scriptsize a_2}\!\!-\cdots-\!\!\textcircled{\scriptsize a_n}\!\!-$, with $a_1, \ldots, a_n \in \mathbb{N}$. The differential is given on the generators of the graphs (viewed as elements of a $\frac{1}{2}$-prop) by

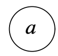
$$d\; -\!\!\textcircled{\scriptsize a}\!\!- \;=\; \sum_{\substack{a=b+c\\ b\geq 1, c\geq 1}} -\!\!\textcircled{\scriptsize b}\!\!-\!\!\textcircled{\scriptsize c}\!\!- \;.$$

2.6.1 PROPOSITION

One has $H^\bullet(C) = \mathrm{span}\langle$  $\rangle$.

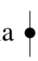
Proof

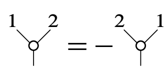
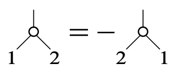
It is well known that the cohomology of the cobar construction $\Omega(T^c(V))$ of the tensor coalgebra $T^c(V)$ generated by any vector space V over a field $\mathbb{K}$ is equal to $\mathbb{K} \oplus V$, so that the cohomology of the reduced cobar construction, $\overline{\Omega}(T^c(V))$, equals V. The complex C is isomorphic to $\overline{\Omega}(T^c(V))$ for a one-dimensional vector space V via the following identification

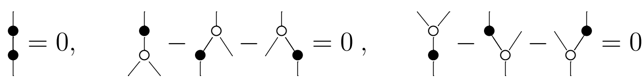
 $\cong V^{\otimes a}$,

and hence the claim follows. □

2.7. An auxiliary dg properad

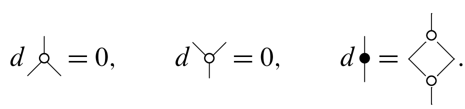
Let $\mathcal{P}$ be a dg properad generated by a degree -1 corolla  and degree 0 corollas,

 and , modulo relations



(9)

and the first three relations in (3). The differential in $\mathcal{P}$ is given on the generators by



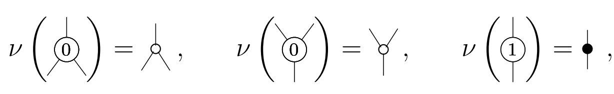
(10)

2.7.1 THEOREM

The surjective morphism of dg properads,

$$\nu : \mathcal{L}ie^\diamond\mathcal{B}_\infty \longrightarrow \mathcal{P}, \tag{11}$$

which sends all generators to zero except for the following ones,



(12)

is a quasi-isomorphism.

Proof

The argument is based on several converging spectral sequences.

Step 1: An exact functor. We define the following functor:

$$F : \text{category of dg } \frac{1}{2}\text{-props} \longrightarrow \text{category of dg properads},$$

by

$$F(s)(m,n) = \bigoplus_{\Gamma \in \overline{\mathrm{Gr}}(m,n)} \Big(\bigotimes_{v \in v(\Gamma)} s\big(\mathrm{Out}(v), \mathrm{In}(v)\big) \otimes \odot H^1(\Gamma, \partial\Gamma) \Big)_{\mathrm{Aut}(\Gamma)},$$

where $\overline{\mathrm{Gr}}(m,n)$ represents the set of all (isomorphism classes of) oriented graphs with n output legs and m input legs that are irreducible in the sense that they do not allow any $\frac{1}{2}$-propic contractions. We consider the relative cohomology $H^1(\Gamma, \partial\Gamma)$ to live in cohomological degree 1. In particular, the graded symmetric product $\odot H^1(\Gamma, \partial\Gamma)$ is finite-dimensional, and the square of any relative cohomology class vanishes. The differential acts trivially on the $H^1(\Gamma, \partial\Gamma)$ part. Our functor F is similar to the Kontsevich functor F discussed in full detail in [29], except for the tensor factor $\odot H^1(\Gamma, \partial\Gamma)$. One defines the properadic compositions on $F(s)$ as in the Kontsevich case, with the tensor factors handled as follows. Suppose we compose elements corresponding to graphs $\Gamma_1, \dots, \Gamma_n$ to an element corresponding to a graph Γ. We first map the tensor factors using the natural maps $H^1(\Gamma_j, \partial\Gamma_j) \to H^1(\Gamma, \partial\Gamma)$, and then multiply them. (To this end, note that if $\Gamma' \subset \Gamma$ is a subgraph, then one has a natural map $H^1(\Gamma', \partial\Gamma') \to H^1(\Gamma, \partial\Gamma)$, and contractions of a graph do not change $H^1(\Gamma, \partial\Gamma)$.)

Our modification of the Kontsevich functor allows treatment of properads $\mathcal{P}$ which might have $\mathcal{P}(1,1)$ nonzero (which is strictly prohibited in the original Kontsevich approach; see steps 3 and 4 below).

2.7.2 LEMMA
The functor F is exact; that is, it preserves cohomology.

Proof
Since the differential preserves the underlying graph, we get

$$H_\bullet\big(F(s)(m,n)\big) = \bigoplus_{\Gamma \in \overline{\mathrm{Gr}}(m,n)} H_\bullet\Big(\Big(\bigotimes_{v \in v(\Gamma)} s\big(\mathrm{Out}(v), \mathrm{In}(v)\big) \otimes \odot H^1(\Gamma, \partial\Gamma) \Big)_{\mathrm{Aut}(\Gamma)}\Big). \tag{13}$$

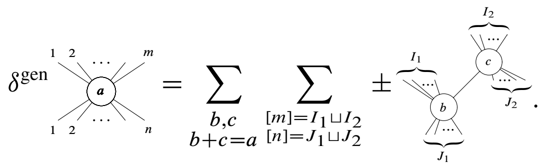
Since the differential commutes with elements of $\mathrm{Aut}(\Gamma)$, $\mathrm{Aut}(\Gamma)$ is finite, and $\mathbb{K}$ is a field of characteristic 0, by Maschke's theorem, we have

$$(13) = \bigoplus_{\Gamma \in \overline{\mathrm{Gr}}(m,n)} \Big(H_\bullet\Big(\bigotimes_{v \in v(\Gamma)} s\big(\mathrm{Out}(v), \mathrm{In}(v)\big) \otimes \odot H^1(\Gamma, \partial\Gamma) \Big)\Big)_{\mathrm{Aut}(\Gamma)}. \tag{14}$$

Applying the Künneth formula twice, together with the fact that the differential is trivial on $H^1(\Gamma, \partial\Gamma)$, we get

$$(14) = \bigoplus_{\Gamma\in\overline{\mathrm{Gr}}(m,n)} \Big(\bigotimes_{v\in v(\Gamma)} H_\bullet\big(s\big(\mathrm{Out}(v),\mathrm{In}(v)\big)\big)\otimes \odot H^1(\Gamma,\partial\Gamma)\Big)_{\mathrm{Aut}(\Gamma)}$$
$$= F\big(H_\bullet(s)\big)(m,n). \qquad \square$$

Step 2: A genus filtration. Consider the genus filtration of $(\mathcal{L}ie^\diamond\mathcal{B}_\infty,\delta)$, and denote by $(\mathrm{gr}\mathcal{L}ie^\diamond\mathcal{B}_\infty,\delta^{\mathrm{gen}})$ the associated graded properad. The differential δ^{gen} in the complex $\mathrm{gr}\mathcal{L}ie^\diamond\mathcal{B}_\infty$ is given by the formula



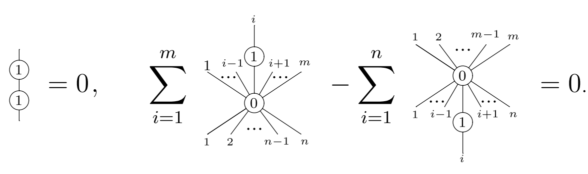
(15)

Consider also the genus filtration of the dg properad $\mathcal{P}$, and denote by $(\mathrm{gr}\mathcal{P},0)$ the associated graded properad. The morphism (11) of filtered complexes induces a sequence of morphisms of the associated graded complexes

$$\nu : \mathrm{gr}\mathcal{L}ie^\diamond\mathcal{B}_\infty \longrightarrow \mathrm{gr}\mathcal{P}. \tag{16}$$

Thanks to the spectral sequence comparison theorem (see [39, p. 126]), we can prove Theorem 2.7.1 if we show that the map ν is a quasi-isomorphism of complexes. We compute below the cohomology $H^\bullet(\mathrm{gr}\mathcal{L}ie^\diamond\mathcal{B}_\infty,\delta^{\mathrm{gen}})$, which will make it evident that the map ν is indeed a quasi-isomorphism.

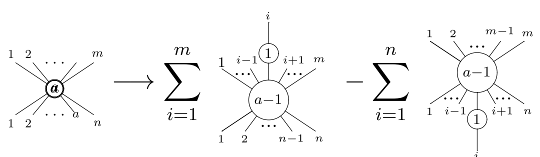
Step 3: An auxiliary prop. Let us consider a properad $\mathcal{Q} = F(\Omega_{\frac{1}{2}}(\mathcal{L}ie\mathcal{B}^{¡}_{\frac{1}{2}}))$, where $\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}$ is the $\frac{1}{2}$-prop governing Lie bialgebras, and $\mathcal{L}ie\mathcal{B}^{¡}_{\frac{1}{2}}$ its Koszul dual. Explicitly, $\mathcal{Q}$ can be understood as generated by corollas as in (6) with either $a=m=n=1$ or $a=0$ and $m+n\geq 3$ subject to the relations



To see this, note that for any graph Γ, $H^1(\Gamma,\partial\Gamma)$ may be identified with the space of formal linear combinations of edges of Γ, modulo the relations that the sum of incoming edges at any vertex equals the sum of outgoing edges. Similarly, $\odot^k H^1(\Gamma,\partial\Gamma)$ may be identified with formal linear combinations of k-fold ("wedge") products of edges, modulo similar relations. Of course, such a product of k edges may be represented combinatorially by putting a marking on those k edges. In our case, this marking is the corolla $\textcircled{1}$, which we may put on edges.

With the above combinatorial description of $\mathcal{Q}$, we see that the map ν above factors naturally through $\mathcal{Q}$, say, $\nu : \mathrm{gr}\mathcal{L}ie^{\diamond}\mathcal{B}_\infty \xrightarrow{p} \mathcal{Q} \xrightarrow{q} \mathrm{gr}\mathcal{P}$. Furthermore, we claim that the right-hand map q is a quasi-isomorphism. First note that $\mathrm{gr}\mathcal{P} = F(\mathcal{L}ie\mathcal{B}_{\frac{1}{2}})$. The $\frac{1}{2}$-prop $\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}$ is Koszul; that is, the natural projection $\Omega_{\frac{1}{2}}(\mathcal{L}ie\mathcal{B}^{\text{¡}}_{\frac{1}{2}}) \twoheadrightarrow \mathcal{L}ie\mathcal{B}_{\frac{1}{2}}$ is a quasi-isomorphism. The result follows by applying the functor F to this map and by Lemma 2.7.2.

Step 4: The map $p : \mathrm{gr}\mathcal{L}ie^{\diamond}\mathcal{B}_\infty \to \mathcal{Q}$ *is a quasi-isomorphism.* Consider a filtration of $\mathrm{gr}\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$ given, for any graph Γ, by the difference $a(\Gamma) - n(\Gamma)$, where $a(\Gamma)$ is the sum of all decorations of non-bivalent vertices, and $n(\Gamma)$ is the sum of valences of non-bivalent vertices. On the zeroth page of this spectral sequence, the differential acts only by splitting bivalent vertices. Then Proposition 2.6.1 tells us that the first page of this spectral sequence consists of graphs with no bivalent vertices such that every vertex is decorated by a number $a \in \mathbb{Z}^+$ and such that every edge has either a decoration $\textcircled{1}$ or no decoration. The differential acts by



(17)

The complex we obtain is precisely

$$\bigoplus_{\Gamma\in\overline{\mathrm{Gr}}(m,n)} \Big(\bigotimes_{v\in v(\Gamma)} \Omega_{\frac{1}{2}}(\mathcal{L}ie\mathcal{B}^{\text{¡}}_{\frac{1}{2}})\big(\mathrm{Out}(v),\mathrm{In}(v)\big) \otimes \odot C^*(\Gamma,\partial\Gamma)\Big)_{\mathrm{Aut}(\Gamma)},$$

where $C^*(\Gamma,\partial\Gamma)$ are the simplicial cochains of Γ relative to its boundary; the differential in this complex is given by the standard differential in $C^*(\Gamma,\partial\Gamma)$. Indeed, we may identify $C^0(\Gamma,\partial\Gamma) \cong \mathbb{K}[V(\Gamma)]$ and $C^1(\Gamma,\partial\Gamma) \cong \mathbb{K}[E(\Gamma)]$. A vertex $v = \textcircled{a_v}$ with weight a_v corresponds to the a_vth power of the cochain representing the vertex, and an edge decorated with the symbol $\textcircled{1}$ corresponds to the cochain representing the edge. The differential d on $C^1(\Gamma,\partial\Gamma)$ is the map dual to the standard boundary map $\partial : C_1(\Gamma,\partial\Gamma) \cong \mathbb{K}[E(\Gamma)] \longrightarrow C_0(\Gamma,\partial\Gamma) \cong \mathbb{K}[V(\Gamma)]$. It is given, on a vertex $v \in V(\Gamma)$, by

$$dv = \sum_{e'_v\in\mathrm{Out}(v)} e'_v - \sum_{e'_v\in\mathrm{In}(v)} e''_v,$$

where $\mathrm{Out}(v)$ is the set of edges outgoing from v and $\mathrm{In}(v)$ is the set of edges ingoing to v. This exactly matches the differential (17) on the first page of the spectral sequence. As $H^0(\Gamma, \partial\Gamma) = 0$, and since the symmetric product functor $\odot$ is exact, we obtain $\mathcal{Q}$ on the second page of the spectral sequence,

$$\begin{aligned} &H^\bullet(\mathrm{gr}\mathcal{L}ie^\diamond\mathcal{B}_\infty) \\ &\cong \bigoplus_{\Gamma\in\overline{\mathrm{Gr}}(m,n)} \Big(\bigotimes_{v\in v(\Gamma)} \Omega_{\frac{1}{2}}(\mathcal{L}ie\mathcal{B}^{\text{¡}}_{\frac{1}{2}})\big(\mathrm{Out}(v), \mathrm{In}(v)\big) \otimes \odot H^1(\Gamma, \partial\Gamma)\Big)_{\mathrm{Aut}(\Gamma)} \\ &= F\big(\Omega_{\frac{1}{2}}(\mathcal{L}ie\mathcal{B}^{\text{¡}}_{\frac{1}{2}})\big) \cong Q, \end{aligned}$$

thus showing that p is a quasi-isomorphism. Hence so is the map ν from (16), and hence Theorem 2.7.1 is shown. ☐

2.8. Auxiliary complexes

Let $\mathcal{A}_n$ be the quadratic algebra generated by $x_1, \ldots, x_n$ with relations $x_i x_{i+1} = x_{i+1} x_i$ for $i = 1, \ldots, n-1$. We denote by $D_n = \mathcal{A}_n^{\text{¡}}$ the Koszul dual coalgebra. Note that $\mathcal{A}_n$ and D_n are weight-graded and that the weight k component of D_n, $D_n^{(k)}$ is zero if $k \geq 3$, while $D_n^{(1)} = \mathrm{span}\{x_1, \ldots, x_n\}$ and $D_n^{(2)} = \mathrm{span}\{u_{1,2} = x_1x_2 - x_2x_1, u_{2,3} = x_2x_3 - x_3x_2, \ldots, u_{n-1,n} = x_{n-1}x_n - x_nx_{n-1}\}$.

2.8.1 PROPOSITION

The algebra $\mathcal{A}_n$ is Koszul. In particular, the canonical projection map

$$A_n := \Omega(D_n) \to \mathcal{A}_n$$

from the cobar construction of D_n is a quasi-isomorphism.

The proof of this proposition is given in Appendix A.

Proposition 2.8.1 in particular implies that the homology of the A_n vanishes in positive degree. The complex A_n is naturally multigraded by the amount of times each index j appears on each word, and the differential respects this multigrading. We are particularly interested in the subcomplex $A_n^{1,1,\ldots,1}$ of A_n that is spanned by words in x_j and $u_{i,i+1}$ such that each index occurs exactly once. Since $A_n^{1,1,\ldots,1}$ is a direct summand of A_n, its homology also vanishes in positive degree.

Let us define a Lie algebra $\mathcal{L}_n = \mathcal{L}ie(x_1, \ldots, x_n)/[x_i, x_{i+1}]$ and a complex $L_n = \mathcal{L}ie(x_1, \ldots, x_n, u_{1,2}, \ldots, u_{n-1,n})$, with $dx_i = 0$ and $d(u_{i,i+1}) = [x_i, x_{i+1}]$. Here $\mathcal{L}ie$ stands for the free Lie algebra functor.

2.8.2 LEMMA

The projection map $L_n \twoheadrightarrow \mathcal{L}_n$ is a quasi-isomorphism.

Proof
It is clear that $H^0(L_n) = \mathcal{L}_n$; therefore, it is enough to see that the homology of L_n vanishes in positive degree. The Poincaré–Birkhoff–Witt theorem[†] gives us an isomorphism

$$\begin{aligned}&\odot\big(\mathcal{L}ie(x_1,\dots,x_n,u_{1,2},\dots,u_{n-1,n})\big)\\&\quad= \odot(L_n) \xrightarrow{\sim} \mathcal{A}ss(x_1,\dots,x_n,u_{1,2},\dots,u_{n-1,n}) = A_n.\end{aligned}$$

This map commutes with the differentials, and therefore we have an isomorphism in homology $H_\bullet(\odot(L_n)) = H_\bullet(A_n)$. Since $\odot$ is an exact functor it commutes with taking homology, and since the homology of A_n vanishes in positive degree by Proposition 2.8.1, the result follows. □

Let us define $A_{n_1,\dots,n_r}$ as the coproduct of $A_{n_1},\dots,A_{n_r}$ in the category of associative algebras; $A_{n_1,\dots,n_k}$ consists of words in $x_1^1, x_2^1, \dots, x_{n_1}^1, x_1^2, \dots, x_{n_2}^2, \dots, x_1^r, \dots, x_{n_r}^r, u_{1,2}^1, \dots, u_{n_1-1,n_1}^1, u_{1,2}^2, \dots, u_{n_r-1,n_r}^r$. We define similarly $L_{n_1,\dots,n_r}$ and $\mathcal{L}_{n_1,\dots,n_r}$.

2.8.3 LEMMA
The homology of $A_{n_1,\dots,n_r}$ vanishes in positive degree.

Proof
Let $\bar{A}_n \subset A_n$ be the kernel of the natural augmentation such that $A_n \cong \mathbb{K} \oplus \bar{A}_n$ as complexes. Similarly, define $\bar{A}_{n_1,\dots,n_r}$. The complex $\bar{A}_{n_1,\dots,n_r}$ splits as

$$\bar{A}_{n_1,\dots,n_r} = \bigoplus_{k\geq 1} \bigoplus_{j_1,\dots,j_k} \bar{A}_{n_{j_1}} \otimes \dots \otimes \bar{A}_{n_{j_r}},$$

where the second sum runs over all strings $(j_1,\dots,j_k) \in [r]^{\times k}$ such that no adjacent indices j_i, j_{i+1} are equal. Since no $\bar{A}_{n_j}$ has homology in positive degrees by Proposition 2.8.1, neither has $A_{n_1,\dots,n_r}$. □

2.8.4 LEMMA
The map $L_{n_1,\dots,n_r} \twoheadrightarrow \mathcal{L}_{n_1,\dots,n_r}$ is a quasi-isomorphism.

[†]Let us describe in more detail how the Poincaré–Birkhoff–Witt theorem applies here. In general, this theorem states that for any graded Lie algebra $\mathfrak{g}$ there is an isomorphism (of coalgebras) $\odot(\mathfrak{g}) \to \mathcal{U}\mathfrak{g}$ between the symmetric coalgebra in $\mathfrak{g}$ and the universal enveloping algebra of $\mathfrak{g}$. In particular, if $\mathfrak{g}$ is a free Lie algebra, then the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ is the free associative algebra in the same generators. This is precisely our situation.

Proof
The same argument from Lemma 2.8.2 holds. ☐

We define the subcomplexes $L^{1,\dots,1}_{n_1,\dots,n_r} \subset L_{n_1,\dots,n_r}$ and $\mathcal{L}^{1,\dots,1}_{n_1,\dots,n_r} \subset \mathcal{L}_{n_1,\dots,n_r}$ spanned by Lie words in which each index occurs exactly once.

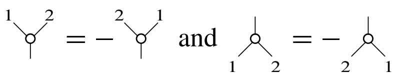
2.8.5 COROLLARY
The map $L^{1,\dots,1}_{n_1,\dots,n_r} \twoheadrightarrow \mathcal{L}^{1,\dots,1}_{n_1,\dots,n_r}$ is a quasi-isomorphism.

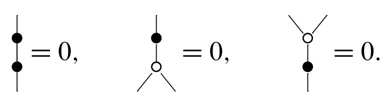
2.9 MAIN THEOREM
The properad $\mathcal{L}ie^{\diamond}\mathcal{B}$ is Koszul; that is, the natural surjection (8) is a quasi-isomorphism.

Proof
The surjection (8) factors through the surjection (11):

$$\pi : \mathcal{L}ie^{\diamond}\mathcal{B}_{\infty} \xrightarrow{\nu} \mathcal{P} \xrightarrow{\rho} \mathcal{L}ie^{\diamond}\mathcal{B}.$$

In view of Theorem 2.7.1, the main theorem is proved once it is shown that the morphism ρ is a quasi-isomorphism. The latter statement is, in turn, proved once it is shown that the cohomology of the nonnegatively graded dg properad $\mathcal{P}$ is concentrated in degree 0. For notational reasons it is suitable to work with the dg prop, $\mathcal{P}\mathcal{P}$, generated by the properad $\mathcal{P}$. We also denote by $\mathcal{L}ieP$ the prop governing Lie algebras and by $\mathcal{L}ieCP$ the prop governing Lie coalgebras.

It is easy to see that the dg prop $\mathcal{P}\mathcal{P} = \{\mathcal{P}\mathcal{P}(m,n)\}$ is isomorphic, as a graded $\mathbb{S}$-bimodule, to the graded prop generated by a degree 1 corolla , degree 0 corollas
, modulo the first three relations of (3) and the following ones:

 (18)

The latter prop can in turn be identified with the following collection of graded vector spaces:

$$W(n,m) := \bigoplus_{N} \big(\mathcal{L}ieP(n,N) \otimes V^{\otimes N} \otimes \mathcal{L}ieCP(N,m)\big)_{\mathbb{S}_N}, \tag{19}$$

where V is a two-dimensional vector space $V_0 \oplus V_1$, where $V_0 = \mathrm{span}\langle$ $\rangle$ and $V_1 = \mathrm{span}\langle$  $\rangle$. The isomorphism $W(n,m) \to \mathcal{P}\mathcal{P}(n,m)$ is realized in the more or less obvious way, by mapping $\mathcal{L}ieP(n,N) \to \mathcal{P}\mathcal{P}(n,N)$, $\mathcal{L}ieCP(N,m) \to \mathcal{P}\mathcal{P}(N,m)$ and composing "in the middle" with either the identity or $\overset{|}{\underset{|}{\bullet}}$. The differential on $W(n,m)$ we define to be that induced by the differential on $\mathcal{PP}(n,m)$, given by the formula (10).

Let us consider a slightly different complex

$$\begin{aligned} V_{n,m} &= \bigoplus_N \big(\mathcal{L}ieP(n,N) \otimes V^{\otimes N} \otimes \mathcal{A}ssCP(N,m)\big)_{\mathbb{S}_N} \\ &\cong \bigoplus_N \bigoplus_{N=n_1+\cdots+n_m} \big(\mathcal{L}ieP(n,N) \otimes V^{\otimes N} \otimes \mathcal{A}ssC(n_1) \\ &\qquad \otimes \cdots \otimes \mathcal{A}ssC(n_m)\big)_{\mathbb{S}_{n_1}\times\cdots\times\mathbb{S}_{n_m}}, \end{aligned} \tag{20}$$

where $\mathcal{A}ssCP$ is the prop governing coassociative coalgebras, and $\mathcal{A}ssC(n_j) = \mathcal{A}ssCP(n_j,1) \cong \mathbb{K}[\mathbb{S}_{n_j}]$.

The operad $\mathcal{A}ss = \mathcal{C}om \circ \mathcal{L}ie = \bigoplus_k (\mathcal{C}om \circ \mathcal{L}ie)^{(k)}$ is, as an $\mathbb{S}$-module, naturally graded with respect to the arity in $\mathcal{C}om$. This decomposition induces a multigrading in $V_{n,m} = \bigoplus_{(k_1,\ldots,k_m)} V_{n;k_1,\ldots,k_m}$. It is clear that this decomposition is actually a splitting of complexes and that in fact the direct summand $V_{n;1,\ldots,1}$ is just $W(n,m)$. Therefore, to show the theorem it suffices to show that the cohomology of $V_{n,m}$ is zero in positive degree.

There is a natural identification $\mathcal{L}ieP(n,N) \cong ((\mathcal{L}ie(y_1,\ldots,y_N))^{\otimes n})^{1,\ldots,1}$ where, as before, we use the notation $1,\ldots,1$ to represent the subspace spanned by tensor products of words such that each index appears exactly once.

The space $(\mathcal{L}ieP(n,N) \otimes \mathcal{A}ssC(n_1) \otimes \cdots \otimes \mathcal{A}ssC(n_m))_{S_{n_1}\times\cdots\times S_{n_k}}$ is isomorphic to $\mathcal{L}ieP(n,N)$, but there is a more natural identification than the one above; namely, the y_j can be gathered by blocks of size n_j, according to the action of $S_1 \times \cdots \times S_{n_m}$ on $\mathcal{L}ieP(n,N)$, and can be relabeled accordingly:

$$y_1, y_2, \ldots, y_N \rightsquigarrow y_1^1, \ldots, y_{n_1}^1, y_1^2, \ldots, y_{n_2}^2, \ldots, y_1^m, \ldots, y_{n_m}^m.$$

Since $V = V_0 \oplus V_1$, there is a natural decomposition of

$$V^{\otimes N} = \bigoplus_\epsilon V_{\epsilon_1^1} \otimes \cdots \otimes V_{\epsilon_{n_1}^1} \otimes V_{\epsilon_1^2} \otimes \cdots \otimes V_{\epsilon_{n_2}^2} \otimes \cdots \otimes V_{\epsilon_1^m} \otimes \cdots \otimes V_{\epsilon_{n_m}^m},$$

where $\epsilon = (\epsilon_1^1, \ldots, \epsilon_{n_m}^m)$ runs through all strings of 0's and 1's of length N. Then,

$$\begin{aligned} &\big(\mathcal{L}ieP(n,N) \otimes V^{\otimes N} \otimes \mathcal{A}ssC(n_1) \otimes \cdots \otimes \mathcal{A}ssC(n_m)\big)_{\mathbb{S}_{n_1}\times\cdots\times\mathbb{S}_{n_m}} \\ &\quad = \bigoplus_\epsilon \big(\mathcal{L}ieP(n,N) \otimes V_{\epsilon_1^1} \otimes \cdots \otimes V_{\epsilon_{n_m}^m} \\ &\qquad \otimes \mathcal{A}ssC(n_1) \otimes \cdots \otimes \mathcal{A}ssC(n_m)\big)_{\mathbb{S}_{n_1}\times\cdots\times\mathbb{S}_{n_m}} \\ &\quad = \bigoplus_\epsilon \big(\mathcal{L}ieP(n,N) \otimes V_{\epsilon_1^1} \otimes \cdots \otimes V_{\epsilon_{n_m}^m}\big). \end{aligned}$$

Given a fixed a string ϵ, we look at the corresponding summand in the above direct sum individually. Our goal is to realize each summand as a subspace of a product of free Lie algebras

$$\Big(\big(\mathcal{L}ie(x^1_1,\dots,x^1_{\tilde n_1},\dots,x^m_1,\dots,x^m_{\tilde n_m},u^1_{1,2},\dots,u^1_{\tilde n_1-1,\tilde n_1},\\ u^2_{1,2},\dots,u^m_{\tilde n_m-1,\tilde n_m})\big)^{\otimes n}\Big)^{1,\dots,1},$$

where $\tilde n_i = n_i + \sum_{j=1}^{n_i}\epsilon^i_j$ and where the superscript will indicate that each index occurs exactly once. We assume that bases of the one-dimensional spaces V_0, V_1 have been fixed. Then, using the bases we may identify

$$\mathcal{L}ieP(n,N)\otimes V_{\epsilon^1_1}\otimes\cdots\otimes V_{\epsilon^m_{n_m}} = \mathcal{L}ieP(n,N).$$

Now, an element $X\in\mathcal{L}ieP(n,N)$ describes a way of taking an n-fold product of Lie words in N generators (or linear combinations thereof), say,

$$X(\underbrace{-,\dots,-}_{n\text{ "slots"}}).$$

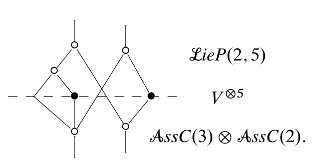
Our map

$$\begin{aligned}\mathcal{L}ieP(n,N) &\cong \mathcal{L}ieP(n,N)\otimes V_{\epsilon^1_1}\otimes\cdots\otimes V_{\epsilon^m_{n_m}}\\ &\to \Big(\big(\mathcal{L}ie(x^1_1,\dots,x^1_{\tilde n_1},\dots,x^m_1,\dots,x^m_{\tilde n_m},u^1_{1,2},\dots,u^1_{\tilde n_1-1,\tilde n_1},\\ &\qquad u^2_{1,2},\dots,u^m_{\tilde n_m-1,\tilde n_m})\big)^{\otimes n}\Big)^{1,\dots,1} \qquad (21)\end{aligned}$$

is then realized by sending $X\in\mathcal{L}ieP(n,N)$ to $X(y^1_1,\dots,y^1_{n_1},\dots,y^m_1,\dots,y^m_{n_m})$, where

$$y^i_j = \begin{cases} x^i_{\tilde j} & \text{if } \epsilon^i_j = 0,\text{ with } \tilde j = j+\sum_{k=1}^{j-1}\epsilon^i_k;\\ u^i_{\tilde j,\tilde j+1} & \text{if } \epsilon^i_j = 1,\text{ with same } \tilde j.\end{cases}$$

For example, consider the following element of $(\mathcal{L}ieP(2,5)\otimes V^{\otimes 5}\otimes \mathcal{A}ssC(3)\otimes \mathcal{A}ssC(2))_{\mathbb{S}_3\times\mathbb{S}_2}$:



In the picture, we understand that the two corollas in the lower half correspond to a triple and a double (co)product in $\mathcal{A}ssC(3)$ and $\mathcal{A}ssC(2)$, (co)multiplying factors from left to right. Then the element in the picture is mapped to the expression $[[x^1_1,u^1_{2,3}],x^2_1]\otimes[x^1_4,u^2_{2,3}]$.

With the map (21), the total space

$$V_{n,m} = \bigoplus_N \bigoplus_\epsilon \bigoplus_{N=n_1+\cdots+n_m} \big(\mathcal{L}ieP(n,N) \otimes V_{\epsilon_1} \otimes \cdots \otimes V_{\epsilon_N} \otimes \mathcal{A}ssC(n_1) \otimes \cdots \otimes \mathcal{A}ssC(n_m)\big)_{\mathbb{S}_{n_1}\times\cdots\times\mathbb{S}_{n_k}}$$

can be seen as a sum of spaces of the form

$$\Big(\big(\mathcal{L}ie(x_1^1,\ldots,x_{\tilde{n}_1}^1,\ldots,x_1^m,\ldots,x_{\tilde{n}_m}^m, u_{1,2}^1,\ldots,u_{\tilde{n}_1-1,\tilde{n}_1}^1,u_{1,2}^2,\ldots,u_{\tilde{n}_m-1,\tilde{n}_m}^m)\big)^{\otimes n}\Big)^{1,\ldots,1}.$$

Under this identification, the differential sends the elements $u^i_{j,j+1}$ to $[x^i_j, x^i_{j+1}]$ and it is zero on the elements x^i_j. Then the differential preserves the $\tilde{n}_i$, and therefore it preserves this direct sum.

We conclude that the complex $V_{n,m}$ splits as a sum of tensor products of complexes of the form $L^{1,\ldots,1}_{p_1,\ldots,p_k}$, so from Corollary 2.8.5 we obtain that its cohomology is concentrated in degree 0. The proof of the main theorem is completed. □

2.10 Remark

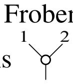
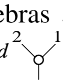
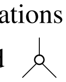
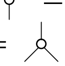
In applications of the theory of involutive Lie bialgebras to string topology, contact topology, and quantum $\mathcal{A}ss_\infty$ algebras, one is often interested in a version of the properad $\mathcal{L}ie^\diamond\mathcal{B}$ in which degrees of Lie and co-Lie operations differ by an even number,

$$\big|[\,,\,]\big| - |\,\triangle\,| = 2d, \quad d \in \mathbb{N}.$$

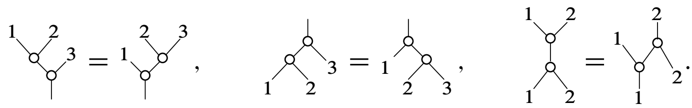
The arguments proving Koszulness of $\mathcal{L}ie^\diamond\mathcal{B}$ work also for such degree-shifted versions of $\mathcal{L}ie^\diamond\mathcal{B}$. The same remark applies to the Koszul dual properads below.

One may also consider versions of the properad $\mathcal{L}ie\mathcal{B}$, where the Lie bracket and cobracket have degrees differing by an odd number and have opposite symmetry. However, in this case the involutivity is trivially satisfied (by symmetry) and does not pose an additional relation. The Koszulness of the corresponding properad is hence much simpler to show, analogously to the Koszulness of $\mathcal{L}ie\mathcal{B}$.

2.11. Properads of Frobenius algebras

The properad of nonunital Frobenius algebras $\mathcal{F}rob_d$ in dimension d is the properad generated by operations  $= (-1)^d$  (graded cocommutative comultiplication) of degree d and  $=$  (graded commutative multiplication) of degree

0, modulo the ideal generated by the following relations:



(22)

For the purposes of this article, we will define the properad of nonunital Frobenius algebras to be

$$\mathcal{F}rob := \mathcal{F}rob_2.$$

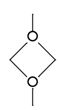
For example, the cohomology $H(\Sigma)$ of any closed Riemann surface Σ is a Frobenius algebra in this sense. Comparing with Section 2.4, we see that the properad $\mathcal{F}rob$ is isomorphic to the Koszul dual properad of $\mathcal{L}ie^{\diamond}\mathcal{B}$, up to a degree shift:

$$\mathcal{L}ie^{\diamond}\mathcal{B}^{¡} \cong \mathcal{F}rob^{*}\{1\}.$$

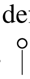
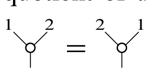
By Koszul duality theory of properads (see [37]), one hence obtains from Theorem 2.9 the following result.

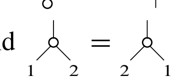
2.11.1 COROLLARY
The properad of nonunital (symmetric) Frobenius algebras $\mathcal{F}rob$ is Koszul.

By adding the additional relation

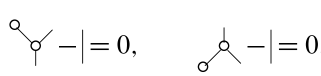

$= 0$

(which is automatic for d odd) to the presentation of $\mathcal{F}rob_d$, we obtain the properad(s) of involutive Frobenius algebras $\mathcal{F}rob_d^{\diamond}$. They are Koszul dual to the operads governing degree-shifted Lie bialgebras (see Remark 2.10), and in particular $\mathcal{L}ie\mathcal{B}^{¡} \cong (\mathcal{F}rob_2^{\diamond})^{*}\{1\}$. It then follows from the Koszulness of $\mathcal{L}ie\mathcal{B}$ (and its degree-shifted relatives) that the properads $\mathcal{F}rob_d^{\diamond}$ are Koszul, as noted in [19], [37], and [20].

The properad $uc\mathcal{F}rob$ of unital-counital Frobenius algebras is, by definition, a quotient of the free properad generated by degree 0 corollas  (unit),  (counit), 

(graded cocommutative comultiplication), and 

(graded commutative multiplication) modulo the ideal generated by the relations (22) and the additional relations

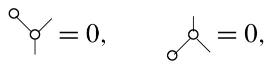


(23)

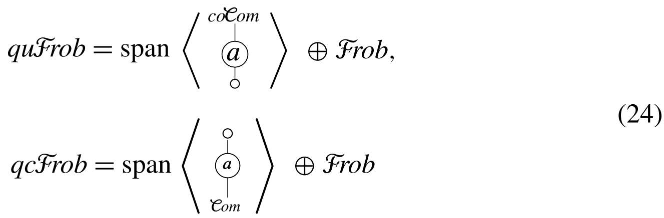
where the vertical line $|$ stands for the unit in the properad $\mathcal{F}rob$. Similarly, one defines a properad $u\mathcal{F}rob$ of unital Frobenius algebras, and a properad $c\mathcal{F}rob$ of counital algebras. Clearly, $u\mathcal{F}rob$ and $c\mathcal{F}rob$ are subproperads of $uc\mathcal{F}rob$.

2.11.2 THEOREM
The properads $u\mathcal{F}rob$, $c\mathcal{F}rob$, and $uc\mathcal{F}rob$ are Koszul.

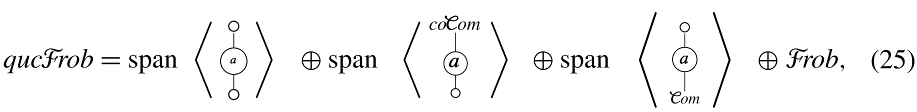
Proof
By curved Koszul duality theory (see [18]), it is enough[†] to prove Koszulness of the associated quadratic properads, $qu\mathcal{F}rob$, $qc\mathcal{F}rob$, and $quc\mathcal{F}rob$, obtained from $u\mathcal{F}rob$, $c\mathcal{F}rob$, and $uc\mathcal{F}rob$, respectively, by replacing inhomogeneous relations (23) by the following ones (see [18]),



so that we have decompositions into direct sums of $\mathbb{S}$-bimodules

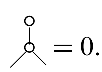


and



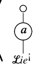
where a stands for the graph given in (5).

Consider, for example, the properad $qc\mathcal{F}rob$ (proofs of Koszulness of properads $qu\mathcal{F}rob$ and $quc\mathcal{F}rob$ can be given by a similar argument). Its Koszul dual properad $qc\mathcal{F}rob^! =: qc\mathcal{L}ie^{\diamond}\mathcal{B}$ is generated by the properads $\mathcal{L}ie^{\diamond}\mathcal{B}$ and $\langle\ \rangle$ modulo the following relation:

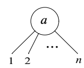
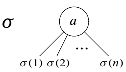


[†]More precisely, one also has to check the technical Conditions (I) and (II) of [18, Section 4.1]. Condition (I) is obvious in our case. For Condition (II), let us temporarily use the notation of [18, Section 4.1]. Condition (II) then states that $(R)^{\leq 2} := (R) \cap \mathcal{F}(V)^{\leq 2} = R$, with $\mathcal{F}(V)^{\leq 2} := \{I \oplus V \oplus \mathcal{F}(V)^{(2)}\}$. This is equivalent to saying that the image of $\mathcal{F}(V)^{\leq 2}$ in $\mathcal{P}$ (i.e., $\mathcal{F}(V)^{\leq 2}/(R)^{\leq 2}$) is isomorphic to $\mathcal{F}(V)^{\leq 2}/R$. But we have an explicit basis of $\mathcal{P}$ in our case given by elements (4) and analogous elements with zero inputs or outputs as in (24) and (25). The subset of basis elements with at most two generators then yields a basis of $\mathcal{F}(V)^{\leq 2}/(R)^{\leq 2}$. Hence we only need to check that the same elements also form a basis of $\mathcal{F}(V)^{\leq 2}/R$, a fact which is again essentially obvious.

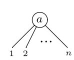
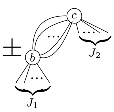
We have

$$qc\mathcal{L}ie^{\diamond}\mathcal{B}^{¡} = (qc\mathcal{F}rob)^{*}\{1\} \cong \mathrm{span}\left\langle \right.$$  $$\left.\right\rangle \oplus \mathcal{L}ie^{\diamond}\mathcal{B}^{¡},$$

where $(\cdots)^{*}$ denotes the genus-graded dual. It will suffice to show that the properad $qc\mathcal{L}ie^{\diamond}\mathcal{B}$ is Koszul. To this end, consider the dg properad $\Omega(qc\mathcal{L}ie^{\diamond}\mathcal{B}^{¡})$ which is a free properad generated by corollas (6) and the following ones,

 $$= (-1)^{\sigma}$$  , (26)

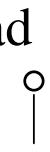
where $a \geq 0$, $n \geq 1$, and $\sigma \in \mathbb{S}_n$ is an arbitrary permutation. The differential is given on corollas (6) by the standard formula (7) and on $(0, n)$-generators by

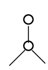
$$d$$  $$= \sum_{a=b+c+l-1} \sum_{\substack{[n]=J_1 \sqcup J_2 \\ \#J_1 \geq 1}} \pm$$  ,

where l counts the number of internal edges connecting the two vertices on the right-hand side. There is a natural morphism of properads

$$\Omega(qc\mathcal{L}ie^{\diamond}\mathcal{B}^{¡}) \longrightarrow qc\mathcal{L}ie^{\diamond}\mathcal{B}$$

which is a quasi-isomorphism if and only if $qc\mathcal{L}ie^{\diamond}\mathcal{B}$ is Koszul. Thus to prove Koszulness of $qc\mathcal{L}ie^{\diamond}\mathcal{B}$, it is enough to establish an isomorphism $H^{\bullet}(\Omega(qc\mathcal{L}ie^{\diamond}\mathcal{B}^{¡})) \cong qc\mathcal{L}ie^{\diamond}\mathcal{B}$ of $\mathbb{S}$-bimodules.

To do this, one may closely follow the proof of Theorem 2.9, adjusting it slightly so as to allow for the additional $(0, n)$ary generators. First, we define a properad $\tilde{\mathcal{P}}$ which is generated by the properad $\mathcal{P}$ of Section 2.7, together with an additional generator of arity $(0, 1)$, in pictures , with the additional relations

 $= 0,$  $= 0.$

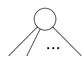
The map $\Omega(qc\mathcal{L}ie^{\diamond}\mathcal{B}^{¡}) \to qc\mathcal{L}ie^{\diamond}\mathcal{B}$ clearly factors through $\tilde{\mathcal{P}}$,

$$\Omega(qc\mathcal{L}ie^{\diamond}\mathcal{B}^{¡}) \to \tilde{\mathcal{P}} \to qc\mathcal{L}ie^{\diamond}\mathcal{B}, \tag{27}$$

and it suffices to show that both of the above maps are quasi-isomorphisms. Consider first the left-hand map. The fact that this map is a quasi-isomorphism may be proved by copying the proof of Theorem 2.7.1, except that now the functor F (as in Section 2.7) is applied not to the cobar construction $\Omega_{\frac{1}{2}}(\mathcal{L}ie\mathcal{B}^{¡}_{\frac{1}{2}})$, but to $\Omega_{\frac{1}{2}}(qc\mathcal{L}ie\mathcal{B}^{¡}_{\frac{1}{2}})$. Here

$$qc\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}(m,n) = \begin{cases} \mathcal{L}ie\mathcal{B}_{\frac{1}{2}}(m,n) & \text{if } m \neq 0, \\ \mathbb{K} & \text{if } m = 0, \end{cases}$$

is the $\frac{1}{2}$-prop governing Lie bialgebras with a counit operation killed by the cobracket. More specifically, $qc\mathcal{L}ie\mathcal{B}^{\text{¡}}_{\frac{1}{2}}(m,n)$ is the same as $\mathcal{L}ie\mathcal{B}^{\text{¡}}_{\frac{1}{2}}(m,n)$ in all arities (m,n) with $m,n > 0$, but $qc\mathcal{L}ie\mathcal{B}^{\text{¡}}_{\frac{1}{2}}(0,n)$ is one-dimensional, the extra operations corresponding to corollas

 .

One can check† that the $\frac{1}{2}$-prop $qc\mathcal{L}ie\mathcal{B}^{\text{¡}}_{\frac{1}{2}}$ is Koszul, that is, that

$$H\big(\Omega_{\frac{1}{2}}(qc\mathcal{L}ie\mathcal{B}^{\text{¡}}_{\frac{1}{2}})\big) \cong qc\mathcal{L}ie\mathcal{B}_{\frac{1}{2}}.$$

The properad $\tilde{\mathcal{P}}$ is obtained by applying the exact functor F to this $\frac{1}{2}$-prop, and hence, by essentially the same arguments as in the proof of Theorem 11, the left-hand map of (27) is a quasi-isomorphism.

Next consider the right-hand map of (27). It can be shown to be a quasi-isomorphism along the lines of the proof of Theorem 2.9. Again, it is clear that the degree 0 cohomology of $\tilde{\mathcal{P}}$ is $qc\mathcal{L}ie^{\diamond}\mathcal{B}$, so it will suffice to show that $H^{>0}(\tilde{\mathcal{P}}) = 0$. First, let $\widetilde{\mathcal{PP}}$ be the prop generated by the properad $\tilde{\mathcal{P}}$. As a dg $\mathbb{S}$-bimodule, it is isomorphic to (see (19))

$$\tilde{W}(n,m) := \bigoplus_{N,M} \big(\mathcal{L}ieP(n,N) \otimes V^{\otimes N} \otimes \mathbb{K}^{\otimes M} \otimes \mathcal{L}ieCP(N+M,m)\big)_{\mathbb{S}_N \times \mathbb{S}_M},$$

where V is as in (19). The above complex $\tilde{W}(n,m)$ is a direct summand of the complex (see (20))

$$\tilde{V}_{n,m} := \bigoplus_{N,M} \big(\mathcal{L}ieP(n,N) \otimes V^{\otimes N} \otimes \mathbb{K}^{\otimes M} \otimes \mathcal{A}ssCP(N+M,m)\big)_{\mathbb{S}_N \times \mathbb{S}_M}$$

by arguments similar to those following (20). Then again, by the Koszulness results of Section 2.8, it follows that the above complex has no cohomology in positive degrees, and hence $\widetilde{\mathcal{PP}}$ also cannot have cohomology in positive degrees. Hence we can conclude that the properad $qc\mathcal{L}ie^{\diamond}\mathcal{B}$ is Koszul. □

†The piece of the $\frac{1}{2}$-prop $\Omega_{\frac{1}{2}}(qc\mathcal{L}ie\mathcal{B}^{\text{¡}}_{\frac{1}{2}})$ involving the additional generators is isomorphic to the complex $(E_1^{\mathsf{Lie}+}, d_1^{\mathsf{Lie}+})$ from [31, p. 344]. According to [31], its cohomology is one-dimensional.

3. Deformation complexes

As one application of the Koszulness of $\mathcal{L}ie^\diamond\mathcal{B}$ and $\mathcal{F}rob$, we obtain minimal models $\mathcal{L}ie^\diamond\mathcal{B}_\infty = \Omega(\mathcal{L}ie^\diamond\mathcal{B}^{¡}) = \Omega(\mathcal{F}rob^*\{1\})$ and $\mathcal{F}rob_\infty = \Omega((\mathcal{L}ie^\diamond\mathcal{B})^*\{1\})$ of these properads, and hence minimal models for their deformation complexes and for the deformation complexes of their algebras.

3.1. A deformation complex of an involutive Lie bialgebra

According to the general theory (see [33]), $\mathcal{L}ie^\diamond\mathcal{B}_\infty$-algebra structures on a dg vector space $(\mathfrak{g}, d)$ can be identified with Maurer–Cartan elements

$$\mathcal{MC}\big(\mathsf{InvLieB}(\mathfrak{g})\big) := \big\{\Gamma \in \mathsf{InvLieB}(\mathfrak{g}) : |\Gamma| = 3 \text{ and } [\Gamma,\Gamma]_{\mathrm{CE}} = 0\big\}$$

of a graded Lie algebra,[†]

$$\mathsf{InvLieB}(\mathfrak{g}) := \mathsf{Def}(\mathcal{L}ie^\diamond\mathcal{B}_\infty \xrightarrow{0} \mathcal{E}nd_{\mathfrak{g}})[-2], \tag{28}$$

which controls deformations of the zero morphism from $\mathcal{L}ie^\diamond\mathcal{B}_\infty$ to the endomorphism properad $\mathcal{E}nd_{\mathfrak{g}} = \{\mathrm{Hom}(\mathfrak{g}^{\otimes n}, \mathfrak{g}^{\otimes m})\}$. As a $\mathbb{Z}$-graded vector space $\mathsf{InvLieB}(\mathfrak{g})$ can be identified with the vector space of homomorphisms of $\mathbb{S}$-bimodules

$$\begin{aligned}
&\mathsf{InvLieB}(\mathfrak{g})\\
&\quad= \mathrm{Hom}_{\mathbb{S}}\big((\mathcal{L}ie^\diamond\mathcal{B})^{¡}, \mathcal{E}nd_{\mathfrak{g}}\big)[-2]\\
&\quad= \prod_{\substack{a\geq 0, m,n\geq 1\\ m+n+a\geq 3}} \mathrm{Hom}_{\mathbb{S}_m\times\mathbb{S}_n}\big(\mathrm{sgn}_n \otimes \mathrm{sgn}_m[m+n+2a-2], \mathrm{Hom}(\mathfrak{g}^{\otimes n}, \mathfrak{g}^{\otimes m})\big)[-2]\\
&\quad= \prod_{\substack{a\geq 0, m,n\geq 1\\ m+n+a\geq 3}} \mathrm{Hom}\big(\odot^n(\mathfrak{g}[-1]), \odot^m(\mathfrak{g}[-1])\big)[-2a]\\
&\quad\subset \widehat{\odot^\bullet}\big(\mathfrak{g}[-1] \oplus \mathfrak{g}^*[-1] \oplus \mathbb{K}[-2]\big) \simeq \mathbb{K}\big[[\eta^i, \psi_i, \hbar]\big],
\end{aligned}$$

where $\hbar$ is a formal parameter of degree 2 (a basis vector of the summand $\mathbb{K}[-2]$ above), and, for a basis $(e_1, e_2, \ldots, e_i, \ldots)$ in $\mathfrak{g}$ and the associated dual basis $(e^1, e^2, \ldots, e^i, \ldots)$ in $\mathfrak{g}^*$ we set $\eta^i := se^i$, $\psi_i := se_i$, where $s : V \to V[-1]$ is the suspension map. Therefore, the Lie algebra $\mathsf{InvLieB}(\mathfrak{g})$ has a canonical structure of a module over the algebra $\mathbb{K}[[\hbar]]$; moreover, for finite-dimensional $\mathfrak{g}$, its elements can be identified with formal power series,[††] f, in variables ψ_i, η^i, and $\hbar$, which satisfy the "boundary" conditions

[†]More precisely, $\mathsf{InvLieB}(\mathfrak{g})$ is a $\mathsf{Lie}\{2\}$-algebra, not a Lie algebra; that is, the Lie bracket has degree -2. We will abuse notation and still call $\mathsf{InvLieB}(\mathfrak{g})$ a Lie algebra.

[††]In fact, this is true for a class of infinite-dimensional vector spaces. Consider a category of graded vector spaces which are inverse limits of finite-dimensional ones (with the corresponding topology and with the completed tensor product), and also a category of graded vector spaces which are direct limits of finite-dimensional

$$f(\psi,\eta,\hbar)|_{\psi_i=0}=0, \qquad f(\psi,\eta,\hbar)|_{\eta^i=0}=0, \qquad f(\psi,\eta,\hbar)|_{\hbar=0}\in I^3, \tag{29}$$

where I is the maximal ideal in $\mathbb{K}[[\psi_i,\eta^i]]$. The Lie brackets in $\mathsf{InvLieB}(\mathfrak{g})$ can be read off either from the coproperad structure in $(\mathcal{L}ie^{\diamond}\mathcal{B})^{¡}$ or directly from the formula (7) for the differential, and are given explicitly by (see [11])

$$[f,g]_\hbar := f *_\hbar g - (-1)^{|f||g|} g *_\hbar f, \tag{30}$$

where (up to Koszul signs)

$$f *_\hbar g := \sum_{k=0}^{\infty} \frac{\hbar^{k-1}}{k!} \sum_{i_1,\dots,i_k} \pm \frac{\partial^k f}{\partial\eta^{i_1}\cdots\eta^{i_k}} \frac{\partial^k g}{\partial\psi_{i_1}\cdots\partial\psi_{i_k}}$$

is an associative product. Note that the differential $d_\mathfrak{g}$ in $\mathfrak{g}$ gives rise to a quadratic element $D_\mathfrak{g}=\sum_{i,j}\pm d^i_j\psi_i\eta^j$ of homological degree 3 in $\mathbb{K}[[\eta^i,\psi_i,\hbar]]$, where d^i_j are the structure constants of $d_\mathfrak{g}$ in the chosen basis, $d_\mathfrak{g}(e_i)=:\sum_j d^j_i e_j$.

Finally, we can identify $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$ structures in a finite-dimensional dg vector space $(\mathfrak{g},d_\mathfrak{g})$ with a homogeneous formal power series

$$\Gamma := D_\mathfrak{g} + f \in \mathbb{K}[[\eta^i,\psi_i,\hbar]]$$

of homological degree 3 such that

$$\Gamma *_\hbar \Gamma = \sum_{k=0}^{\infty} \frac{\hbar^{k-1}}{k!} \sum_{i_1,\dots,i_k} \pm \frac{\partial^k \Gamma}{\partial\eta^{i_1}\cdots\eta^{i_k}} \frac{\partial^k \Gamma}{\partial\psi_{i_1}\cdots\partial\psi_{i_k}} = 0, \tag{31}$$

and the summand f satisfies boundary conditions (29).

For example, let

$$\big(\triangle : V\to\wedge^2 V, [\,,\,]:\wedge^2 V\to V\big)$$

be a Lie bialgebra structure in a vector space V which we assume for simplicity to be concentrated in degree 0. Let C^k_{ij} and Φ^{ij}_k be the associated structure constants

$$[x_i,x_j] =: \sum_{k\in I} C^k_{ij} x_k,$$
$$\triangle(x_k) =: \sum_{i,j\in I} \Phi^{ij}_k x_i\wedge x_j.$$

Then it is easy to check that all the involutive Lie bialgebra axioms (3) get encoded into a single equation $\Gamma *_\hbar \Gamma = 0$ for $\Gamma := \sum_{i,j,k\in I}(C^k_{ij}\psi_k\eta^i\eta^j + \Phi^{ij}_k\eta^k\psi_i\psi_j)$.

---

ones. If $\mathfrak{g}$ belongs to one of these categories, then $\mathfrak{g}^*$ belongs (almost by definition) to the other, and we have isomorphisms of the type $(\mathfrak{g}\otimes\mathfrak{g})^*=\mathfrak{g}^*\otimes\mathfrak{g}^*$ which are required for the "local coordinate" formulas to work.

Note that all the above formulas taken modulo the ideal generated by the formal variable $\hbar$ give us a Lie algebra

$$\mathsf{LieB}(\mathfrak{g}) := \mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty \xrightarrow{0} \mathcal{E}nd_{\mathfrak{g}})[-2] \cong \mathbb{K}[[\psi_i, \eta^i]] \tag{32}$$

controlling the deformation theory of (not necessarily involutive) Lie bialgebra structures in a dg space $\mathfrak{g}$. Lie brackets in (32) are given in coordinates by the standard Poisson formula

$$\{f, g\} = \sum_{i \in I} (-1)^{|f||\eta^i|} \frac{\partial f}{\partial \psi_i} \frac{\partial g}{\partial \eta^i} - (-1)^{|f||\psi_i|} \frac{\partial f}{\partial \eta^i} \frac{\partial g}{\partial \psi_i} \tag{33}$$

for any $f, g \in \mathbb{K}[[\psi_i, \eta^i]]$. Formal power series, $f \in \mathbb{K}[[\psi_i, \eta^i]]$, which have homological degree 3 and satisfy the equations

$$\{f, f\} = 0, \qquad f(\psi, \eta)|_{\psi_i = 0} = 0, \qquad f(\psi, \eta)|_{\eta^i = 0} = 0,$$

are in one-to-one correspondence with strongly homotopy Lie bialgebra structures in a finite-dimensional dg vector space $\mathfrak{g}$.

3.2. Deformation complexes of properads

The deformation complex of a properad $\mathcal{P}$ is by definition the dg Lie algebra $\mathrm{Der}(\tilde{\mathcal{P}})$ of derivations of a cofibrant resolution $\tilde{\mathcal{P}} \xrightarrow{\sim} \mathcal{P}$. (See the remarks at the end of the Introduction for our slightly nonstandard definition of $\mathrm{Der}(\dots)$, and [35, Section 5.1] for similar considerations in the operadic setting.) It may be identified as a complex with the deformation complex of the identity map $\tilde{\mathcal{P}} \to \tilde{\mathcal{P}}$ (which controls deformations of $\mathcal{P}$-algebras) up to a degree shift:

$$\mathrm{Der}(\tilde{\mathcal{P}}) \cong \mathsf{Def}(\tilde{\mathcal{P}} \to \tilde{\mathcal{P}})[1].$$

Note, however, that both $\mathrm{Der}(\mathcal{P})$ and $\mathsf{Def}(\tilde{\mathcal{P}} \to \tilde{\mathcal{P}})$ have natural dg Lie (or $\mathcal{L}ie_\infty$) algebra structures that are *not* preserved by the above map. Furthermore, there is a quasi-isomorphism of dg Lie algebras

$$\mathsf{Def}(\tilde{\mathcal{P}} \to \tilde{\mathcal{P}}) \to \mathsf{Def}(\tilde{\mathcal{P}} \to \mathcal{P}). \tag{34}$$

The zeroth cohomology $H^0(\mathrm{Der}(\tilde{\mathcal{P}}))$ is of particular importance. It is a differential graded Lie algebra whose elements act on the space of $\tilde{\mathcal{P}}$-algebra structures on any vector space. We will see in the examples we are interested in that this dg Lie algebra is very rich and that it acts nontrivially in general.

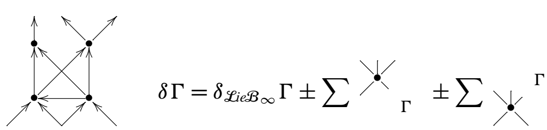
Using the Koszulness of the properads $\mathcal{L}ie\mathcal{B}$ and $\mathcal{F}rob^\diamond$ from [29] and [23] and the Koszulness of $\mathcal{L}ie^\diamond\mathcal{B}$ and $\mathcal{F}rob$ from Theorem 2.9 and Corollary 2.11.1, we can write down the following models for the deformation complexes:

$$\begin{aligned}
\mathrm{Der}(\mathcal{L}ie\mathcal{B}_\infty) &= \prod_{n,m\geq 1} \mathrm{Hom}_{\mathbb{S}_n\times\mathbb{S}_m}\big((\mathcal{F}rob_2^\diamond)^*\{1\}(n,m), \mathcal{L}ie\mathcal{B}_\infty(n,m)\big)[1] \\
&\cong \prod_{n,m\geq 1} \big(\mathcal{L}ie\mathcal{B}_\infty(n,m)\otimes \mathrm{sgn}_n \otimes \mathrm{sgn}_m\big)^{\mathbb{S}_n\times\mathbb{S}_m}[3-n-m], \\
\mathrm{Der}(\mathcal{L}ie^\diamond\mathcal{B}_\infty) &= \prod_{n,m\geq 1} \mathrm{Hom}_{\mathbb{S}_n\times\mathbb{S}_m}\big((\mathcal{F}rob_2)^*\{1\}(n,m), \mathcal{L}ie^\diamond\mathcal{B}_\infty(n,m)\big)[1] \\
&\cong \prod_{n,m\geq 1} \big(\mathcal{L}ie^\diamond\mathcal{B}_\infty(n,m)\otimes \mathrm{sgn}_n \otimes \mathrm{sgn}_m\big)^{\mathbb{S}_n\times\mathbb{S}_m}[3-n-m][[\hbar]], \\
\mathrm{Der}(\mathcal{F}rob_\infty) &= \prod_{n,m\geq 1} \mathrm{Hom}_{\mathbb{S}_n\times\mathbb{S}_m}\big((\mathcal{L}ie^\diamond\mathcal{B}_2)^*\{1\}(n,m), \mathcal{F}rob_\infty(n,m)\big)[1], \\
\mathrm{Der}(\mathcal{F}rob_\infty^\diamond) &= \prod_{n,m\geq 1} \mathrm{Hom}_{\mathbb{S}_n\times\mathbb{S}_m}\big((\mathcal{L}ie\mathcal{B}_2)^*\{1\}(n,m), \mathcal{F}rob_\infty^\diamond(n,m)\big)[1].
\end{aligned}$$

Here $\hbar$ is a formal variable of degree 2, and $\mathcal{F}rob_2^\diamond/\mathcal{L}ie^\diamond\mathcal{B}_2$ and $\mathcal{F}rob_2^\diamond/\mathcal{L}ie\mathcal{B}_2$ are analogues of (involutive) Frobenius/Lie bialgebras properads with the $(2,1)$-generator placed in degree 0 and the $(1,2)$-generator placed in degree 2. Each of the models on the right has a natural combinatorial interpretation as a graph complex (see also [29, Section 1.7]). For example, $\mathrm{Der}(\mathcal{L}ie\mathcal{B}_\infty)$ may be interpreted as a complex of directed graphs which have incoming and outgoing legs but have no closed paths of directed edges. The differential is obtained by splitting vertices and by attaching new vertices at one of the external legs (see Figure 1).

Similarly, $\mathrm{Der}(\mathcal{L}ie^\diamond\mathcal{B}_\infty)$ may be interpreted as a complex of $\hbar$-power series of graphs with weighted vertices. The differential is obtained by splitting vertices and attaching vertices at external legs as indicated in Figure 2.

The Lie bracket is combinatorially obtained by inserting graphs into vertices of another. We leave it to the reader to work out the structure of the graph complexes and the differentials for the complexes $\mathrm{Der}(\mathcal{F}rob_\infty)$ and $\mathrm{Der}(\mathcal{F}rob_\infty^\diamond)$.



$$\delta\Gamma = \delta_{\mathcal{L}ie\mathcal{B}_\infty}\Gamma \pm \sum \;\; \pm \sum$$

Figure 1. A graphical interpretation of an example element of $\mathrm{Der}(\mathcal{L}ie\mathcal{B}_\infty)$ (left), and the pictorial description of the differential (right). For the two right-most terms, one sums over all possible ways of attaching an additional vertex to an external leg of Γ, as indicated by the picture.

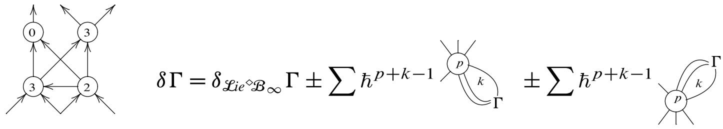


Figure 2. A graphical interpretation of an element of $\mathrm{Der}(\mathcal{L}ie^{\diamond}\mathcal{B}_{\infty})$, and the pictorial description of the differential. In the two terms on the right, one sums over all ways of attaching a new vertex to some subset of the incoming or outgoing legs (k many), and sums over all possible decorations p of the added vertex, with an appropriate power of $\hbar$ as prefactor. Note that the power of $\hbar$ counts the number of loops added to the graph, if we count a vertex decorated by p as contributing p loops.

The cohomology of all these graph complexes is hard to compute. We may, however, simplify the computation by using formula (34) and equivalently compute instead

$$\begin{aligned}
\mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty \to \mathcal{L}ie\mathcal{B}) &= \prod_{n,m} \mathrm{Hom}_{\mathbb{S}_n\times\mathbb{S}_m}\big((\mathcal{F}rob_2^{\diamond})^*\{1\}(n,m), \mathcal{L}ie\mathcal{B}(n,m)\big)\\
&\cong \prod_{n,m} \big(\mathcal{L}ie\mathcal{B}(n,m)\otimes \mathrm{sgn}_n \otimes \mathrm{sgn}_m\big)^{\mathbb{S}_n\times\mathbb{S}_m}[2-n-m],\\
\mathsf{Def}(\mathcal{L}ie^{\diamond}\mathcal{B}_\infty \to \mathcal{L}ie^{\diamond}\mathcal{B}) &= \prod_{n,m} \mathrm{Hom}_{\mathbb{S}_n\times\mathbb{S}_m}\big((\mathcal{F}rob_2)^*\{1\}(n,m), \mathcal{L}ie^{\diamond}\mathcal{B}(n,m)\big)\\
&\cong \prod_{n,m} \big(\mathcal{L}ie^{\diamond}\mathcal{B}(n,m)\otimes \mathrm{sgn}_n \otimes \mathrm{sgn}_m\big)^{\mathbb{S}_n\times\mathbb{S}_m}[2-n-m][[\hbar]],\\
\mathsf{Def}(\mathcal{F}rob_\infty \to \mathcal{F}rob) &= \prod_{n,m} \mathrm{Hom}_{\mathbb{S}_m\times\mathbb{S}_n}\big((\mathcal{L}ie^{\diamond}\mathcal{B}_2)^*\{1\}(n,m), \mathcal{F}rob(n,m)\big),\\
\mathsf{Def}(\mathcal{F}rob_\infty^{\diamond} \to \mathcal{F}rob^{\diamond}) &= \prod_{n,m} \mathrm{Hom}_{\mathbb{S}_n\times\mathbb{S}_m}\big((\mathcal{L}ie\mathcal{B}_2)^*\{1\}(n,m), \mathcal{F}rob^{\diamond}(n,m)\big).
\end{aligned}$$

Note, however, that in passing from $\mathrm{Der}(\dots)$ to the (quasi-isomorphic) simpler complexes $\mathsf{Def}(\dots)$ above we lose the dg Lie algebra structure, or rather there is a different Lie algebra structure on the above complexes. The above complexes may again be interpreted as graph complexes. For example, $\mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty \to \mathcal{L}ie\mathcal{B})$ consists of oriented trivalent graphs with incoming and outgoing legs, modulo the Jacobi and Drinfeld five-term relations. The differential is obtained by attaching a trivalent vertex at one external leg in all possible ways.

Finally, we note that of the above four deformation complexes, only two are essentially different. For example, note that $\mathrm{Hom}_{\mathbb{S}_n\times\mathbb{S}_m}(\mathcal{L}ie\mathcal{B}^*\{1\}(n,m), \mathcal{F}rob^{\diamond}(n,m))$ is just a completion of

$$\begin{aligned}&\mathrm{Hom}_{\mathbb{S}_n\times\mathbb{S}_m}\big((\mathcal{F}rob^{\diamond})^*\{1\}(n,m),\mathcal{L}ie\mathcal{B}(n,m)\big)\\&\quad\cong\big(\mathcal{F}rob^{\diamond}(n,m)\otimes\mathrm{sgn}_n\otimes\mathrm{sgn}_m\big)\otimes_{\mathbb{S}_n\times\mathbb{S}_n}\mathcal{L}ie\mathcal{B}(n,m)[n-m].\end{aligned}$$

Specifically, the completion is with respect to the genus grading of $\mathcal{L}ie\mathcal{B}$, and the differential preserves the genus grading. Hence the cohomology of one complex is just the completion of the cohomology of the other with respect to the genus grading.

Similar arguments show that the cohomologies $\mathsf{Def}(\mathcal{L}ie^{\diamond}\mathcal{B}_\infty\to\mathcal{L}ie^{\diamond}\mathcal{B})$ and $\mathsf{Def}(\mathcal{F}rob_\infty\to\mathcal{F}rob)$ are the same up to completion issues. Here the differential does not preserve the genus but does preserve the quantity (genus)-($\hbar$-degree). Hence it suffices to discuss one of each pair of deformation complexes. We will discuss $\mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty\to\mathcal{L}ie\mathcal{B})$ and $\mathsf{Def}(\mathcal{L}ie^{\diamond}\mathcal{B}_\infty\to\mathcal{L}ie^{\diamond}\mathcal{B})$ in the next section.

4. Oriented graph complexes and the $\mathfrak{grt}_1$ action

The goal of this section is to reduce the computation of the above deformation complexes to the computation of the cohomology of M. Kontsevich's graph complex. By a result of one of the authors (see [41]), the degree 0 cohomology of this graph complex agrees with the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_1$. This will allow us to conclude that the Grothendieck–Teichmüller group universally acts on $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$-structures. This extends the well-known result that the Grothendieck–Teichmüller group acts on Lie bialgebra structures.

4.1. Grothendieck–Teichmüller group

The profinite and prounipotent Grothendieck–Teichmüller groups were introduced by Drinfeld in his study of braid groups and quasi-Hopf algebras. They turned out to be one of the most interesting and mysterious objects in modern mathematics. The profinite Grothendieck–Teichmüller group $\widehat{\mathrm{GT}}$ plays an important role in number theory and algebraic geometry. The prounipotent Grothendieck–Teichmüller group GT (and its graded version GRT) over a field of characteristic 0 appeared in Pavel Etingof's and David Kazhdan's solution of Drinfeld's quantization conjecture for Lie bialgebras. Maxim Kontsevich's and Dmitry Tamarkin's formality theory unravels the role of the group GRT in the deformation quantization of Poisson structures. Later, Anton Alekseev and Charles Torossian applied GRT to the Kashiwara–Vergne problem in Lie theory. Because the Grothendieck–Teichmüller group unifies different fields, its appearance in a mathematical theory is oftentimes followed by a breakthrough in that theory. We refer to Furusho's lecture note [14] for precise definitions and references.

In this paper, we consider the Grothendieck–Teichmüller group GRT_1, which is the kernel of the canonical morphism of groups $\mathrm{GRT}\to\mathbb{K}^*$. As GRT_1 is prounipotent, it is of the form $\exp(\mathfrak{grt}_1)$ for some Lie algebra $\mathfrak{grt}_1$, whose definition can be found, for example, in [41, Section 6]. Therefore, understanding representations of

GRT_1 is the same as understanding representations of the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_1$.

4.2. Completed versions of $\mathcal{L}ie^{\diamond}\mathcal{B}$ and $\mathcal{L}ie\mathcal{B}$

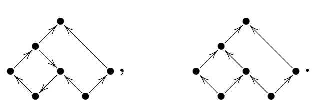
The properads $\mathcal{L}ie\mathcal{B}$ and $\mathcal{L}ie^{\diamond}\mathcal{B}$ are naturally graded by the genus of the graphs describing the operations. We denote by $\widehat{\mathcal{L}ie^{\diamond}\mathcal{B}}$ and $\widehat{\mathcal{L}ie\mathcal{B}}$ the completions with respect to this grading. Similarly, we denote by $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ the completion of $\mathcal{L}ie\mathcal{B}_\infty$ with respect to the genus grading. The natural map $\widehat{\mathcal{L}ie\mathcal{B}}_\infty \to \widehat{\mathcal{L}ie\mathcal{B}}$ is a quasi-isomorphism. Furthermore, we denote by $\widehat{\mathcal{L}ie^{\diamond}\mathcal{B}}_\infty$ the completion of $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$ with respect to the genus plus the total weight-grading, that is, with respect to the grading $\|\cdot\|$ described in Section 2.5. Then the map $\widehat{\mathcal{L}ie^{\diamond}\mathcal{B}}_\infty \to \widehat{\mathcal{L}ie^{\diamond}\mathcal{B}}$ is a quasi-isomorphism.

We call a continuous representation of $\widehat{\mathcal{L}ie\mathcal{B}}$ (resp., $\widehat{\mathcal{L}ie^{\diamond}\mathcal{B}}$) a *genus complete* (involutive) Lie bialgebra. Here the topology on $\widehat{\mathcal{L}ie\mathcal{B}}$ (resp., $\widehat{\mathcal{L}ie^{\diamond}\mathcal{B}}$) is the one induced by the genus filtration (resp., the filtration $\|\cdot\|$). For example, the involutive Lie bialgebra discussed in Section 2.2.1 is clearly genus complete since both the cobracket and the bracket reduce the lengths of the cyclic words.

Abusing notation slightly, we denote by $\mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty)$ (resp., $\mathrm{Der}(\widehat{\mathcal{L}ie^{\diamond}\mathcal{B}}_\infty)$) the complex of *continuous* derivations. The subproperads $\mathcal{L}ie\mathcal{B}_\infty \subset \widehat{\mathcal{L}ie\mathcal{B}}_\infty$ and $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty \subset \widehat{\mathcal{L}ie^{\diamond}\mathcal{B}}_\infty$ are dense by definition and hence any continuous derivation is determined by its restriction to these subproperads. It also follows that the above complexes of derivations are isomorphic as complexes to $\mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty \to \widehat{\mathcal{L}ie\mathcal{B}}_\infty)[1]$ and $\mathsf{Def}(\mathcal{L}ie^{\diamond}\mathcal{B}_\infty \to \widehat{\mathcal{L}ie^{\diamond}\mathcal{B}}_\infty)[1]$. Finally, we note that the cohomology of these complexes is merely the completion of the cohomology of the complexes $\mathrm{Der}(\mathcal{L}ie\mathcal{B}_\infty)$ and $\mathrm{Der}(\mathcal{L}ie^{\diamond}\mathcal{B}_\infty)$, since the differential respects the gradings.

4.3. An operad of graphs $\mathcal{G}ra^{\uparrow}$

A graph is called *directed* if its edges are equipped with directions as in the following examples:



A directed graph is called *oriented* or *acyclic* if it contains no directed *closed* paths of edges. For example, the second graph above is oriented, while the first one is not. For arbitrary integers $n \geq 1$ and $l \geq 0$, let $\mathsf{G}^{\uparrow}_{n,l}$ stand for the set of connected oriented graphs, $\{\Gamma\}$, with n vertices and l edges such that the vertices of Γ are labeled by elements of $[n] := \{1,\ldots,n\}$; that is, an isomorphism $V(\Gamma) \to [n]$ is fixed. We allow graphs with multiple edges between two vertices throughout.

Let $\mathbb{K}\langle \mathsf{G}^{\uparrow}_{n,l}\rangle$ be the vector space over a field $\mathbb{K}$ of characteristic 0 which is spanned by graphs from $\mathsf{G}^{\uparrow}_{n,l}$, and consider a $\mathbb{Z}$-graded $\mathbb{S}_n$-module

$$\mathcal{G}ra^{\uparrow}(n) := \bigoplus_{l=0}^{\infty} \mathbb{K}\langle \mathsf{G}^{\uparrow}_{n,l}\rangle[2l].$$

For example, $\overset{1}{\bullet}\!\longrightarrow\!\overset{2}{\bullet}$ is a degree -2 element in $\mathcal{G}ra^{\uparrow}(2)$. The $\mathbb{S}$-module, $\mathcal{G}ra^{\uparrow} := \{\mathcal{G}ra(n)^{\uparrow}\}_{n\geq 1}$, is naturally an operad with the operadic compositions given by

$$\begin{aligned} \circ_i : \mathcal{G}ra^{\uparrow}(n) \otimes \mathcal{G}ra^{\uparrow}(m) &\longrightarrow \mathcal{G}ra^{\uparrow}(m+n-1), \\ \Gamma_1 \otimes \Gamma_2 &\longrightarrow \sum_{\Gamma \in \mathsf{G}^{i}_{\Gamma_1,\Gamma_2}} \Gamma, \end{aligned} \tag{35}$$

where $\mathsf{G}^{i}_{\Gamma_1,\Gamma_2}$ is the subset of $\mathsf{G}^{\uparrow}_{n+m-1,\#E(\Gamma_1)+\#E(\Gamma_2)}$ consisting of graphs, Γ, satisfying the condition: the full subgraph of Γ spanned by the vertices labeled by the set $\{i, i+1, \ldots, i+m-1\}$ is isomorphic to Γ_2, and the quotient graph Γ/Γ_2 (which is obtained from Γ by contracting that subgraph Γ_2 to a single vertex) is isomorphic to Γ_1 (see, e.g., [32, Section 7] or [41, Section 2] for explicit examples of these kind of operadic compositions). The unique element in $\mathsf{G}^{\uparrow}_{1,0}$ serves as the unit in the operad $\mathcal{G}ra^{\uparrow}$.

4.3.1. A representation of $\mathcal{G}ra^{\uparrow}$ *in* $\mathsf{LieB}(\mathfrak{g})$

For any graded vector space $\mathfrak{g}$, the operad $\mathcal{G}ra^{\uparrow}$ has a natural representation in the associated graded vector space $\mathsf{LieB}(\mathfrak{g})$ (see (32)),

$$\begin{aligned} \rho : \mathcal{G}ra^{\uparrow}(n) &\longrightarrow \mathcal{E}nd_{\mathsf{LieB}(\mathfrak{g})}(n) = \mathrm{Hom}\big(\mathsf{LieB}(\mathfrak{g})^{\otimes n}, \mathsf{LieB}(\mathfrak{g})\big), \\ \Gamma &\longrightarrow \Phi_{\Gamma} \end{aligned} \tag{36}$$

given by the formula

$$\begin{aligned} \Phi_{\Gamma} : \otimes^n \mathsf{LieB}(\mathfrak{g}) &\longrightarrow \mathsf{LieB}(\mathfrak{g}), \\ \gamma_1 \otimes \cdots \otimes \gamma_n &\longrightarrow \Phi_{\Gamma}(\gamma_1, \ldots, \gamma_n) \\ &:= \mu\Big(\Big(\prod_{e \in E(\Gamma)} \Delta_e\Big)\gamma_1(\psi,\eta) \otimes \gamma_2(\psi,\eta) \otimes \cdots \otimes \gamma_n(\psi,\eta)\Big), \end{aligned}$$

where, for an edge $e = \overset{a}{\bullet}\!\longrightarrow\!\overset{b}{\bullet}$ connecting a vertex labeled by $a \in [n]$ and a vertex labeled by $b \in [n]$, we set

$$\Delta_e(\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_n)$$
$$= \begin{cases} \sum_{i\in I} (-1)^{|\eta^i|(|\gamma_a|+|\gamma_{a+1}|+\cdots+|\gamma_{b-1}|)} \gamma_1 \otimes \cdots \otimes \frac{\partial \gamma_a}{\partial \psi_i} \otimes \cdots \otimes \frac{\partial \gamma_b}{\partial \eta^i} \otimes \cdots \otimes \gamma_n \\ \quad \text{for } a < b, \\ \sum_{i\in I} (-1)^{|\psi_i|(|\gamma_b|+|\gamma_{b+1}|+\cdots+|\gamma_{a-1}|+|\eta^i|)} \gamma_1 \otimes \cdots \otimes \frac{\partial \gamma_b}{\partial \eta^i} \otimes \cdots \otimes \frac{\partial \gamma_a}{\partial \psi_i} \otimes \cdots \otimes \gamma_n \\ \quad \text{for } b < a, \end{cases}$$

and where μ is the standard multiplication map in the ring $\mathsf{LieB}(\mathfrak{g}) \subset \mathbb{K}[[\psi_i, \eta^i]]$:

$$\mu : \mathsf{LieB}(\mathfrak{g})^{\otimes n} \longrightarrow \mathsf{LieB}(\mathfrak{g}),$$
$$\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_n \longrightarrow \gamma_1\gamma_2 \cdots \gamma_n.$$

Note that this representation makes sense for both finite- and *infinite*-dimensional vector spaces $\mathfrak{g}$, as graphs from $\mathcal{G}ra^{\uparrow}$ do not contain oriented cycles.

4.3.1 Remark
The above action of $\mathcal{G}ra^{\uparrow}$ on $\mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty \xrightarrow{0} \mathcal{E}nd_{\mathfrak{g}})[-2]$ only uses the properadic compositions in $\mathcal{E}nd_{\mathfrak{g}}$ and no further data. It follows that the same formulas may in fact be used to define an action of $\mathcal{G}ra^{\uparrow}$ on the deformation complex

$$\mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty \xrightarrow{0} \mathcal{P})[-2] \cong \prod_{m,n} \big(\mathcal{P}(m,n) \otimes \mathrm{sgn}_m \otimes \mathrm{sgn}_n\big)^{\mathbb{S}_m\times\mathbb{S}_n}[-m-n]$$

for any properad $\mathcal{P}$. To give a more concrete description of the action, let us identify $(\mathbb{S}_m \times \mathbb{S}_n)$-coinvariants with invariants by symmetrization, and let us describe an action on the space of coinvariants

$$\prod_{m,n} \big(\mathcal{P}(m,n) \otimes \mathrm{sgn}_m \otimes \mathrm{sgn}_n\big)_{\mathbb{S}_m\times\mathbb{S}_n}[-m-n]$$

instead. Specifically, let $\Gamma \in \mathcal{G}ra^{\uparrow}(n)$ be a graph with n vertices, and let

$$x_j \in \big(\mathcal{P}(m_j,n_j) \otimes \mathrm{sgn}_{m_j} \otimes \mathrm{sgn}_{n_j}\big)_{\mathbb{S}_m\times\mathbb{S}_n}$$

for $j = 1,\ldots,n$. If some vertex j of Γ has more than n_j outgoing or more than m_j incoming edges, then we define the action to be trivial: $\Gamma(x_1,\ldots,x_n) = 0$. Otherwise, we want to interpret the directed graph Γ as a properadic composition pattern. For notational simplicity, we assume that the x_j are actual elements of $\mathcal{P}(m_j,n_j)$, representing the corresponding elements of the coinvariant space. Suppose that for each vertex j of Γ, an injective map from the set of (say, k_j many) incoming half-edges at j to $\{1,\ldots,m_j\}$ and an injective map from the set of (say, l_j many) outgoing half-edges at j to $\{1,\ldots,n_j\}$ is fixed. Denote the collection of those maps (for all j) by f for convenience. Then we may define

$$\Gamma_f(x_1,\dots,x_n) \in \mathcal{P}\Big(\sum_{j=1}^{n}(m_j-k_j), \sum_{j=1}^{n}(n_j-l_j)\Big)$$

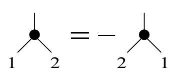
obtained by using the appropriate properadic composition. (The overall inputs and outputs are to be ordered according to the numbering of the vertices.) Then we define our desired action to be

$$\Gamma(x_1,\dots,x_n) := \sum_f \pm\Gamma_f(x_1,\dots,x_n),$$

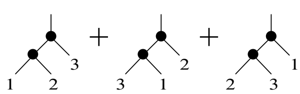
where the f in the sum runs over assignments of half-edges to inputs/outputs as above. The sign can be determined by considering half-edges and inputs/outputs of the x_j as odd objects. The sign is then the sign of the permutation bringing each half-edge "to the left of" the input/output it is assigned to via f.

4.4. An oriented graph complex

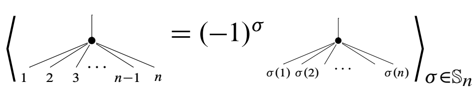
Let $\mathcal{L}ie\{2\}$ be a (degree-shifted) operad of Lie algebras, and let $\mathcal{L}ie_\infty\{2\}$ be its minimal resolution. Thus $\mathcal{L}ie\{2\}$ is a quadratic operad generated by a degree -2 skew-symmetric binary operation



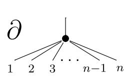
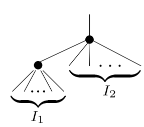
modulo the Jacobi relations


$= 0,$ (37)

while $\mathcal{L}ie_\infty\{2\}$ is the free operad generated by an $\mathbb{S}$-module $E = \{E(n)\}_{n\geq 2}$,

$E(n) := \mathrm{sgn}_n[3n-4] =$  ,

and equipped with the following differential:


$$= \sum_{\substack{[n]=I_1\sqcup I_2\\ \#I_1\geq 2, \#I_2\geq 1}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1||I_2|}$$

, (38)

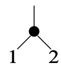
where $\sigma(I_1 \sqcup I_2)$ is the sign of the shuffle $[n] \to [I_1 \sqcup I_2]$.

4.4.1 PROPOSITION (see [40, Section 2.1])

There is a morphism of operads

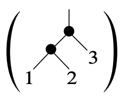
$$\varphi : \mathcal{L}ie\{2\} \longrightarrow \mathcal{G}ra^{\uparrow}$$

given on the generators by



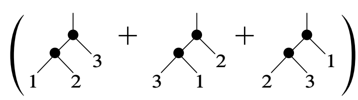
$$\longrightarrow \quad \overset{1}{\bullet}\!\longrightarrow\!\overset{2}{\bullet} - \overset{2}{\bullet}\!\longrightarrow\!\overset{1}{\bullet}.$$

Proof
Using the definition of the operadic composition in $\mathcal{G}ra^{\uparrow}$, we get

φ 

$$= \overset{1}{\bullet}\!\to\!\overset{2}{\bullet}\!\to\!\overset{3}{\bullet} - \overset{2}{\bullet}\!\to\!\overset{1}{\bullet}\!\to\!\overset{3}{\bullet} + \overset{2}{\bullet}\!\leftarrow\!\overset{1}{\bullet}\!\to\!\overset{3}{\bullet} - \overset{1}{\bullet}\!\leftarrow\!\overset{2}{\bullet}\!\to\!\overset{3}{\bullet}$$
$$+ \overset{1}{\bullet}\!\leftarrow\!\overset{2}{\bullet}\!\leftarrow\!\overset{3}{\bullet} - \overset{2}{\bullet}\!\leftarrow\!\overset{1}{\bullet}\!\leftarrow\!\overset{3}{\bullet} + \overset{2}{\bullet}\!\to\!\overset{1}{\bullet}\!\leftarrow\!\overset{3}{\bullet} - \overset{1}{\bullet}\!\to\!\overset{2}{\bullet}\!\leftarrow\!\overset{3}{\bullet}, \tag{39}$$

which implies

φ  $= 0.$

□

All possible morphisms of dg operads $\mathcal{L}ie_\infty\{2\} \longrightarrow \mathcal{G}ra^{\uparrow}$ can be usefully encoded as Maurer–Cartan elements in the graded Lie algebra

$$\mathsf{fGC}_3^{\mathrm{or}} := \mathsf{Def}\big(\mathcal{L}ie_\infty\{2\} \stackrel{0}{\longrightarrow} \mathcal{G}ra^{\uparrow}\big),$$

which controls deformation theory of the zero morphism (see [33]). As a graded vector space,

$$\mathsf{fGC}_3^{\mathrm{or}} \cong \prod_{n\geq 2} \mathrm{Hom}_{\mathbb{S}_n}\big(E(n), \mathcal{G}ra^{\uparrow}(n)\big)[-1] = \prod_{n\geq 2} \mathcal{G}ra^{\uparrow}(n)^{\mathbb{S}_n}[3-3n], \tag{40}$$

so that its elements can be understood as ($\mathbb{K}$-linear series of) graphs Γ from $\mathcal{G}ra^{\uparrow}$ whose vertex labels are skew-symmetrized (so that we can often forget numerical labels of vertices in our pictures), and which are assigned the homological degree

$$|\Gamma| = 3\#V(\Gamma) - 3 - 2\#E(\Gamma),$$

where $V(\Gamma)$ (resp., $E(\Gamma)$) stands for the set of vertices (resp., edges) of Γ.

The Lie brackets, $[\,,\,]_{\mathsf{gra}}$, in $\mathsf{fGC}_3^{\mathrm{or}}$ can be read either from the differential (38) or, equivalently, from the following explicit Lie algebra structure (see [21]) associated with the degree-shifted operad $\mathcal{G}ra_3^{\uparrow}\{3\}$ (and which makes sense for any operad):

$$[\,,\,] : \mathsf{P} \otimes \mathsf{P} \longrightarrow \mathsf{P},$$
$$\big(a \in \mathcal{P}(n), b \in \mathcal{P}(m)\big) \longrightarrow [a,b] := \sum_{i=1}^{n} a \circ_i b - (-1)^{|a||b|} \sum_{i=1}^{m} b \circ_i a,$$

where $\mathsf{P} := \prod_{n\geq 1} \mathcal{G}ra^{\uparrow}(n)[3-3n]$. These Lie brackets in P induce Lie brackets, $[\, , \,]_{\mathsf{gra}}$, in the subspace of $\mathbb{S}$-coinvariants (see [21]). By the isomorphism of invariants and coinvariants, we obtain a Lie bracket on the space of invariants

$$\mathsf{P}^{\mathbb{S}} := \prod_{n\geq 1} \mathcal{G}ra^{\uparrow}(n)[3-3n]^{\mathbb{S}_n} = \mathsf{fGC}_3^{\mathrm{or}}$$

via the standard symmetrization map $\mathsf{P} \to \mathsf{P}^{\mathbb{S}}$.

The graph

$$\bullet\!\longrightarrow\!\bullet := \overset{1}{\bullet}\!\longrightarrow\!\overset{2}{\bullet} - \overset{2}{\bullet}\!\longrightarrow\!\overset{1}{\bullet}$$

is a degree $2 \cdot 3 - 3 - 2 = 1$ element in $\mathsf{fGC}_3^{\mathrm{or}}$, which, in fact, is a Maurer–Cartan element

$$\left[\bullet\!\longrightarrow\!\bullet, \bullet\!\longrightarrow\!\bullet\right] = \text{skew-symmetrization of the RHS in (39)} = 0,$$

which represents the above morphism φ in the Lie algebra $\mathsf{fGC}_3^{\mathrm{or}}$. This element makes, therefore, $\mathsf{fGC}_3^{\mathrm{or}}$ into a *differential* graded Lie algebra with the differential

$$d\Gamma := \left[\bullet\!\longrightarrow\!\bullet, \Gamma\right]_{\mathsf{gra}}. \tag{41}$$

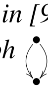
Let $\mathsf{GC}_3^{\mathrm{or}}$ be a subspace of $\mathsf{fGC}_3^{\mathrm{or}}$ spanned by connected graphs whose vertices are at least bivalent, and if bivalent do not have one incoming and one outgoing edge. It is easy to see that this is a dg Lie subalgebra.

4.4.1 Remark
The definition of $\mathsf{GC}_3^{\mathrm{or}}$ in [41] differs slightly from the one here, as all bivalent vertices are allowed in that work. However, it is easy to check that this extra condition does not change the cohomology.

The cohomology of the *oriented graph complex* $(\mathsf{GC}_3^{\mathrm{or}}, d)$ was partially computed in [41] and [42].

4.4.2 THEOREM ([42, Theorem 1], [41, Theorem 1, Proposition 3.4])

(i) $H^0(\mathsf{GC}_3^{\mathrm{or}}, d) = \mathfrak{grt}_1$, *where* $\mathfrak{grt}_1$ *is the Lie algebra of the prounipotent Grothendieck–Teichmüller group* GRT_1 *introduced by Drinfeld in [9].*

(ii) $H^{-1}(\mathsf{GC}_3^{\mathrm{or}}, d) \cong \mathbb{K}$. *The single class is represented by the graph* .

(iii) $H^i(\mathsf{GC}_3^{\mathrm{or}}, d) = 0$ *for all* $i \leq -2$.

4.5. Action on $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$

There is a natural action of $\mathsf{GC}_3^{\mathrm{or}}$ on the properad $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ by properadic derivations. Specifically, for any graph Γ we define the derivation $F(\Gamma) \in \mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty)$ sending the generator $\mu_{m,n}$ of $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ to the linear combination of graphs

$$\mu_{m,n} \cdot \Gamma = \sum \begin{matrix} \overbrace{\backslash\backslash\cdots//}^{m\times} \\ \Gamma \\ \underbrace{//\cdots\backslash\backslash}_{n\times} \end{matrix}, \tag{42}$$

where the sum is taken over all ways of attaching the incoming and outgoing legs such that all vertices are at least trivalent and have at least one incoming and one outgoing edge.

4.5.1 LEMMA

The above formula (42) defines a right action of $\mathsf{GC}_3^{\mathrm{or}}$ *on* $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$.

Sketch of proof

We denote by $\bullet$ the pre-Lie product on the deformation complex $\mathsf{Def}(\mathcal{L}ie_\infty\{2\} \xrightarrow{0} \mathcal{G}ra^{\uparrow}) \supset \mathsf{GC}_3^{\mathrm{or}}$, so that the Lie bracket on $\mathsf{GC}_3^{\mathrm{or}}$ may be written as $[\Gamma, \Gamma'] = \Gamma \bullet \Gamma' \pm \Gamma' \bullet \Gamma$ for $\Gamma, \Gamma' \in \mathsf{GC}_3^{\mathrm{or}}$. Note that

$$\mu_{m,n} \cdot (\Gamma \bullet \Gamma') = (\mu_{m,n}^k \cdot \Gamma) \cdot \Gamma',$$

where it is important that we excluded graphs with bivalent vertices with one incoming and one outgoing edge from the definition of $\mathsf{GC}_3^{\mathrm{or}}$. It follows that the formula above defines an action of the graded Lie algebra $\mathsf{GC}_3^{\mathrm{or}}$. We leave it to the reader to check that this action also commutes with the differential. □

Of course, by a change of sign the right action may be transformed into a left action, and hence we obtain a map of Lie algebras

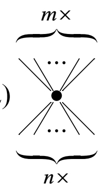
$$F : \mathsf{GC}_3^{\mathrm{or}} \to \mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty).$$

Interpreting the right-hand side as a graph complex as in Section 3.2, the map F sends a graph $\Gamma \in \mathsf{GC}_3^{\mathrm{or}}$ to the series of graphs

$$\pm \sum_{m,n} \sum \begin{matrix} \overbrace{\backslash\backslash\cdots//}^{m\times} \\ \Gamma \\ \underbrace{//\cdots\backslash\backslash}_{n\times} \end{matrix}.$$

4.5.2 Remark

It can be shown that the map $F : \mathsf{GC}_3^{\mathrm{or}} \to \mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty)$ is a quasi-isomorphism, up to one class in $\mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty)$ represented by the series

$$\sum_{m,n}(m+n-2)$$  .

The result will not be used directly in this paper. The proof is an adaptation of the proof of [42, Proposition 3] and is given in [3].[†]

4.6. GRT_1 *action on Lie bialgebra structures*

The action of the Lie algebra of closed degree 0 cocycles $\mathsf{GC}_{3,\mathrm{cl}}^{\mathrm{or}} \subset \mathsf{GC}_3^{\mathrm{or}}$ on $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ by derivations may be integrated to an action of the exponential group $\mathrm{Exp}\,\mathsf{GC}_{3,\mathrm{cl}}^{\mathrm{or}}$ on $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ by (continuous) automorphisms. Hence this exponential group acts on the set of $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ algebra structures on any dg vector space $\mathfrak{g}$, that is, on the set of morphisms of properads

$$\widehat{\mathcal{L}ie\mathcal{B}}_\infty \to \mathsf{End}_{\mathfrak{g}},$$

by precomposition. Furthermore, it follows that the cohomology Lie algebra $H^0(\mathsf{GC}_3^{\mathrm{or}}) \cong \mathfrak{grt}_1$ maps into the Lie algebra of continuous derivations up to homotopy $H^0(\mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty))$ and the exponential group $\mathrm{Exp}\,H^0(\mathsf{GC}_3^{\mathrm{or}}) \cong \mathrm{GRT}_1$ maps into the set of homotopy classes of continuous automorphisms of $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$.

4.6.1 Remark

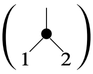
Note also that one may define a noncomplete version $\mathsf{GC}_{3,\mathrm{inc}}^{\mathrm{or}}$ of the graph complex $\mathsf{GC}_3^{\mathrm{or}}$ by merely replacing the direct product by a direct sum in (40). The zeroth cohomology of $\mathsf{GC}_{3,\mathrm{inc}}^{\mathrm{or}}$ is a noncomplete version of the Grothendieck–Teichmüller group. Furthermore, $\mathsf{GC}_{3,\mathrm{inc}}^{\mathrm{or}}$ acts on the noncompleted operad $\mathcal{L}ie\mathcal{B}_\infty$ by derivations, using the formulas (42) of Section 4.5, and hence also on $\mathcal{L}ie\mathcal{B}_\infty$ algebra structures. However, these actions can in general not be integrated, whence we work with the completed properad $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ above.

Finally, let us describe the action $\mathsf{GC}_{3,\mathrm{cl}}^{\mathrm{or}}$ on Lie bialgebra structures in yet another form. We have a sequence of morphisms of dg Lie algebras

[†]A sketch of the proof was contained as an appendix in the preprint version of this article but was removed following the suggestion of a referee.

$$\mathsf{GC}_3^{\mathrm{or}} \longrightarrow \mathsf{Def}\big(\mathcal{L}ie_\infty\{2\} \to \mathcal{G}ra^{\uparrow}\big) \longrightarrow \mathsf{Def}\big(\mathcal{L}ie_\infty\{2\} \xrightarrow{\{\,,\,\}} \mathcal{E}nd_{\mathsf{LieB}(\mathfrak{g})}\big),$$

where the first arrow is just the inclusion, and the second arrow is induced by the canonical representation (36) and which obviously satisfies

$$\rho \circ \varphi \Big($$  $$\Big) = \{\,,\,\} \in \mathrm{Hom}\big(\wedge^2 \mathsf{LieB}, \mathsf{LieB}[-2]\big) \subset \mathsf{Def}\big(\mathcal{L}ie_\infty\{2\} \xrightarrow{\{\,,\,\}} \mathcal{E}nd_{\mathsf{LieB}(\mathfrak{g})}\big).$$

The dg Lie algebra

$$\mathsf{Def}\big(\mathcal{L}ie_\infty\{2\} \xrightarrow{\{\,,\,\}} \mathcal{E}nd_{\mathsf{LieB}(\mathfrak{g})}\big) =: \mathrm{CE}^\bullet\big(\mathsf{LieB}(\mathfrak{g})\big)$$

is nothing but the classical Chevalley–Eilenberg complex controlling deformations of the Poisson brackets (33) in $\mathsf{LieB}(\mathfrak{g})$. In particular, for any closed degree 0 element $g \in \mathsf{GC}_3^{\mathrm{or}}$, and especially for representatives of elements of $\mathfrak{grt}_1$ in $\mathsf{GC}_3^{\mathrm{or}}$, we obtain a Lie_∞ derivation of $\mathsf{LieB}(\mathfrak{g})$. This derivation may be integrated into a Lie_∞ automorphism $\exp(ad_g)$. For a Maurer–Cartan element γ in $\mathsf{LieB}(\mathfrak{g})$ corresponding to a $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ structure on $\mathfrak{g}$, the series

$$\gamma \longrightarrow \exp(ad_g)\gamma$$

converges and redefines another $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ structure on $\mathfrak{g}$.

4.6.2 Remark

By reasons of degree, the above action on $\widehat{\mathcal{L}ie\mathcal{B}}_\infty$ structures maps $\widehat{\mathcal{L}ie\mathcal{B}}$ structures again to $\widehat{\mathcal{L}ie\mathcal{B}}$ structures. In other words, no higher homotopies are created if there were none before.

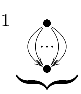
4.7. Another oriented graph complex

We introduce next a new oriented graph complex and then use Theorem 4.4.2 to partially compute its cohomology; we then deduce formulas for an action of GRT_1 on *involutive* Lie bialgebra structures.

Let $\hbar$ be a formal variable of homological degree 2. The Lie brackets $[\,,\,]_{\mathrm{gra}}$ in $\mathsf{GC}_3^{\mathrm{or}}$ extend $\hbar$-linearly to the topological vector space $\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$.

4.8 PROPOSITION

The element

$$\Phi_\hbar := \sum_{k=1}^{\infty} \hbar^{k-1}$$ 

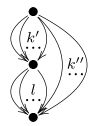
k edges

is a Maurer–Cartan element in the Lie algebra $(\mathsf{fGC}_3^{\mathrm{or}}[[\hbar]], [\,,\,]_{\mathsf{gra}})$.

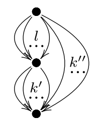
Convention
Here we adopt the convention that a picture of an unlabeled graph with black vertices will stand for the element of $\mathsf{fGC}_3^{\mathrm{or}}/\mathsf{GC}_3^{\mathrm{or}}$ (i.e., a symmetrically labeled graph) by summing over all labelings of vertices and dividing by the order of the symmetry group of the graph. In particular, this means that the kth term in the above formula for $\Phi_\hbar$ carries an implicit prefactor $\frac{1}{k!}$. This convention will kill many prefactors arising in calculations.

Proof of Proposition 4.8
We have

$$\frac{1}{2}[\Phi_\hbar, \Phi_\hbar] = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \hbar^{k+l-2} \sum_{k=k'+k''}$$



$$- \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \hbar^{k+l-2} \sum_{k=k'+k''}$$



$$= 0.$$

□

Hence the degree 1 continuous map

$$d_\hbar : \mathsf{GC}_3^{\mathrm{or}}[[\hbar]] \longrightarrow \mathsf{GC}_3^{\mathrm{or}}[[\hbar]],$$
$$\Gamma \longrightarrow d_\hbar\Gamma := [\Phi_\hbar, \Gamma]_{\mathrm{gra}}$$

is a differential in $\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$. The induced differential, d, in $\mathsf{GC}_3^{\mathrm{or}} = \mathsf{GC}_3^{\mathrm{or}}[[\hbar]]/\hbar\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$ is precisely the original differential (41).

4.9. Action on $\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty$
The dg Lie algebra $(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]], d_\hbar)$ acts naturally on the properad $\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty$ by continuous properadic derivations. More precisely, let $\Gamma \in \mathsf{GC}_3^{\mathrm{or}}$ be a graph. Then to the element $\hbar^N\Gamma \in \mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$, we assign the derivation of $\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty$ that sends the generator $\mu_{m,n}^k$ to zero if $k < N$ and to

$$\mu_{m,n}^k \cdot (\hbar^N\Gamma) = \mathrm{mark}_{k-N}(\mu_{m,n}\cdot\Gamma), \tag{43}$$

where $\mu_{m,n}\cdot\Gamma$ is a series of graphs obtained attaching external legs to Γ in all possible ways as in (42), and the operation mark_{k-N} assigns weights to the vertices in all possible ways such that the weights sum to $k-N$.

4.9.1 LEMMA
The above formula (43) defines a right action of $(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]], d_\hbar)$ *on* $\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty$.

Sketch of proof
The proof is similar to that of Lemma 4.5.1 after noting that

$$\mu^k_{m,n} \cdot (\hbar^N \Gamma \bullet \hbar^M \Gamma') = (\mu^k_{m,n} \cdot \hbar^N \Gamma) \cdot \hbar^M \Gamma'$$

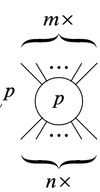
for all M, N and $\Gamma, \Gamma' \in \mathsf{GC}_3^{\mathrm{or}}$. □

Again, by a change of sign the right action may be transformed into a left action and hence we obtain a map of Lie algebras:

$$F_\hbar \colon \mathsf{GC}_3^{\mathrm{or}}[[\hbar]] \to \mathrm{Der}(\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty).$$

4.9.2 Remark
It can be seen that the map $F_\hbar$ is a quasi-isomorphism, up to classes $T\mathbb{K}[[\hbar]] \subset \mathrm{Der}(\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty)$, where

$$T = \sum_{m,n,p} (m+n+2p-2)\hbar^p$$



The result will not be used in this paper, and the proof will appear elsewhere (see [3]).

4.10. Action on involutive Lie bialgebra structures
By the previous section, the Lie algebra of degree 0 cocycles in $\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$ acts on the properad $\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty$ by derivations. The action may be integrated to an action of the corresponding exponential group on $\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty$ by continuous automorphisms, and hence also on the set of $\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty$ algebra structures on some dg vector space by precomposition. Furthermore, the cohomology Lie algebra $H^0(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]])$ maps into the Lie algebra of continuous derivations up to homotopy $H^0(\mathrm{Der}(\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty))$, while the exponential group $\exp H^0(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]])$ maps into the set of homotopy classes of continuous automorphisms of $\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty$. By precomposition, we also have a map of Lie algebras $H^0(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]) \to H^0(\mathsf{Def}(\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty \to \mathsf{End}_\mathfrak{g}))$ for any $\widehat{\mathcal{L}ie^\diamond\mathcal{B}}_\infty$ algebra $\mathfrak{g}$ and an action of $\exp H^0(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]])$ on the set of such algebra structures up to homotopy. Let us encode these findings in the following corollary.

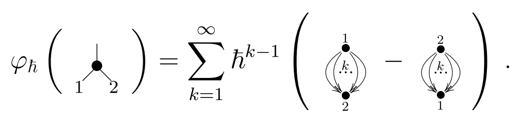
4.10.1 COROLLARY
The Lie algebra $H^0(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]])$ *and its exponential group* $\exp(H^0(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]))$ *canonically act on the set of homotopy classes of graded complete strong homotopy involutive Lie bialgebra structures on the dg vector space* $\mathfrak{g}$.

4.10.2 Remark
Note again that, analogously to Remark 4.6.1, we may define a noncomplete version of the graph complex $\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$ which acts on the noncomplete properad $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$ by derivations, and hence also on ordinary Lie bialgebra structures. However, these actions can in general not be integrated, whence we prefer to work with the complete version of the graph complex and $\widehat{\mathcal{L}ie^{\diamond}\mathcal{B}}_\infty$.

Finally, let us give yet another description of the action of $\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$ on involutive Lie bialgebra structures, strengthening the above result a little. As usual, the deformation complex of the zero morphism of dg props $\mathsf{InvLieB}(\mathfrak{g}) := \mathsf{Def}(\mathcal{L}ie^{\diamond}\mathcal{B}_\infty \stackrel{0}{\to} \mathcal{E}nd_{\mathfrak{g}})$ has a canonical dg Lie algebra structure, with the Lie bracket $[\ ,\ ]_\hbar$ given explicitly by (30), such that the Maurer–Cartan elements are in one-to-one correspondence with $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$-structures on the dg vector space $\mathfrak{g}$. The Maurer–Cartan element $\Phi_\hbar$ in $\mathsf{fGC}_3^{\mathrm{or}}[[\hbar]]$ corresponds to a continuous morphism of operads

$$\varphi_\hbar : \mathcal{L}ie\{2\}[[\hbar]] \longrightarrow \mathcal{G}ra^{\uparrow}[[\hbar]]$$

given on the generator of $\mathcal{L}ie\{2\}$ by the formula



The representation (36) of the operad $\mathcal{G}ra^{\uparrow}$ in the deformation complex $\mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty \stackrel{0}{\to} \mathcal{P})[-2]$ extends $\hbar$-linearly to a representation $\rho_\hbar$ of $\mathcal{G}ra^{\uparrow}[[\hbar]]$ in $\mathsf{Def}(\mathcal{L}ie^{\diamond}\mathcal{B}_\infty \stackrel{0}{\to} \mathcal{P})[-2]$ for any properad $\mathcal{P}$. Furthermore, it is almost immediate to see that the action of $\mathsf{Lie}\{2\}$ on the latter deformation complex factors through the map $\mathsf{Lie}\{2\} \to \mathcal{G}ra^{\uparrow}[[\hbar]]$.

It follows that one has a morphism of dg Lie algebras induced by $\varphi_\hbar$,

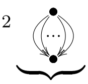
$$\begin{aligned}(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]], d_\hbar) &\longrightarrow \mathsf{Def}\big(\mathcal{L}ie_\infty\{2\}[[\hbar]] \xrightarrow{\varphi_\hbar} \mathcal{G}ra^{\uparrow}[[\hbar]]\big)\\ &\xrightarrow{\rho_\hbar} \mathsf{Def}\big(\mathcal{L}ie_\infty\{2\}[[\hbar]] \xrightarrow{[\,,\,]_\hbar} \mathcal{E}nd_D\big) =: \mathrm{CE}^\bullet(D),\end{aligned}$$

from the graph complex $(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]], d_\hbar)$ into the Chevalley–Eilenberg dg Lie algebra of $D := \mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty \stackrel{0}{\to} \mathcal{P})[-2]$. In particular, this implies the following.

4.11 THEOREM
We have $H^0(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]], d_\hbar) \simeq H^0(\mathsf{GC}_3^{\mathrm{or}}, d_0) \simeq \mathfrak{grt}_1$ *as Lie algebras. Moreover,* $H^i(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]], d_\hbar) = 0$ *for all* $i \leq -2$ *and* $H^{-1}(\mathsf{GC}_3^{\mathrm{or}}[[\hbar]], d_\hbar) \cong \mathbb{K}$*, with the single*

class being represented by

$$\sum_{k=2}^{\infty}(k-1)\hbar^{k-2}$$



k edges .

Proof
First note that the element above is exactly $\frac{d}{d\hbar}\Phi_\hbar$, and the fact that it is closed follows easily by differentiating the Maurer–Cartan equation

$$0=\frac{d}{d\hbar}[\Phi_\hbar,\Phi_\hbar]_{\mathrm{gra}}=2\Big[\Phi_\hbar,\frac{d}{d\hbar}\Phi_\hbar\Big]_{\mathrm{gra}}=2d_\hbar\Big(\frac{d}{d\hbar}\Phi_\hbar\Big).$$

It is easy to see that the cocyle $\frac{d}{d\hbar}\Phi_\hbar$ cannot be exact, by just considering the leading term in $\hbar$, which is given by the following graph:

. (44)

Let us write

$$d_\hbar=\sum_{k=1}^{\infty}\hbar^{k-1}d_k.$$

Consider a decreasing filtration of $\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$ by the powers in $\hbar$. The first term of the associated spectral sequence is

$$\mathcal{E}_1=\bigoplus_{i\in\mathbb{Z}}\mathcal{E}_1^i,\qquad \mathcal{E}_1^i=\bigoplus_{p\geq 0}H^{i-2p}(\mathsf{GC}_3^{\mathrm{or}},d_0)\hbar^p$$

with the differential equal to $\hbar d_1$. The main result of [42] states that $H^0(\mathsf{GC}_3^{\mathrm{or}},d_0)\simeq\mathfrak{grt}_1$, $H^{\leq -2}(\mathsf{GC}_3^{\mathrm{or}},d_0)=0$, and $H^{-1}(\mathsf{GC}_3^{\mathrm{or}},d_0)\cong\mathbb{K}$, with the single class being represented by (44). The desired results follow by reasons of degree. First, there is clearly no cohomology in $\mathcal{E}_1$ of degree ≤ -2, so that there can be no such cohomology in $\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$. The single class in $\mathcal{E}_1$ of degree -1 may, as we just saw above, be extended to a cocycle in $\mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$, and hence will be killed by all further differentials in the spectral sequence. Hence no elements of degree 0 in $\mathcal{E}_1$ can be rendered exact on later pages of the spectral sequence. Hence the only thing that remains to be shown is that the degree 0 elements in $\mathcal{E}_1$ can be extended to cocycles, that is, that they are closed on all further pages of the spectral sequence. However, the differential on later pages will necessarily increase the number of $\hbar$'s occurring. Hence the differential on later pages will map the degree 0 part of $\mathcal{E}_1$ (i.e., $\mathfrak{grt}_1$) into (subquotients of)

$H^{1-2p}(\mathsf{GC}_3^{\mathrm{or}}, d_0)$ for $p \geq 1$. But by the aforementioned vanishing result of [42], there are no such classes, except possibly for $p = 1$, when $H^{-1}(\mathsf{GC}_3^{\mathrm{or}}, d_0) \cong \mathbb{K}$. However, the relevant cocycle is represented by $\hbar$ times the 2-vertex graph (44), which cannot be "hit" because the differential increases the number of vertices by one, and all elements of $\mathfrak{grt}_1$ are represented by graphs with multiple (in fact, more than 6) vertices. (See also the following section.) □

4.11.1 Remark

The above result in particular provides us with an action of the group GRT_1 on the set of homotopy classes of $\widehat{\mathcal{L}ie^{\diamond}\mathcal{B}}_\infty$-structures on an arbitrary differential graded vector space $\mathfrak{g}$.

4.11.2. Iterative construction of graph representatives of elements of $\mathfrak{grt}_1$

The above Theorem 4.11 says that any degree 0 graph $\Gamma \in \mathsf{GC}_3^{\mathrm{or}}$ satisfying the cocycle condition $d_0\Gamma_0 = 0$ can be extended to a formal power series

$$\Gamma_\hbar = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \cdots$$

satisfying the cocycle condition $d_\hbar\Gamma_\hbar = 0$. Let us show how this inductive extension works in detail. The equation $d_\hbar^2 = 0$ implies, for any $n \geq 0$, $\sum_{\substack{n=i+j\\ i,j\geq 0}} d_i d_j = 0$, which in turn reads

$$d_0^2 = 0,$$
$$d_0 d_1 + d_1 d_0 = 0,$$
$$d_0 d_2 + d_2 d_0 + d_1^2 = 0 \qquad \text{and so on.}$$

Thus the equation $d_0\Gamma_0 = 0$ implies

$$0 = d_1 d_0 \Gamma_0 = -d_0 d_1 \Gamma_0.$$

The oriented graph $d_1\Gamma_0 \in \mathsf{GC}_3^{\mathrm{or}}$ has degree -1 and $H^{-1}(\mathsf{GC}_3^{\mathrm{or}}, d_0) \cong \mathbb{K}$. Since the one cohomology class cannot be hit (its leading term has necessarily only two vertices), there exists a degree -2 graph Γ_1 such that $d_1\Gamma_0 = -d_0\Gamma_1$ so that

$$d_\hbar(\Gamma_0 + \hbar\Gamma_1) = 0 \bmod \hbar^2.$$

Assume by induction that we constructed a degree 0 polynomial

$$\Gamma_0 + \hbar\Gamma_1 + \cdots + \hbar^n\Gamma_n \in \mathsf{GC}_3^{\mathrm{or}}[[\hbar]]$$

such that

$$d_\hbar(\Gamma_0 + \hbar\Gamma_1 + \cdots + \hbar^n\Gamma_n) = 0 \bmod \hbar^{n+1}. \tag{45}$$

Let us show that there exists an oriented graph Γ_{n+1} of degree $-2n-2$ such that

$$d_\hbar(\Gamma_0 + \hbar\Gamma_1 + \cdots + \hbar^n\Gamma_n + \hbar^{n+1}\Gamma_{n+1}) = 0 \bmod \hbar^{n+2}$$

or, equivalently, such that

$$d_0\Gamma_{n+1} + d_{n+1}\Gamma_0 + \sum_{\substack{n+1=i+j\\ i,j\geq 1}} d_i\Gamma_j = 0. \tag{46}$$

Equation (45) implies, for any $j \leq n$,

$$d_0\Gamma_j + d_j\Gamma_0 + \sum_{\substack{j=p+q\\ p,q\geq 1}} d_p\Gamma_q = 0.$$

We have

$$\begin{aligned} 0 = d_{n+1}d_0\Gamma_0 &= -d_0d_{n+1}\Gamma_0 - \sum_{\substack{n+1=i+j\\ i,j\geq 1}} d_id_j\Gamma_0 \\ &= -d_0d_{n+1}\Gamma_0 + \sum_{\substack{n+1=i+j\\ i,j\geq 1}} d_id_0\Gamma_j + \sum_{\substack{n+1=i+p+q\\ i,p,q\geq 1}} d_id_p\Gamma_q \\ &= -d_0d_{n+1}\Gamma_0 - \sum_{\substack{n+1=i+j\\ i,j\geq 1}} d_0d_i\Gamma_j - \sum_{\substack{n+1=i+p+q\\ i,p,q\geq 1}} d_id_p\Gamma_q + \sum_{\substack{n+1=i+p+q\\ i,p,q\geq 1}} d_id_p\Gamma_q \\ &= -d_0\Big(d_{n+1}\Gamma_0 + \sum_{\substack{n+1=i+j\\ i,j\geq 1}} d_i\Gamma_j\Big). \end{aligned}$$

As $H^{-1-2n}(\mathsf{GC}_3^{\mathrm{or}}, d_0) = 0$ for all $n \geq 1$, there exists a degree $-2-2n$ graph Γ_{n+1} such that the required equation (46) is satisfied. This completes an inductive construction of $\Gamma_\hbar$ from Γ_0.

4.12. Deformations of Frobenius algebra structures

Note that the complexes $\mathsf{Def}(\mathcal{F}rob_\infty^\diamond \to \mathcal{F}rob^\diamond)$ and $\mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty \to \widehat{\mathcal{L}ie\mathcal{B}})$ are isomorphic. (This is because deformation complexes of Koszul dual properads are isomorphic, up to completion issues due to dualizing infinite-dimensional vector spaces.) We hence have a zigzag of (quasi-)isomorphisms of complexes

$$\begin{aligned} \mathrm{Der}(\mathcal{F}rob_\infty^\diamond) \to \mathsf{Def}(\mathcal{F}rob_\infty^\diamond \to \mathcal{F}rob^\diamond)[1] \\ \cong \mathsf{Def}(\mathcal{L}ie\mathcal{B}_\infty \to \widehat{\mathcal{L}ie\mathcal{B}})[1] \leftarrow \mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty). \end{aligned}$$

In particular, we obtain a map†

†In fact, the first arrow is an injection and almost an isomorphism by Remark 4.5.2.

$$\mathfrak{grt}_1 \to H^0\big(\mathrm{Der}(\widehat{\mathcal{L}ie\mathcal{B}}_\infty)\big) \cong H^0\big(\mathrm{Der}(\mathcal{F}rob^\diamond_\infty)\big).$$

Hence we obtain a large class of homotopy nontrivial derivations of the properad $\mathcal{F}rob^\diamond_\infty$ and, accordingly, a large class of potentially homotopy nontrivial universal deformations of any $\mathcal{F}rob^\diamond_\infty$ algebra.

4.12.1 Remark
From the above map $\mathfrak{grt}_1 \to H^0(\mathrm{Der}(\mathcal{F}rob^\diamond_\infty))$, we obtain a map $\mathfrak{grt}_1 \to H^1(\mathsf{Def}(\mathcal{F}rob^\diamond_\infty \to \mathsf{End}_A))$ for any $\mathcal{F}rob^\diamond_\infty$ algebra A, and hence a large class of universal deformations of $\mathcal{F}rob^\diamond_\infty$ structures on A.

Next, consider the Frobenius properad $\mathcal{F}rob$, and let $\widehat{\mathcal{F}rob}$ be its genus completion. Analogously to Section 4.2, let $\widehat{\mathcal{F}rob}_\infty$ be the completion of $\mathcal{F}rob_\infty$ with respect to the total genus, and let $\mathrm{Der}(\widehat{\mathcal{F}rob}_\infty)$ be the continuous derivations. Note that the complex $\mathsf{Def}(\mathcal{F}rob_\infty \to \widehat{\mathcal{F}rob})$ is isomorphic to the complex $\mathsf{Def}(\mathcal{L}ie^\diamond\mathcal{B}_\infty \to \widehat{\mathcal{L}ie^\diamond\mathcal{B}})$. We hence obtain a zigzag of quasi-isomorphisms:

$$\begin{aligned}\mathrm{Der}(\widehat{\mathcal{F}rob}_\infty) \to \mathsf{Def}(\mathcal{F}rob_\infty \to \widehat{\mathcal{F}rob})[1] &\cong \mathsf{Def}(\mathcal{L}ie^\diamond\mathcal{B}_\infty \to \widehat{\mathcal{L}ie^\diamond\mathcal{B}})[1] \\ &\leftarrow \mathrm{Der}(\widehat{\mathcal{L}ie^\diamond\mathcal{B}_\infty}).\end{aligned}$$

In particular, we obtain a map

$$\mathfrak{grt}_1 \to H^0\big(\mathrm{Der}(\widehat{\mathcal{L}ie^\diamond\mathcal{B}_\infty})\big) \cong H^0\big(\mathrm{Der}(\widehat{\mathcal{F}rob}_\infty)\big).$$

Consider the explicit construction of representatives of $\mathfrak{grt}_1$-elements of Section 4.11.2. The $\hbar^n$-correction term Γ_n to some graph cohomology class Γ of genus g has genus $g+n$. It follows that the map $\mathfrak{grt}_1 \to H^0(\mathrm{Der}(\widehat{\mathcal{F}rob}_\infty))$ in fact factors through $H^0(\mathrm{Der}(\mathcal{F}rob_\infty))$, and in particular we have a map

$$\mathfrak{grt}_1 \to H^0\big(\mathrm{Der}(\mathcal{F}rob_\infty)\big),$$

and hence a map from $\mathfrak{grt}_1$ into the deformation complex of any $\mathcal{F}rob_\infty$ algebra.

4.12.2 Remark
From the map $\mathfrak{grt}_1 \to H^0(\mathrm{Der}(\mathcal{F}rob_\infty))$, we obtain a map $\mathfrak{grt}_1 \to H^1(\mathsf{Def}(\mathcal{F}rob_\infty \to \mathsf{End}_A))$ for any $\mathcal{F}rob_\infty$ algebra A, and hence a large class of universal deformations of $\mathcal{F}rob_\infty$ structures on A.

5. Involutive Lie bialgebras as homotopy Batalin–Vilkovisky algebras

Let $\mathfrak{g}$ be a Lie bialgebra. Then it is a well-known fact that the Chevalley–Eilenberg complex of $\mathfrak{g}$ (i.e., the cobar construction of the Lie coalgebra $\mathfrak{g}$) $\mathrm{CE}(\mathfrak{g}) = \odot^\bullet \mathfrak{g}[-1]$

carries a Gerstenhaber algebra structure. Specifically, the commutative algebra structure is the obvious one. To define the Lie bracket (of degree -1) it is sufficient to define it on the generators $\mathfrak{g}[-1]$, where it is given by the Lie bracket on $\mathfrak{g}$.

Similarly, if $\mathfrak{g}$ is a $\mathcal{L}ie^{\diamond}\mathcal{B}$ algebra, then $\mathrm{CE}(\mathfrak{g}) = \odot^{\bullet}\mathfrak{g}[-1]$ carries a natural Batalin–Vilkovisky (BV) algebra structure. The product and Lie bracket are as before. The BV operator Δ is defined on a word $x_1 \cdots x_n$ as

$$\Delta(x_1 \cdots x_n) = -\sum_{i<j} (-1)^{i+j} [x_i, x_j] x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n.$$

The involutivity condition is needed for the BV operator to be compatible with the differential. Now suppose that $\mathfrak{g}$ is a $\mathcal{L}ie\mathcal{B}_\infty$ algebra. We call it *good* if, for any fixed m, only finitely many of the generating operations $\mu_{m,n} \in \mathrm{Hom}(\mathfrak{g}^{\otimes n}, \mathfrak{g}^{\otimes m})$ are nonzero. Then one may define the Chevalley complex $\mathrm{CE}(\mathfrak{g}) = \odot^{\bullet}\mathfrak{g}[-1]$ of $\mathfrak{g}$ as a $\mathcal{L}ie_\infty$ coalgebra. It is known (see, e.g., [42, Remark 1]) that $\mathrm{CE}(\mathfrak{g}) = \odot^{\bullet}\mathfrak{g}[-1]$ carries a natural homotopy Gerstenhaber structure. In this section, we show that, similarly, if $\mathfrak{g}$ is a good $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$ algebra, then the Chevalley–Eilenberg complex $\mathrm{CE}(\mathfrak{g})$ carries a natural homotopy BV algebra structure.

5.1. The order of an operator

Let V be a graded commutative algebra. For a linear operator $D : V \to V$, define a collection

$$\begin{aligned} F_n^D : \otimes^n V &\longrightarrow V, \\ v_1 \otimes \cdots \otimes v_n &\longrightarrow F_n^D(v_1, \ldots, v_n) \end{aligned}$$

of linear maps by induction: $F_1^D = D$,

$$\begin{aligned} &F_{n+1}^D(v_1, \ldots, v_{n-1}, v_n, v_{n+1}) \\ &\quad = F_n^D(v_1, \ldots, v_{n-1}, v_n \cdot v_{n+1}) - F_n^D(v_1, \ldots, v_n) \cdot v_{n+1} \\ &\qquad - (-1)^{|v_n||v_{n+1}|} F_n^D(v_1, \ldots, v_{n-1}, v_{n+1}) \cdot v_n. \end{aligned}$$

The operator D is said to have *order* $\leq n$ if $F_{n+1}^D = 0$.

The operators F_n^D are in fact graded symmetric; moreover, if D is a differential in V (i.e., $|D| = 1$ and $D^2 = 0$), then the collection $\{F_n^D : \odot^n V \to V\}_{n\geq 1}$ defines a $\mathcal{L}ie_\infty$-structure on the space $V[-1]$ (see [24]). Indeed, consider a graded Lie algebra

$$\mathrm{CoDer}(\otimes^{\bullet \geq 1} V) \cong \prod_{n\geq 1} \mathrm{Hom}_{\mathbb{K}}(\otimes^n V, V)$$

of coderivations of the tensor coalgebra $\otimes^{\bullet\geq 1} V$. As the differential $D : V \to V$ is a Maurer–Cartan element in this Lie algebra and the multiplication $\mu : \odot^2 V \to V$ is its degree 0 element, we can gauge transform D,

$$D \longrightarrow F^D := e^{-\mu} D e^{\mu} = \sum_{n=0}^{\infty} \frac{1}{n!}\big[\ldots\big[[D,\mu],\mu\big],\ldots,\mu\big],$$

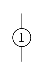
into a less trivial codifferential whose associated components

$$F^D = \left\{ F^D_{n+1} = \frac{1}{n!}\big[\ldots\big[[D,\mu],\mu\big],\ldots,\mu\big] : \odot^{n+1} V \to V \right\}_{n\geq 0}$$

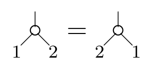
coincide precisely with the above defined tensors F^D_{n+1} which measure a failure of D to respect the multiplication operation in V. There is a standard symmetrization functor which associates to any $\mathcal{A}_\infty$ algebra an associated $\mathcal{L}ie_\infty$ algebra; as the tensors F^D_{n+1} are already graded symmetric (see [24]), the collection $\{F^D_{n+1}\}_{n\geq 1}$ gives us a $\mathcal{L}ie_\infty$ structure in $V[-1]$, as required. (Some of these arguments appeared also in [10], [27].)

5.2. *Batalin–Vilkovisky algebras*

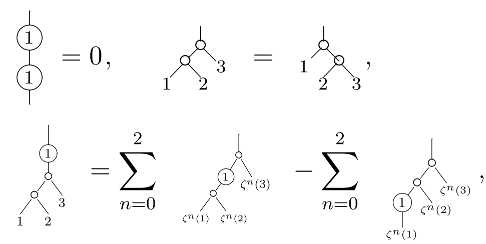
A *Batalin–Vilkovisky algebra* is, by definition, a graded commutative algebra V equipped with a degree -1 operator $\Delta : V \to V$ of order ≤ 2 such that $\Delta^2 = 0$. Denote by $\mathcal{BV}$ the operad whose representations are Batalin–Vilkovisky algebras. This is, therefore, a graded operad generated by corollas



and



of homological degrees -1 and 0, respectively, modulo the following relations:



where ζ is the cyclic permutation (123). A nice nonminimal cofibrant resolution of the operad $\mathcal{BV}$ has been constructed in [15]. We denote this resolution by $\mathcal{BV}^K_\infty$ in this paper, K standing for *Koszul*. The minimal resolution, $\mathcal{BV}_\infty$, has been constructed in [12].

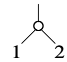
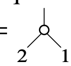
5.3. *An operad* $\mathcal{BV}^{\mathrm{com}}_\infty$

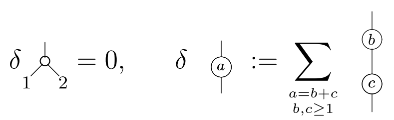
A $\mathcal{BV}^{\mathrm{com}}_\infty$-algebra is, by definition (see [24]), a differential graded commutative algebra (V,d) equipped with a countable collection of homogeneous linear maps $\{\Delta_a :$

$V \to V, |\Delta_a| = 1 - 2a\}_{a\geq 1}$ such that each Δ_a is of order $\leq a+1$ and the equations

$$\sum_{a=0}^{n} \Delta_a \circ \Delta_{n-a} = 0 \tag{47}$$

hold for any $n \in \mathbb{N}$, where $\Delta_0 := -d$.

Let $\mathcal{BV}_\infty^{\mathrm{com}}$ be the dg operad of $\mathcal{BV}_\infty^{\mathrm{com}}$-algebras. This operad is a quotient of the free operad generated by one binary operation in degree 0,  = , and a countable family of unary operations, $\{$$\}_{a\geq 1}$ (of homological degree $1-2a$), modulo the ideal I generated by the associativity relations for the binary operation  and the compatibility relations between the latter and unary operations encoding the requirement that each unary operation  be of order $\leq a+1$ with respect to the multiplication operation. The differential δ in the operad $\mathcal{BV}_\infty^{\mathrm{com}}$ is given by



(48)

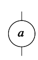
There is an explicit morphism of dg operads (see [15, Proposition 23])[†]

$$\mathcal{BV}_\infty^{K} \longrightarrow \mathcal{BV}_\infty^{\mathrm{com}},$$

which implies the existence of a morphism of dg operads $\mathcal{BV}_\infty \to \mathcal{BV}_\infty^{\mathrm{com}}$. The existence of such a morphism follows also from the following theorem, whose proof is given in Appendix B.

5.3.1 THEOREM
The dg operad $\mathcal{BV}_\infty^{\mathrm{com}}$ is formal with the cohomology operad $H^\bullet(\mathcal{BV}_\infty^{\mathrm{com}})$ isomorphic to the operad, $\mathcal{BV}$, of Batalin–Vilkovisky algebras; that is, there is a canonical surjective quasi-isomorphism of operads

$$\pi : \mathcal{BV}_\infty^{\mathrm{com}} \longrightarrow \mathcal{BV}$$

which sends to zero all generators  *with $a \geq 2$.*

5.4. *From strongly homotopy involutive Lie bialgebras to $\mathcal{BV}_\infty$-algebras*

We call a $\mathcal{L}ie^\diamond\mathcal{B}_\infty$ algebra $\mathfrak{g}$ *good* if for any fixed m and k, only finitely many of the operations $\mu_{m,n}^k \in \mathrm{Hom}(\mathfrak{g}^{\otimes n}, \mathfrak{g}^{\otimes m})$ are nonzero. In this case, we define the Chevalley–Eilenberg complex $\mathrm{CE}(\mathfrak{g}) = \odot^\bullet(\mathfrak{g}[-1])$ of $\mathfrak{g}$ as a $\mathcal{L}ie_\infty$ coalgebra. More

[†]We are grateful to Bruno Vallette for pointing out this result to us.

concretely, for a finite-dimensional $\mathfrak{g}$ we may understand the $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$ algebra structure as a formal power series $\Gamma_\hbar = \Gamma_\hbar(\psi_i, \eta^i, \hbar)$ as explained in Section 3.1. Using similar notation, we may understand the space $\mathrm{CE}(\mathfrak{g})$ as the space of polynomials in the variables ψ_i. Then the differential Δ_0 on $\mathrm{CE}(\mathfrak{g})$ is given by the formula

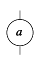
$$\Delta_0 := \sum_i \frac{\partial \Gamma_\hbar}{\partial \eta^i}\Big|_{\hbar=\eta^i=0} \frac{\partial}{\partial \psi_i}.$$

5.4.1 PROPOSITION

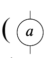
Let $\mathfrak{g}$ be a good $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$ algebra, with the $\mathcal{L}ie^{\diamond}\mathcal{B}_\infty$ algebra structure being defined as a power series $\Gamma_\hbar = \Gamma_\hbar(\psi_i, \eta^i, \hbar)$ as explained in Section 3.1. Then there is a natural $\mathcal{BV}^{\mathrm{com}}_\infty$ algebra structure ρ on the complex $\mathrm{CE}(\mathfrak{g})$ *given by the formulas*

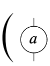
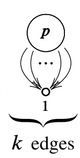
$\rho\Big($  $\Big) :=$ the standard multiplication in $\odot^\bullet(\mathfrak{g}[-1])[[\hbar]]$

and, for any $a \geq 1$,

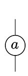
$\rho\Big($  $\Big) :=$ $$\sum_{\substack{p+k=a+1\\ k\geq 1, p\geq 0}} \frac{1}{p!k!} \sum_{i_1,\dots,i_k} \frac{\partial^{a+1}\Gamma_\hbar}{\partial^p \hbar \partial\eta^{i_1}\cdots\partial\eta^{i_k}}\Big|_{\hbar=\eta^i=0} \frac{\partial^k}{\partial\psi_{i_1}\cdots\partial\psi_{i_k}}.$$

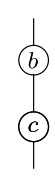
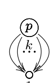
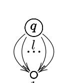
Proof

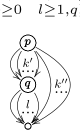
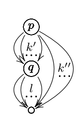
It is clear that $\Delta_a := \rho($  $)$ is an operator of order $\leq a+1$ with respect to the standard multiplication in the graded commutative algebra $\odot^\bullet(\mathfrak{g}[-1])$. The verification that the operators $\{\Delta_a\}_{a\geq 0}$ satisfy identities (47) is best done pictorially. We represent the expression on the right-hand side by the picture

$\rho\Big($  $\Big) = \sum_{\substack{a+1=p+k\\ k\geq 1, p\geq 0}}$  .

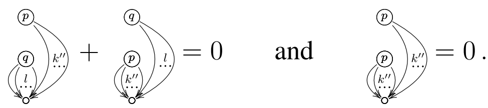
Then we compute

$\rho\Big(\delta$  $\Big)$

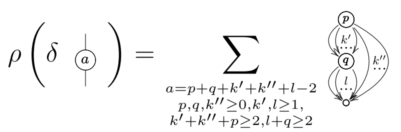
$= \sum_{\substack{a=b+c\\ b,c\geq 1}} \rho\Big($  $\Big) = \sum_{\substack{a=b+c\\ b,c\geq 1}} \sum_{\substack{b+1=p+k\\ k\geq 1, p\geq 0}} \sum_{\substack{c+1=q+l\\ l\geq 1, q\geq 0}} \Big($  $\Big) \circ_1 \Big($  $\Big)$

$= \sum_{\substack{a=b+c\\ b,c\geq 1}} \sum_{\substack{b+1=p+k\\ k\geq 1}} \sum_{\substack{c+1=q+l\\ l\geq 1}} \sum_{k=k'+k''}$  $= \sum_{\substack{a=p+q+k'+k''+l-2\\ p,q,k',k''\geq 0, k'+k'', l\geq 1,\\ k'+k''+p\geq 2, l+q\geq 2}}$  .

Here we hide the binomial prefactors in the notation by assuming that a picture is preceded by a factor $\frac{1}{p!q!|G|}$, where G is the symmetry group of the picture. Note that terms with $k' = 0$ can be dropped from the sum on the right-hand side by symmetry. Indeed, the piece $k' = 0$ of the sum is symmetric under interchange of (p, k'') and (q, l), and can hence be written as a linear combination of anticommutators of anticommuting operators. Pictorially, the signs are best verified by thinking of the vertices in the pictures as being odd, so that, in particular, graphs which have symmetries acting by an odd permutation on the vertices vanish, and we have

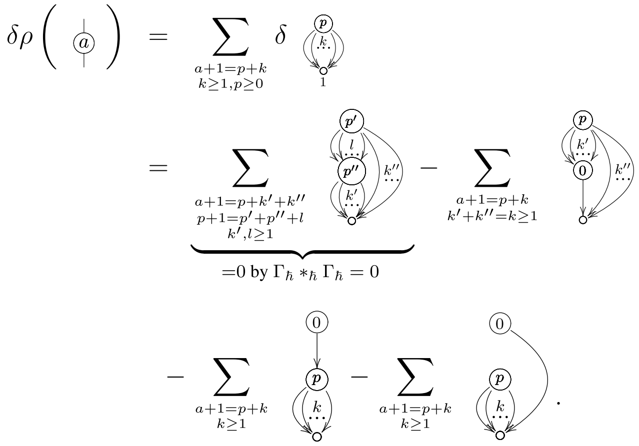


We hence find



(49)

On the other hand,†



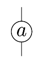
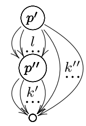
There are several cancellations in this expression. First, the terms with $k' = 0$ from the second sum cancel the fourth sum by the same symmetry argument as above.

†To see the vanishing of the first term in the second line, consider the (zero) differential operator

$$f \mapsto \left((\Gamma_\hbar *_\hbar \Gamma_\hbar) *_\hbar f\right)_{\eta^i=0}$$

for $f \in K[[\psi^i, \hbar]]$. The depicted terms then correspond to the coefficient of $\hbar^a$ of this vanishing differential operator.

The remaining terms of the second sum together with the third sum kill those terms of the first sum for which either $p''=0$, $k'=1$ or $p'=0$, $l=1$, $k''=0$. We hence find that

$$\delta\rho\Big($$  $$\Big) = \sum_{\substack{a=p'+p''+l+k'+k''-2\\ k',l\geq 1\\ p''+k'\geq 2,\, p'+l+k''\geq 2}}$$  .

Comparing this formula with (49), we see that both expressions agree, up to a relabeling of the summation indices. □

Appendices

A. Proof of Proposition 2.8.1

In this section, we show that the quadratic algebra $\mathcal{A}_n$ of Section 2.8 is Koszul. In fact, we show the equivalent statement that the Koszul dual algebra $B_n = \mathcal{A}_n^!$ is Koszul. Specifically, B_n is the algebra generated by $V = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$ with relations $x_i x_j = 0$ if $|i-j| \neq 1$ and $x_i x_{i+1} = -x_{i+1}x_i$.

We denote by $C_n \otimes_\kappa B_n$ the Koszul complex of B_n (see [22] for details), that is, the complex $(C_n \otimes B_n, d = d_\kappa)$, where $C_n = B_n^{¡}$ is the coalgebra generated by the elements x_i in degree 1, with quadratic correlations $R = \operatorname{span}(\{x_i x_j \mid |i-j| \neq 1\} \cup \{x_i x_{i+1} + x_{i+1}x_i\})$, and the differential is induced by the degree -1 map $\kappa : C_n \to B_n$ that is zero everywhere except on V, where it identifies $V \subset C_n$ with $V \subset B_n$. Informally, the differential acts by "jumping" the tensor product over the x_i on its left, producing a sign coming from the degree in C_n.

Note that B_n and C_n are weight-graded and that the weight k component of B_n, $B_n^{(k)}$ is zero if $k \geq 3$.

The result will follow from the acyclicity of the Koszul complex, which will in turn be shown by constructing a contracting homotopy h.

Let $l \geq 1$, let w be a word of length $l-1$ on the variables x_i, and let $1 \leq a, b \leq n$ be indices such that $|a-b| = 1$. We define a degree 1 map $h\colon V^{\otimes l} \otimes B_n^{(2)} \to V^{\otimes l+1} \otimes B_n^{(1)}$ for $n \geq 1$ by

$$h(wx_k \otimes x_a x_b) = \begin{cases} \frac{(-1)^l}{2}(wx_k x_a \otimes x_b - wx_k x_b \otimes x_a) \\ \quad \text{if } |k-a| \neq 1 \text{ and } |k-b| \neq 1, \\ (-1)^l wx_k x_a \otimes x_b \\ \quad \text{if } |k-b| = 1. \end{cases}$$

Note that all the cases are covered because $|a-b|=1 \wedge |k-b|=1 \Rightarrow |k-a| \neq 1$. Moreover, due to the antisymmetry in $B_n^{(2)}$, if $|k-a|=1$, we have $h(wx_k \otimes x_a x_b) = -(-1)^l wx_k x_b \otimes x_a$.

If $l=0$, we define $h\colon B_n^{(2)} \to V \otimes B_n^{(1)}$ using the first formula from above; that is, we consider the nondefined differences to be different from 1 and ignore the nonexistent variables.

A.0.1 LEMMA

The map h restricts to a function $C_n^{(l)} \otimes B_n^{(2)} \to C_n^{(l+1)} \otimes B_n^{(1)}$ that satisfies $dh = \mathrm{id}_{C_n^{(l)} \otimes B_n^{(2)}}$.

Proof

Recall that $C_n^{(1)} = V$, and $C_n^{(l)} = \bigcap_{a+b=l-2} V^{\otimes a} R V^{\otimes b}$ for $l \geq 2$.

First note that h maps $V^{\otimes a} R V^{\otimes b} \otimes B_n^{(2)}$ to $V^{\otimes a} R V^{\otimes b+1} \otimes B_n^{(1)}$, since it leaves the elements in R unaltered. It is also clear by the construction of h that the image of an element of $V^{\otimes l} \otimes B_n^{(2)}$ lands in $V^{\otimes l-1} R \otimes B_n^{(1)}$; therefore h restricts indeed to a map $C_n^{(l)} \otimes B_n^{(2)} \to C_n^{(l+1)} \otimes B_n^{(1)}$.

To check the identity $dh = \mathrm{id}_{C_n^{(l)} \otimes B_n^{(2)}}$, suppose first that both $|k-a|$ and $|k-b|$ are different from 1:

$$\begin{aligned} dh(wx_k \otimes x_a x_b) &= \frac{1}{2} d\big((-1)^l wx_k x_a \otimes x_b - (-1)^l wx_k x_b \otimes x_a\big) \\ &= \frac{1}{2} wx_k \otimes x_a x_b - \frac{1}{2} wx_k \otimes x_b x_a = wx_k \otimes x_a x_b. \end{aligned}$$

If $|k-b|=1$, $dh(wx_k \otimes x_a x_b) = d((-1)^l wx_k x_a \otimes x_b) = wx_k \otimes x_a x_b$. An analogous calculation holds if $|k-a|=1$. $\square$

To define $h\colon C_n^{(l)} \otimes B_n^{(1)} \to C_n^{(l+1)} \otimes B_n^{(0)} = C^{(l+1)}$, as before we define it on $V^{\otimes l} \otimes B_n^{(1)}$ and verify that it restricts properly.

Let $l \geq 2$, and let us denote by w some word on x_i of length $l-2$. We define $h\colon V^{\otimes l} \otimes B_n^{(1)} \to V^{\otimes l+1}$ by

$$h(wx_k x_a \otimes x_b) = \begin{cases} (-1)^l wx_k x_a x_b & \text{if } |a-b| \neq 1, \\ \frac{(-1)^l}{2} wx_k(x_a x_b + x_b x_a) & \text{if } |a-b| = 1, |a-k| \neq 1 \text{ and} \\ & \quad |b-k| \neq 1, \\ 0 & \text{if } |a-b|=1 \text{ and } |b-k|=1, \\ (-1)^l wx_k(x_a x_b + x_b x_a) & \text{if } |a-b|=1 \text{ and } |a-k|=1. \end{cases}$$

Interpret this definition for $l < 2$ in the following way. Whenever some difference is not defined because a or k are not defined, take the case in the definition where the absolute value of the difference is different from 1 and ignore the nonexistent variables.

A.0.2 LEMMA
We have that h restricts to a function $C_n^{(l)} \otimes B_n^{(1)} \to C_n^{(l+1)}$ that satisfies $dh + hd = \mathrm{id}_{C_n^{(l)} \otimes B_n^{(1)}}$.

Proof
Note that, by construction, the image of h sits inside $V^{\otimes l-1}R$. The R part in $V^{\otimes a}RV^{\otimes b}$ is left unaltered by h if b is at least 1; hence $V^{\otimes a}RV^{\otimes b} \otimes B_n^{(1)}$ is sent to $V^{\otimes a}RV^{\otimes b+1}$.

Let us suppose that l is at least 2, and let us check that $V^{\otimes l-2}R \otimes B_n^{(1)}$ is sent to $V^{\otimes l-2}RV$.

Let w be a word in the variables x_i of length $l-2$. We have that $V^{\otimes l-2}R \otimes B_n^{(1)}$ is spanned by elements of the form $wx_kx_a \otimes x_b$, with $|k-a| \neq 1$, and by elements of the form $w(x_kx_a + x_ax_k) \otimes x_b$, with $|k-a| = 1$.

Let us consider first the first type of elements. If $|a-b| \neq 1$, $h(wx_kx_a \otimes x_b) = (-1)^l wx_kx_ax_b \in V^{\otimes l-2}RV$.

If, on the other hand, $|a-b| = 1$, then either $|b-k| = 1$ and the image via h is zero, or $|b-k| \neq 1$ and both summands of $h(wx_kx_a \otimes x_b) = \frac{(-1)^l}{2} wx_kx_ax_b + \frac{(-1)^l}{2} wx_kx_bx_a$ belong to $V^{\otimes l-2}RV$.

Let us now consider the elements of the form $w(x_kx_a + x_ax_k) \otimes x_b$ with $|k-a| = 1$. If both $|b-k|$ and $|b-a|$ are different from 1, then $h(w(x_kx_a + x_ax_k) \otimes x_b) = (-1)^l w(x_kx_a + x_ax_k)x_b$ is in $V^{\otimes l-2}RV$.

Otherwise, let us assume without loss of generality that $|b-a| = 1$ (and therefore $|b-k| \neq 1$). Then

$$\begin{aligned} &h\big(w(x_kx_a + x_ax_k) \otimes x_b\big) \\ &\quad = (-1)^l \big(wx_k(x_ax_b + x_bx_a) + wx_ax_kx_b\big) \\ &\quad = (-1)^l \big(w(x_kx_a + x_ax_k)x_b + wx_kx_bx_a\big) \in V^{\otimes l-2}RV. \end{aligned}$$

Let us now show the homotopy equation. As before, we consider a generic element $wx_kx_a \otimes x_b \in V^{\otimes l} \otimes B_n^{(1)}$ and we divide the verification into various cases.

If $|a-b| \neq 1$,

$$dh(wx_kx_a \otimes x_b) + hd(wx_kx_a \otimes x_b) = wx_kx_a \otimes x_b + 0.$$

If $|a-b| = 1$ and $|a-k| \neq 1$ and $|b-k| \neq 1$,

$$\begin{aligned}&(dh+hd)(wx_kx_a\otimes x_b)\\&\quad=\frac{1}{2}(wx_kx_a\otimes x_b+wx_kx_b\otimes x_a)+\frac{1}{2}(wx_kx_a\otimes x_b-wx_kx_b\otimes x_a)\\&\quad=wx_kx_a\otimes x_b.\end{aligned}$$

If $|a-b|=1$ and $|b-k|=1$,

$$(dh+hd)(wx_kx_a\otimes x_b)=0+wx_kx_a\otimes x_b.$$

If $|a-b|=1$ and $|a-k|=1$,

$$\begin{aligned}(dh+hd)(wx_kx_a\otimes x_b)&=(wx_kx_a\otimes x_b+wx_kx_b\otimes x_a)-wx_kx_b\otimes x_a\\&=wx_kx_a\otimes x_b.\end{aligned}$$

The cases with $l<2$ can be easily checked. □

With the construction of the map h finished, Proposition 2.8.1 follows from the next lemma, which, together with the previous lemmas in this section, shows that h is a contracting homotopy.

A.0.3 LEMMA
It holds that $hd=\mathrm{id}_{C_n^l\otimes B_n^{(0)}}$.

Proof
If $l<2$, the statement is clear. Let $l\geq 2$, and let w be a word in the variables x_i of length $l-2$.

It suffices to check the equation on elements of the form wx_ax_b, with $|a-b|\neq 1$, and on elements of the form $w(x_ax_b+x_bx_a)$, with $|a-b|=1$.

For the first type of elements, it is clear that $hd(wx_ax_b)=wx_ax_b$.

For the second type of elements, if l is at least 3 and $w=w'x_k$ with both $|a-k|$ and $|b-k|$ different from 1, then

$$hd\big(w(x_ax_b+x_bx_a)\big)=\frac{1}{2}w(x_ax_b+x_bx_a)+\frac{1}{2}w(x_bx_a+x_ax_b)=w(x_ax_b+x_bx_a).$$

The same calculation holds if $l=2$.

For the remaining case where (without loss of generality) $|a-k|=1$, we have

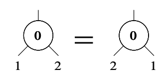
$$hd(wx_ax_b+wx_bx_a)=w(x_ax_b+x_bx_a)+0.$$ □

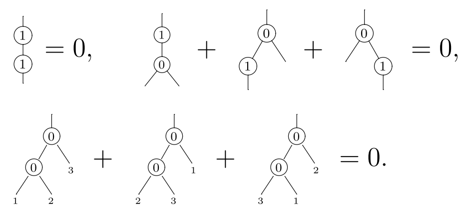
A.1 Remark
After the submission of this manuscript, Jan-Erik Roos provided us with a nicer and

shorter proof of the Koszulness of the algebras $\mathcal{A}_n$. If one uses the ordering of the generators $x_2, x_1, x_4, x_3, \ldots, x_{2k}, x_{2k-1}$ (for $n = 2k$ even) or $x_2, x_1, x_4, x_3, \ldots, x_{2k}$, $x_{2k-1}x_{2k+1}$ (for $n = 2k + 1$ odd) instead of the standard ordering, then there is a finite quadratic Gröbner basis. Alternatively, one can see using the above ordering of the generators that the relations form a confluent rewriting system and hence $\mathcal{A}_n$ is Koszul.

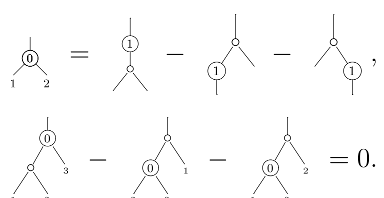
B. Computation of the cohomology of the operad $\mathcal{BV}_\infty^{\mathrm{com}}$

B.1. An equivalent definition of the operad $\mathcal{BV}$

Let $\mathcal{L}^\diamond$ be an operad generated by two degree -1 corollas,  and , subject to the following relations:

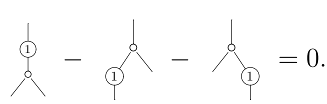


Let $\mathcal{C}om$ be the operad of commutative algebras with the generator controlling the graded commutative multiplication denoted by . Define an operad $\mathcal{BV}$ of Batalin–Vilkovisky algebras as the free operad generated by operads $\mathcal{L}^\diamond$ and $\mathcal{C}om$ modulo the following relations:



(50)

In fact, the second relation in (50) follows from the previous ones. We keep the relation in order to define, following [15], an operad $q\mathcal{BV}$ as an operad freely generated by $\mathcal{L}^\diamond$ and $\mathcal{C}om$ modulo a version of relations (50) in which the first relation is replaced by the following one:



Being a quotient of a free operad, the operad $\mathcal{BV}$ inherits an increasing filtration by the number of vertices in the trees. It is clear that there is a morphism

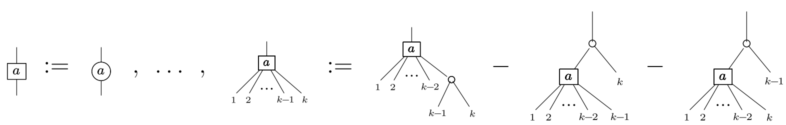
$$g : q\mathcal{BV} \longrightarrow \mathrm{gr}(\mathcal{BV})$$

from $q\mathcal{BV}$ into the associated graded operad.

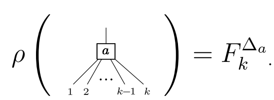
B.2 PROPOSITION ([15, Theorem 7])
The morphism $g : q\mathcal{BV} \longrightarrow \mathrm{gr}(\mathcal{BV})$ *is an isomorphism.*

B.3 Remark
The relations in the operad $q\mathcal{BV}$ are homogeneous. It is easy to see that, as an $\mathbb{S}$-module, $q\mathcal{BV}$ is isomorphic to $\mathcal{C}om \circ \mathcal{L}^{\diamond}$, the vector space spanned by graphs from $\mathcal{C}om$ whose legs are decorated with elements from $\mathcal{L}^{\diamond}$.

B.4. An auxiliary dg operad
For any natural number $a \geq 1$, define by induction (over the number, $k = 1, 2, \dots, a+1$, of input legs) a collection of $a+1$ elements

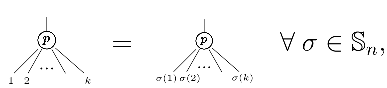

(51)

of the operad $\mathcal{BV}_\infty^{\mathrm{com}}$. If $\rho : \mathcal{BV}_\infty^{\mathrm{com}} \to \mathcal{E}nd_V$ is a representation, then, in the notation of Section 5.1,

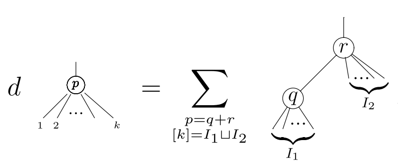


Note that $F_k^{\Delta_a}$ identically vanishes for $k \geq a+2$ as the operator Δ_a is, by its definition, of order $\leq a+1$; this is the reason why we defined the above elements of $\mathcal{BV}_\infty^{\mathrm{com}}$ only in the range $1 \leq k \leq a+1$. For all other k these elements vanish identically due to the relations between the generators of $\mathcal{BV}_\infty^{\mathrm{com}}$.

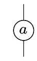
Consider next a free operad, $\mathcal{L}_\infty^{\diamond}$, generated, for all integers $p \geq 0$, $k \geq 1$ with $p + k \geq 2$, by the symmetric corollas



of homological degree $3 - 2k - 2p$, and equipped with the following differential:



Representations, $\rho : \mathcal{L}^{\diamond}_{\infty} \to \mathcal{E}nd_V$, of this operad in a dg vector space (V, d) are the same thing as continuous representations of the operad $\mathcal{L}ie_{\infty}\{1\}[[\hbar]]$ in the topological vector space $V[[\hbar]]$ equipped with the differential

$$-d + \sum_{p\geq 1} \hbar^p \Delta_p, \qquad \Delta_p := \rho\Big($$ 

$$\Big),$$

where the formal parameter $\hbar$ is assumed to have homological degree 2.

B.4.1 PROPOSITION

The cohomology of the dg operad $\mathcal{L}^{\diamond}_{\infty}$ is the operad $\mathcal{L}^{\diamond}$ defined in Section B.1; that is, $\mathcal{L}^{\diamond}_{\infty}$ is a minimal resolution of $\mathcal{L}^{\diamond}$.

Proof

The dg operad $\mathcal{L}^{\diamond}_{\infty}\{1\}$ is a direct summand of the graded properad $\mathrm{gr}\mathcal{L}ie^{\diamond}\mathcal{B}_{\infty}$ associated with the genus filtration of the properad $\mathcal{L}ie^{\diamond}\mathcal{B}_{\infty}$. Hence the required result follows from the proof of Proposition 2.7.1. $\square$

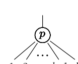
We are interested in the operad $\mathcal{L}^{\diamond}_{\infty}$ because of the following property.

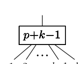
B.4.2 LEMMA

There is a monomorphism of dg operads,

$$\chi : \mathcal{L}^{\diamond}_{\infty} \longrightarrow \mathcal{BV}^{\mathrm{com}}_{\infty},$$

given on generators as follows:

$$\chi\Bigg($$ 

$$\Bigg) =$$ 

.

Proof

For notational reasons, we prove the proposition in terms of representations: for any representation $\rho : \mathcal{BV}^{\mathrm{com}}_{\infty} \to \mathcal{E}nd_V$, we construct an associated representation $\rho' : \mathcal{L}^{\diamond}_{\infty} \to \mathcal{E}nd_V$ such that $\rho' = \rho \circ \chi$.

Let $\{\Delta_a : V \to V[1-2a], \mu : \odot^2 V \to V\}_{a\geq 1}$ be a $\mathcal{BV}^{\mathrm{com}}_{\infty}$-structure in a dg vector space (V, d). Then

$$\Delta := -d + \sum_{a\geq 1} \hbar^a \Delta_a$$

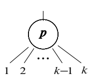
is a degree 1 differential in the graded vector space $V[[\hbar]]$, $\hbar$ being a formal parameter of homological degree 2. As explained in Section 5.1, this differential makes the graded commutative algebra $(V[[\hbar]], \mu)$ into a $\mathcal{L}ie_\infty\{1\}[[\hbar]]$ algebra over the ring $\mathbb{K}[[\hbar]]$, with higher Lie brackets given by

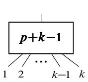
$$F_k^\Delta := \frac{1}{(k-1)!} \underbrace{\big[\ldots\big[[\Delta, \mu], \mu\big], \ldots, \mu\big]}_{k-1 \text{ brackets}}$$
$$= \frac{\hbar^{k-1}}{(k-1)!} \sum_{p=0}^{\infty} \hbar^p \underbrace{\big[\ldots\big[[\Delta_{p+k-1}, \mu], \mu\big], \ldots, \mu\big]}_{k-1 \text{ brackets}}.$$

It is well known that $\mathcal{L}ie_\infty$-algebra structures are homogeneous in the sense that if $\{\mu_n\}_{n\geq 1}$ is a $\mathcal{L}ie_\infty$-algebra structure in some vector space, then, for any $\lambda \in \mathbb{K}$, the collection $\{\lambda^{n-1}\mu_n\}_{n\geq 1}$ is again a $\mathcal{L}ie_\infty$-algebra structure in the same space. Therefore, the rescaled collection of operations

$$\hat{F}_k^\Delta := \frac{1}{(k-1)!} \sum_{p=0}^{\infty} \hbar^p \underbrace{\big[\ldots\big[[\Delta_{p+k-1}, \mu], \mu\big], \ldots, \mu\big]}_{k-1 \text{ brackets}}$$

also defines a continuous representation of $\mathcal{L}ie_\infty\{1\}[[\hbar]]$ in the dg space $(V[[\hbar]], \Delta)$, and hence a representation of $\mathcal{L}_\infty^\diamond$ in the space (V, d) given on the generators as follows:

$\rho'\Big($  $\Big) = \frac{1}{(k-1)!} \underbrace{\big[\ldots\big[[\Delta_{p+k-1}, \mu], \mu\big], \ldots, \mu\big]}_{k-1 \text{ brackets}}.$

As $\rho\Big($  $\Big)$ equals $\frac{1}{(k-1)!} \underbrace{[\ldots[[\Delta_{p+k-1}, \mu], \mu], \ldots, \mu]}_{k-1 \text{ brackets}}$ as well, the proof is completed. □

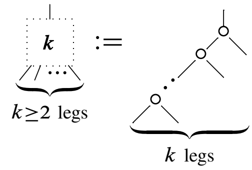
Finally, we can give the proof of Theorem 5.3.1.

Proof of Theorem 5.3.1
The map π obviously induces a morphism of operads:

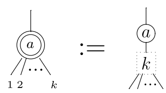
$$[\pi] : H^\bullet(\mathcal{BV}_\infty^{\mathrm{com}}) \longrightarrow \mathcal{BV}.$$

Therefore, to prove the theorem it is enough to show that $[\pi]$ induces an isomorphism of $\mathbb{S}$-modules.

Denote the following (equivalence class of a) graph in $\mathcal{BV}_\infty^{\mathrm{com}}$ by

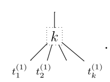


and call it a *dashed square vertex*. Consider an operad, $\mathcal{O}$, freely generated by the operad $\mathcal{C}om$ and a countable family of unary operations, $\{$ ⓐ $\}_{a\geq 1}$ of homological degree $1-2a$ equipped with the differential (48); the properad $\mathcal{BV}_{\infty}^{\mathrm{com}}$ is the quotient of $\mathcal{O}$ by the ideal encoding the requirement that each unary operation ⓐ be of order $\leq a+1$ with respect to the multiplication operation. Let $t^{(1)}$ be any tree built from the following "corollas,"



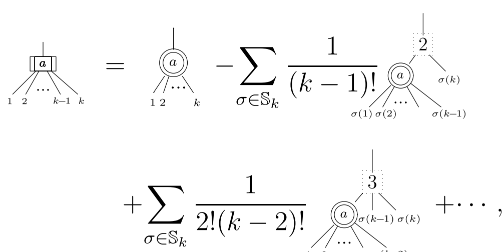
with $k\geq 1, a\geq 1$

(where we assume implicitly that for $k=1$ the left-hand side corolla equals ⓐ), and let $t^{(2)}$ be any graph obtained by attaching to one or more (or none) input legs of a dashed square vertex a tree of the type $t^{(1)}$, for example,

$t^{(2)} =$ 

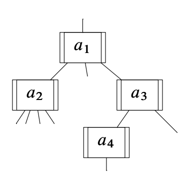
The family $\{t^{(1)}, t^{(2)}\}$ forms a basis of $\mathcal{O}$ as an $\mathbb{S}$-module.

Define for any $a, k\geq 1$ a linear combination



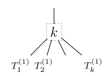
(52)

and consider (i) a set $\{T^{(1)}\}$ of all possible trees generated by these "square" corollas, for example,

$T^{(1)} =$ 

,

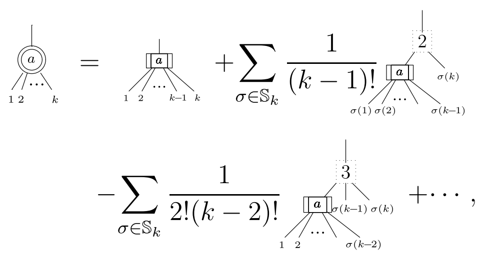
and also (ii) a set $\{T^{(2)}\}$ of all possible trees obtained by attaching to (some) legs of a dashed square vertex trees from the set $\{T^{(1)}\}$, for example,

$T^{(2)} =$  .

Formulas (52) define a natural linear map of $\mathbb{S}$-modules:

$$\phi : \mathrm{span}\langle T^{(1)}, T^{(2)}\rangle \longrightarrow \mathrm{span}\langle t^{(1)}, t^{(2)}\rangle = \mathcal{O}.$$

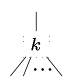
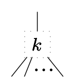
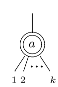
The expressions (52) can be (inductively) inverted,

 (53)

and hence give us, again by induction, a linear map

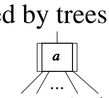
$$\psi : \mathrm{span}\langle t^{(1)}, t^{(2)}\rangle \longrightarrow \mathrm{span}\langle T^{(1)}, T^{(2)}\rangle$$

as follows. On 1-vertex trees from the family $\{t^{(1)}, t^{(2)}\}$ the map ψ is given by

$\psi\Big($  $\Big) =$  , $\quad \psi\Big($  $\Big) =$ the RHS of (53).

Assume that the map ψ is constructed on n-vertex trees from the family $\{t^{(1)}, t^{(2)}\}$. Let t be a tree with $n+1$ vertices. The complement to the root vertex of t is a disjoint union of trees, $\{t'\}$, with at most n vertices. To get $\psi(t)$, apply first ψ to the subtrees t' to get a linear combination of trees, $\sum t''$, where each t'' is obtained by attaching to (some) input legs of a 1-vertex tree v from $\{t^{(1)}, t^{(2)}\}$ an element of the set $\{T^{(1)}, T^{(2)}\}$; finally, apply ψ to the root vertex v of each summand t''. By construction, $\psi \circ \phi = \mathrm{Id}$ and $\phi \circ \psi = \mathrm{Id}$ so that the map ϕ is an isomorphism of $\mathbb{S}$-modules

$$\mathcal{O} \cong \mathrm{span}\langle T^{(1)}, T^{(2)}\rangle.$$

The ideal I in $\mathcal{O}$ defining the operad $\mathcal{BV}_\infty^{\mathrm{com}}$ now takes a very simple form—this is an $\mathbb{S}$-submodule of $\mathcal{O}$ spanned by trees from the family $\{T^{(1)}, T^{(2)}\}$ which contain at least one "bad" square vertex  with $a < k-1$. If $\{T_+^{(1)}, T_+^{(2)}\} \subset \{T^{(1)}, T^{(2)}\}$

is the subset of trees containing no bad vertices, then we can write an isomorphism of $\mathbb{S}$-modules:

$$\mathcal{BV}_\infty^{\mathrm{com}} \cong \mathrm{span}\langle T_+^{(1)}\rangle \oplus \mathrm{span}\langle T_+^{(2)}\rangle.$$

The submodule $\mathrm{span}\langle T_+^{(1)}\rangle$ is the image of the dg operad $\mathcal{L}_\infty^\diamond$ under the monomorphism χ (see Lemma B.4.2) so that the above sum is a direct sum of *complexes*, and we get eventually an isomorphism of complexes,

$$\mathcal{BV}_\infty^{\mathrm{com}} \cong \mathcal{C}om \circ \mathcal{L}_\infty^\diamond,$$

with the above splitting corresponding to the augmentation splitting of the operad $\mathcal{C}om$:

$$\mathcal{C}om = \mathrm{span}\langle 1\rangle \oplus \overline{\mathcal{C}om}.$$

Therefore, by Proposition B.4.1 and Remark B.3,

$$H^\bullet(\mathcal{BV}_\infty^{\mathrm{com}}) \cong \mathcal{C}om \circ \mathcal{L}^\diamond \cong q\mathcal{BV}.$$

By Proposition B.2, we get isomorphisms of $\mathbb{S}$-modules,

$$H^\bullet(\mathcal{BV}_\infty^{\mathrm{com}}) \cong \mathrm{Gr}(\mathcal{BV}) \cong \mathcal{BV},$$

which completes the proof of the theorem. □

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