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BV formality

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ABSTRACT

We prove a stronger version of the Kontsevich Formality Theorem for orientable manifolds, relating the Batalin–Vilkovisky (BV) algebra of multivector fields and the homotopy BV algebra of multidifferential operators of the manifold.

Contents

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[1] Most of this work was carried out while the author was at the University of Zurich.

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1. Introduction

Given a manifold M, the space of multidifferential operators of M, $D_{\text{poly}}(M)$ is a smooth version of the Hochschild complex of the functions on M. Both $D_{\text{poly}}(M)$ and the space $T_{\text{poly}}(M)$ of multivector fields of M are (shifted) differential graded Lie algebras. These two objects are related by the Hochschild–Kostant–Rosenberg Theorem that provides us with a quasi-isomorphism $T_{\text{poly}}(M) \to D_{\text{poly}}(M)$. However, this map not compatible with the Lie structure.

Searching for a canonical formal quantization of Poisson manifolds, in [15] M. Kontsevich establishes the existence of a homotopy Lie quasi-isomorphism $T_{\text{poly}}(M) \to D_{\text{poly}}(M)$ extending the Hochschild–Kostant–Rosenberg map. This map, nowadays called Kontsevich's Formality morphism, has a very explicit description involving integrals over configuration spaces of points when $M = \mathbb{R}^d$.

Taking the wedge product into consideration T_{poly} is a Gerstenhaber algebra, and even if D_{poly} is not a Gerstenhaber algebra, its homology is in a standard way. It is natural to ask whether one can put a homotopy Gerstenhaber algebra structure on D_{poly} that induces the usual Gerstenhaber algebra in the cohomology (Deligne's conjecture) and find a Formality morphism satisfying the Gerstenhaber structure up to homotopy. This question has been answered affirmatively by D. Tamarkin [21,13].

In [24], T. Willwacher uses a different model for the Gerstenhaber operad, the Braces operad, that acts naturally on D_{poly} given the nature of the formulas. Willwacher proves in [24] a homotopy Braces version of the Formality morphism.

In this paper we intend to take the final step on this chain of results by showing a BV version of the Formality Theorem(s). As described in Section 2, we can endow both $T_{\text{poly}}(\mathbb{R}^d)$ and the cohomology of $D_{\text{poly}}(\mathbb{R}^d)$ with a degree -1 operator, extending the previous Gerstenhaber structures to BV algebra structures.

The cyclic structure of $D_{\text{poly}}(\mathbb{R}^d)$ leads to the construction of CBr, the Cyclic Braces operad which is a refinement of the Braces operad. We show that the operad CBr is quasi-isomorphic to BV, the operad governing BV algebras, and the action of CBr on $D_{\text{poly}}(\mathbb{R}^d)$ descends to the canonical BV algebra structure on $H(D_{\text{poly}}(\mathbb{R}^d))$. In section 5 we show that the BV action on $T_{\text{poly}}(\mathbb{R}^d)$ can be lifted to an action of CBr_∞, a resolution of CBr and we show the first main Theorem.

Theorem 1. *There exists a CBr_∞ quasi-isomorphism $T_{\text{poly}}(\mathbb{R}^d) \to D_{\text{poly}}(\mathbb{R}^d)$.*

The components of this morphism are defined through integrals similarly to Kontsevich's case.

The formality of the Cyclic Braces operad implies that in the previous Theorem CBr_∞ can be replaced any other cofibrant resolution of BV, namely its minimal model or the Koszul resolution of BV.

The approach to the proof of this result uses a framework similar to [24]. Many intermediate results needed can actually be seen as framed generalizations of [24] and some of the proofs from sections 5 and 6 can be partially adapted. When that is the case we point out the explicit result that we are adapting/generalizing.

If we require orientability of the manifold M, the spaces $T_{\mathrm{poly}}(M)$ and $H(D_{\mathrm{poly}}(M))$ still have natural BV structures. Using formal geometry techniques, together with the formalism of twisting of bimodules, in Section 6 we show a global version Theorem 1.

Theorem 2. *Let M be an oriented manifold. There exists a CBr_∞ quasi-isomorphism $T_{\mathrm{poly}}(M) \to D_{\mathrm{poly}}(M)$ extending Kontsevich's Formality morphism.*

We remark that the fact that CBr is quasi-isomorphic to BV is, although not usually stated in this way, also known as the "cyclic Deligne conjecture" [14] and section 5.3 presents an alternative proof of this known result.

Some applications of theorem to string topology are also expected. In the study of the homology of free loop space $LM = Map(S^1, M)$ string topologists are interested on the BV structure on the Hochschild cohomology of the differential forms $\Omega(M)$. Since $\Omega(M)$ can be interpreted as $C^\infty(\Pi TM)$, where ΠTM is the odd tangent bundle, this result might be used to understand this Hochschild complex by reducing to the study of the corresponding multivector fields [6,2].

1.1. Acknowledgments

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1.2. Notation and conventions

In this paper we work over the field $\mathbb{R}$ of real numbers, even though the "algebraic" results hold in any field of characteristic zero.

All algebraic objects are differential graded, or dg for short, unless otherwise stated so.

If $\mathcal{P}$ is a 2-colored operad, we denote the space of operations with m inputs in color 1, n inputs in color 2 and output in color i by $\mathcal{P}^i(m,n)$ and we might denote $\mathcal{P}$ by $\left(\mathcal{P}^1, \mathcal{P}^2\right)$.

2. Preliminaries

2.1. BV algebras

Let us recall the definition of a BV algebra and also fix degree conventions.

Definition 1. A Batalin–Vilkovisky algebra or BV algebra is a quadruplet $(A, \cdot, [\ ,\ ], \Delta)$, such that:

- $(A, \cdot)$ is a (differential graded) commutative associative algebra,
- $(A, [\ ,\ ])$ is a 1-shifted Lie algebra (i.e., the bracket has degree -1),
- $(A, \cdot, [\ ,\ ])$ is a Gerstenhaber algebra, i.e., for all $a \in A$ of degree $|a|$, the operator $[a, -]$ is a derivation of degree $|a| - 1$.
- $\Delta\colon A \to A$ is a unary linear operator of degree -1 such that Δ is a derivation of the bracket,
- The bracket is the failure of Δ being a derivation for the product, i.e.,

$$[-, -] = \Delta \circ (- \cdot -) - (\Delta(-) \cdot -) - (- \cdot \Delta(-)).$$

We denote by BV, the operad governing BV algebras.

2.2. Hochschild cochain complex

In this section we recall the basics of Hochschild cohomology. For a more detailed introduction, along with the missing proofs, see [18].

Let A be a non-graded associative algebra.

For $f\colon A^{\otimes m} \to A$ and $g\colon A^{\otimes n} \to A$, we define $f \circ_i g\colon A^{\otimes m+n-1} \to A$, for $i = 1, \ldots, m$, to be the insertion of g at the i-th slot of f,

$$f \circ_i g(a_1, \ldots, a_{m+n-1}) = f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+n-1}), \ldots, a_{m+n-1}).$$

Lemma 3. *Let $f\colon A^{\otimes m} \to A$ and $g\colon A^{\otimes n} \to A$. The operation $f \circ g\colon A^{\otimes m+n-1} \to A$ given by*

$$f \circ g = \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} f \circ_i g,$$

defines a pre-Lie product (of degree -1).

This defines a -1 shifted graded Lie algebra structure on $\prod_{n\geq 0} \operatorname{Hom}(A^{\otimes n}, A)$. Let $\mu\colon A^{\otimes 2} \to A$ be the multiplication of the algebra.

Since A is an associative algebra, we have

$$[\mu,\mu](a_1,a_2,a_3) = 2\mu(a_1,\mu(a_2,a_3)) - 2\mu(\mu(a_1,a_2),a_3) = 0,$$

i.e., μ is a Maurer–Cartan element of the Lie algebra $\prod_{n\geq 0}\mathrm{Hom}(A^{\otimes n},A)$.

Definition 2. The Hochschild cochain complex of A, $(C^\bullet(A),d)$ is defined by

$$C^n(A) = \mathrm{Hom}(A^{\otimes n},A); \qquad d = [\mu,\cdot].$$

Explicitly, for $f \in C^n(A)$ and $a_i \in A$, the differential is given by $df(a_1,\dots,a_{n+1}) =$

$$= a_1 f(a_2,\dots,a_{n+1}) + \sum_{i=1}^{n-1}(-1)^{i-1} f(a_1,\dots,a_i a_{i+1},\dots,a_n) + (-1)^n f(a_1,\dots,a_n)a_{n+1}.$$

Definition 3. The Hochschild cohomology of an associative algebra A is the cohomology of the complex $C^\bullet(A)$ and is denoted by $HH^\bullet(A)$.

Definition 4. Let $f \in C^m(A)$ and $g \in C^n(A)$. The cup product on Hochshild cochains $f \cup g \in C^{m+n}(A)$ is defined by

$$f \cup g(a_1 \dots, a_{m+n}) = f(a_1,\dots,a_m)\cdot g(a_{m+1},\dots,a_{n+m}).$$

The cup product is trivially associative but, in general, non-commutative and it does not satisfy the desired compatibility with the Lie bracket. However, as M. Gerstenhaber showed (cf. [7]), this is rectified at the cohomological level.

Proposition 4. *The cup product and the Lie bracket above defined, induce a Gerstenhaber algebra structure on $HH^\bullet(A)$.*

2.3. Multidifferential operators

Let M be an oriented manifold. One of the central objects of this paper is the space of multidifferential operators of M, which are a smooth analog of the Hochschild cochain complex.

Definition 5. Let $A = C^\infty(M)$, the algebra of smooth functions of M. The space of multidifferential operators $\overline{D_{\mathrm{poly}}}^\bullet(M)$ is a subcomplex of $C^\bullet(A)$, given by

$$\overline{D_{\mathrm{poly}}}^n(M) = \left\{ D\colon C^\infty(M)^{\otimes n} \to C^\infty(M) \,\middle|\, D \overset{\text{locally}}{=} \sum f \frac{\partial}{\partial x_{I_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{I_n}} \right\},$$

where the I_j are finite sequences of indices between 1 and $dim(M)$ and $\frac{\partial}{\partial x_{I_j}}$ is the multi-index notation representing the composition of partial derivatives.

The space $D_{\text{poly}}(M)$ of *normalized cochains*, or just D_{poly} if there is no ambiguity, the subspace of $\overline{D_{\text{poly}}}(M)$ consisting of multidifferential operators vanishing on constant functions.

We will now describe an action of the group $C_{n+1} = \langle \sigma_n | \sigma_n^{n+1} = e \rangle$ on D^n_{poly}.

Since every multidifferential operator is uniquely determined by evaluation on the compactly supported functions $C_c^\infty(M)$, then, D^n_{poly}, for $n \geq 1$ can be seen as a subspace of $\text{Hom}(C_c^\infty(M)^{\otimes n}, C_c^\infty(M))$. One can equally see D_{poly} as a subspace of $\text{Hom}(C_c^\infty(M)^{\otimes n+1}, \mathbb{R})$ in the following way:

Let us denote by vol the given volume form M. We identify $D \in D^n_{\text{poly}} \subset \text{Hom}(C_c^\infty(M)^{\otimes n}, C_c^\infty(M))$, with

$$\left[f_1 \otimes \cdots \otimes f_{n+1} \mapsto \int_M f_1 D(f_2, \ldots, f_{n+1}) \,\text{vol} \right] \in \text{Hom}(C_c^\infty(M)^{\otimes n+1}, \mathbb{R}).$$

The reverse identification can be obtained by integrating by parts in order to remove differential operators from f_1.

From now on we drop the M as the domain of integration and the vol to make the notation lighter.

There is an action of C_{n+1} on $D^n_{\text{poly}} \subset \text{Hom}(C_c^\infty(M)^{\otimes n+1}, \mathbb{R})$ is given by the cyclic permutation of the inputs.

$$\int f_1 D(f_2, \ldots, f_{n+1}) = \int f_2 D^\sigma(f_3 \ldots, f_{n+1}, f_1).$$

Definition 6. The Connes' B operator on D_{poly}, is the map $B\colon D^\bullet_{\text{poly}} \to D^{\bullet-1}_{\text{poly}}$ defined for all $D \in D^n_{\text{poly}}$ by

$$B(D)(f_1, \ldots, f_{n-1}) = \sum_{k=0}^{n-1} (-1)^k D^{\sigma^k}(1, f_1, \ldots, f_{n-1}), \quad \forall f_i \in C^\infty(M).$$

Proposition 5. *The B operator induces a well defined map in the cohomology of D_{poly}. Together with the Lie bracket and cup product defined in the previous section induces a BV-algebra structure on $H^\bullet(D_{\text{poly}})$.*

The proposition can be proved "by hand" (cf. [18] where this is done in the non-differential setting), but it will also follow from the result that the operad CBr, whose homology is the BV operad, acts on D_{poly}.

2.4. Multivector fields

Definition 7. Let M be an oriented manifold. The graded vector space $T_{\text{poly}}(M)$ or just T_{poly} of multivector fields on M is

$$T^\bullet_{\text{poly}} = \Gamma(M, \bigwedge^\bullet T_M),$$

where T_M is the tangent bundle of M.

T_{poly} has a natural Gerstenhaber algebra structure by taking as product the wedge product of multivector fields and as bracket, the Schouten–Nijenhuis bracket, i.e., the unique $\mathbb{R}$-linear bracket satisfying

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(|X|-1)(|Y|-1)} Y \wedge [X, Z], \ \forall X, Y, Z \in T^\bullet_{\text{poly}}$$

that restricts to the usual Lie bracket of vector fields.

We can define a map $f\colon T^\bullet_{\text{poly}}(M) \to \Omega^{n-\bullet}_{dR}(M)$ that sends a multivector field to its contraction with the volume form of M.

This map is easily checked to be an isomorphism of vector spaces. We define the divergence operator div to be the pullback of the de Rham differential via f, i.e. $\text{div} = f^{-1} \circ d_{dR} \circ f$.

A series of straightforward calculations proves the following:

Proposition 6. *The space $T^\bullet_{\text{poly}}(M)$, with the wedge product, the Schouten–Nijenhuis bracket and the divergence operator forms a BV-algebra.*

3. Cyclic Swiss Cheese type operads

3.1. Cyclic operads

The standard notion of an operad is used in order to describe operations on a certain vector space with a given number of inputs and one output. A symmetric operad is used when one wants to take into consideration the symmetries on the inputs. The notion of a *cyclic operad* [10,19], introduced by Getzler and Kapranov, arises when one considers the output as an *additional input* that can be cyclically permuted along with the remaining inputs. This can arise naturally in many situations, for example, when one is given a finite dimensional vector space V equipped with a non-degenerate symmetric bilinear form, the space $\text{Hom}(V^{\otimes n}, V)$ can be identified with $\text{Hom}(V^{\otimes n+1}, \mathbb{R})$.

Definition 8. A non-symmetric cyclic operad on a symmetric monoidal category $(\mathcal{C}, \otimes, I, s)$ is the data of a non-symmetric operad $\mathcal{P}$ and a right action of $C_{n+1} = \langle \sigma_n | \sigma_n^{n+1} = e \rangle$, the symmetric group of order $n + 1$ on $\mathcal{P}(n)$ satisfying the following axioms:

a) The cyclic action on the unit in $\mathcal{P}(1)$ is trivial.
b) For every $m, n \geq 1$, the diagram

$$\begin{tikzcd}
\mathcal{P}(m) \otimes \mathcal{P}(n) \arrow[r, "\circ_1"] \arrow[d, "\sigma_m \otimes \sigma_n"] & \mathcal{P}(m+n-1) \arrow[dd, "\sigma_{m+n-1}"] \\
\mathcal{P}(m) \otimes \mathcal{P}(n) \arrow[d, "s"] & \\
\mathcal{P}(n) \otimes \mathcal{P}(m) \arrow[r, "\circ_n"] & \mathcal{P}(m+n-1)
\end{tikzcd}$$

commutes.
c) For every $m, n \geq 1$ and $2 \leq i \leq m$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{P}(m) \otimes \mathcal{P}(n) \arrow[r, "\circ_i"] \arrow[d, "\sigma_m \otimes \mathrm{id}_{\mathcal{P}(n)}"] & \mathcal{P}(m+n-1) \arrow[d, "\sigma_{m+n-1}"] \\
\mathcal{P}(m) \otimes \mathcal{P}(n) \arrow[r, "\circ_{i-1}"] & \mathcal{P}(m+n-1)
\end{tikzcd}.$$

3.2. Operad of Cyclic Swiss Cheese type

Definition 9. Let $\mathcal{P}$ be a 2-colored operad that is non-symmetric in color 2. We say that $\mathcal{P}$ is of Swiss Cheese type if $\mathcal{P}^1(m, n) = 0$ if $n > 0$.

Furthermore, $\mathcal{P}$ is said to be of Cyclic Swiss Cheese type if the following additional conditions hold:

- The cyclic group of order $n + 1$, C_{n+1} acts on the right on $\mathcal{P}^2(m, n)$ satisfying the same axioms as the axioms of a non-symmetric cyclic operad,
- The cyclic action is $\mathcal{P}^1$ equivariant,
- There is a distinguished element $\mathbb{1}_{\mathcal{P}} \in \mathcal{P}^2(0, 0)$.

For simplicity of notation we denote $\mathcal{P}^1(m, 0)$ by $\mathcal{P}^1(m)$. Using the distinguished element $\mathbb{1}_{\mathcal{P}}$ we define the "forgetful" map $\mathrm{Forget}_\infty \colon \mathcal{P}^2(m, n) \to \mathcal{P}^2(m, n-1)$ by $\mathrm{Forget}_\infty(p) = p^{\sigma_n}(\mathrm{id}_{\mathcal{P}^1}, \dots, \mathrm{id}_{\mathcal{P}^1}; \mathrm{id}_{\mathcal{P}^2}, \dots, \mathrm{id}_{\mathcal{P}^2}, \mathbb{1}_{\mathcal{P}})$.

A morphism $\mathcal{P} \to \mathcal{Q}$ of Cyclic Swiss Cheese type operads is a colored operad morphism that is equivariant with respect to the cyclic action and sends $\mathbb{1}_{\mathcal{P}}$ to $\mathbb{1}_{\mathcal{Q}}$.

3.3. Examples

3.3.1. Multidifferential operators as an operad

Let M be an oriented manifold. The operad of multidifferential operators $\tilde{D}_{\mathrm{poly}}(M)$, or just $\tilde{D}_{\mathrm{poly}}$, is a differential graded operad concentrated in degree zero with zero differential given by

$$\tilde{D}^n_{\mathrm{poly}} := \tilde{D}_{\mathrm{poly}}(n) = \left\{ D\colon C^\infty(M)^{\otimes n} \to C^\infty(M) \middle| D \overset{\mathrm{loc.}}{=} \sum f \frac{\partial}{\partial x_{I_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{I_n}} \right\},$$ [1]

where the multidifferential operators are required to vanish on constant functions.

The operadic structure is the one induced by the endomorphisms operad of $C^\infty(M)$, i.e., given by composition of operators. As any other operad, $\tilde{D}_{\mathrm{poly}}$ can be seen as a 2-colored operad simply by declaring that there are no operations with inputs or outputs in color 1. To endow $\tilde{D}_{\mathrm{poly}}$ with a Cyclic Swiss Cheese Operad type structure we use the cyclic action defined in Section 2.3 and the distinguished element $\mathbb{1} \in \tilde{D}^0_{\mathrm{poly}} = C^\infty(M)$ is defined to be the constant function 1.

For every $D \in \tilde{D}^n_{\mathrm{poly}} \subset \mathrm{Hom}(C^\infty_c(\mathbb{R}^d)^{\otimes n}, C^\infty_c(\mathbb{R}^d))$ we have

$$\mathrm{Forget}_\infty(D) = \int D(\cdot)\,\mathrm{vol} \in \mathrm{Hom}(C^\infty_c(\mathbb{R}^d)^{\otimes n}, \mathbb{R}).$$

3.3.2. Configurations of framed points

In this section we consider a framed analog of the original Swiss Cheese operad, or rather the homotopy equivalent compactified version, both originally introduced by Voronov [22]. The Fulton–MacPherson topological operad FM_2, introduced by Getzler and Jones [9] is constructed in such a way that the n-ary space $\mathsf{FM}_2(n)$ is a compactification of the configuration space of points labeled $1, \ldots, n$ in $\mathbb{R}^2$, modulo scaling and translation. The spaces $\mathsf{FM}_2(n)$ are manifolds with corners with each boundary stratum representing a set of points that got infinitely close.

The first few terms are

- $\mathsf{FM}_2(0) = \emptyset$,
- $\mathsf{FM}_2(1) = \{*\}$,
- $\mathsf{FM}_2(2) = S^1$.

The operadic composition $\circ_i$ is given by inserting a configuration at the boundary stratum at the point labeled by i. For details on this construction see also [3, Part IV] or [15].

Definition 10. Let $\mathcal{P}$ be a topological operad such that there is an action of topological group G on every space $\mathcal{P}(n)$ and the operadic compositions are G-equivariant. The semi-direct product $G \ltimes \mathcal{P}$ is a topological operad with n-spaces

$$(G \ltimes \mathcal{P})(n) = G^n \times \mathcal{P}(n),$$

and composition given by

[1] This is almost the object introduced in Section 2.3. The tilde is a reminder that there is no grading or differential.

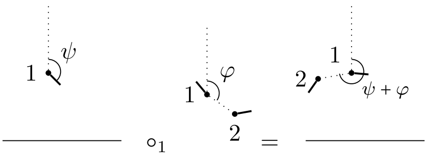


Fig. 1. Composition of an element of FFM_2 with an element in $\mathsf{F}\mathbb{H}$.

$$(\overline{g}, p) \circ_i (\overline{g'}, p') = (g_1, \ldots, g_{i-1}, g_i g'_1, \ldots, g_i g'_m, g_{i+1}, \ldots, g_n, p \circ_i (g_i \cdot p')),$$

where $\overline{g} = (g_1, \ldots, g_n)$ and $\overline{g'} = (g'_1, \ldots, g'_m)$.

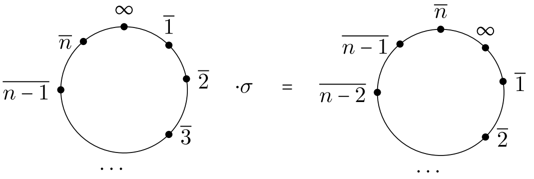
The topological group S^1 acts on FM_2 by rotations. We define the Framed Fulton–MacPherson topological operad FFM_2 to be the semi-direct product $S_1 \ltimes \mathsf{FM}_2$. Equivalently, $\mathsf{FFM}_2(n)$ is the compactification of the configuration space of points modulo scaling and translation such that at every point we assign a frame, i.e., an element of S^1. When the operadic composition is performed, the configuration inserted rotates according to the frame on the point of insertion.

We denote by $\mathbb{H}_{m,n}$, the space of configurations of m points in the upper half plane labeled by $1, \ldots, m$ and n points at the boundary, labeled by $\overline{1}, \ldots, \overline{n}$, modulo scaling and horizontal translations, with a similar compactification. Similarly, $\mathsf{F}\mathbb{H}_{m,n}$ shall be the compactification of the space of configurations of m framed points in the upper half plane and n non-framed points at the boundary. These spaces are considered unital in the sense that $\mathsf{F}\mathbb{H}_{0,0}$ is topologically a point, instead of the empty space.

Together they form a Swiss Cheese type topological operad $\mathcal{P}$, with $\mathcal{P}^1 = \mathsf{FFM}_2$ and $\mathcal{P}^2 = \mathsf{F}\mathbb{H}$ with composition of color 2 being insertion of the corespondent configuration in the boundary stratum and composition of color 1 on the vertex labeled by i being the insertion at the boundary stratum at the point i after applying the corresponding rotation given by the frame of i. We shall consider that a framing pointing upwards represents the identity of S^1, see Fig. 1.

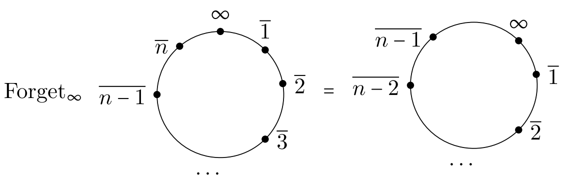
In fact they can be endowed with a Cyclic Swiss Cheese type operad structure.

The open upper half plane is isomorphic to the Poicaré disk via a conformal (angle preserving) map. This isomorphism sends the boundary of the plane to the boundary of the disk except one point, that we label by ∞. We define the cyclic action of C_{n+1} in $\mathsf{FFM}_2(m, n)$ by cyclic permutation of the point labeled by infinity with the other points at the boundary.



The element $\mathbb{1}$ is defined to be the unique point in $\mathsf{F}\mathbb{H}_{0,0}$. Insertion of this element represents forgetting a certain point at the boundary.

The forgetful map is defined by forgetting the point at infinity and labeling the first point as the new ∞ point and the previous $\overline{n}$ becomes the new $\overline{n-1}$.



3.3.3. Two kinds of graphs

A directed graph Γ is the data of a finite set of vertices, $V(\Gamma)$, a set of edges $E(\Gamma)$ and two maps from $E(\Gamma)$ to $V(\Gamma)$ (a source and target map). Notice that tadpoles (edges connecting a vertex to itself) and multiple edges are allowed. However, in all types of graphs we will use in this paper, graphs with multiple edges will vanish for symmetry reasons.

Let $\mathsf{BVKGra}'(m,n)$ be the graded vector space spanned by directed graphs with m vertices of *type I* labeled with the numbers $\{1,\dots,m\}$, n labeled with the numbers $\{\overline{1},\dots,\overline{n}\}$ of *type II* and edges labeled with the numbers $\{1,\dots,\#\text{edges}\}$, such that there are no edges starting on a vertex of type II. The degree of a graph is $-\#$edges, i.e., every edge has degree -1. For every non-negative integer d, there is an action of $\mathbb{S}_d$ on $\mathsf{BVKGra}'_{-d}(n)$ by permutation of the labels of the edges.

We define the space BVKGra of BV Kontsevich Graphs by

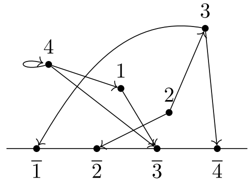
$$\mathsf{BVKGra}(m,n) := \bigoplus_d \mathsf{BVKGra}'_{-d}(m,n) \otimes_{\mathbb{S}_d} \mathrm{sgn}_d,$$

where sgn_d is the sign representation.

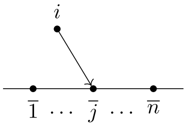
We define the space of BV Graphs, $\mathsf{BVGra}(n) := \mathsf{BVKGra}(n,0)$. There is a natural $\mathbb{S}_n$ action by permutation of the labels and we define a symmetric operad structure in BVGra by setting the composition $\Gamma_1 \circ_i \Gamma_2$ to be the insertion of Γ_2 in the i-th vertex of Γ_1 and sum over all possible ways of connecting the edges incident to i to Γ_2.

We can form a Swiss Cheese type operad by setting BVGra to be the operations in color 1 and BVKGra to be the operations in color 2, considering the symmetric action permuting the labels of type I vertices and ignoring the symmetric action of type II vertices. The partial compositions are given as in BVGra, i.e., by insertion on the corresponding vertex and connecting in all possible ways.

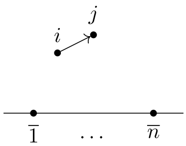
The type II vertices in BVKGra will be later seen as *boundary vertices* when we relate BVKGra with $\mathsf{F}\mathbb{H}$, and since we wish to distinguish between $\mathsf{BVGra}(\cdot)$ and $\mathsf{BVKGra}(\cdot,0)$, we draw the latter with a line passing by the type II vertices.



The space $\mathsf{BVKGra}(m,n)$ forms a graded commutative algebra with product of two graphs defined by superposing the vertices and taking the union of the edges. This algebra is generated by one edge graphs

$\Gamma^i_j :=$ 

, with $1 \leq i \leq m$ and $1 \leq j \leq n$ and

$\Gamma^{i,j} :=$ 

, with $1 \leq i, j \leq m$. For simplicity, the dependance of m and n is dropped from the notation.

We define the cyclic action of C_{n+1} on one-edge graphs of $\mathsf{BVKGra}(m,n)$ by $\sigma(\Gamma^i_j) = \Gamma^i_{j-1}$ if $j \neq 1$ and $\sigma(\Gamma^i_1) = -\sum_{k=1}^n \Gamma^i_k - \sum_{k=1}^m \Gamma^{i,k}$. The action of σ on $\Gamma^{i,j} \in \mathsf{BVKGra}(m,n)$ is defined by $\sigma(\Gamma^{i,j}) = \Gamma^{i,j}$, for $1 \leq i, j \leq m$.

Since $\sigma^2(\Gamma^i_1) = \Gamma^i_m$, we have that σ^{n+1} acts as the identity in every one-edge graph, therefore the action of C_{n+1} is well defined.

We extend this action to $\mathsf{BVKGra}(m,n)$ by declaring that the action distributes over a product of graphs (i.e., making the cyclic action a morphism of unital algebras).

The element $\mathbb{1} \in \mathsf{BVKGra}(0,0)$ is the empty graph, the unique graph with no vertices. The insertion $\Gamma \circ_{\overline{j}} \mathbb{1}$ is zero if there is any edge incident to the vertex labeled by $\overline{j}$ or, if there are no such edges, it forgets the vertex labeled by j.

4. The Cyclic Braces operad

In [16], Kontsevich and Soibelman introduced an operad that they call minimal operad that acts naturally on the Hochschild cochain complex of A_∞ algebras. They show that this operad is quasi-isomorphic to Ger, the operad governing Gerstenhaber algebras (see also [20]). In this paper we call this operad Br, standing for Braces. In this section we introduce the Cyclic Braces operad, which is a refinement of the Braces operad that is meant to take into account the a unit and a cyclic action. A similar operad was constructed by Ward in [23].

4.1. The Cyclic Planar Trees operad

Let $\mathsf{CPT}'(n)$ be the graded vector space spanned by rooted planar trees with vertices labeled with the numbers in $\{1, \ldots, n\}$ with the additional feature that every vertex can have additional edges connecting to a symbol $\mathbb{1}$ (formally this is an augmentation of the vertex set) and every vertex has a marked edge, that can be one of the additional edges.[2] The non-root edges (including the ones connecting to $\mathbb{1}$) are labeled by the numbers $\{1, \ldots, \#\text{edges}\}$. The degree of a rooted planar tree is $-\#\text{edges}$. For every non-negative integer d, there is an action of $\mathbb{S}_d$ on $\mathsf{CPT}'_{-d}(n)$ by permuting the labels of the edges.

We define the operad CPT of Cyclic Planar Trees by

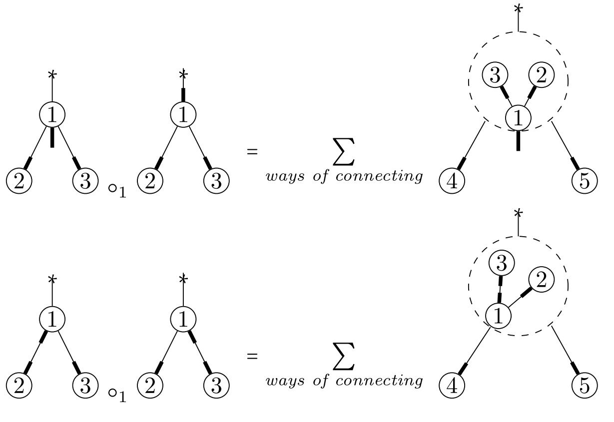
$$\mathsf{CPT}(n) := \bigoplus_d \mathsf{CPT}''_{-d}(n) \otimes_{\mathbb{S}_d} \operatorname{sgn}_d,$$

where sgn_d is the sign representation and CPT'' is the quotient of CPT' by trees in which there is a vertex is connected to an element $\mathbb{1}$ whose mark is not pointing towards $\mathbb{1}$.

The operadic composition $T_1 \circ_j T_2$ is given by inserting the tree T_2 in the vertex labeled j of the tree T_1, orienting the root of T_2 with the marking at the vertex j of T_1, forgetting both the root and the mark at the vertex j and reconnecting all incident edges in all planar possible ways.

Since it unambiguous, for simplicity of the drawing we draw only a mark between two edges when some vertex is connected to $\mathbb{1}$.

Example 1. Examples of insertion:



The operad is generated by T_n^i, $i = 1, \ldots, n$ and $T_n^{i,i+1}$, $i = 1, \ldots, n$, see Fig. 2.

[2] In fact, we want at most one edge connecting to $\mathbb{1}$ per vertex and for vertices having an edge connecting to $\mathbb{1}$ we want to force the marked edge to be that one, but imposing this condition directly would not be stable by the composition that we define below. This is resolved by considering a quotient of CPT', rather than a subspace.

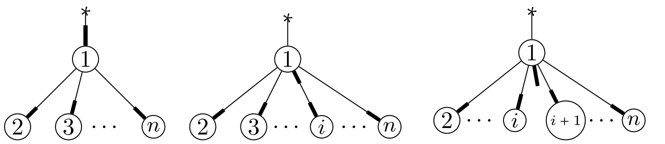


Fig. 2. T_n^1, T_n^i and $T_n^{i,i+1}$, from left to right.

Remark 7. Notice that since the edges appear unlabeled, the trees above are only well defined up to sign. To make the sign well defined one has to choose some convention of labeling the edges when those appear unlabeled. For example one might choose to label the root edge by 1 and travel around the graph to the left until getting back to the root labeling all edges consecutively. When performing an operadic compositions the labeling will have to be taken back to the conventioned labeling via a permutation which might cause the appearance of negative signs.

4.2. *Algebras over* CPT

The operad CPT acts naturally on spaces with cyclic structure.

Proposition 8. *Let $\mathcal{P}$ be an operad of Cyclic Swiss Cheese type. Its total space, $\prod_n \mathcal{P}^2(\cdot, n)[-n]$ forms a* $\mathsf{CPT} - \mathcal{P}^1$*-bimodule.*

Proof. To describe the left action of CPT we use the following multi-insertion notation:

For $p_1, p_2, \dots, p_n \in \mathcal{P}^2$, p_1 in arity N, we say that I is a planar insertion of $p_2, \dots, p_n$ in p_1 if I is an N-tuple containing each $p_2, \dots, p_n$ exactly once, in that order and the other entries are filled with $\mathrm{id}_{\mathcal{P}^2}$. For $i = 1, \dots, n$, we define $i(I)$ as the position of p_i in I. By $p_1(I)$, we mean the operadic composition given by I (ignoring insertions in color 1).

The action of $T_n^1 \in \mathsf{CPT}$ is given by braces operations, i.e., $T_n^1(p_1, p_2, \dots, p_n) = p_1\{p_2, \dots, p_n\}$. The action of $T_n^i \in \mathsf{CPT}$, for $i = 1, \dots, n$ is given by a composition of the braces operation and a permutation of C_{N+1} "turning the mark in the direction of the root". Explicitly, if σ is the generator of the cyclic group, $T_n^i(p_1, p_2, \dots, p_n) = \sum_I p_1^{\sigma^{-i(I)}}(I)$, where the sum runs over all possible planar insertions I of $p_2, \dots, p_n$ in p_1.

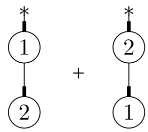
The action of $T_n^{i,i+1}$ is given by $T_n^{i,i+1}(p_1, p_2, \dots, p_n) = \sum_I \sum_{k=i(I)}^{(i+1)(I)} \mathrm{Forget}_\infty(p_1^{\sigma^{-k}})(I)$, where the first sum runs over all possible planar insertions I of $p_2, \dots, p_n$ in p_1. This corresponds to the insertion of the element $\mathbb{1}_{\mathcal{P}}$ in the marked space and the permutation sending the mark back to the direction of the root. $\square$

Lemma 9. *A morphism of Cyclic Swiss Cheese type operads induces a morphism of bimodules.*

Proof. Since a morphism of Cyclic Swiss Cheese type operads is in particular a morphism of colored operads, the induced map on the total space is a morphism of right bimodules. Since the definition of the action of CPT uses only the cyclic action and Forget_∞ and by hypothesis a morphism of Cyclic Swiss Cheese type operads commutes with these maps, the induced map on the total spaces is a left module morphism. $\square$

4.3. The operad CBr

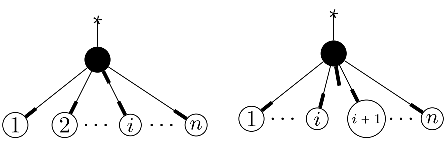
We now finish the construction of the Cyclic Braces operad via operadic twisting. There is a map $F\colon \mathsf{Lie}\{1\} \to \mathsf{CPT}$ sending the Lie bracket to

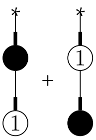


Using F we consider the (dg) operad given by the operadic twisting of CPT, $Tw\mathsf{CPT}$ (see the Appendix for details).

The space $Tw\mathsf{CPT}(n) = \left(\prod_k \mathsf{CPT}(n+k) \otimes \mathbb{K}[-2]^{\otimes k}\right)_{\mathbb{S}_k}$ is made out of trees, similar to the ones in CPT but with vertices of two different kinds. There are n *external* vertices, labeled from 1 to n and k *internal* unlabeled vertices, that we draw as a full black vertex. The degree of each edge or marked space is -1, the degree of an external vertex is 0 and the degree of an internal vertex is 2.

This operad is generated by elements as in Fig. 2 together with $T_n'^{i}$ and $T_n'^{i,i+1}$, $i = 0, \dots, n$:



The differential has two pieces, the first is computed by taking the operadic Lie bracket with  $= T_1'^{1} + T_2^{1} \circ_1 T_0'^{1}$, which amounts to split an internal vertex at every external vertex, but subtracting some combinations with one 1-valent or 2-valent internal vertex. The second piece just splits an internal vertex out of every internal vertex.

Lemma 10. *The subspace* $(Tw\mathsf{CPT})' \subset Tw\mathsf{CPT}$ *spanned by trees whose internal vertices are at least 3-valent is a suboperad of* $Tw\mathsf{CPT}$.

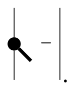
Proof. The composition of trees in $(Tw\mathsf{CPT})'$ cannot create internal vertices with valence 1 or 2.

The differential, however can create both kinds of vertices, so we must check that these contributions are canceled.

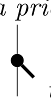
1-valent internal vertices can be created at every internal vertex by splitting it and reconnecting all edges incident edges to one of the internal vertices. Similarly, 1-valent internal vertices can be created at an external vertex when inserting $T_1'^1 + T_2^1 \circ_1 T_0'^1$ at that vertex and then reconnect to the external vertex. These contributions are all canceled by the remaining term of the differential consisting in inserting the tree in $T_1'^1 + T_2^1 \circ_1 T_0'^1$.

To see that 2-valent internal vertices contributions are canceled, it is enough to notice that every time such a vertex is created, it will be canceled by a similar contribution on the other adjacent vertex. □

Definition 11. We define the Cyclic Braces operad as $\mathsf{CBr} := (Tw\mathsf{CPT})'/J$, where J is the operadic ideal generated by $T_n'^i - T_n'^{i-1},\ T_n'^{i,i+1},\ i = 0, \ldots, n$ and


(1)

Remark 11. The $T_n'^i - T_n'^{i-1}$ in J mean that in CBr the marks at internal vertices are irrelevant. We will therefore not draw them in pictures and we will denote the image of $T_n'^i$ in CBr just by T_n'.

Convention 12. *Since J is not homogeneous with respect to the number of (internal) vertices, the number of (internal and therefore the total number of) vertices of a cyclic braces tree is a priori ill defined. We shall consider that whenever we have subsection of a tree like this*  *that there is only one edge and no vertices.*

4.4. The homology of the Cyclic Braces operad

In this subsection we show that the homology of CBr is the BV operad. For this, we make use of the operad Br whose homology, as mentioned in the beginning of this section, is the operad Ger.

Definition 12. The operad Br is defined as the suboperad of CBr generated by T_n^1 and T_n', or equivalently, the suboperad spanned by trees whose marks at every vertex are pointing towards the root.

In Br we "forget" that there are marks at vertices, therefore when referring to this operad we use the notation T_n instead of T_n^1 and we do not draw the marks in the pictures.

Two trees in CBr are said to have the same shape if when one forgets about the marks at vertices and connections to $\mathbb{1}$, they are the same. For example, T_n^i and $T_n^{i,i+1}$ have the same shape.

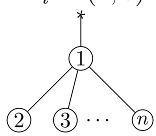
Let us consider the map $f = \oplus_n f_n \colon \mathsf{Br}(n) \otimes (\mathbb{K} \oplus \mathbb{K}[1])^{\otimes n} \to \mathsf{CBr}(n)$ sending $T \otimes \epsilon$, where T is braces tree and $\epsilon = \epsilon_1 \otimes \cdots \otimes \epsilon_n \in (\mathbb{K} \oplus \mathbb{K}[1])^{\otimes n}$, to the a sum of cyclic braces trees of the same shape, according to the following rules:

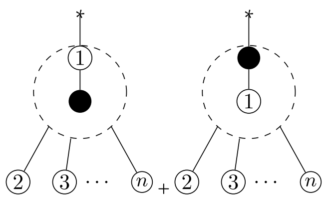
If the $\epsilon_i = (1, 0)$, the vertex labeled by i is sent to the same vertex with the marking pointing in the direction of the root.

If the $\epsilon_i = (0, 1)$, the vertex labeled by i is sent to a sum over all possible ways of inserting an edge connecting to $\mathbb{1}$.

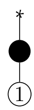
Lemma 13. *f is a quasi-isomorphism of chain complexes.*

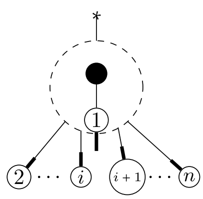
Proof. Since marked spaces have degree -1, f preserves the degree. Since the differential acts by derivations, it is enough to check that f commutes with the differentials on every vertex i and this is clearly the case if $\epsilon_i = (1, 0)$.

Let us consider the case of $T_n =$  $\in \mathsf{Br}$ with $\epsilon_1 = (0, 1)$.

$dT_n = \sum_{\substack{\text{ways of}\\ \text{connecting}}}$  , (2)

where the sum runs over all planar possible ways of connecting the incident edges such that the internal vertex is at least trivalent.

We have $f(T_n) = \sum_{i=1}^{n} T_n^{i,i+1}$, following the notation in Fig. 2. If we compute $df(T_n)$, the part of the differential given by the insertion of  on every $T_n^{i,i+1}$ is canceled over all the sum.

Therefore $df(T_n) = \sum_i \sum_{\substack{\text{ways of}\\ \text{connecting}}}$ . To see that $df(T_n) = f(dT_n)$, we note that there are two possibilities. Every summand of $df(T_n)$ has either the internal vertex connected to the root vertex or the vertex labeled by 1 connected to the root vertex. If the root is connected to the internal vertex, we find that same summand on the image by f of the second type of trees on equation (2), and similarly if the root is connected to the vertex 1.

Conversely, all trees that we get when we compute $f(dT_n)$ appear only once (due to the planar ordering of edges and marks around a vertex) and can be obtained as a summand in $df(T_n)$.

To show that f is a quasi-isomorphism, we filter CBr and Br by the number of internal vertices (see Remark 11). The map f is compatible with these filtrations and on the zeroth page of the corresponding spectral sequence in CBr one obtains the only piece of the differential that does not increase the number of internal vertices. Explicitly $d_0(T_n^{i,i+1}) = T_n^{i+1} - T_n^i$ and $d_0(T_n^j) = 0$. On the correspondent spectral sequence in Br one obtains the zero differential.

The differential d_0 respects the shape of the tree. Therefore the complex (CBr, d_0) splits as

$$\mathsf{CBr}(n) = \bigoplus_{\text{Shape } S} V_S,$$

where the sum runs over all possible shapes S of trees with n external vertices and V_S is the subcomplex spanned by all trees with the shape S.

The differential acts on the tree by acting on every vertex by means of the Leibniz rule, therefore if V_S^i represents the space of the i-th vertices of the trees with the given shape, then each V_S splits as a complex as $V_S = \bigotimes_{i=1}^n V_S^i$ (up to some degree shift).

But (V_S^i, d_0) is isomorphic to the simplicial complex of the k-gon, where k is the valence of the vertex i (again, up to some degree shift).

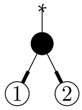
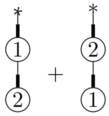
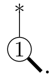
Therefore $H(\mathsf{CBr}, d_0) = \bigoplus_{\text{Shape } S} \left(\bigotimes_{i=1}^n H(V_S^i) \right) [k_S] = \bigoplus_{\text{Shape } S} \left(\bigotimes_{i=1}^n (\mathbb{K} \oplus \mathbb{K}[1]) \right) [k_S]$, where k_S is a degree shift dependent only on the shape of the tree.

Then, at the level of the homology of the zeroth pages of the spectral sequences we get and induced map $\mathsf{Br}(n) \otimes (\mathbb{K} \oplus \mathbb{K}[1])^{\otimes n} \to \bigoplus_{\text{Shape } S} (\mathbb{K} \oplus \mathbb{K}[1])^{\otimes n}[k_S]$.

Since clearly every possible shape of Cyclic Braces trees has a unique representative that is a Braces tree, this induced map is an isomorphism. Therefore f induces a quasi-isomorphism on the zeroth page of the spectral sequence, which implies that f is a quasi-isomorphism between the original complexes. $\square$

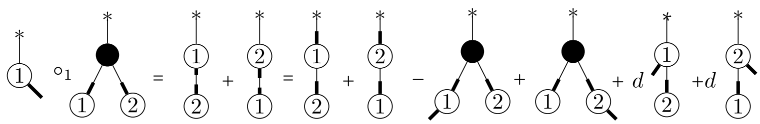
Corollary 14. *The homology of* CBr *is* BV*, the operad governing BV algebras.*

Proof. As a consequence of Lemma 13 we have $H(\mathsf{CBr}(n)) = H(\mathsf{Br}(n) \otimes (\mathbb{K} \oplus \mathbb{K}[1])^{\otimes n}) = H(\mathsf{Br}(n)) \otimes (\mathbb{K} \oplus \mathbb{K}[1])^{\otimes n} = \mathsf{Ger}(n) \otimes (\mathbb{K} \oplus \mathbb{K}[1])^{\otimes n}$. This establishes an isomorphism of graded vector spaces (but not necessarily of operads) $\mathsf{BV}(n) \to H(\mathsf{CBr}(n))$.

This isomorphism maps the commutative product $\cdot$ to , the Lie bracket $[\ ,\ ]$ to  and the BV operator Δ to .

To see that this is an indeed an isomorphism of operads it suffices to check that the relations of the BV operad are satisfied in $H(\mathsf{CBr})$, i.e., these elements of CBr satisfy the relations from Definition 1 up to exact terms.

The relation $\Delta \circ \cdot = [\ ,\ ] + \cdot \circ_1 \Delta + \cdot \circ_2 \Delta$ is satisfied in $H(\mathsf{CBr})$ since



The other relations can be easily checked so it follows that as an operad $H(\mathsf{CBr})$ is canonically isomorphic to BV. $\square$

Remark 15. Notice that the operad CBr is similar to Ward's operad $\mathcal{TS}_\infty$ [23]. In fact there is an isomorphism of operads $\mathsf{CBr} \to \mathcal{TS}_\infty$ given by contracting connected internal vertices into a single internal vertex and labeling it by the correspondent A_∞ operation. Since in [23] it is shown $H(\mathcal{TS}_\infty) = \mathsf{BV}$, this provides an alternative proof of the previous Corollary.

5. Operadic bimodule maps

Given an operad $\mathcal{P}$ and a resolution $\mathcal{P}_\infty \circlearrowright \mathcal{P}_\infty^{\mathrm{bimod}} \circlearrowleft \mathcal{P}_\infty$ of the canonical bimodule $\mathcal{P} \circlearrowright \mathcal{P} \circlearrowleft \mathcal{P}$, an infinity morphism of $\mathcal{P}_\infty$ algebras A and B, can be expressed as the following bimodule map:

$$\begin{tikzcd}
\mathcal{P}_\infty \arrow[d] & \circlearrowright & \mathcal{P}_\infty^{\mathrm{bimod}} \arrow[d] & \circlearrowleft & \mathcal{P}_\infty \arrow[d] \\
\operatorname{End} B & \circlearrowright & \operatorname{Hom}(A^{\otimes \bullet}, B) & \circlearrowleft & \operatorname{End} A,
\end{tikzcd}$$

where by $\operatorname{End} A$ we mean the operadic endomorphisms $\operatorname{End} A(n) = \operatorname{Hom}(A^{\otimes n}, A)$ and the bimodule structure on $\operatorname{Hom}(A^{\otimes \bullet}, B)$ is the natural one using composition of maps. In this section we prove Theorem 1 by expressing it in terms of a morphism of bimodules.

5.1. *Chains*$(\mathsf{F}\mathbb{H}_{m,n}) \to \mathsf{BVKGra}$

The topological operad of Cyclic Swiss Cheese type $(\mathsf{FFM}_2, \mathsf{F}\mathbb{H}_{m,n})$ introduced in 3.3.2 is in fact an operad on the category of semi-algebraic manifolds [12,17]. We consider the

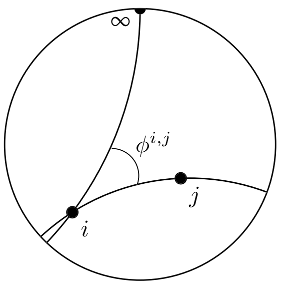


Fig. 3. The hyperbolic angle $\phi^{i,j}$.

functor *Chains* of semi-algebraic chains. This functor is monoidal so it induces a functor from semi-algebraic Cyclic Swiss Cheese type operads to dg Cyclic Swiss Cheese type operads.

In this section we define a morphism of Cyclic Swiss Cheese type operads

$$(Chains(\mathsf{FFM}_2), Chains(\mathsf{F}\mathbb{H}_{m,n})) \to (\mathsf{BVGra}, \mathsf{BVKGra})\,. \tag{3}$$

We start by defining a map $f_2\colon \mathsf{BVKGra}^* \to \Omega(\mathsf{F}\mathbb{H}_{m,n})$, where Ω is the functor sending a semi-algebraic manifold to its algebra of semi-algebraic forms.

Notice that $\mathsf{F}\mathbb{H}_{m,n}$ is a quotient of the configuration space of m points in the upper half plane and n points at the boundary by a group of conformal maps. The identification of $\mathbb{H}$ with the Poincaré Disk necessary for the definition of the cyclic action and the forgetful map is also conformal. Therefore, given a point p in the upper half plane and a point q either in the upper half plane or at the boundary the angle between the hyperbolic line passing by the point at ∞ and p and the hyperbolic line passing by the points p and q is well defined (up to a multiple of 2π).

We define $d\phi^i_j \in \Omega^1(\mathsf{F}\mathbb{H}_{m,n})$, for $1 \leq i \leq m$ and $1 \leq j \leq n$ as the 1-form given by the angle made by the hyperbolic line defined by the point at ∞ and the point labeled by i and the hyperbolic line defined by the point labeled by i and the point labeled by $\overline{j}$.

Similarly, $1 \leq i \neq j \leq m$, we define $d\phi^{i,j} \in \Omega^1(\mathsf{F}\mathbb{H}_{m,n})$ as the 1-form given by the angle defined by the line passing by ∞ and i and the line passing by i and j.

Finally, we define $d\phi^{i,i} \in \Omega^1(\mathsf{F}\mathbb{H}_{m,n})$ as the 1-form corresponding to the angle between the line passing by ∞ and i and the frame at i (see Fig. 3).

There is a canonical basis of $\mathsf{BVKGra}(m,n)$ given by the graphs and, by abuse of notation, we denote by the same graphs the dual basis of $\mathsf{BVKGra}^*(m,n)$.

Following the notation in 3.3.3, we define $f_2(\Gamma^i_j) := \frac{d\phi^i_j}{2\pi}$ for $1 \leq i \leq m$, $1 \leq j \leq n$ and $f_2(\Gamma^{i,j}) := \frac{d\phi^{i,j}}{2\pi}$ for $i \neq j$ between 1 and m.

$\mathsf{BVKGra}^*(m,n)$ admits a similar algebra structure by defining the product of two graphs as the superposition of edges. We extend the map f_2 to BVKGra^* by requiring it to be a morphism of unital algebras.

A C_{n+1} action on $\mathsf{BVKGra}^*(m,n)$ can be defined via the pullback of the cyclic action on $\mathsf{BVKGra}(m,n)$. Notice that this is not the standard definition of an action of a group on the dual space (one normally uses the pullback via the inverse of the map), but since C_{n+1} is abelian no problems arise from this.

$\Omega(\mathsf{F}\mathbb{H}_{m,n})$ inherits a C_{n+1} cyclic action from the cyclic action in $\mathsf{F}\mathbb{H}_{m,n}$ (also by pullback).

Lemma 16. *The map $f_2\colon \mathsf{BVKGra}^*(m,n) \to \Omega(\mathsf{F}\mathbb{H}_{m,n})$ is C_{n+1} equivariant.*

Proof. Notice actually that the algebra structure on $\mathsf{BVKGra}(m,n)$ is in fact the exterior algebra $\bigwedge V$, where V is the (finite dimensional) vector space concentrated in degree -1 spanned by all graphs with exactly one edge.

We had defined the cyclic action on V, extended this action to $\bigwedge V$ by requiring the action to commute with the product and defined an action on $(\bigwedge V)^* = \mathsf{BVKGra}(m,n)$. Alternatively, the cyclic action on V induces a cyclic action on V^* which induces a cyclic action on $\bigwedge V^*$. Under the identification $\bigwedge V^* = (\bigwedge V)^*$ these two actions are the same. This is an immediate consequence of the fact that if $e_1, \dots, e_n$ are part of a basis of V and $e_1^*, \dots, e_n^*$ are the corresponding parts of the dual basis, then $e_1 \wedge \dots \wedge e_n$ is dual to $e_1^* \wedge \dots \wedge e_n^*$.

This allows us to conclude that the cyclic action on $\mathsf{BVKGra}^*(m,n)$ commutes with the product of graphs.

It is therefore enough to show that f_2 is equivariant with respect to one-edge graphs.

The cyclic action of $C_{n+1} = \langle \sigma \rangle$ on one-edge graphs in $\mathsf{BVKGra}^*(m,n)$ is given by $\left(\Gamma^{i,j}\right)^\sigma = \Gamma^{i,j} - \Gamma^i_1$ and $\left(\Gamma^i_j\right)^\sigma = \Gamma^i_{j+1} - \Gamma^i_1$ with the convention that $\Gamma^i_{n+1} = 0$.

Since the cyclic action on $\mathsf{F}\mathbb{H}_{m,n}$ is by rotation of the n points at the boundary with the point ∞, we have $\left(d\phi^i_j\right)^\sigma = d(\phi^i_j \cdot \sigma) = d(\phi^i_{j+1} - \phi^i_1)$ and similarly $\left(d\phi^{i,j}\right)^\sigma = \left(d\phi^{i,j} - d\phi^i_1\right)$, therefore f_2 commutes with the action. $\square$

Analogously, a map $f_1\colon \mathsf{BVGra}^*(n) \to \Omega(\mathsf{FFM}_2)(n)$ can be defined on one-edge graphs by considering the angle with the vertical and extending as a morphism of algebras.

Remark 17. It is easy to check on generators that these maps produce a map of colored cooperads

$$(f_1, f_2)\colon\ (\mathsf{BVGra}^*, \mathsf{BVKGra}^*) \to (\Omega(\mathsf{FFM}_2), \Omega(\mathsf{F}\mathbb{H}_{m,n}))\,.$$

Let us sketch the verification for the case of $\Gamma^{1,2} \in \mathsf{BVKGra}^*(2,0)$.

The composition map in $(\mathsf{FFM}_2, \mathsf{F}\mathbb{H})$ is done by insertion at the boundary stratum with an appropriate rotation given by the framing. Since the cocomposition map is given by the pullback of the composition map, the part of the cocomposition given by $\Omega(\mathsf{F}\mathbb{H}) \to \Omega(\mathsf{F}\mathbb{H}) \otimes \bigotimes \Omega(\mathsf{FFM}_2)$ sends $d\phi^{1,2} \in \mathsf{F}\mathbb{H}(2,0)$ to $d\phi^{1,1} \otimes 1 + 1 \otimes d\phi^{1,2} \in \Omega(\mathsf{F}\mathbb{H}(1,0)) \otimes \Omega(\mathsf{FFM}_2(2))$ (recall Fig. 1).

The corresponding cocomposition in BVKGra^* sends $\Gamma^{1,2}$ to $\left(\Gamma^{1,1} \otimes \overset{1}{\bullet}\ \ \overset{2}{\bullet}\right) + \left(1 \otimes \overset{1}{\bullet}\!\longrightarrow\!\overset{2}{\bullet}\right) \in \mathsf{BVKGra}^*(1,0) \otimes \mathsf{BVGra}^*(2)$, therefore the diagrams commute. The general case for $\Gamma^{i,j} \in$

$\mathsf{BVKGra}^*(m,n)$ is similar and all the remaining cases are as simple or even simpler to check.

Remark 18. The functor Ω is not comonoidal since the canonical map $\Omega(A)\otimes\Omega(B)\to\Omega(A\times B)$ goes "in the wrong direction", therefore $\Omega(\mathsf{FFM}_2)$ is not a cooperad but still satisfies cooperad-like relations (see [17]). Nevertheless, by abuse of language throughout this paper we will these spaces as cooperads and refer to maps such as $\mathsf{BVGra}^*\to\Omega(\mathsf{FFM}_2)$ as maps of (colored) cooperads if they satisfy a compatibility relation such as commutativity of the following diagram:

$$\begin{tikzcd}
\mathsf{BVGra}^*(n) \arrow[r] \arrow[dd] & \Omega(\mathsf{FFM}_2(n)) \arrow[d] \\
 & \Omega(\mathsf{FFM}_2(n-k+1)\times\mathsf{FFM}_2(k)) \\
\mathsf{BVGra}^*(n-k+1)\otimes\mathsf{BVGra}^*(k) \arrow[r] & \Omega(\mathsf{FFM}_2(n-k+1))\otimes\Omega(\mathsf{FFM}_2(k)) \arrow[u]
\end{tikzcd}.$$

We define a map $g_1\colon Chains(\mathsf{FFM}_2)\to\Omega^*(\mathsf{FFM}_2)$ that maps every elementary semi-algebraic chain $c\in Chains(\mathsf{FFM}_2)$ to the linear form $\omega\mapsto\int_c\omega$. Similarly we define $g_2\colon Chains(\mathsf{F}\mathbb{H})\to\Omega^*(\mathsf{F}\mathbb{H})$ sending a chain to integration over that chain.

Clearly $\mathsf{BVKGra}(m,n)$ is finite dimensional for a fixed degree, therefore the double dual of $\mathsf{BVKGra}(m,n)$ can be identified with the original space.

Finally, the map of Cyclic Swiss Cheese type operads (3) that we were searching is defined as the composition

$$(Chains(\mathsf{FFM}_2), Chains(\mathsf{F}\mathbb{H}))\xrightarrow{(g_1,g_2)}(\Omega^*(\mathsf{FFM}_2),\Omega^*(\mathsf{F}\mathbb{H}))\xrightarrow{(f_1^*,f_2^*)}(\mathsf{BVGra},\mathsf{BVKGra})\,.$$

This is a colored operad map as a consequence of Remark 17, it commutes with the cyclic action as a consequence of Lemma 16 and by hand one checks that $\mathbb{1}_{Chains(\mathsf{F}\mathbb{H})}$ is sent to $\mathbb{1}_{\mathsf{BVKGra}}$.

Explicitly, given a chain $c\in Chains(\mathsf{FFM}_2)$, we have $f_1^*\circ g_1(c)=\sum_\Gamma\Gamma\int_c f_1(\Gamma)$, where Γ runs through all the graphs in BVGra. This sum is finite because the integral is zero every time the degree of Γ differs from the degree of the chain c.

Recall section 4.1 where we saw that given a Cyclic Swiss Cheese type operad $\mathcal{P}$ one can endow the total space $\prod_n\mathcal{P}^2(\cdot,n)[n]$ with a $\mathsf{CPT}-\mathcal{P}^1$-bimodule structure. Moreover, morphism of Cyclic Swiss Cheese type operads induce morphisms of bimodules. Therefore we obtain a bimodule map

$$\begin{tikzcd}
\mathsf{CPT} \arrow[d, "\mathrm{id}"] & \circlearrowright & \prod_n Chains(\mathsf{F}\mathbb{H}_{\bullet,n})[-n] \arrow[d] & \circlearrowleft & Chains(\mathsf{FFM}_2) \arrow[d] \\
\mathsf{CPT} & \circlearrowright & \prod_n \mathsf{BVKGra}(\cdot,n)[-n] & \circlearrowleft & \mathsf{BVGra}.
\end{tikzcd}$$

We choose a Maurer–Cartan element $\mu \in (\prod_n Chains(\mathsf{F}\mathbb{H}_{0,n})[-n])_2$ to be $\mu = \prod_{n\geq 2} c_n$, where c_n is the fundamental chain of the space $\mathsf{F}\mathbb{H}_{0,n}$.

It is easy to see that the image of c_n is zero for $n > 2$ and for $n = 2$ is the single graph in $\mathsf{BVKGra}(0,2)[-2]$ with no edges.

By twisting both $\prod_n Chains(\mathsf{F}\mathbb{H}_{\bullet,n})[-n]$ and $\prod_n \mathsf{BVKGra}(\cdot,n)[-n]$ with respect to μ and its image, we get a map of $Tw\mathsf{CPT}$-modules $\prod_n Chains^{\mu}(\mathsf{F}\mathbb{H}_{\bullet,n})[-n] \to \prod_n \mathsf{BVKGra}^{\mu}(\cdot,n)[-n]$ where the superscript μ indicates that there is a changed differential induced by the Maurer–Cartan elements. Since the ideal generated by (1) acts as zero, we can restrict our action to the subquotient CBr, of $Tw\mathsf{CPT}$, thus obtaining a morphism of left CBr-modules.

Since the right action of $Chains(\mathsf{FFM}_2)$ on $Chains(\mathsf{F}\mathbb{H})$ is on the non-boundary points, and analogously, the action of BVGra on BVKGra is on the type II vertices, it is clear that the morphism commutes with the right action. We obtain then the following bimodule map:

$$\begin{tikzcd} \mathsf{CBr} \arrow[d] & \circlearrowright & \prod_n Chains^{\mu}(\mathsf{F}\mathbb{H}_{\bullet,n})[-n] \arrow[d] & \circlearrowleft & Chains(\mathsf{FFM}_2) \arrow[d] \\ \mathsf{CBr} & \circlearrowright & \prod_n \mathsf{BVKGra}^{\mu}(\cdot,n)[-n] & \circlearrowleft & \mathsf{BVGra}. \end{tikzcd} \tag{4}$$

The projection map $p_{m,n}\colon \mathsf{F}\mathbb{H}_{m,n} \to \mathsf{F}\mathbb{H}_{m,0}$ that forgets the points at the boundary induces a strongly continuous chain [12] $p^{-1}_{m,n}\colon \mathsf{F}\mathbb{H}_{m,0} \to Chains(\mathsf{F}\mathbb{H}_{m,n})$. Intuitively the image of a configuration of points in $\mathsf{F}\mathbb{H}_{m,0}$ is the same configuration of points but with n points at the real line that are freely allowed to move. If we consider the complex $Chains(\mathsf{F}\mathbb{H}_{\bullet,0}) = \bigoplus_{m\geq 1} Chains(\mathsf{F}\mathbb{H}_{m,0})$, this induces a degree preserving map

$$p^{-1}\colon Chains(\mathsf{F}\mathbb{H}_{\bullet,0}) \to \prod_{n\geq 0} Chains^{\mu}(\mathsf{F}\mathbb{H}_{\bullet,n})[-n].$$

The next Lemma essentially follows [24, Appendix A.2].

Lemma 19. *p^{-1} is a morphism of right $Chains(\mathsf{FFM}_2)$-modules and its image is a $\mathsf{CBr} - Chains(\mathsf{FFM}_2)$-subbimodule.*

Proof. The morphism clearly commutes with the right action. Let us check that p^{-1} commutes with the differentials.

Let $c \in Chains(\mathsf{F}\mathbb{H}_{m,0})$.

The boundary term $\partial p^{-1}_{m,n}(c)$ has two kind of components. When at least two points at the upper half plane get infinitely close, giving us the term $p^{-1}_{m,n}(\partial c)$, and when points at the real line get infinitely close, giving us $\pm p^{f\partial}_{m,n}(c)$, where the $f\partial$ superscript represents that we are considering the boundary at every fiber.

Then, we have $p^{-1}(\partial c) = \prod_{n\geq 0} p^{-1}_{m,n}(\partial c) = \prod_{n\geq 0} \partial p^{-1}_{m,n}(c) \pm p^{f\partial}_{m,n}(c)$. The first summand corresponds to the normal differential in $Chains(\mathsf{F}\mathbb{H}_{m,n})$ and the second summand is precisely the extra piece of the differential induced by the twisting.

It remains to check the stability under the left CBr action. It is enough to check the stability under the action of the generators $T^i_n, T^{i,i+1}_n, T'_n$ and $T'^{i,i+1}_n$.

Let $c_1,\dots,c_n \in Chains(\mathsf{F}\mathbb{H}_{\bullet,0})$ of arbitrary degree. It is not hard to see that

$$p^{-1}\circ p\left(T^1_n(p^{-1}(c_1),\dots,p^{-1}(c_n))\right) = T^1_n(p^{-1}(c_1),\dots,p^{-1}(c_n)).$$

This follows essentially from the fact that on the right hand side the projection in $Chains^\mu(\mathsf{F}\mathbb{H}_{\bullet,k})[-k]$ is the sum over all the possibilities of distributing k_i points on the boundary stratum of c_i, for $i = 2,\dots,n$ and k_1 boundary points not infinitely close to any of these chains, with $k_1 + ... + k_n = k$, whereas the left hand is taking all of these possibilities into account at once.

For the remaining T^i_n, the stability follows from the remark that if a chain is in the image of p^{-1}, then any cyclic permutation of it is still in the image of p^{-1}. Since forgetting one of the boundary points of a chain in the image of p^{-1} leaves it in the image of p^{-1}, we get stability under the action of $T^{j,j+1}_n$.

The other generators follow from similar arguments. $\square$

p^{-1} is right inverse to the projection map, therefore it is an embedding of right $Chains(\mathsf{FFM}_2)$-modules. We can therefore transport back the left CBr action on its image, making p^{-1} a morphism of $\mathsf{CBr} - Chains(\mathsf{FFM}_2)$-bimodules.

By composition with the map (4), we obtain the following bimodule map:

$$\begin{array}{ccccc} \mathsf{CBr} & \circlearrowright & Chains(\mathsf{F}\mathbb{H}_{\bullet,0}) & \circlearrowleft & Chains(\mathsf{FFM}_2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{CBr} & \circlearrowright & \prod_n \mathsf{BVKGra}^\mu(\cdot,n)[-n] & \circlearrowleft & \mathsf{BVGra}. \end{array} \tag{5}$$

5.2. A representation on the colored vector space $D_{\mathrm{poly}} \oplus T_{\mathrm{poly}}$

In this section we drop the $\mathbb{R}^d$ from the notation T_{poly}, $\tilde{D}_{\mathrm{poly}}$ and D_{poly}, for simplicity. In Section 6 we globalize the results obtained here.

Let $x_1,\dots,x_n$ be coordinates in $\mathbb{R}^n$ and let $\xi_1,\dots,\xi_n$ be the corresponding basis of vector fields. Following [15] we define an action of BVGra on the graded algebra of multivector fields T_{poly} in $\mathbb{R}^d$ by setting

$$\Gamma(X_1,\dots,X_k) = \left(\prod_{(i,j)\in\Gamma}\sum_{l=1}^d \frac{\partial}{\partial x^{(j)}_l}\wedge\frac{\partial}{\partial \xi^{(i)}_l}\right)(X_1\wedge\dots\wedge X_k),$$

where $\Gamma \in \mathsf{BVGra}(k)$, $X_1,\dots,X_k$ are multivector fields, the product runs over all edges of Γ in the order given by the numbering of edges and the superscripts (i) and (j) mean that the partial derivative is being taken on the i-th and j-th component of $X_1, \ldots, X_k$. This is equivalent to an operad morphism $\mathsf{BVGra} \to \operatorname{End} T_{\mathrm{poly}}$.

Seeing Γ as an element of $\mathsf{BVGra}(m+n)$ and, using the action of BVGra in T_{poly}, together with the fact that C^∞ functions are degree zero multivector fields we define a map $g\colon \mathsf{BVKGra}(m,n) \to \operatorname{Hom}(T_{\mathrm{poly}}^{\otimes m} \otimes C_c^\infty(\mathbb{R}^d)^{\otimes n}, C_c^\infty(\mathbb{R}^d))$ by

$$g(\Gamma)(X_1,\ldots,X_m)(f_1,\ldots,f_n) = \Gamma(X_1,\ldots,X_m,f_1,\ldots,f_n).^{3} \tag{6}$$

These two maps form a colored operad morphism from $(\mathsf{BVGra}, \mathsf{BVKGra})$ to the Swiss Cheese type operad $\left(\operatorname{End} T_{\mathrm{poly}}, \operatorname{Hom}(T_{\mathrm{poly}}^{\otimes m} \otimes C_c^\infty(\mathbb{R}^d)^{\otimes n}, C_c^\infty(\mathbb{R}^d))\right)$, a suboperad of the colored operad $\operatorname{End}\left(T_{\mathrm{poly}} \oplus C_c^\infty(\mathbb{R}^d)\right)$.

The Tensor–Hom adjunction allows us to rewrite $\operatorname{Hom}(T_{\mathrm{poly}}^{\otimes m} \otimes C_c^\infty(\mathbb{R}^d)^{\otimes n}, C_c^\infty(\mathbb{R}^d))$ as $\operatorname{Hom}\left(T_{\mathrm{poly}}^{\otimes m}, \operatorname{Hom}\left(C_c^\infty(\mathbb{R}^d)^{\otimes n}, C_c^\infty(\mathbb{R}^d)\right)\right)$ and the bilinear form $\int\colon C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\mathbb{R}^d) \to \mathbb{R}$ induces a map

$$\operatorname{Hom}\left(T_{\mathrm{poly}}^{\otimes m}, \operatorname{Hom}\left(C_c^\infty(\mathbb{R}^d)^{\otimes n}, C_c^\infty(\mathbb{R}^d)\right)\right) \to \operatorname{Hom}\left(T_{\mathrm{poly}}^{\otimes m}, \operatorname{Hom}\left(C_c^\infty(\mathbb{R}^d)^{\otimes n+1}, \mathbb{R}\right)\right). \tag{7}$$

There is a natural C_{n+1} action on $\operatorname{Hom}\left(T_{\mathrm{poly}}^{\otimes m}, \operatorname{Hom}\left(C_c^\infty(\mathbb{R}^d)^{\otimes n+1}, \mathbb{R}\right)\right)$ given by the action on $C_c^\infty(\mathbb{R}^d)^{\otimes n+1}$ and also a distinguished element $\mathbb{1}$ map given by the insertion of the constant function $\equiv 1$ on the first input of $\operatorname{Hom}\left(C_c^\infty(\mathbb{R}^d)^{\otimes n+1}, \mathbb{R}\right)$.

Lemma 20. *With the above described map and cyclic action, the composition of the maps (6) and (7) induces a morphism of Cyclic Swiss Cheese type operads*

$$(\mathsf{BVGra}, \mathsf{BVKGra}) \to \left(\operatorname{End} T_{\mathrm{poly}}, \operatorname{Hom}\left(T_{\mathrm{poly}}^{\otimes \bullet}, \operatorname{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes \bullet+1}, \mathbb{R})\right)\right).$$

Proof. It is clear that the map is a morphism of colored operads and it sends one distinguished element to the other. It is enough to check the compatibility with the cyclic action.

Notice that the image of a graph under the morphism

$$\mathsf{BVKGra}(m,n) \to \operatorname{Hom}\left(T_{\mathrm{poly}}^{\otimes m}, \operatorname{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes n+1}, \mathbb{R})\right)$$

actually lands inside of $\operatorname{Hom}\left(T_{\mathrm{poly}}^{\otimes m}, \tilde{D}_{\mathrm{poly}}(n)\right)$ and this space is an algebra with product given by the product of functions.

It is clear by the definition of this morphism that it commutes with products, therefore to check the compatibility with the cyclic action it is enough to check it on graphs with just one edge.

[3] We set all $\xi_i = 0$.

Let $\Gamma^i_j \in \mathsf{BVKGra}(m,n)$. Recall that the action of the generator σ of C_{n+1} on Γ^i_j is $\sigma(\Gamma^i_j) = \Gamma^i_{j-1}$ if $j \neq 1$ and $\sigma(\Gamma^i_1) = -\sum_{k=1}^n \Gamma^i_k - \sum_{k=1}^m \Gamma^{i,k}$. The action of σ on $\Gamma^{i,j} \in \mathsf{BVKGra}(m,n)$ is $\sigma(\Gamma^{i,j}) = \Gamma^{i,j}$, for $1 \leq i,j \leq m$.

Let $X_1,\dots,X_m \in T_{\text{poly}}$ and let $f_0,\dots,f_n \in C^\infty(\mathbb{R}^d)$.

Notice that $g(\Gamma^i_1)(X_1,\dots,X_m)$ can only be non-zero if all the X_j, for $j \neq i$ are in ${T_{\text{poly}}}^0 = C^\infty(\mathbb{R}^d)$ and $X_i \in {T_{\text{poly}}}^1 = \Gamma(\mathbb{R}^d, T_{\mathbb{R}^d})$.

The operator $\left(g(\Gamma^i_1)(X_1,\dots,X_m)\right)^\sigma$ is defined by

$$\int f_0 g(\Gamma^i_1)(X_1,\dots,X_m)(f_1,\dots,f_n) = \int f_1 \left(g(\Gamma^i_1)(X_1,\dots,X_m)\right)^\sigma (f_2\dots,f_n,f_0),$$

i.e., by “taking the derivatives from f_1”.

Let us write $X_i = \sum_{k=1}^d \psi_k \frac{\partial}{\partial x_k}$. Expanding the first integral we have

$$\int f_0 g(\Gamma^i_1)(X_1,\dots,X_m)(f_1,\dots,f_n) =$$

$$\sum_{k=1}^d \int \frac{\partial f_1}{\partial x_k}\psi_k X_1 \dots \hat{X}_i \dots X_m f_2 \dots f_n f_0 =$$

$$-\sum_{k=1}^d \int f_1 \frac{\partial \psi_k}{\partial x_k} X_1 \dots \hat{X}_i \dots X_m f_0 f_2 \dots f_n + f_1 \psi_k \frac{\partial X_1}{\partial x_k} X_2 \dots \hat{X}_i \dots X_m f_2 \dots f_n f_0 +$$

$$+ \dots + f_1 \psi_k X_1 \dots \hat{X}_i, \dots X_m f_2 \dots f_n \frac{\partial f_0}{\partial x_k}.$$

Therefore

$$\left(g(\Gamma^i_1)(X_1,\dots,X_m)\right)^\sigma (a_1,\dots,a_n) =$$

$$-\Gamma^{i,i}(X_1,\dots,X_m,a_1,\dots,a_n) - \sum_{k=1,k\neq i}^m \Gamma^{i,k}(X_1,\dots,X_m,a_1,\dots,a_n)$$

$$-\sum_{k=1}^n \Gamma^i_k(X_1,\dots,X_m,a_1,\dots,a_n) =$$

$$g(-\sum_{k=1}^m \Gamma^{i,k} - \sum_{k=1}^n \Gamma^i_k)(X_1,\dots,X_m)(a_1,\dots,a_n) =$$

$$g(\Gamma^i_1 \cdot \sigma)(X_1,\dots,X_m)(a_1,\dots,a_n).$$

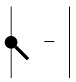
The verification for the case $\Gamma^{i,j}$ is trivial and the case Γ^i_j with $j \neq 1$ is also immediate because there is only permutation of variables involved. $\square$

We obtain then a bimodule map

$$\begin{tikzcd} \mathsf{CPT} \arrow[d, "\mathrm{id}"] & \circlearrowright & \prod_n \mathsf{BVKGra}(\cdot, n)[-n] \arrow[d] & \circlearrowleft & \mathsf{BVGra} \arrow[d] \\ \mathsf{CPT} & \circlearrowright & \prod_n \operatorname{Hom}\left(T_{\mathrm{poly}}^{\otimes \bullet}, \operatorname{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes n+1}, \mathbb{R})\right)[-n] & \circlearrowleft & \operatorname{End} T_{\mathrm{poly}}. \end{tikzcd} \tag{8}$$

The image of the Maurer–Cartan element  $\in \mathsf{BVKGra}(0,2)[-2]$ is the element induced by the multiplication map $\mu\colon C_c^\infty(\mathbb{R}^d)^{\otimes 2} \to C_c^\infty(\mathbb{R}^d)$.

By twisting with respect to these Maurer–Cartan elements we obtain a map of $Tw\mathsf{CPT}$ from $\prod_n \mathsf{BVKGra}^\mu(\cdot, n)[-n]$ to $\operatorname{Hom}^\mu\left(T_{\mathrm{poly}}^{\otimes \bullet}, \prod_n \operatorname{Hom}(C_c^\infty(\mathbb{R}^d)^{\otimes n+1}, \mathbb{R})[-n]\right)$. Notice that in this last space, the differential coming from the twisting is the same as the one induced by the Hochschild differential and the degrees also agree with the Hochschild complex. In fact, the image of the map (8) lands in $\operatorname{Hom}\left(T_{\mathrm{poly}}^{\otimes \bullet}, D_{\mathrm{poly}}\right)$.

Since  $\in Tw\mathsf{CPT}$ acts trivially on both spaces, this induces an action of its subquotient CBr, therefore we obtain the following maps of bimodules:

$$\begin{tikzcd} \mathsf{CBr} \arrow[d] & \circlearrowright & \prod_n \mathsf{BVKGra}^\mu(\cdot, n)[-n] \arrow[d] & \circlearrowleft & \mathsf{BVGra} \arrow[d] \\ \mathsf{CBr} & \circlearrowright & \operatorname{Hom}(T_{\mathrm{poly}}^{\otimes \bullet}, D_{\mathrm{poly}}) & \circlearrowleft & \operatorname{End} T_{\mathrm{poly}}. \end{tikzcd} \tag{9}$$

Also, the CBr action on $\operatorname{Hom}(T_{\mathrm{poly}}^{\otimes \bullet}, D_{\mathrm{poly}})$ comes from the action of CBr on D_{poly} (as seen in 3.3.1), which translates into an operadic morphism $\mathsf{CBr} \to \operatorname{End} D_{\mathrm{poly}}$. Thus, by composition with the map (5) we obtain

$$\begin{tikzcd} \mathsf{CBr} \arrow[d] & \circlearrowright & Chains(\mathsf{F}\mathbb{H}_{\bullet,0}) \arrow[d] & \circlearrowleft & Chains(\mathsf{FFM}_2) \arrow[d] \\ \operatorname{End} D_{\mathrm{poly}} & \circlearrowright & \operatorname{Hom}(T_{\mathrm{poly}}^{\otimes \bullet}, D_{\mathrm{poly}}) & \circlearrowleft & \operatorname{End} T_{\mathrm{poly}}. \end{tikzcd}$$

5.3. A zig-zag of quasi-torsors

Let us recall the definition of an *operadic quasi-torsor* from [1]:

Definition 13. Let $\mathcal{P}$ and $\mathcal{Q}$ be two differential graded operads and let M be a $\mathcal{P} - \mathcal{Q}$ operadic differential graded bimodule, i.e., there are compatible actions

$$\mathcal{P} \circlearrowright M \circlearrowleft \mathcal{Q}.$$

We say that $\mathcal{M}$ is a $\mathcal{P}$–$\mathcal{Q}$ quasi-torsor if there is an element $\mathbf{1} \in M^0(1)$ such that the canonical maps

$$\begin{aligned} l\colon \mathcal{P} &\to \mathcal{M} & r\colon \mathcal{Q} &\to \mathcal{M} \\ p &\mapsto p \circ (\mathbf{1}, \dots, \mathbf{1}) & q &\mapsto \mathbf{1} \circ q \end{aligned} \tag{10}$$

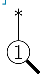
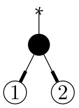
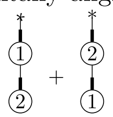
are quasi-isomorphisms.

Lemma 21. *$Chains(\mathsf{F}\mathbb{H}_{\bullet,0})$ is a $\mathsf{CBr} - Chains(\mathsf{FFM}_2)$ quasi-torsor.*

Proof. Let us consider the element $\mathbf{1} \in Chains_0(\mathsf{F}\mathbb{H}_{1,0})$ corresponding to a single point on the upper half plane with frame is pointing upwards.

Let $i\colon \mathsf{FFM}_2 \to \mathsf{F}\mathbb{H}_{\bullet,0}$ be the map that sends a configuration in $c \in \mathsf{FFM}_2$ to the configuration in $\mathsf{F}\mathbb{H}_{\bullet,0}$ given by one boundary stratum on the upper half plane with c on it. It is clear that i is a homotopy equivalence (with homotopy inverse being the map that "forgets" the boundary of the upper half plane). The map $r\colon Chains(\mathsf{FFM}_2) \to Chains(\mathsf{F}\mathbb{H}_{\bullet,0})$, as defined in equation (10) is the image of i via the functor $Chains$. Since i is a homotopy equivalence, r is a quasi-isomorphism.

It was shown in [8] that $H(\mathsf{FFM}_2) = \mathsf{BV}$.

The map l sends  $\in \mathsf{CBr}_{-1}(1)$ to the fundamental chain of the circle. It sends  to the zero chain consisting of two horizontally aligned points in the upper half plane with frames pointing upwards. And it sends  to the 1-chain corresponding to two points rotating around each other.

Since the homologies of CBr and of $\mathsf{F}\mathbb{H}_{\bullet,0}$ are both BV and l sends (representatives of) generators to (representatives of) generators, l is a quasi-isomorphism. $\square$

The main Theorem of [1] states that if the $\mathcal{P} - \mathcal{Q}$-bimodule M is an operadic quasi-torsor, then there is a zig-zag of quasi-isomorphisms connecting $\mathcal{P} \circlearrowright M \circlearrowleft \mathcal{Q}$ to the canonical bimodule $\mathcal{P} \circlearrowright \mathcal{P} \circlearrowleft \mathcal{P}$.

It follows then from Lemma 21 that there is a zig-zag of bimodules

$$\begin{array}{ccccc} \mathsf{CBr} & \circlearrowright & \mathsf{CBr} & \circlearrowleft & \mathsf{CBr} \\ \downarrow & & \downarrow & & \downarrow \\ \cdots & \circlearrowright & \cdots & \circlearrowleft & \cdots \\ \uparrow & & \uparrow & & \uparrow \\ \mathsf{CBr} & \circlearrowright & Chains(\mathsf{F}\mathbb{H}_{\bullet,0}) & \circlearrowleft & Chains(\mathsf{FFM}_2) \end{array}.$$

Let $\mathsf{CBr}_\infty^{\mathrm{bimod}}$ be a cofibrant resolution of the canonical bimodule CBr. $\mathsf{CBr}_\infty^{\mathrm{bimod}}$ is a $\mathsf{CBr}_\infty - \mathsf{CBr}_\infty$-bimodule, where CBr_∞ is a cofibrant resolution of the operad CBr.

Finally, the zig-zag can be lifted up to homotopy to a bimodule map

$$\begin{tikzcd} \mathsf{CBr}_\infty \arrow[d] & \circlearrowright & \mathsf{CBr}_\infty^{\mathrm{bimod}} \arrow[d] & \circlearrowleft & \mathsf{CBr}_\infty \arrow[d] \\ \operatorname{End} D_{\mathrm{poly}} & \circlearrowright & \operatorname{Hom}(T_{\mathrm{poly}}^{\otimes\bullet}, D_{\mathrm{poly}}) & \circlearrowleft & \operatorname{End} T_{\mathrm{poly}}, \end{tikzcd}$$

giving us the desired quasi-isomorphism and thus proving Theorem 1.

It also follows from Lemma 21 and [1] that CBr is quasi-isomorphic to $Chains(\mathsf{FFM}_2)$. Due to the formality of FFM_2 [11], it follows that we can replace CBr_∞ in Theorem 1 by any cofibrant replacement of the operad BV.

6. Globalization

Let M be a d-dimensional oriented manifold. In this section we globalize the BV_∞ quasi-isomorphism $T_{\mathrm{poly}}(\mathbb{R}^d) \to D_{\mathrm{poly}}(\mathbb{R}^d)$ from Theorem 1 to a quasi-isomorphism $T_{\mathrm{poly}}(M) \to D_{\mathrm{poly}}(M)$, thus proving Theorem 2. To do this we use standard formal geometry techniques.

6.1. The idea

We refer the reader to the paper [4], from which we borrow the notation.

Theorem 1 is valid if we replace $\mathbb{R}^d$ by $\mathbb{R}^d_{\mathrm{formal}}$, its formal completion at the origin, i.e., the space whose ring of functions is given by formal power series on the coordinates $x_1, \dots, x_d$.

We consider $\mathcal{T}_{\mathrm{poly}}$ (resp. $\mathcal{D}_{\mathrm{poly}}$), the vector bundle on M of fiberwise formal multivector fields (resp. multidifferential operators) tangent to the fibers. We can then construct the vector bundles $\Omega(\mathcal{T}_{\mathrm{poly}}, M)$ of forms valued in $\mathcal{T}_{\mathrm{poly}}$ and $\Omega(\mathcal{D}_{\mathrm{poly}}, M)$ of forms valued in $\mathcal{D}_{\mathrm{poly}}$ with appropriate differentials.

The fibers of the bundles $\mathcal{T}_{\mathrm{poly}}$ and $\mathcal{D}_{\mathrm{poly}}$ are isomorphic to $T_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ and $D_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$, respectively. Therefore, the formal version of the formality map can be used to find a vector bundle CBr_∞ quasi-isomorphism

$$U^f : \Omega(\mathcal{T}_{\mathrm{poly}}, M) \to \Omega(\mathcal{D}_{\mathrm{poly}}, M).^4 \tag{11}$$

These two vector bundles can be related with $T_{\mathrm{poly}}(M)$ and $D_{\mathrm{poly}}(M)$. In fact, with an appropriate change of differential that comes from a choice of a flat connection, $\Omega(\mathcal{T}_{\mathrm{poly}}, M)$ becomes a resolution of $T_{\mathrm{poly}}(M)$ and $\Omega(\mathcal{D}_{\mathrm{poly}}, M)$ becomes a resolution of $D_{\mathrm{poly}}(M)$. This change of differential can be seen locally as a twist via a Maurer–Cartan element B in $\Omega^1(\mathcal{T}^1_{\mathrm{poly}}, U) = \Omega^1(\mathcal{D}^1_{\mathrm{poly}}, U)$. However, the linear part of B (in the fiber coordinates) is not globally well defined.

---

[4] Using the fact that the formality morphism is invariant by linear transformation of coordinates.

In general, if one wants to twist an algebra A over an operad $\mathcal{P}$ (with an implicit map $\mathsf{Lie}_1 \to \mathcal{P}$), one obtains that the twisted algebra A^μ is not an algebra over $\mathcal{P}$ but over $Tw\mathcal{P}$. However, if $\mathcal{P}$ is *natively twistable*, i.e., there exists an operad morphism $\mathcal{P} \to Tw\mathcal{P}$ such that $\mathcal{P} \to Tw\mathcal{P} \to \mathcal{P}$ is the identity, then $\mathcal{P}$ still acts on A. The twist of an operad is always natively twistable [24, Lemma 94]. We would therefore like to replace the third or fourth row of diagram (12) with a row that arises from a twist.

This would be simple (we could just twist one of them), if not for the fact that the linear part of B is not well defined. Indeed, this causes action of $Tw\mathsf{BVGra}$ on the twisted $\Omega(\mathcal{T}_{\mathrm{poly}})^B$ to be ill-defined if the linear part of the Maurer–Cartan element has to be used and this occurs whenever there exist internal vertices with exactly one outgoing edge and at most one incoming edge (since more incoming edges would kill the linear part). To circumvent this problem, instead of replacing BVGra by $Tw\mathsf{BVGra}$ we consider a related object $\mathsf{BVGraphs}$. And similar with BVKGra leads us to define $\mathsf{BVKGraphs}$.

To show that the relevant maps factor through these objects we need Kontsevich's vanishing lemmas [15] that hold for his formality morphism. The way we constructed the BV_∞ formality morphism from Theorem 1 depends on some choice of lifts and might not be in general true that the vanishing lemmas hold, but they certainly hold if we choose the BV_∞ formality morphism in such a way that it extends Kontsevich's original formality morphism.

6.2. An extension of Kontsevich's L_∞ morphism

In this section we show that the CBr_∞ formality morphism from Theorem 1 can be obtained in such a way that it extends Kontsevich's original L_∞ morphism [15].

We have the following chain of maps:

$$
\begin{array}{ccccc}
\mathsf{holie}_1 & \circlearrowright & {\mathsf{hoLie}_1}^{\mathrm{bimod}} & \circlearrowleft & \mathsf{hoLie}_1 \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{CBr}_\infty & \circlearrowright & \mathsf{CBr}_\infty^{\mathrm{bimod}} & \circlearrowleft & \mathsf{CBr}_\infty \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{CBr} & \circlearrowright & Chains(\mathsf{F}\mathbb{H}_{\bullet,0}) & \circlearrowleft & Chains(\mathsf{FFM}_2) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{CBr} & \circlearrowright & \prod_n \mathsf{BVKGra}^\mu(\cdot, n)[-n] & \circlearrowleft & \mathsf{BVGra} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{End} D_{\mathrm{poly}} & \circlearrowright & \operatorname{Hom}(T_{\mathrm{poly}}^{\otimes \bullet}, D_{\mathrm{poly}}) & \circlearrowleft & \operatorname{End} T_{\mathrm{poly}},
\end{array}
\tag{12}
$$

where $\mathsf{hoLie}_1 = \Omega(\mathsf{Lie}\{1\}^\vee)$, the first downwards maps are induced by the inclusion $\mathsf{Lie} \to \mathsf{CBr}$ and the other maps follow from the proof of Theorem 1. Showing that our morphism extends Kontsevich's formality morphism amounts to showing that the full composition of the maps in (12) gives Kontsevich's map. This is clear for the left column.

For the other two columns the argument is similar so we will only prove it for the right column given that the notation is simpler. Let us call μ_n the generator of $\mathsf{Lie}\{1\}^\vee(n)$.

We recall that in [15] the construction of U_n, the L_∞ components of the formality morphism are constructed by sending μ_n to the fundamental chain of $\mathbb{H}_{n,0}$. We wish then to show the commutativity of the following diagram, where the uppers horizontal maps represent Kontsevich's approach and Gra is the suboperad of BVGra in which tadpoles are not admitted.

$$\begin{tikzcd} \mathsf{holie}_1 \arrow[r] \arrow[d] & Chains(\mathsf{FM}_2) \arrow[r] & \mathsf{Gra} \arrow[r] & \mathrm{End}(T_{\mathrm{poly}}). \\ \mathsf{CBr}_\infty \arrow[r] & Chains(\mathsf{FFM}_2) \arrow[r] & \mathsf{BVGra} \arrow[ur] & \end{tikzcd} \tag{13}$$

As semi-algebraic manifolds, $\mathsf{FFM}_2(n) = \mathsf{FM}_2(n) \times (S^1)^{\times n}$, therefore there exists an inclusion map $i\colon \mathsf{FM}_2 \to \mathsf{FFM}_2$ that is the identity on the FM_2 component and constant equal to the vertical direction in the S^1 components.

Naming the relevant maps, diagram (13) becomes

$$\begin{tikzcd} \mathsf{holie}_1 \arrow[r, "f"] \arrow[d, "i_L"] & Chains(\mathsf{FM}_2) \arrow[r] \arrow[d, "i_*"] & \mathsf{Gra} \arrow[r] \arrow[d, hook] & \mathrm{End}(T_{\mathrm{poly}}). \\ \mathsf{CBr}_\infty \arrow[r, "g"] & Chains(\mathsf{FFM}_2) \arrow[r] & \mathsf{BVGra} \arrow[ur] & \end{tikzcd} \tag{14}$$

It is clear that the right triangle diagram and the adjacent square diagram are commutative. To conclude the commutativity of the exterior diagram it is enough to show that the left square is commutative but this need not be the case. Fortunately this can be rectified if one is careful when constructing the map g as a lift over quasi-isomorphisms. We sketch here the argument that is nothing but an adapted version or the argument of Lemmas 12 and 13 in [1].

The fact that $\mathsf{Lie}\{1\}$ can be seen embedded in CBr via the map F in section 4.3 implies that the generators μ_n of holie_1 can be seen as part of the generators of CBr_∞ (via the map i_L) and the map f sends μ_n to the fundamental chain of $\mathsf{FM}_2(n)$.

To construct g one starts with a filtration $0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathsf{CBr}_\infty$ such that when differentiating the generators we fall in the previous degree of the filtration and then we construct the map recursively using the following diagram:

$$\begin{tikzcd} & & F \arrow[d, two heads] \arrow[dr, two heads] & \\ \mathsf{CBr}_\infty \arrow[r] \arrow[urr, dotted, "g'"] & \mathsf{CBr} \arrow[r] & E & Chains(\mathsf{FFM}_2) \arrow[l] \end{tikzcd}$$

where all maps are quasi-isomorphisms, E is the operad through which the zig-zag connecting CBr and $Chains(\mathsf{FFM}_2)$ goes and F is the operad resulting from the "surjective trick", i.e., an operad that surjects both onto E and $Chains(\mathsf{FFM}_2)$ such that the depicted triangle commutes up to homotopy. At every stage we wish to map μ_n to a pre-image of the fundamental chain of FM_2 (seen inside of FFM_2) and essentially one has to check that $dg'(\mu_n) = g'(d\mu_n)$, but this follows from the fact that the boundary of the fundamental chain of $\mathsf{FM}_2(n)$ is computed the same way as the cocomposition of μ_n in $\mathsf{Lie}\{1\}^\vee$.

6.3. The bimodule BVKGraphs

In this section we construct the appropriate twisted and "corrected" versions of BVGra (resp. BVKGra)), $\mathsf{BVGraphs}$ (resp. $\mathsf{BVKGraphs}$). The construction of the full bimodule $\mathsf{BVKGraphs}$ is a generalization of [24, Section 8] where Willwacher works with graphs without tadpoles.

Notice that due to the chain of morphisms (12) there is a morphism of bimodules

$$\begin{tikzcd} \mathsf{hoLie}_1 \arrow[d] & \circlearrowright & {\mathsf{hoLie}_1}^{\mathrm{bimod}} \arrow[d] & \circlearrowleft & \mathsf{hoLie}_1 \arrow[d] \\ \mathsf{CBr} & \circlearrowright & \prod_n \mathsf{BVKGra}^\mu(\cdot, n)[-n] & \circlearrowleft & \mathsf{BVGra}\cdot \end{tikzcd}$$

Using the formalism of twisting of bimodules described in the Appendix we can perform the bimodule twisting with respect to this morphism, thus obtaining the operadic bimodule $Tw\mathsf{CBr} \circlearrowright Tw \prod_n \mathsf{BVKGra}^\mu(\cdot, n)[-n] \circlearrowleft Tw\mathsf{BVGra}$.

The elements in $Tw\mathsf{BVGra}(n)$ can be seen as linear combinations of directed graphs with at least n vertices, where from these, n of them are labeled by numbers from 1 to n and the remaining ones are indistinguishable. The labeled vertices are called external vertices and the unlabeled ones are called internal vertices. In a similar way, the elements of $Tw \prod_n \mathsf{BVKGra}^\mu(m, n)[-n]$ consist of the same kind of graphs, but where now the type I vertices come in two flavors, the indistinguishable internal vertices and the m labeled external vertices.

Proposition/Definition 22. *The operad* $\mathsf{BVGraphs}$ *is defined to be the quotient* $\mathsf{BVGraphs}^{\circlearrowright}/I$ *where:*

– $\mathsf{BVGraphs}^{\circlearrowright} \subset Tw\mathsf{BVGra}$ *is the suboperad spanned by graphs having no 0- or 1-valent internal vertices and having no 2-valent internal vertices with exactly one incoming and one outcoming edges,*

– *I is the suboperad of* $\mathsf{BVGraphs}^{\circlearrowright}$ *spanned by graphs containing at least one tadpole on an internal vertex.*

Proof. Since the operadic insertion is done at external vertices it is clear that conditions defining $\mathsf{BVGraphs}^{\circlearrowright}$ are preserved by operadic composition, but we need to check that S is preserved under the action of the differential.

The differential d in $Tw\mathsf{BVGra}$ has the form $d = d_1 + d_2$, where d_1 is defined by $d_1\Gamma = (\circ\!\!\longrightarrow\!\!\bullet + \circ\!\!\longleftarrow\!\!\bullet) \circ \Gamma - \sum_i \pm\Gamma \circ_i (\circ\!\!\longrightarrow\!\!\bullet + \circ\!\!\longleftarrow\!\!\bullet)$, and d_2 acts by replacing every internal vertex by $\bullet\!\!\longrightarrow\!\!\bullet$.

This means that the differential acts by splitting internal vertices out of every vertex.

The d_2 component of the differential produces 1-valent internal vertices when all incident edges are reconnected to only one of the internal vertices. Similarly, the second summand in d_1 produces a 1-valent internal vertex whenever all incident edges are reconnected to the external vertex. All of these factors are canceled out by the first summand of the definition of d_1.

The creation of internal vertices with exactly one incoming and one outcoming edges happens only when after taking the differential in one vertex, there is exactly one other vertex that connects to the split internal vertex. However this term will be canceled out when the differential is taken on this other vertex.

It remains to see that I is a suboperad. Since the composition is done at external vertices, this cannot destroy tadpoles at internal vertices. The only way the action of the differential could destroy a tadpole at an internal vertex would be by acting with d_2 on that tadpole and reconnecting the two new internal vertices. But this would create a multiple edge and graphs with multiple edges vanish. □

Given an operad $\mathcal{P}$ with a morphism from $\mathsf{holie}_1 \to \mathcal{P}$, there is a canonical projection $Tw\mathcal{P} \to \mathcal{P}$, as described in the Appendix. We prove now a Lemma that will be useful to show that the operad morphism $Chains(\mathsf{FFM}_2) \to \mathsf{BVGra}$ factors through $\mathsf{BVGraphs}$. The proof of this Lemma is essentially in [24, Appendix D.3].

Lemma 23. *$Chains(\mathsf{FFM}_2)$ is natively twistable.*

Proof. We need to construct an operad map $\iota\colon Chains(\mathsf{FFM}_2) \to Tw\ Chains(\mathsf{FFM}_2)$ that is a right inverse to the canonical projection.

Let $\mathsf{FFM}_2^k(n+k)$ be the subspace of $\mathsf{FFM}_2(n+k)$ whose last k points have their frame constantly pointing upwards.

The bundle maps $\pi_{n,k}\colon \mathsf{FFM}_2^k(n+k) \to \mathsf{FFM}_2(n)$ defined by "forgetting" the last k points define a map at the level of chains

$$\pi_{k,n}^{-1}\colon Chains(\mathsf{FFM}_2(n)) \to Chains(\mathsf{FFM}_2^k(n+k)) \subset Chains(\mathsf{FFM}_2(n+k)).$$

Notice that this map lands in the $\mathbb{S}_k$ invariant subspace $Chains(\mathsf{FFM}_2(n+k))^{\mathbb{S}_k}$.

Let $c \in Chains(\mathsf{FFM}_2)(n)$. To define $\iota(c)$ it is enough to define its projection in $Chains(\mathsf{FFM}_2)(n+k)^{\mathbb{S}_k}$. We define this projection to be $\pi_{k,n}^{-1}$.

To see that this is an operad map, we need to check that $\iota(c \circ_i c') = \iota(c) \circ_i \iota(c')$. This equality follows from the observation that fixed a boundary stratum of a configuration of points, having k points varying freely is the same as i points inside that boundary stratum and $k-i$ outside, for $i = 0, \ldots, k$. □

The operad morphism $Chains(\mathsf{FFM}_2) \to \mathsf{BVGra}$ and the functoriality of Tw and the canonical projections $Tw\mathcal{P} \to \mathcal{P}$ give us the following commutative square

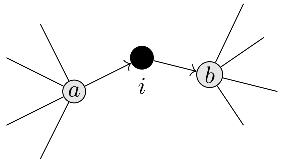


Fig. 4. An internal vertex connected to two (internal or external) vertices.

$$\begin{array}{ccc} Tw\ Chains(\mathsf{FFM}_2) & \longrightarrow & Tw\mathsf{BVGra} \\ \downarrow & & \downarrow \\ Chains(\mathsf{FFM}_2) & \longrightarrow & \mathsf{BVGra} \end{array}$$

As a corollary of the previous Lemma, the operad morphism $Chains(\mathsf{FFM}_2) \to \mathsf{BVGra}$ factors as $Chains(\mathsf{FFM}_2) \to Tw\mathsf{BVGra} \to \mathsf{BVGra}$. Explicitly, the first map is given by

$$c \in Chains(\mathsf{FFM}_2)(n) \mapsto \sum_\Gamma \Gamma \int_{\pi_\Gamma^{-1}(c)} f_1(\Gamma), \tag{15}$$

where $f_1(\Gamma)$ is the form associated to the graph Γ, as defined in Section 5.1 and for Γ a graph with n external and m internal vertices, $\pi_\Gamma^{-1}(c)$ is the chain in $Chains(\mathsf{FFM}_2)(n+m)$ in which the m points corresponding to the internal vertices vary freely in $\mathbb{R}^d$ while their frame is constantly pointing upwards.

Proposition 24. *The operad morphism given by the composition* $Chains(\mathsf{FFM}_2) \to Tw\mathsf{BVGra} \to \mathrm{End}(T_{\mathrm{poly}})$ *factors through* $\mathsf{BVGraphs}$.

Proof. We first check that the morphism (15) lands inside $\mathsf{BVGraphs}^{\circlearrowleft}$ and for this one must check that the coefficient of the graphs that are "forbidden" in $\mathsf{BVGraphs}^{\circlearrowleft}$ is zero. This is clear if the graph contains a 1-valent internal vertex, since the computation of the coefficient involves an integral of a 1-form (corresponding to the incident edge) over a 2 dimensional space.

Suppose the graph Γ contains an internal vertex with exactly one incoming and one outcoming edges. Let us call this vertex i and let us also call a and b the vertices to which these two edges connect (see Fig. 4).

By Fubini's Theorem for fibrations, the integral $\int_{\pi_\Gamma^{-1}(c)} f_1(\Gamma)$ can be rewritten as

$$\int \left(\int_{X_{z_a,z_b}} d\phi_{ai} d\phi_{ib} \right) \ldots,$$

where X_{z_a,z_b} is the space of configurations in which the points labeled by a and b are in positions z_a and z_b, and the point labeled by i moves freely. It suffices therefore to show that the integral

$$\int_{X_{z_a,z_b}} d\phi_{ai}d\phi_{ib} \tag{16}$$

vanishes. To check this, notice that by (the fibration integral version of) Stokes Theorem, we have

$$\underbrace{d\int_{Y_{z_a,z_b}} d\phi_{ai}d\phi_{ij}d\phi_{jb}}_{0} = \int_{Y_{z_a,z_b}} \underbrace{d(d\phi_{ai}d\phi_{ij}d\phi_{jb})}_{0} \pm \int_{\partial Y_{z_a,z_b}} d\phi_{ai}d\phi_{ij}d\phi_{jb},$$

where Y_{z_a,z_b} is the configuration space of four points (i, j, a and b) where a and b are fixed at z_a and z_b and the points labeled by i and j are free. The integral on the left hand side vanishes by degree reasons. The boundary terms on the right hand side vanish except on the following cases:

- The boundary stratum in which a and i are infinitely close,
- The boundary stratum in which i and j are infinitely close,
- The boundary stratum in which j and b are infinitely close.

In each of these cases, the result is an integral of the form of integral (16) (possibly with different signs), therefore it is zero.

To see that the map actually factors through the quotient $\mathsf{BVGraphs}$, one must prove that if a graph Γ contains an internal vertex with a tadpole, then it is sent zero. This is clear since as T_{poly} is not twisted yet the action of graphs with internal vertices is zero. But even after twisting by the Maurer–Cartan element B the map still factors through $\mathsf{BVGraphs}$ as $\operatorname{div}B = 0$. $\square$

As a consequence of Lemma 29, the canonical projections $Tw\mathsf{CBr} \to \mathsf{CBr}$ and $Tw\mathsf{BVGraphs}^{\circlearrowleft} \to \mathsf{BVGraphs}^{\circlearrowleft}$ admit right inverses. This defines a $\mathsf{CBr}-\mathsf{BVGraphs}^{\circlearrowleft}$ bimodule structure on $Tw\prod_n \mathsf{BVKGra}^{\mu}(\cdot,n)[-n]$. Elements of this bimodule are (sequences of) graphs with type I and type II vertices as before, but now there are two kinds of type I vertices. Using the same designations as in CBr we refer to the labeled type I vertices as external vertices and the indistinguishable type I vertices as internal vertices.

Proposition/Definition 25. *The* $\mathsf{CBr} - \mathsf{BVGraphs}^{\circlearrowleft}$ *bimodule* $Tw\prod_n \mathsf{BVKGra}^{\mu}(\cdot,n)[-n]$ *has a subquotient that we call* $\mathsf{BVKGraphs}$ *constructed in the following way:*

We first consider the quotient Q of $Tw\prod_n \mathsf{BVKGra}^{\mu}(\cdot,n)[-n]$ *by the ideal J consisting of graphs with tadpoles on type I internal vertices and then the subspace of Q spanned by the graphs with the following properties:*

(1) There is at least one type I external vertex,
(2) There are no 0-valent type I internal vertices
(3) There are no 1-valent type I internal vertices with an outgoing edge,
(4) There are no 2-valent type I internal vertices with one incoming and one outgoing edge.

Moreover, this induces a well defined $\mathsf{CBr} - \mathsf{BVGraphs}$ bimodule on $\mathsf{BVKGraphs}$.

Proof. The ideal J is stable by the differential as in Proposition/Definition 22, since the additional piece of the differential does not interact with type I vertices. The right $\mathsf{BVGraphs}^{\circlearrowleft}$ action is on external vertices so it cannot destroy tadpoles on internal vertices. The left action does not affect edges between type I vertices so J is also stable by the left action. It follows that Q has a well defined $\mathsf{CBr}-\mathsf{BVGraphs}^{\circlearrowleft}$ bimodule structure and it is clear that the right $\mathsf{BVGraphs}^{\circlearrowleft}$ action descends to a right $\mathsf{BVGraphs}$ action.

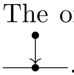
We must check that $\mathsf{BVKGraphs}$ is preserved by the differential, the left CBr and right $\mathsf{BVGraphs}$ actions. This is clear for the right $\mathsf{BVGraphs}$ action.

To check that $\mathsf{BVKGraphs}$ is closed under the action on CBr we start by considering the action of the generator T_n^1. Let $\Gamma_1, \dots, \Gamma_n$ be graphs in $\mathsf{BVKGraphs}$. The element $T_n^1(\Gamma_1, \dots, \Gamma_n)$ is determined by inserting $\Gamma_2, \dots, \Gamma_n$ at the type II vertices of Γ_1, therefore every type I vertex in $T_n^1(\Gamma_1, \dots, \Gamma_n)$ can be identified as coming from one of the Γ_i. Since there are only incoming edges at type II vertices, the action of T_n^1 can increase or maintain the number of incoming edges at a type I vertex but it can only maintain the number of outgoing edges at every type I vertex, thus proving that properties *(2)*, *(3)* and *(4)* are preserved. Property *(1)* is clearly preserved.

The action of T_n^j is given by insertions of the Γ_i in the type II vertices on cyclic permutations of Γ_1, using the cyclic action of BVKGra described in section 3.3.3. Since the cyclic action preserves properties *(1)–(4)*, $\mathsf{BVKGraphs}$ is closed under the action of T_n^j.

The insertion of the empty graph $\mathbb{1} \in \mathsf{BVKGra}(0,0)$ on some type II vertex of another graph has two possible outcomes. Either there is an incoming edge and the insertion of $\mathbb{1}$ at that vertex is 0 or there were no incoming edges and the insertion of $\mathbb{1}$ forgets the vertex. In both cases properties *(1)–(4)* are preserved, therefore $\mathsf{BVKGraphs}$ is closed under the action of $T_n^{j,j+1}$.

To show that $\mathsf{BVKGraphs}$ is closed under the action of T'^j_n, it is enough to check that summands of the Maurer–Cartan element by which $\prod \mathsf{BVKGra}^\mu(\cdot, n)[-n]$ was twisted (image of the generators of ${\mathsf{holie}_1}^{\mathrm{bimod}}$) satisfy the following two properties:

(a) The only graph containing a 1-valent type I internal vertex is the 2 vertex graph , with coefficient 1.
(b) There are no graphs with vertices like the ones in property *(4)*.

To verify these properties we recall that the map $\mathsf{holie_1}^{\mathrm{bimod}} \to \prod \mathsf{BVKGra}^{\mu}(\cdot, n)[-n]$ involves at some step the integration of differential forms over $\mathsf{F}\mathbb{H}_{\bullet,0}$. Then, property (a) follows from degree reasons and property (b) has a proof similar to Proposition 24.

It remains to check that the differential preserves $\mathsf{BVKGraphs}$.

The differential is composed of the following pieces:

- The original splitting of type II vertices,
- Insertion of $\circ\!\!\longrightarrow\!\!\bullet + \circ\!\!\longleftarrow\!\!\bullet$ at type I external vertices,
- Insertion of $\bullet\!\!\longrightarrow\!\!\bullet$ at type I internal vertices,
- Bracket with the image of the generators of $\mathsf{holie_1}^{\mathrm{bimod}}$.

The first piece of the differential clearly preserves $\mathsf{BVKGraphs}$. Properties *(1)* and *(2)* are trivially preserved by all pieces of the differential. It remains to check properties *(3)* and *(4)*. The remaining pieces of the differential can produce vertices like *(3)* and *(4)*, so we must verify that these graphs cancel. There are 3 possibilities to obtain a vertex of the kind *(3)* with the differential:

Using the second piece of the differential on a graph $\Gamma \in \mathsf{BVKGraphs}$, at every external vertex we get a forbidden 1-valent vertex connecting to it, corresponding to inserting $\circ\!\!\longleftarrow\!\!\bullet$ and reconnecting all the originally incident edges to the external vertex. Similarly, for every internal vertex of Γ, the second piece of the differential produces one 1-valent internal vertex with one outgoing edge connecting to it.

Due to property (a), the only "problematic" graphs that may arise from the fourth piece of the differential are coming from bracket with $\underline{\overset{\bullet}{\downarrow}}$. The bracket with this element gives $\overset{\overset{\bullet}{\downarrow}}{\Gamma} \pm \sum_i \Gamma \circ_{\bar{i}} \underline{\overset{\bullet}{\downarrow}}$ where on the first summand we connect the internal vertex to every possible (type I or II) vertex of Γ and on the second summand the $\circ_i$ represents an insertion at the vertices of Γ of type II. In $\overset{\overset{\bullet}{\downarrow}}{\Gamma}$, the edges connecting to type I vertices in Γ are all canceled out with the second and third pieces of the differential as described above. The edges connecting to type II vertices are canceled by the terms in $\sum_i \Gamma \circ_{\bar{i}} \underline{\overset{\bullet}{\downarrow}}$ in which all the incident edges to $\bar{i}$ are reconnected to the type II vertex after the insertion.

To check that the differential preserves property *(4)*, one can see that everytime an internal vertex having property *(4)* is created due to type I internal or external vertex splitting, this is term is canceled by a splitting on the other adjacent vertex to the 2-valent vertex that was created. This also holds for splitting of vertices adjacent to type II vertices, but in that case the cancellation is done with a term coming from $\sum_i \Gamma \circ_{\bar{i}} \underline{\overset{\bullet}{\downarrow}}$. Due to property (b), no more forbidden graphs are produced by the fourth piece of the differential. $\square$

Lemma 26. *$Chains(\mathbb{H}_{\bullet,0})$ is natively twistable.*

The construction of the map $Chains(\mathbb{H}_{\bullet,0}) \to Tw\ Chains(\mathbb{H}_{\bullet,0})$ is identical to Lemma 23 and the compatibility with the left and right actions is immediate.

As a consequence, the bimodule morphism $Chains(\mathbb{H}_{\bullet,0}) \to \mathsf{BVKGra}$ factors through $Tw \prod_n \mathsf{BVKGra}^{\mu}(\cdot, n)[-n]$. The explicit formula is given by

$$c \in Chains(\mathbb{H})(n) \mapsto \sum_{\Gamma} \Gamma \int_{\pi_{\Gamma}^{-1}(c)} f_2(\Gamma), \tag{17}$$

where $f_2(\Gamma)$ is the form associated to the graph Γ, as defined in Section 5.1 and if Γ is a graph with n external and m internal type I vertices and k type II vertices, $\pi_{\Gamma}^{-1}(c)$ is the chain in $Chains(\mathbb{H}_{n+m,k})$ in which the m points corresponding to the internal vertices vary freely in the upper half plane while their frame is constantly pointing upwards.

Proposition 27. *The bimodule morphism $Chains(\mathbb{H}_{\bullet,0}) \to \mathsf{BVKGra}$ factors through* $\mathsf{BVKGraphs}$.

The proof is essentially the same as the one of Proposition 24.

6.4. The twist

As a consequence of the previous section we have the following map of bimodules representing the last layer of the formality morphism:

$$\begin{tikzcd} \mathsf{CBr} \arrow[d] & \circlearrowright & \mathsf{BVKGraphs} \arrow[d] & \circlearrowleft & \mathsf{BVGraphs} \arrow[d] \\ \operatorname{End} D_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) & \circlearrowright & \operatorname{Hom}(T_{\text{poly}}^{\otimes \bullet}(\mathbb{R}^d_{\text{formal}}), D_{\text{poly}}(\mathbb{R}^d_{\text{formal}})) & \circlearrowleft & \operatorname{End} T_{\text{poly}}(\mathbb{R}^d_{\text{formal}}). \end{tikzcd}$$

As described in section 6.1, the fibers of the vector bundles $\mathcal{T}_{\text{poly}}$ and $\mathcal{D}_{\text{poly}}$ are isomorphic to $T_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ and $D_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$, therefore this induces the following map of bimodules:

$$\begin{tikzcd} \mathsf{CBr} \arrow[d] & \circlearrowright & \mathsf{BVKGraphs} \arrow[d] & \circlearrowleft & \mathsf{BVGraphs} \arrow[d] \\ \operatorname{End} \Omega(\mathcal{D}_{\text{poly}}) & \circlearrowright & \operatorname{Hom}(\Omega(\mathcal{T}_{\text{poly}})^{\otimes \bullet}, \Omega(\mathcal{D}_{\text{poly}})) & \circlearrowleft & \operatorname{End} \Omega(\mathcal{T}_{\text{poly}}). \end{tikzcd}$$

Since the CBr_∞ formality morphism from Theorem 1 is an extension of Kontsevich's L_∞ formality morphism (see section 6.2), its L_∞ part satisfies properties P1)–P5) from section 7 in [15]. In particular, property P4) implies that for $n \geq 2$, $U_n(B, \dots, B) = 0$ and thus $B' = \sum_{n=1}^{\infty} \frac{1}{n!} U_n(B, \dots, B) = U_1(B) = B$, under the identification $\Omega^1(\mathcal{T}^1_{\text{poly}}) = \Omega^1(\mathcal{D}^1_{\text{poly}})$.

On the other hand, the bimodule $\mathsf{BVKGraphs}$ is obtained from a twist therefore it is natively twistable.

Therefore, following the Appendix, we obtain a map of bimodules:

$$\begin{array}{ccccc} \mathsf{CBr} & \circlearrowright & \mathsf{BVKGraphs} & \circlearrowleft & \mathsf{BVGraphs} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{End}\Omega(\mathcal{D}_{\mathrm{poly}})^B & \circlearrowright & \operatorname{Hom}\Big(\big(\Omega(\mathcal{T}_{\mathrm{poly}})^B\big)^{\otimes\bullet}, \Omega(\mathcal{D}_{\mathrm{poly}})^B\big)\Big) & \circlearrowleft & \operatorname{End}\Omega(\mathcal{T}_{\mathrm{poly}})^B, \end{array}$$

where the superscript B indicates that we are considering the twisted differential. For this twist it is important that $\mathsf{BVKGraphs}$ forbids 1-valent internal vertices with an outgoing edge and 2-valent internal vertices with one incoming and one outgoing edges, since the linear part of B is not globally well-defined.

Composing with this map with bimodule maps $\mathsf{CBr}_\infty^{\mathrm{bimod}} \to Chains(\mathbb{H}_{\bullet,0}) \to \mathsf{BVKGraphs}$, we obtain the desired global CBr_∞ quasi-isomorphism.

Appendix A. Twisting

In this Appendix we give an overview on the theory of operadic twisting following [5] that we need for this paper and we define a notion of twisting of operadic bimodules, which is not more than an adaptation of the same theory. We advise the reader to read the third section of [5] if they are not familiar with twisting of operads.

We make the standard assumptions used in the context of twisting with respect to Maurer–Cartan elements. Namely, all algebras $\mathfrak{g}$ (over hoLie_1 or another operad) are equipped with complete decreasing filtrations $\mathfrak{g} = F_0\mathfrak{g} \supset F_1\mathfrak{g} \supset \dots$, such that the operations are compatible with the filtration and $\mathfrak{g} = \varprojlim_i \mathfrak{g}/F_i\mathfrak{g}$. These assumptions are made so that infinite sums (going deeper in the filtration) are allowed.

Let $\mathfrak{g}$ is a hoLie_1 algebra, an element $\mu \in F_1\mathfrak{g}$ of degree 2 is said to be *Maurer–Cartan* element of $\mathfrak{g}$ if it satisfies the Maurer–Cartan equation:

$$d\mu + \sum_{n=2}^{\infty} \frac{1}{n!} l_n(\mu, \dots, \mu) = 0,$$

where l_n are the generating operations in hoLie_1.

Given such a Maurer–Cartan element, one can construct the twisted hoLie_1 algebra $\mathfrak{g}^\mu$, that is as a graded vector space just $\mathfrak{g}$, but with a changed (called twisted) differential, denoted by d^μ, that is defined by $d^\mu(x) = dx + \sum_{n=1}^\infty \frac{1}{n!} l_{n+1}(\mu, \dots, \mu, x)$, and new brackets given by $l_n^\mu(x_1, \dots, x_k) = \sum_{n=1}^\infty \frac{1}{n!} l_{n+k}(\mu, \dots, \mu, x_1, \dots, x_k)$.

A.1. Twisting of operads

Let $\mathcal{P}$ be an operad and let us assume the existence of an operad morphism $F\colon \mathsf{hoLie}_1 \to \mathcal{P}$. If $\mathfrak{g}$ is a $\mathcal{P}$-algebra, it inherits a hoLie_1-algebra structure thanks to F. Therefore it makes sense to talk about Maurer–Cartan elements of $\mathfrak{g}$. If μ be a Maurer–Cartan element of $\mathfrak{g}$, the twisted algebra $\mathfrak{g}^\mu$ is no longer necessarily a $\mathcal{P}$ algebra. It is, however, an algebra over the operad $Tw\mathcal{P}$, whose construction depends on the map F.

As an $\mathbb{S}$-module, we have

$$TwP(p) = \prod_{r \geq 0} \left(\mathcal{P}(r+p) \otimes \mathbb{K}[-2r]\right)^{\mathbb{S}_r},$$

where $\mathbb{S}_r$ here is the subgroup of $\mathbb{S}_{r+p}$ fixing the last p entries. The r non-symmetric inputs should be thought as representing the insertion of r Maurer–Cartan elements. The composition is defined using the original composition in $\mathcal{P}$, but summing over shuffles to ensure that it lands in the invariants over the action of $\mathbb{S}_{r_1+r_2}$.

To describe the differential we need an auxiliary dg Lie algebra:

$$\mathcal{L}_{\mathcal{P}} := \mathrm{Conv}(\mathsf{Lie}\{1\}^\vee, \mathcal{P}) = \prod_{n \geq 1} \mathcal{P}(n)^{\mathbb{S}_n}[2-2n].$$

The Lie algebra $\mathcal{L}_{\mathcal{P}}$ acts on $Tw\mathcal{P}$, by composition on the non-symmetric inputs. $Tw\mathcal{P}(1)$ acts on $Tw\mathcal{P}$ by inner derivations.

There is an obvious degree zero map $\kappa\colon \mathcal{L}_{\mathcal{P}} \to \mathcal{T}w\mathcal{P}(1)$.

The map F induces a Maurer–Cartan element $\tilde{F}$, and the final differential is $d_{Tw} = d_{\mathcal{P}} + d_{\tilde{F}} + d_{\kappa(\tilde{F})}$, where the first piece is induced by the original differential in $\mathcal{P}$, the second one comes from the L_P action and the third one comes from the $Tw\mathcal{P}(1)$ action.

The fact that this is a differential is essentially a consequence of the following Proposition:

Proposition 28. *[5, Prop. 3.3] The map*

$$\mathcal{L}_{\mathcal{P}} \to \mathcal{L}_{\mathcal{P}} \ltimes Tw\mathcal{P}(1)$$
$$v \mapsto v + \kappa(v)$$

is a morphism of Lie algebras.

The action of $Tw\mathcal{P}$ on $\mathfrak{g}^\mu$ is given by inserting Maurer–Cartan elements in the non-symmetric slots. Explicitly, let $p \in Tw\mathcal{P}(n)$ and let $x_1, \ldots, x_n \mathfrak{g}^\mu$.

$$p(x_1, \ldots, x_n) := \sum_{r=0}^{\infty} \frac{1}{r!} p_r(\mu, \ldots, \mu, x_1, \ldots, x_n),$$

where p_r is the projection of p in the factor $(\mathcal{P}(r+n) \otimes \mathbb{K}[-2r])^{\mathbb{S}_r}$.

There is a natural operad projection map $Tw\mathcal{P} \to \mathcal{P}$, sending $\prod_{r\geq 1}(\mathcal{P}(r+n) \otimes \mathbb{K}[-2r])^{\mathbb{S}_r}$ to zero. At the algebra level this tells us that not only twisted $\mathfrak{g}^\mu$ but the original $\mathfrak{g}$ are naturally $Tw\mathcal{P}$ algebras.

On the other direction, an operad $\mathcal{P}$ is said to be *natively twistable* if there exists an operad morphism $\mathcal{P} \to Tw\mathcal{P}$ such that $\mathcal{P} \to Tw\mathcal{P} \to \mathcal{P}$ is the identity. In this case, the twist of a $\mathcal{P}$-algebra is still a $\mathcal{P}$-algebra.

Lemma 29. *[24, Lemma 16] Let $\mathcal{P}$ be an operad (with an implicit map $\mathsf{hoLie}_1 \to \mathcal{P}$). $Tw\mathcal{P}$ is natively twistable.*

Notice that $Tw\,Tw\mathcal{P}(n) = \prod_{r_1,r_2\geq 0} \left((\mathcal{P}((n+r_1)+r_2) \otimes \mathbb{K}[-2r_1] \otimes \mathbb{K}[-2r_2])^{\mathbb{S}_{r_1}} \right)^{\mathbb{S}_{r_2}} = \prod_{r_1,r_2\geq 0} (\mathcal{P}(n+r_1+r_2) \otimes \mathbb{K}[-2(r_1+r_2)])^{\mathbb{S}_{r_1}\times\mathbb{S}_{r_2}} = \prod_{r\geq 0} \prod_{r_1+r_2=r} (\mathcal{P}(n+r) \otimes \mathbb{K}[-2r])^{\mathbb{S}_{r_1}\times\mathbb{S}_{r_2}}$.

For $p \in Tw\mathcal{P}(n)$, the map $Tw\mathcal{P} \to Tw\,Tw\mathcal{P}$ is defined by the inclusion of $p_r \in (\mathcal{P}(n+r) \otimes \mathbb{K}[-2r])^{\mathbb{S}_r}$ in the factors of $Tw\,Tw\mathcal{P}(n)$ in which $r_1 + r_2 = r$ and zero if $r_1 + r_2 \neq r$.

A.2. Twisting of bimodules

Let $\mathfrak{g}$ and $\mathfrak{h}$ be hoLie_1 algebras. Given an infinity morphism from $\mathfrak{g}$ to $\mathfrak{h}$, we define a *Maurer–Cartan* element of this morphism to be a pair (μ, μ'), where μ is a Maurer–Cartan element of $\mathfrak{g}$ and μ' is a Maurer–Cartan element of $\mathfrak{h}$ such that the hoLie_1 morphism sends μ to μ'.[5]

Let $\mathcal{P}$ and $\mathcal{Q}$ be (dg) operads and $\mathcal{M}$ be a $\mathcal{P} - \mathcal{Q}$ operadic bimodule, that we assume to come with an implicit bimodule morphism $F\colon {\mathsf{hoLie}_1}^{\mathrm{bimod}} \to \mathcal{M}$.

$$\begin{tikzcd} \mathsf{hoLie}_1 \arrow[d, "F_{\mathcal{P}}"] & \circlearrowright & {\mathsf{hoLie}_1}^{\mathrm{bimod}} \arrow[d, "F"] & \circlearrowleft & \mathsf{hoLie}_1 \arrow[d, "F_{\mathcal{Q}}"] \\ \mathcal{P} & \circlearrowright & \mathcal{M} & \circlearrowleft & \mathcal{Q} \end{tikzcd}$$

Let $\mathfrak{g}$ be a $\mathcal{P}$ algebra and let $\mathfrak{h}$ be a $\mathcal{Q}$ algebra. Due to the map F, a morphism of bimodules

$$\begin{tikzcd} \mathcal{P} \arrow[d] & \circlearrowright & \mathcal{M} \arrow[d] & \circlearrowleft & \mathcal{Q} \arrow[d] \\ \operatorname{End}\mathfrak{h} & \circlearrowright & \operatorname{Hom}(\mathfrak{g}^{\otimes\bullet}, \mathfrak{h}) & \circlearrowleft & \operatorname{End}\mathfrak{g} \end{tikzcd} \tag{18}$$

determines a hoLie_1 infinity morphism from $\mathfrak{g}$ to $\mathfrak{h}$. We wish to construct a $Tw\mathcal{P} - Tw\mathcal{Q}$ bimodule $\mathcal{M}$ such that for every (μ, μ'), Maurer–Cartan element of this morphism, there is a natural map of bimodules

$$\begin{tikzcd} Tw\mathcal{P} \arrow[d] & \circlearrowright & Tw\mathcal{M} \arrow[d] & \circlearrowleft & Tw\mathcal{Q} \arrow[d] \\ \operatorname{End}\mathfrak{h}^{\mu'} & \circlearrowright & \operatorname{Hom}(\mathfrak{g}^{\mu\otimes\bullet}, \mathfrak{h}^{\mu'}) & \circlearrowleft & \operatorname{End}\mathfrak{g}^{\mu} \end{tikzcd}$$

We start by giving the description of $Tw\mathcal{M}$ as an $\mathbb{S}$-module.

[5] Evidently for a fixed hoLie_1 infinity morphism, μ determines a unique μ'.

Definition 14. The $Tw\mathcal{P} - Tw\mathcal{Q}$ bimodule $Tw\mathcal{M}$ is the space

$$Tw\mathcal{M}(n) = \prod_{r\geq 0} \left(\mathcal{M}(r+n) \otimes \mathbb{K}[-2r]\right)^{\mathbb{S}_r},$$

with differential d_{Tw}, where $\mathbb{S}_r$ here is the subgroup of $\mathbb{S}_{r+n}$ fixing the last n entries.

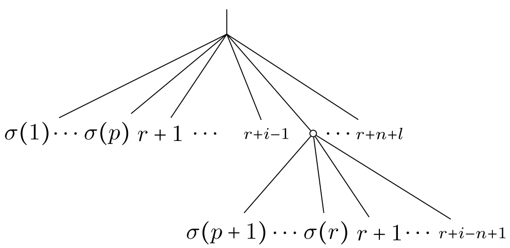
We need now to clarify the left and right actions, as well as the differential.

Let $m \in Tw\mathcal{M}(n) = \prod_{r\geq 0} \left(\mathcal{M}(p+r) \otimes \mathbb{K}[-2r]\right)^{\mathbb{S}_r}$. We denote by m_r it's projection in $\left(\mathcal{M}(p+r) \otimes \mathbb{K}[-2r]\right)^{\mathbb{S}_r}$ and for $p \in TwP$, $q \in Tw\mathcal{Q}$ we use a similar notation p_r, q_r.

The right $Tw\mathcal{Q}$ action on $\mathcal{M}$ is defined in the following way: Let $m \in Tw\mathcal{M}(n)$ and $q \in Tw\mathcal{Q}(l)$.

$$(m \circ_i q)_r := \sum_{p=0}^{r} \sum_{\sigma\in Sh_{p,r-p}} \gamma_{i,\sigma}(m_p, q_{r-p}),$$

where $Sh_{p,r-p} \subset \mathbb{S}_r$ are the $(p, r-p)$ shuffles $\gamma_{i,\sigma}$ is the composition given by the following tree



We write $d_{Tw} = d_{\mathcal{M}} + d_R + d_L$, where $d_{\mathcal{M}}$ is the differential induced by the differential in $\mathcal{M}$.

The Lie Algebra $\mathcal{L}_{\mathcal{Q}}$ acts on $(Tw\mathcal{M}, d_{\mathcal{M}})$ by operadic derivations. The proof of this is the same as [5, Proposition 3.2].

The Lie Algebra $Tw\mathcal{Q}(1)$ acts on the right on $Tw\mathcal{M}$ by

$$m \cdot q = \sum_{i=1}^{n} m \circ_i q,$$

where $m \in Tw\mathcal{M}(n)$ and $q \in Tw\mathcal{Q}$.

Multiplying by a minus sign, the previous right action becomes a left action, thus inducing a dg Lie algebra action $\mathcal{L}_{\mathcal{Q}} \ltimes Tw\mathcal{Q}(1) \circlearrowright (Tw\mathcal{M}, d_{\mathcal{M}})$.

The map $F_{\mathcal{Q}} \colon \mathsf{hoLie}_1 \to \mathcal{Q}$ gives us a Maurer–Cartan element in $\mathcal{L}_{\mathcal{Q}}$. Due to Lemma 28 we can twist $(Tw\mathcal{M}, d_{\mathcal{M}})$ with respect to this Maurer–Cartan element, giving us the module $(Tw\mathcal{M}, d_{\mathcal{M}} + d_R)$.

There is an obvious left $\mathcal{P}$ action on $(Tw\mathcal{M}, d_{\mathcal{M}})$, using the original $\mathcal{P}$ action on $\mathcal{M}$. It is easy to see that $\mathcal{P}$ also acts on $(Tw\mathcal{M}, d_{\mathcal{M}} + d_R)$.

Indeed, the equation of compatibility with the differential

$$(d_{\mathcal{M}} + d_R)(p \circ_i m) = d_{\mathcal{P}} p \circ_i m + (-1)^{|p|} p \circ_i (d_{\mathcal{M}} + d_R) m$$

is equivalent to $d_R(p \circ_i m) = (-1)^{|p|} p \circ_i d_R m$, and the associativity axiom involving the left and right actions of an operadic bimodule, together with the fact that d_R uses right compositions ensures that this equality holds for all $p \in Tw\mathcal{P}$ and $m \in Tw\mathcal{M}$.

The map $F\colon \mathsf{hoLie}_1{}^{\mathrm{bimod}} \to \mathcal{M}$ gives us a Maurer–Cartan element in $\prod_r \mathrm{Hom}_{\mathbb{S}_r}(\mathbb{K}[2r], \mathcal{M}(r)) = \prod_r (\mathcal{M}(r) \otimes \mathbb{K}[-2r])^{\mathbb{S}_r} = Tw\mathcal{M}(0)$. Twisting with respect to this Maurer–Cartan element we obtain a left action of $Tw\mathcal{P}$ on $(Tw\mathcal{M}, (d_{\mathcal{M}} + d_R) + d_L)$.

Using a similar argument of compatibility with the differential, we see that $Tw\mathcal{Q}$ acts on the right on $(Tw\mathcal{M}, d_{\mathcal{M}} + d_R + d_L) = (Tw\mathcal{M}, d_{Tw})$. The associativity of the left $Tw\mathcal{P}$ and right $Tw\mathcal{Q}$ actions is clear and so we finished the construction of the bimodule $Tw\mathcal{M}$.

A.2.1. The action on $\mathrm{Hom}(\mathfrak{g}^{\mu\otimes\bullet}, \mathfrak{h}^{\mu'})$

As described in the beginning of the section, we wish now to construct a map of bimodules

$$\begin{array}{ccccc} Tw\mathcal{P} & \circlearrowright & Tw\mathcal{M} & \circlearrowleft & Tw\mathcal{Q} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{End}\,\mathfrak{h}^{\mu'} & \circlearrowright & \mathrm{Hom}(\mathfrak{g}^{\mu\otimes\bullet}, \mathfrak{h}^{\mu'}) & \circlearrowleft & \mathrm{End}\,\mathfrak{g}^{\mu} \end{array} \tag{19}$$

The two outer maps are the maps induced by the usual twisting of operads. For the main map, informally we do the usual procedure of inserting the Maurer–Cartan element on the non-symmetric slots. Formally, if $m \in Tw\mathcal{M}(n)$,

$$m(x_1, \ldots, x_n) = \sum_{r=0}^{\infty} \frac{1}{r!} m_r(\mu, \ldots, \mu, x_1, \ldots, x_n), \quad x_i \in \mathfrak{g},$$

where we identify an element of $\mathcal{M}$ (resp. $Tw\mathcal{M}$) with its image in $\mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ (resp. $\mathrm{Hom}(\mathfrak{g}^{\mu}, \mathfrak{h}^{\mu'})$).

The only thing that remains to be checked is the commutativity of the left and right squares, as well as the compatibility with the differential of the central vertical map. Let as call $l_r^{\mathcal{P}}$ the image of the r-ary generator of hoLie_1 in $\mathcal{P}$, and we define similarly $l_r^{\mathcal{Q}}$ and $l_r^{\mathcal{M}}$.

Due to the original bimodule morphism (18), the right square is trivially commutative and the commutativity of the left square is a simple consequence (18) together with the hypothesis $\sum_r \frac{1}{r!} l_r^{\mathcal{M}}(\mu, \ldots, \mu) = \mu'$. Also, thanks to this equation, when we evaluate $d_L m$ in $\mathrm{Hom}(g^{\mu}, h^{\mu'})$, mu' will replace the Maurer–Cartan element of $Tw\mathcal{M}$.

We wish to show that for $m \in Tw\mathcal{M}(n)$ and $x_1, \ldots, x_n \in \mathfrak{g}^{\mu}$,

$$d^{\mu'}(m(x_1,\dots,x_n)) = (d_{\mathcal{M}}m + d_L m + d_R m)(x_1,\dots,x_n)$$
$$+\sum_{i=1}^{n}(-1)^{|m|+|x_1|+\dots+|x_{i-1}|}m(x_1,\dots,d^{\mu}x_i,\dots,x_n).$$

Keep in mind in the following computations that m_r has r non-symmetric inputs and n symmetric inputs, whereas $l_r^{\mathcal{P}/\mathcal{Q}}$ will be of arity r but will have $r-1$ non-symmetric inputs. Expanding the right hand side we get

$$\sum_{r\geq 0}\frac{1}{r!}d_{\mathcal{M}}m_r(\mu,\dots,\mu,x_1,\dots,x_n)$$
$$+\sum_{k\geq 2,r\geq 0}\frac{1}{(k-1)!r!}l_k^{\mathcal{P}}(\mu',\dots,\mu',m_r(\mu,\dots,\mu,x_1,\dots,x_n))+$$
$$-\sum_{r\geq 0,k\geq 2}\frac{r}{r!k!}m_r(l_k^{\mathcal{Q}}(\mu,\dots,\mu),\dots,\mu,x_1,\dots,x_n)+$$
$$-\sum_{r\geq 0,k\geq 2}\frac{(-1)^{|x_1|+\dots+|x_{i-1}|}}{r!(k-1)!}m_r(\mu,\dots,\mu,x_1,\dots,l_k^{\mathcal{Q}}(\mu,\dots,\mu,x_i),\dots,x_n)+$$
$$-\sum_{i=1}^{n}\sum_{r\geq 0}\frac{(-1)^{|x_1|+\dots+|x_{i-1}|}}{r!}m_r(\mu,\dots,\mu,x_1,\dots,dx_i,\dots,x_n)+$$
$$\sum_{i=1}^{n}\sum_{r\geq 0,k\geq 2}\frac{(-1)^{|x_1|+\dots+|x_{i-1}|}}{r!(k-1)!}m_r(\mu,\dots,\mu,x_1,\dots,l_k^{\mathcal{Q}}(\mu,\dots,\mu,x_i),\dots,x_n).$$

Using the Maurer–Cartan equation, the third summand simplifies to

$$\sum_{r\geq 0}\frac{r}{r!}m_r(d\mu,\mu,\dots,\mu,x_1,\dots,x_n),$$

therefore, the first, third and fifth summands add up to $d(m(x_1,\dots,x_n))$, while the fourth and sixth summands cancel out, leaving us precisely with $d^{\mu'}(m(x_1,\dots,x_n))$.

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